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NONLINEAR PROBLEMS OF THE THEORY OF HETEROGENEOUS SLIGHTLY CURVED SHELLS

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

An account is given in the book of the variational method of the solution of physically and geometrically nonlinear problems of the theory of heterogeneous slightly curved shells. Examined are the bending and supercritical behavior of plates and conical and spherical cupolas of variable thickness in a temperature field, taking into account the dependence of the elastic parameters on temperature. The bending, stability in general and load-bearing capacity of flexible isotropic elastic-plastic shells with different criteria of plasticity, taking into account / compressibility and hardening are studied. The effect of the plastic heterogeneity caused by heat treatment, surface work hardening and irradiation by fast neutron flux is investigated. Some problems of the dynamic behavior of flexible shells are solved. Calculations are performed in high approximations. Considerable attention is given to the construction of a machine algorithm and to the checking of the convergence of iterative processes.

The book is intended for scientific and technical personnel who are engaged in the problems of the theory of shells. It can be useful to instructors, graduate students and the students of colleges.

EDITOR'S COMMENTS

Research on the effect of the physical and geometric non-linearity on the behavior of slightly curved shells is a current and sufficiently complex theoretical problem. Recently considerable attention has been given to the development of different methods of calculation which, in supplementing each other, make it possible to satisfy completely the demands of practice.

In the book proposed to the reader an account is given of the variational method of the solution of nonlinear problems of the theory of slightly curved shells, especially convenient in cases of elastic and plastic heterogeneities which appear as a result of those or other physical effects on the material. Starting points in the construction of this method were works on the variation equations of the mixed type of N. A. Alumyae, Kh. M. Mushtari, K. Z. Galimov and according to the method of the variable paremeters of elasticity of I. A. Birger. Largely used is the experience accumulated in the Kharkov branch of the Institute of Mechanics of the Academy of Sciences of the Ukrainian SSR (now the branch of the Institute of Technical Thermophysics of the Academy of Sciences of the Ukrainian SSR) of the use of variational methods for the solution of linear problems of the applied theory of elasticity. The construction of an effective algorithm and the implementation of calculation on a computer are examined The work carried out by the author in this direction made it possible to calculate in high approximations. This again made it possible to be convinced of the fact that the variational methods can give not only qualitative but also good quantitative results.

Considerable attention is given in the book to the direct study of the bending of slightly curved shells of rotation and circular plates with a separate and joint account of the geometric and physical nonlinearities. An analysis of the effect of the variability of thickness, plastic heterogeneity, temperature field and elastic heterogeneity caused by them is given. By the method of direct determination of critical states new data on the stability of shells in general are obtained. Investigated are the greatest loads which the flexible elasto-plastic slightly curved shell can receive. The nonlinear problem of the behavior of a shell under the action of a dynamic load is examined.

The data presented by the author develop the nonlinear theory of shells and can be used in the practice of design.

CHAPTER I

STATE OF THE PROBLEM

The rapid development of the nonlinear theory of heterogeneous shells is caused by the urgent necessities of practice. The wide application of new materials and the use of shells in unusual conditions with a great intensity of external actions urgently require the further perfection of methods of calculation.

In the book geometrically and physically nonlinear problems for homogeneous and heterogeneous isotropic shells within limits of the accuracy of the technical theory are examined. This direction covers a wide range of questions connected with the strength, stability and load-bearing capacity of slightly curved shells and plates.

Calculation of the geometric nonlinearity is necessary not only for the search of critical loads which determine the boundary of the stability of the structural element, but also for the precise calculation of stresses in the subcritical area. The joint calculation of the geometric and physical nonlinearity makes it possible, without introducing the concept of a plastic-rigid body, in more detail to study the interconnection of load-bearing capacity and stability. The development of methods of calculation of heterogeneous flexible shells makes it possible to establish the features of their behavior and utilize the obtained information in the designing.

The brief survey given below is not meant to be complete, and a number of trends is not reflected in it. Specifically, the studies of the nonlinear concentration developed by the school of G. N. Savin, thermoplastic problems, and problems of stability in particular are not dealt with. The selection of the material is dictated by its connection with the basic content of the book and the method developed in it. Thus, the known methods of the solution of problems are estimated from the viewpoint of their use in the case of a heterogeneous body.

1. The Methods of the Solution of Geometrically Nonlinear Problems

An outline on the history of development of the theory of flexible plates and shells is given in A. S. Vol'mir's work [22], and therefore let us note only the main development stages of this theory.

The bases of the theory of flexible plates were laid by the well-known Russian Scientist I. G. Bubnov. T. Karman gave the general equations for the plates. In 1949 V. Z. Vlasov obtained the system of differential equations of the theory of flexible slightly curved shells. Nonlinear equations of the axisymmetric deformation of the flexible slightly curved shells of rotation were derived by D. Yu. Panov and V. I. Feodos'yev. The system of differential equations for flexible slightly curved heterogeneous anisotropic shells was obtained in Cartesian coordinates by I. Stavskiy [135], and for plates of small deflection - N. A. Lobkova and L. A. Il'in [74]. A great contribution to the substantiation and development of the geometrically nonlinear theory was made by S. A. Alekseyev, A. S. Vol'mir, I. I. Vorovich, K. Z. Galimov, Kh. M. Mushtari, V. V. Novozhilov, A. V. Pogorelov, and V. I. Feodos'yev.

In the initial period of development of the nonlinear theory, the basic method of the solution of the problems, just as in other fields of nonlinear mechanics, was the small parameter method. However, it was soon clarified that the region of its use was limited. Over a prolonged period of time a large part of the problems was solved by P. F. Papkovich's method. According to this method the deflection should be searched in the form of the sum of the products of the coordinate functions, which satisfy all the boundary conditions, by the indefinite parameters. By substituting such a sum into the equation of the compatibility of deformations in the middle surface (linear relative to the function of stresses) and integrating it analytically, we find the expression of the function of stresses in terms of the unknown parameters. This makes it possible further by the Bubnov-Galerkin method to solve the equation of equilibrium in the projection on the normal to the middle surface. By precisely this way (and, for the most part, in first approximations) many problems were solved by A. S. Vol'mir, V. I. Feodos'yev, E. I. Grigolyuk, M. A. Koltunov and other researchers. According to P. F. Papkovich's method, in general it is possible to obtain the system of algebraic equations of the third power. The use of a computer permitted using P. F. Papkovich's method in high approximations.

K. Z. Galimov [26] proposed the method opposite to P. F. Papkovich's method. By varying only the function of stresses and assigning it in series after substitution of the function into the equation of equilibrium we obtain the expression of deflection by ϕ. This makes it possible further by the Bubnov-Galerkin method to solve the equation of compatibility. By using for calculating one object the methods of P. F. Papkovich and K. Z. Galimov, it is possible to obtain a two-sided estimate of the solution.

It should be noted that in the case of the heterogeneity of the material or nonuniform shell, the equations of equilibrium and compatibility will include the first and second derivatives of rigidities - functions of the coordinates - and the structure of these equations will be noticeably complicated. This will make the analytical solution of the equation of compatibility (equilibrium) impossible and will make the numerical solution of the equation of equilibrium (compatibility) by Bubnov-Galerkin method difficult.

Having somewhat greater possibilities is the direct use of the Galerkin method to the system of equations of flexible slightly curved shells at which the deflection w and function of stresses ϕ are assigned independently in the form of a series with undetermined coefficients. This makes it possible to reduce the problem to the system of quadratic algebraic equations. A certain inconvenience of this method is the need for the assignment of coordinate functions which accurately satisfy all the boundary conditions and the algorithmic difficulties in the case of the heterogeneity of the material.

Among the methods of algebraization (reduction of differential equations to algebraic) of geometrically nonlinear problems, the net-point method occupies an important place. The basic results of applying this method belong to M. S. Kornishin [63]. Numerical data for diverse cases of the bending of the plates and slightly curved shells of constant thickness, calculated by the net-point method, are given in a monograph [64]. The net-point method is widely used also in works of American scientists [7, 83, 115] and others.

The net-point method reduces to systems of quadratic algebraic equations with slightly filled matrices of coefficients. The automation of the composition and solution of such systems is difficult. Just as when using the method of Bubnov-Galerkin, calculation of the heterogeneity of the material complicates the algorithm.

Diverse variants of methods of the Bubnov-Galerkin type and successive approximations are developed in I. V. Svirskiy's work [97]. Some results for plates and panels of rectangular form are obtained when using the method of V. Z. Vlasov, which consists in the reduction of partial differential equations to the system of ordinary diffential equations [91].

A unique means of the solution of nonlinear boundary value problems for systems of ordinary differential equations was proposed by N. V. Valishvili [16, 17]. Its method does not require the approximate transition to algebraic equations but consists of the combination of the usual iterative method of the solution of systems of transcendental equations with the repeated solution to the Cauchy problem. This made it possible to consider the axisymmetric deformation of spherical slightly curved shells whose characteristic, the load-deflection, has a complex form. Earlier load - deflection graph with loops was obtained by Meskoll [78]. Similar dependences appear under the assumption of axial symmetry in shells with the lift above the plane of approximately more than five thicknesses. However, such shells lose stability in axially nonsymmetric forms [22]. The value of N. V. Valishvili's method consists in the fact that the accuracy of the results is determined only by the accuracy of the solution of the Cauchy problem.

The method of calculation of the knocking of shells, based on the geometric analysis of the bending of surfaces, was proposed by A. V. Pogorelov [92].

Ritz's method, which is rather widely used for the solution of linear problems, is hardly used for geometrically nonlinear problems. The solution by this method of nonlinear problems, formulated by means of a mixed type of the variation equation, comprises the basic contents of the book. Some basic information about this approach is given in Section 4 of this chapter.

Without discussing the methods of the solution of the systems of algebraic equations of geometrically nonlinear problems (a survey of these methods is given in work [97]), let us note that the effect of great deflections is investigated mainly for plates and slightly curved shells of constant thickness with a uniform isotropic material.

Devoted to an account of the temperature field, heterogeneity, anisotropy and variability of the thickness in a nonlinear formulation are the single works fulfilled for the most part in the first approximation.

2. Physically Nonlinear Plates and Slightly Curved Shells

Surveys of the examinations of the bending of uniform plates and shells with nonlinear connections between the deformations and stresses [71, 103] attest to the fact that in the majority of the cases the deformation theory of plasticity is used. This theory is comparatively simple, and although it is precise only for the direct means of loading, recently it was possible to justify its use in a wider region (see Chapter VI). Furthermore, it is necessary to consider that at present still there are no theories which are more reliably checked than the deformation theory [57].

For calculating the plates is widely utilized the net-point method in conjunction with the method of elastic solutions [104]. A large role in the development of the theory of elasto-plastic plates is played by works of A. S. Grigor'yev [32-36].

Numerical results on the bending of slightly curved elasto-plastic shells (even in a geometrically linear formulation) are hardly obtained. Interesting is M. S. Ganeyeva's work [27], where by the net-point method with small nonlinearity (after Kauderer) circular plates and slightly curved spherical shells under different loads and conditions of the attachment are calculated.

The equations of physically nonlinear slightly curved shells of constant thickness, according to the method of supplementary loads, were derived by I. A. Tsurpal and N. A. Shul'ga [116]. However, in derivation it is assumed that Poisson's ratio does not depend on state of the strain at the point. Used is the physically nonlinear theory of Kauderer [54], the relations of which are fulfilled in the narrow zone of deformations.

A numerical experiment in the analysis of the different means of linearization was conducted by Ye. M. Kuznetsova [65].

In the example of cylindrical bending of the plate, she established that, although the Newton's method leads to a more rapid convergence than do the methods of secants and chords, for it the best initial approximation is necessary. For the implementation of calculation by a computer, the most convenient is the method of variable parameters of elasticity.

The theory of the plasticity of heterogeneous bodies is developed basically in the works of V. Ol'shak [132] and his students. A survey on this problem can be found in works [87, 88]. For the first time the study of the bending of plastically heterogeneous circular and circular plates is conducted in works [133]. A number of problems of the bending of plastically heterogeneous circular plates was solved by A. S. Grigoryev [34, 36]. The method of the elastic solutions in conjunction with finite-difference method is used for calculating a square heterogeneous plate in work [105].

3. Flexible Nonlinear Elastic Plates and Slightly Curved Shells

In order to explain at which relative thickness it is necessary to consider the effect of plastic deformations on stability, let us assume that $\sigma_s/E = 2 \cdot 10^{-3}$ and use formula (16.9) [22] for the breaking stress of the complete spherical shell $\sigma_{кр} \approx 0.6Eh/R$. When $\sigma_{кр} > \sigma_s$ we have $h/R > 1/300$. Since the upper critical loads of the slightly curved shells are of the order $q_в$ of the complete spherical shell, the obtained estimate can also be attributed to the slightly curved shells. Shells of this thickness are widely utilized in practice, and therefore in many instances in the calculation for stability it is necessary to have solutions of the physically and geometrically nonlinear problem. It is shown below that if with plastic deformations the geometric nonlinearity is not considered, it is possible to draw false (not only quantitatively, but also qualitatively) conclusions relative to the behavior of the shell under a load.

The physically nonlinear problems with the deflection comparable with the thickness have attracted the attention of researchers for a long time, but a large part of the works appeared after 1960. The summer school on this problem conducted in the city of Tartu (1966) showed the importance of the subject and made it possible to reveal the problems least studied. Specifically, it has been established that considerable attention should be given to the development of methods of calculation and to the analysis of the behavior of elasto-plastic flexible slightly curved shells.[1] Differential equations of such shells

[1]In not one of the 32 works mentioned in Yu. R. Lepik's survey [72] is the calculation of the flexible elasto-plastic slightly curved shell reduced to a number.

of constant thickness with the incompressible (not only in the plastic but also the elastic region) material were obtained by N. F. Yershov [38]. Using the principle of possible displacements, N. F. Yershov examined a flexible rectangular plate [39]. Somewhat earlier P. A. Lukash [75] proposed the energy method of the calculation of shells of an incompressible material.

By the Ritz method, in the first approximation, B. V. Ponomarev [93] calculated a square plate of an incompressible material. The potential of the elastic body was assigned in the form of a polynomial according to even degrees of deformation intensity. This made it possible to compute accurately the coefficients of the resolving algebraic equation.

The finite-difference method with a large quantity of points (approximately 100) and the process of interations according to Newton-Rafson are used in work [106]; as a condition of plasticity Tresca's criterion with strengthening is accepted. For calculating slightly curved panels rectangular in design, K. Soonets [101] used the method of elastic solutions in conjunction with the net-point method and the condition of incompressibility only in the plastic region.

V. V. Sorokin [102] combines the methods of variable parameters of elasticity and Bubnov-Galerkin, solving at each stage the problem of the loading of a linearly elastic nonuniform shell. In the algorithm the rigidities are approximate by the sum of the products of orthogonal functions. The calculation of a circular cylindrical shell is given. Deflection is represented by three terms of the series.

The step-by-step method [125] is used by V. I. Feodos'yev and S. M. Chernyakov [112] for calculating an uncurved spherical nonuniform shell of an incompressible material with the retention of three terms of the series for deflection and meridional displacement. The leading paremeter is taken as the deflection under the force applied in the center.

An analysis of the literature on the examination of geometrically and physically nonlinear plates and shells showed the need for the development of the theory of heterogeneous slightly curved shells in a nonlinear formulation. Of interest is the construction of method and algorithm convenient for implementation by the computers, which will make it possible to obtain in high approximations solutions to the problems with elastic and plastic heterogeneity, the complete calculation of the compressibility, the variability of the thickness, and the assigned form of the connection between the intensities of deformations and stresses. Presented below is one of the possible ways for achieving this goal: the use of a variational equation of the mixed type, the method of variable parameters of elasticity and methods of Ritz and Newton-Kantorovich.

4. Mixed-Type Variational Equation in the Nonlinear Theory of Shells

The possibility of the direct use Ritz's method for the solution of nonlinear problems of the theory of slightly curved shells is closely connected with the construction of functionals or variational equations in which all the unknown functions should undergo variation. Various forms of variational equations, based on the variational principles of mechanics which differ mainly by the selection of the varied functions, are obtained in the works of L. Ya. Aynola [3], K. Z. Galimov [25], Kh. M. Mushtari, K. Z. Galimov [81], Reissner [134], R. Z. Murtazin, I. G. Teregulov [82], and others.

The type of variational equation convenient for applications is of the mixed-type equation relative to the deflection and function of the stresses. Apparently, for the first time such an equation for the nonlinear problem is given by N. A. Alumyae [4]. For flexible plates it was constructed by L. I. Balabukh, and for slightly curved shells it is given in work [81]. The generalization of this equation, connected with an account of the initial chamber and different external actions is given in works [45, 47]. These equations correspond to the variation principle, which is intermediate between the principles of Lagrange and Castigliano, since in them the displacement and stresses in median surface are varied. A feature of such equations consists in the fact that the functional, which is under the sign of the variation, is not equal to the total energy of system, although a variation in the functional coincides with a variation in the total energy. To solve the physically nonlinear problems it is especially important that in the construction of the mixed-type equation it is not necessary to vary the integrals in terms of the thickness of mechanical parameters of the material determined by the state of strain at the point, i.e., in the final analysis - depending on the unknown functions.

The indicated and other positive properties of mixed-type equations became clear when using Ritz's method in linear problems [44, 46, 113]. The experience obtained in their solution made the assumption plausible about the fact that in nonlinear cases the use of Ritz's method is effective. In this connection let us give L. M. Kachanov's proposition [58]: "At one time it seemed that the development of electronic digital computers will make it possible to be limited by the structurally simplest computing methods, in particular, will make it possible to solve boundary value problems by the net-point method. However, the experience accumulated did not confirm this for equations in partial derivatives and showed that other methods, variational, are more effective."

The comparative estimate of the Ritz and net-point methods [80] is constantly being reexamined, and final judgment on this question is hardly possible; however, it is clear that the development and use of variational methods, just as the net-point method, is useful.

The theoretical substantiation of the use of variational methods in nonlinear problems is most fully given in S. G. Mikhlin's work [79], where questions of the selection of coordinate functions, connected with stability of Ritz's algebraic systems and some methods of their solution are examined.

One of the basic conditions in the effective use of the Ritz's method for obtaining results in high approximations is the complete automation of the calculations, which include the construction and solution of the Ritz's systems. The appearance of a digital computer and the machine implementation of these processes made it possible to obtain results acceptable in accuracy by the Ritz's method. The importance of the calculation of coefficients of algebraic equations by the computer was noted in the article of A. S. Vol'mir [21]. The first in this field were sources [44, 99]. The means utilized in the calculation of coefficients of the equations [44] (precise calculation of the integrands in nodes and the numerical integration over the region) makes it possible to solve the heterogeneous and physically nonlinear problems simply. The polynomial representation of all the values entering into the calculation [99] is not always convenient and frequently is also inexpedient.

Experience showed that in the calculation of plates and shells by the computer, it is possible to obtain solutions with the necessary degree of accuracy [20]. The execution of calculations of flexible shells in high approximations is caused, in particular, by the growing interest in the pattern of the stressed state in the supercritical region [15].

The use of variational methods has obtained wide promise in connection with the appearance of the R-function theory created by V. L. Rvachev [95]. The use of these functions makes it possible to construct a solution in the form of a series which accurately satisfies all the boundary conditions on the boundary of a region of complex form.

The method of calculation proposed in this book is convenient for calculating the physical nonlinearity and heterogeneity of a material and from a systematic point of view is the single method of the solution of the geometrically and physically nonlinear problems.

CHAPTER II

THEORY OF FLEXIBLE PHYSICALLY NONLINEAR HETEROGENEOUS SLIGHTLY CURVED SHELLS

1. Statement of the Problem. Assumptions

Let us consider the shell (Fig. 1) the middle surface of which is limited by the closed line Γ. Let us refer the middle surface to the orthogonal system of curvilinear coordinates α, β, without requiring the agreement of coordinate lines with lines of the principal curvatures. Let us assume that Lamé's parameters A and B and the radii of curvature R_1', R_{12} and R_2' of the middle surface are continuous together with their first-order derivatives of function α, β. Unlike the main radii R_1 and R_2, the prime denotes radii of curvature in directions α, β.

Let us designate the coordinate normal to the middle surface by the letter γ. Positive directions of the coordinates are shown on Fig. 2. By plotting in positive and negative directions of the γ-coordinate on half-thickness h/2, we form the body of the shell. We consider

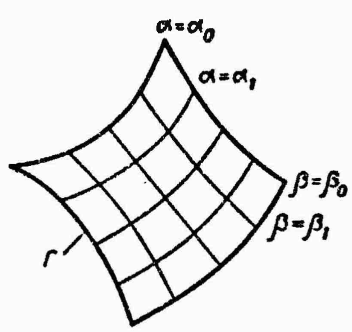


Fig. 1.

that the function $h(\alpha, \beta)$ does not have discontinuities of the first kind, and the maximum thickness $h_{max} \equiv h_0$ is considerably less than the smallest main radius of curvature R_{min}. Without determining more accurately the possible values of h_0/R_{min}, we assume that this ratio can be disregarded in comparison with unity. Shells with such a ratio h_0/R_{min} are called thin.

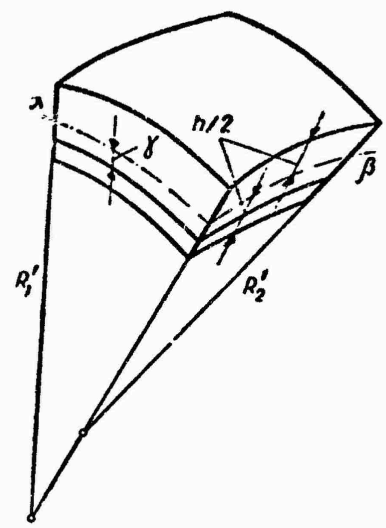


Fig. 2.

Let us designate displacements in directions α, β, and γ respectively by u, v, and w. By virtue of a comparative flexibility of the shell in direction γ, the deflection w is commensurable with the thickness h, so that the relation $w/h \ll 1$ is not fulfilled. This fact leads to the so-called geometric nonlinearity of the resolving equations. However, as will be shown below, not the value w/h but derivatives of deflection in terms of the coordinates enters directly into the nonlinear terms. Therefore the relation $w \sim h$ only indirectly determines the consequences of the different rigidity of the shell in different directions. The shells, with the calculation of which consider the geometric nonlinearity, are called flexible.

We take all components of the displacement considerably less than the characteristic dimension of the shell in the middle surface. Let us introduce the initial chamber $w_0(\alpha, \beta)$. This function assigns the inadequacies of the initial form of the middle surface before deformation. Let us assume that it is continuous, together with its first and second derivatives in terms of α, β and is commensurable with the thickness.

Deformations in the middle surface ε_1, ε_2, ε_{12} are assumed to be negligible in comparison with unity. However, this does not mean that the connection between the stresses and strains should be linear. We characterize the changes in the curvatures by the parameters κ_1, κ_2, and κ_{12}.

Used in this work is the technical theory of shells based on assumptions about the fact that the effect of displacements u and v on parameters of the change in the curvature and first-order derivatives of deflection and function of stresses with factor $\frac{AB}{R_1R_2}$ can be disregarded in comparison with higher derivatives.

The material from which the shell is made is considered to be isotropic (resistance to deformation in any direction equally), but heterogeneous so that the moduli of expansion E, shear G, volume strain K, coefficient of lateral deformation ν, yield point σ_s, and coefficient of linear expansion t are functions of α, β, and γ. For an account of the theory the very fact of the heterogeneity of the material is especially important. The methods of its assignment and the possibility of a change in the heterogeneity in the process of deformation will be discussed further.

The strength of a material is characterized by the form of the connections between the strains and stresses σ_1, σ_2, and σ_{12}. We assume in general that these connections are nonlinear. The problems in which considered is the indicated phenomenon are called physically nonlinear. Nonlinear properties with small elastic deformations are possessed, for example, by high-alloy steels, cast iron and a number of other materials.

Further everywhere it is assumed that physical parameters of the material E, G, K, and ν are single-valued functions of the point and state of strain in it. Such a position always takes place when using the nonlinear theory of elastic (reversible) deformations or, coinciding with it in the absence unloading, the theory of small elasto-plastic deformations. We will characterize the state of strain of the point as the volume strain ε_0 and strain intensity ε_1.

The theory given below is based on the hypothesis of direct normals, according to which the points, which were located before deformation on the normal to the middle surface, remain on it after deformation. Actually, this hypothesis requires neglecting the shears in normal sections in comparison with the angles of turn of the normals.

In the usual formulation of the hypothesis of direct normals, this denotes also the retention of length of the normal element, which is equivalent to the neglecting of deformations $\varepsilon_{\gamma\gamma}$ in comparison with unity. However, entering into expressions for ε_0 and ε_1 is this component of the vector of deformations along with the others commensurable with it, and in these expressions the value $\varepsilon_{\gamma\gamma}$ cannot be disregarded. For determining $\varepsilon_{\gamma\gamma}$ (in the calculation of ε_0 and ε_1), we use the condition of plane stressed state $\sigma_{\gamma\gamma} = 0$.

The shell can be loaded by forces distributed on the edge which are transverse Q^0_n, normal T^0_n and tangent to the edge T^0_l in the middle surface and by the bending moment M^0_n. The load Z distributed on the surface is considered to be normal to the middle surface. In the case h ≠ const we disregard the tangential components of the pressure normal to surfaces $\gamma = \pm\frac{1}{2}h(\alpha, \beta)$, since derivatives $\frac{1}{A}\cdot\frac{\partial h}{\partial \alpha}$, $\frac{1}{B}\cdot\frac{\partial h}{\partial \beta}$ are considered to be small in comparison with unity. We will consider also body forces in the middle surface with the potential U and temperature field T, which depends on the three coordinates.

2. Fundamental Principles

A detailed account of the theory of shells is the subject of many monographs and enormous periodical literature, and therefore fundamental principles are given here without derivation.

In accordance with the hypothesis of the direct normals, the deformations of a shell at an arbitrary point have the form

$$e_1^{(\gamma)} = \varepsilon_1 + \gamma\varkappa_1, \quad e_2^{(\gamma)} = \varepsilon_2 + \gamma\varkappa_2, \quad e_{12}^{(\gamma)} = \varepsilon_{12} + 2\gamma\varkappa_{12}. \tag{2.1}$$

Deformations in the middle surface [22]

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{A}\cdot\frac{\partial u}{\partial\alpha} + \frac{1}{AB}\cdot\frac{\partial A}{\partial\beta}v - \frac{w}{R_1} + \frac{1}{2}\left(\frac{1}{A}\cdot\frac{\partial w_1}{\partial\alpha}\right)^2 - \\
&\quad - \frac{1}{2}\left(\frac{1}{A}\cdot\frac{\partial w_0}{\partial\alpha}\right)^2; \\
\varepsilon_2 &= \frac{1}{B}\cdot\frac{\partial v}{\partial\beta} + \frac{1}{AB}\cdot\frac{\partial B}{\partial\alpha}u - \frac{w}{R_2} + \frac{1}{2}\left(\frac{1}{B}\cdot\frac{\partial w_1}{\partial\beta}\right)^2 - \\
&\quad - \frac{1}{2}\left(\frac{1}{B}\cdot\frac{\partial w_0}{\partial\beta}\right)^2; \\
\varepsilon_{12} &= \frac{A}{B}\cdot\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + \frac{B}{A}\cdot\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + \frac{2w}{R_{12}} + \\
&\quad + \frac{1}{A}\cdot\frac{\partial w_1}{\partial\alpha}\cdot\frac{1}{B}\cdot\frac{\partial w_1}{\partial\beta} - \frac{1}{A}\cdot\frac{\partial w_0}{\partial\alpha}\cdot\frac{1}{B}\cdot\frac{\partial w_0}{\partial\beta},
\end{aligned} \tag{2.2}$$

where $w_1 = w + w_0$.

Expressions (2.2) include the quadratic terms which consider the geometric nonlinearity caused by the fact that the squares of turning angles of the normal to middle surface are commensurable with elongations and shears. Formulas (2.2) are obtained from more general formulas (VI.37) [84], if we consider that the angles of turning are negligible in comparison with unity.

The parameters of the change in curvature of the middle surface are:

$$\left.\begin{aligned}
\varkappa_1 &= -\frac{1}{A}\cdot\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{1}{AB}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial w}{\partial\beta}=\\
&= -w^{\bullet}_{\alpha\alpha};\\
\varkappa_2 &= -\frac{1}{B}\cdot\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial w}{\partial\beta}-\frac{1}{AB}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}=\\
&= -w^{\bullet}_{\beta\beta};\\
\varkappa_{12} &= -\frac{1}{A}\cdot\frac{\partial}{\partial\alpha}\cdot\frac{1}{B}\cdot\frac{\partial w}{\partial\beta}+\frac{1}{AB}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}=-w^{\bullet}_{\alpha\beta}.
\end{aligned}\right\} \qquad (2.3)$$

Expressions (2.3) have an approximate nature even within limits of the linear theory. In precise formulas (12.53) [22] terms $\frac{u}{R_1}$, $\frac{v}{R_2}$ are also taken into account. The deletion of these terms is one of the assumptions of the technical theory of shells which is utilized in this work. In the case of plates and also in the examination of the axisymmetric deformation ($v \equiv 0$) of cylindrical and coniform shells ($R_1 \equiv \infty$), the indicated terms are accurately equal to zero. In expressions (2.3) there are no nonlinear terms in connection with the disregard of values of turning angles in comparison with unity.

Let us present the relationships between stresses and strains in the form

$$\left.\begin{aligned}
\varepsilon_1^{(\gamma)} &= \frac{1}{E}(\sigma_1-\nu\sigma_2)+tT;\\
\varepsilon_2^{(\gamma)} &= \frac{1}{E}(\sigma_2-\nu\sigma_1)+tT;\\
\varepsilon_{12}^{(\gamma)} &= 2\,\frac{1+\nu}{E}\,\sigma_{12}.
\end{aligned}\right\} \qquad (2.4)$$

These expressions coincide only in form with the formulation of Hooke's law for the plane stressed state. It is important to emphasize that understood by E and ν here are not fixed Young's modulus E_0 and Poisson's ratio ν_0 of the linear theory but the functions determined and in a nonlinear manner connected with the state of strain at the point in question: $E = E(\alpha, \beta, \gamma, \varepsilon_0, \varepsilon_1)$, $\nu = \nu(\alpha, \beta, \gamma, \varepsilon_0, \varepsilon_1)$. Based on such a representation of formulas (2.4) is the method of variable parameters of elasticity [10, 55] of the solution of the nonlinearly elastic and elasto-plastic problems. Since for obtaining the resolving equations of the proposed theory knowledge of the specific form of functions E and ν is not required, then here let us indicate only their general variants: $E = E(\alpha, \beta, \gamma)$, $\nu = \nu(\alpha, \beta, \gamma)$, and $E = E(\varepsilon_0, \varepsilon_1)$, $\nu = \nu(\varepsilon_0, \varepsilon_1)$, which correspond to the heterogeneous physically linear material and uniform physically nonlinear material before deformation.

It is interesting to note that since (in this strained state) $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \gamma)$ and $\varepsilon_1 = \varepsilon_1(\alpha, \beta, \gamma)$, in general E and ν are complex functions of the coordinates. The values of these functions change with a change in the state of strain at the point. Thus, a physically nonlinear body can be considered as a body whose heterogeneity nonlinearly depends on the state of strain, although its properties are physically linear.

From formulas (2.4) it follows that

$$\left.\begin{aligned}\sigma_1 &= \frac{E}{1-\nu^2}[e_1 + \nu e_2 + \gamma(\varkappa_1 + \nu\varkappa_2)] - \frac{Et}{1-\nu}T;\\ \sigma_2 &= \frac{E}{1-\nu^2}[e_2 + \nu e_1 + \gamma(\varkappa_2 + \nu\varkappa_1)] - \frac{Et}{1-\nu}T;\\ \sigma_{12} &= \frac{E}{2(1+\nu)}(e_{12} + 2\gamma\varkappa_{12}).\end{aligned}\right\} \quad (2.5)$$

Integrating the stresses with respect to γ, we obtain stresses in the middle surface

$$\left.\begin{aligned}T_1 &= B_1e_1 + B_2e_2 + C_1\varkappa_1 + C_2\varkappa_2 - T_t;\\ T_2 &= B_2e_1 + B_1e_2 + C_2\varkappa_1 + C_1\varkappa_2 - T_t;\\ T_{12} &= \frac{1}{2}D_{00}e_{12} + D_{10}\varkappa_{12}.\end{aligned}\right| \quad (2.6)$$

Multiplying the stresses by γ and integrating over the thickness, we find the bending and torsional moments

$$\left.\begin{aligned} M_1 &= C_1 e_1 + C_2 e_2 + D_1 \varkappa_1 + D_2 \varkappa_2 - M_t; \\ M_2 &= C_2 e_1 + C_1 e_2 + D_2 \varkappa_1 + D_1 \varkappa_2 - M_t; \\ M_{12} &= \frac{1}{2} D_{10} e_{12} + D_{20} \varkappa_{12}. \end{aligned}\right\} \qquad (2.7)$$

The positive directions of forces and moments are shown on Figs. 3 and 4. Coefficients in expressions (2.6) and (2.7) are functions α and β and they are connected with the integrals

$$D_{bl}(\alpha, \beta) = \int_{-h/2}^{h/2} \frac{E}{1 + c\nu} \gamma^b d\gamma, \; b = 0, 1, 2; \; c = 1, -1; \qquad (2.8)$$

$$l = \frac{1}{2}(1 - c)$$

and formulas

$$B_{1,2} = \frac{1}{2}(D_{01} \pm D_{00}); \; C_{1,2} = \frac{1}{2}(D_{11} \pm D_{10});$$

$$D_{1,2} = \frac{1}{2}(D_{21} \pm D_{20}).$$

The temperature components of stresses and moments have the form

$$T_t = \int_{-h/2}^{h/2} \frac{EtT}{1 - \nu} d\gamma; \quad M_t = \int_{-h/2}^{h/2} \frac{EtT}{1 - \nu} \gamma d\gamma. \qquad (2.9)$$

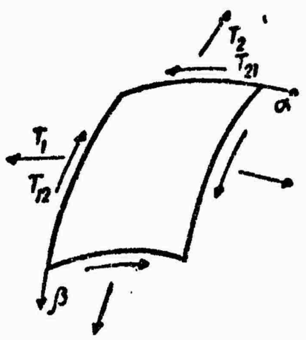


Fig. 3.

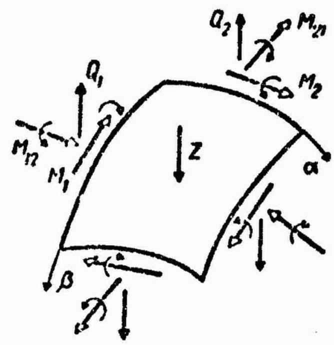


Fig. 4.

From formulas (2.6) let us determine deformations in the middle surface

$$
\left.\begin{aligned}
e_1 &= A_{qq}T_1 + B_{qq}T_2 - B_{\varkappa q}\varkappa_1 - A_{\varkappa q}\varkappa_2 + \frac{1}{D_{01}}T_t;\\
e_2 &= B_{qq}T_1 + A_{qq}T_2 - A_{\varkappa q}\varkappa_1 - B_{\varkappa q}\varkappa_2 + \frac{1}{D_{01}}T_t;\\
e_{12} &= \frac{2}{D_{66}}T_{12} - 2\frac{D_{16}}{D_{66}}\varkappa_{12} = 2[(A_{qq} - B_{qq})T_{12} +\\
&\quad + (A_{\varkappa q} - B_{\varkappa q})\varkappa_{12}].
\end{aligned}\right| \qquad (2.10)
$$

Let us introduce the function of stresses in the middle surface by means of relations

$$T_1 = \varphi^{\cdot\cdot}_{\beta\beta} + U;\ T_{12} = -\varphi^{\cdot\cdot}_{\alpha\beta};\ T_2 = \varphi^{\cdot\cdot}_{\alpha\alpha} + U. \qquad (2.11)$$

Brief designations of the second-order derivatives in curvilinear coordinates indicated in formulas (2.3) are used here. Function U is the potential of body forces in the middle surface whose projections on the axes α and β are

$$X = -\frac{1}{A}\cdot\frac{\partial}{\partial\alpha}U;\quad Y = -\frac{1}{B}\cdot\frac{\partial}{\partial\beta}U. \qquad (2.12)$$

Let us recall the expressions of the known operators

$$
\begin{aligned}
\Delta &= \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\cdot\frac{B}{A}\cdot\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\cdot\frac{A}{B}\cdot\frac{\partial}{\partial\beta}\right);\\
\Delta_k &= \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\cdot\frac{1}{R_2'}\cdot\frac{B}{A}\cdot\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha}\cdot\frac{1}{R_{12}}\cdot\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\beta}\cdot\frac{1}{R_{12}}\cdot\frac{\partial}{\partial\alpha} +\right.\\
&\quad \left. + \frac{\partial}{\partial\beta}\cdot\frac{1}{R_1'}\cdot\frac{A}{B}\cdot\frac{\partial}{\partial\beta}\right)
\end{aligned} \qquad (2.13)
$$

and let us introduce the operators

$$
\begin{aligned}
L_1(\psi, \eta) &= \psi^{\cdot\cdot}_{\alpha\alpha}\eta^{\cdot\cdot}_{\alpha\alpha} + 2\psi^{\cdot\cdot}_{\alpha\beta}\eta^{\cdot\cdot}_{\alpha\beta} + \psi^{\cdot\cdot}_{\beta\beta}\eta^{\cdot\cdot}_{\beta\beta};\\
L_2(\psi, \eta) &= \psi^{\cdot\cdot}_{\alpha\alpha}\eta^{\cdot\cdot}_{\beta\beta} - 2\psi^{\cdot\cdot}_{\alpha\beta}\eta^{\cdot\cdot}_{\alpha\beta} + \psi^{\cdot\cdot}_{\beta\beta}\eta^{\cdot\cdot}_{\alpha\alpha},
\end{aligned} \qquad (2.14)
$$

the second of which is used in the theory of plates and shells: function ψ, just as η, takes the values w and φ.

3. Variational Equation

In accordance with the principle of possible displacements [67, 76], the variation of the total energy δW of a deformed shell is equal to zero in the equilibrium state:

$$\delta W = \delta V + \delta U_{и} + \delta U_{c} = 0. \tag{2.15}$$

Let us transform equation (2.15) by using the relationships given in Section 2 of this chapter.

The variation of the work of the applied forces is

$$\delta V = -\delta \int (Xu + Yv + Zw)\,ds - \delta \int (T_n^0 u_n + T_l^0 u_l - - M_n^0 w_n' + Q_n^0 w)\,dl. \tag{2.16}$$

Here the first integral is taken on the area of the middle surface, and the second is taken on its edge; u_n and u_l are the components of displacement in the middle surface normal and tangent to the edge; w_n' is the derivative of deflection in a direction normal to the edge of the shell. We consider further that the sections of the edge on which not assigned are w and w_n' are free from loads T_n^0 and T_1.

The variation of the potential energy, which appears as a result of bending strain, is

$$\delta U_{и} = \iint (M_1 \delta \varkappa_1 + M_2 \delta \varkappa_2 + 2M_{12} \delta \varkappa_{12})\,ds. \tag{2.17}$$

The variation of energy which corresponds to the deformation in the middle surface

$$\delta U_c = \iint (T_1 \delta \varepsilon_1 + T_2 \delta \varepsilon_2 + T_{12} \delta \varepsilon_{12})\,ds, \tag{2.18}$$

let us present in the form

$$\delta U_c = \delta \iint (T_1 \varepsilon_1 + T_2 \varepsilon_2 + T_{12} \varepsilon_{12})\,ds - - \iint (\varepsilon_1 \delta T_1 + \varepsilon_2 \delta T_2 + \varepsilon_{12} \delta T_{12})\,ds. \tag{2.19}$$

This conversion will involve important consequences. Actually, in the initial equation (2.15) only displacements undergo variation, while introduced into expression (2.19) are variations in stresses in the middle surface. Formally expressions

(2.18) and (2.19) are equal, but the use of formula (2.19) leads in summation to the mixed type of variation equation. In it the displacement (deflection) and stresses (function of stresses in the middle surface) will be varied.

Let us consider the sum

$$\delta G = \delta\left[\iint (T_1\varepsilon_1 + T_2\varepsilon_2 + T_{12}\varepsilon_{12} - Xu - Yv)\,ds - \right.$$
$$\left. - \int (T_n^0 u_n + T_t^0 u_t)\,dl\right] \tag{2.20}$$

of the first integral from formula (2.19) with a variation of the work of tangential components of applied forces. By substituting instead of the deformations their expressions by formulas (2.2) and integrating in parts, we obtain

$$\delta G = -\delta\iint \left\{\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_1) - \frac{\partial B}{\partial\alpha}T_2 + \frac{1}{A}\cdot\frac{\partial}{\partial\beta}(A^2T_{12})\right] + \right.$$
$$\left. + X\right\} u\,ds - \delta\iint\left\{\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AT_2) - \right.\right.$$
$$\left.\left. - \frac{\partial A}{\partial\beta}T_1 + \frac{1}{B}\cdot\frac{\partial}{\partial\alpha}(B^2T_{12})\right] + Y\right\} v\,ds +$$
$$+ \frac{1}{2}\delta\iint\left\{T_1\left[\left(\frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}\right)^2 + 2\frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}\cdot\frac{1}{A}\cdot\frac{\partial w_0}{\partial\alpha}\right] + \right.$$
$$+ T_2\left[\left(\frac{1}{B}\cdot\frac{\partial w}{\partial\beta}\right)^2 + 2\frac{1}{B}\cdot\frac{\partial w}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial w_0}{\partial\beta}\right] +$$
$$+ 2T_{12}\left(\frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}\cdot\frac{1}{B}\cdot\frac{\partial w}{\partial\beta} + \frac{1}{A}\cdot\frac{\partial w}{\partial\alpha}\cdot\frac{1}{B}\cdot\frac{\partial w_0}{\partial\beta} + \right.$$
$$\left.\left. + \frac{1}{B}\cdot\frac{\partial w}{\partial\beta}\cdot\frac{1}{A}\cdot\frac{\partial w_0}{\partial\alpha}\right)\right\} ds -$$
$$- \delta\iint\left(\frac{T_1}{R_1} - 2\frac{T_{12}}{R_{12}} + \frac{T_2}{R_2}\right) w\,ds +$$
$$+ \delta\int [(T_n - T_n^0)u_n + (T_t - T_t^0)u_t]\,dl. \tag{2.21}$$

By substituting the stresses according to formulas (2.11), disregarding terms $\frac{AB}{R_1R_2}\frac{\partial\varphi}{\partial\alpha}$, $\frac{AB}{R_1R_2}\frac{\partial\varphi}{\partial\beta}$ in comparison with higher derivatives of ϕ (without the factor $\frac{AB}{R_1R_2}$), and subordinating the function ϕ to the boundary conditions

$$T_n = T_n^0;\ T_t = T_t^0. \tag{2.22}$$

we find that the first and second integrals on the region and integral on the outline in δG vanish. Let us note that two first integrands in (2.21) are products of the left sides of equations of equilibrium in the middle surface of the technical theory ((19.3) [84]) by the displacements. The absence in these equations of projections of transverse forces is the direct consequence of the neglecting of terms of the type $\frac{u}{R_1}$ and $\frac{v}{R_2}$ in expressions for curvature (2.3) (see also [85], page 96).

The rejection of first-order derivatives of ϕ with the factor $\frac{AB}{R_1 R_2}$ is the second assumption of the technical theory of shells. If only one of the main radii of curvature is equal to infinity, this does not cause error.

By applying the integration in parts to the third term of (2.21), and to the fourth term - the Kodatstsi - Gauss formulas ((8.6) [29]), we obtain

$$\delta G = -\frac{1}{2}\delta\iint\Big[2\Delta_k\varphi + L_2(w_2, \varphi) - F(w_2) + \\ + 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)U\Big] w\,ds + \frac{1}{2}\delta\int\left(T_n\frac{\partial w}{\partial n} + T_l\frac{\partial w}{\partial l}\right)w\,dl, \qquad (2.23)$$

where $w_2 = w + 2w_0$;

$$F(\psi) = X\frac{1}{A}\cdot\frac{\partial\psi}{\partial\alpha} + Y\frac{1}{B}\cdot\frac{\partial\psi}{\partial\beta} - U\Delta\psi.$$

The contour integral in expression (2.23) is equal to zero, since on the fastened part of the edge $w = 0$, and on the free part $T_n = T_l = 0$ according to formulas (2.22) and the condition accepted at the beginning of the section. Let us note that in (2.23) the physical parameters of the material are not included.

By substituting into the sum

$$N = \iint(M_1\delta\varkappa_1 + M_2\delta\varkappa_2 + 2M_{12}\delta\varkappa_{12})\,ds - \iint(\varepsilon_1\delta T_1 + \\ + \varepsilon_2\delta T_2 + \varepsilon_{12}\delta T_{12})\,ds$$

of the right side of equality (2.17) and the second term from (2.19) the moments and deformations in the middle surface according to formulas (2.7) and (2.10), taking into account the variation of the work of the loads Z, M_n^0, and Q_n^0 and relation (2.23), and replacing the stresses by their expressions with the function ϕ, we reduce equation (2.15) to the form

$$\iint \left[\frac{1}{2} R(w, w) - R(w, \varphi) - \frac{1}{2} R(\varphi, \varphi) + M_i^0 \delta \Delta w - T_i^0 \delta \Delta \varphi \right] ds - \delta \iint \left[\Delta_k \varphi + L_2 \left(\frac{1}{2} w + w_0, \varphi\right) - \frac{1}{2} F(w_2) + \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) U + Z \right] w\, ds + \delta \oint \left(M_n^0 \frac{\partial w}{\partial n} - Q_n^0 w\right) dl = 0. \qquad (2.24)$$

where

$$M_i^0 = M_i - D_{11} T_i^0; \quad T_i^0 = -\frac{1}{D_{01}} (T_i + U).$$

The operator R has the form

$$R(\psi, \eta) = A_{\psi\eta} \delta L_1 (\psi, \eta) + B_{\psi\eta} \delta L_2 (\psi, \eta). \qquad (2.25)$$

and coefficients

$$\left.\begin{matrix} A_{ww} \\ B_{ww} \end{matrix}\right\} = \frac{1}{2}\left[(D_{21} \pm D_{20}) - \left(\frac{D_{11}^2}{D_{01}} \pm \frac{D_{10}^2}{D_{00}}\right)\right];$$

$$\left.\begin{matrix} A_{w\varphi} \\ B_{w\varphi} \end{matrix}\right\} = \frac{1}{2}\left(\frac{D_{11}}{D_{01}} \pm \frac{D_{10}}{D_{00}}\right); \quad \left.\begin{matrix} A_{\varphi\varphi} \\ B_{\varphi\varphi} \end{matrix}\right\} = \frac{1}{2}\left(\frac{1}{D_{01}} \pm \frac{1}{D_{00}}\right) \qquad (2.26)$$

are the complex functions of coordinates α and β and depend on the state of strain of each element of the shell. This is caused by the fact that the integrals D_{bl} (2.8) include functions E and ν of the coordinates and state of strain.

Let us note that in the derivation of equation (2.24) it was not required to fulfil the operation of the variation of functions containing the physical parameters E and ν.

The variation equation (2.24) is basic one in this work. This equation is of the mixed type, since it contains independently varied unknown functions w and ϕ.

In obtaining this equation, besides the usual assumptions of the classical theory of shells, the condition of heterogeneity of the material is accepted, and, most importantly, assumptions of the technical theory are introduced. Usually the theory with such assumptions is called the theory of slightly curved shells, since the rejected terms have in the denominator radii of curvature (relatively larger near the slightly curved shells); however, the field of application of this theory is considerably more widespread. The technical theory gives good results if the stressed state of the shell rapidly changes along the coordinates [29], although the shell cannot be slightly curved. Therefore, the use in the name of the chapter of the term "slightly curved" bears somewhat of a conditional nature, and in actuality possibilities of the theory are not exhausted only by slightly curved shells.

The variational equation (2.24) is the natural generalization of the equations obtained in sources [45, 47]. For the case of uniform physically linear shells, the mixed type of variational equations are given previously by N. A. Alumyae [4, 5]. Later an equation of such type was obtained by Lyu Shi-nin [73], who took into account only the geometric nonlinearity and did not introduce the initial chamber, potential U and temperature.

All the indicated equations have the form of an equality to zero of the variation of the functional, while in equation (2.24) the sign of the variation cannot be removed from under the first integral. This is connected with the dependence of functions $A_{\psi\eta}$ and $B_{\psi\eta}$, which enter into the operator R (2.25), on E and ν, which in the physically nonlinear case are determined by the state of strain described in terms of unknowns, w and ϕ. In the examination of the physically linear problems $A_{\psi\eta}$ and $B_{\psi\eta}$ become the assigned functions of the coordinates, the sign of the variation can be taken out, and the equation acquires a more customary form.

It is advantageous to use equation (2.24) directly (without transition to the system of differential equations) for the solution of heterogeneous and physically nonlinear problems. Its main advantage is the absence of derivatives of functions which depend on the physical parameters of the material. On the contrary, these functions enter only under the integral sign. Taking their complex nature into account, we note that this fact undoubtedly contributes to an increase in the accuracy of the solutions obtained by numerical methods.

The mixed type variational equations relative to the deflection and the function of forces have a number of other advantages in comparison with the variational equations relative to the three displacements u, v, and w or by the equations used in the principle of Reissner [134]:

1) only two functions are sought (one should, however, consider that function ϕ can be introduced only in the technical theory of shells);

2) the function of stresses is considered to be independently varied, and therefore there is no need to integrate preliminarily the equation of compatibility in order to express ϕ by w;

3) at any selection of coordinate functions for ϕ, the equations of equilibrium in the middle surface are satisfied automatically, which increases the accuracy of the solution;

4) the coordinate functions cannot satisfy the natural boundary conditions, which is especially important in the calculation of shells which have free sections of the edge;

5) methods of the solution of variational equations of mixed type are convenient for an algorithmization, programming and complete automation of count by a computer.

The different special cases of equation (2.24) will be examined in the subsequent chapters. Here let us indicate only the simplification which appears under parity condition E and ν with respect to γ, i.e., when $E(\alpha, \beta, \gamma) = E(\alpha, \beta, -\gamma)$ and $\nu(\alpha, \beta, \gamma) = \nu(\alpha, \beta, -\gamma)$. Then from formula (2.8) it follows that $D_{10} = D_{11} = 0$, and formulas (2.26) give $\left.\begin{matrix}A_{ww}\\B_{ww}\end{matrix}\right\} = \frac{1}{2}(D_{21} \pm D_{20})$. $A_{w\phi} = B_{w\phi} = 0$. The latter leads to the idenity $R(w, \phi) \equiv 0$. The obtained formulas are fulfilled for a linearly elastic heterogeneous material with the structure symmetric relative to the middle surface of the shell. For a nonlinearly elastic material (with the same structure before deformation), these formulas take place with a symmetric strained state relative to the surface $\gamma = 0$ or with antisymmetric state, if E and ν are even functions of invariants of the strain tensor.

Let us note also that equation (2.24) can be used in the calculation of laminar shells, since the discontinuities of the first kind of functions $E(\gamma)$ and $\nu(\gamma)$ do not break the existence of integral D_{bl}. In this case it is necessary, of course, to consider that the layers should possess a commensurable shear stiffness and that this theory is based on the hypothesis of direct normals.

4. Differential Equations

Although for the solution of specific problems in the work the variational equation (2.24) is used, nevertheless, the completeness of the theory requires the obtaining and a system of differential equations corresponding to equation (2.24). It is important that in this case there will also be obtained the natural and main boundary conditions the knowledge of which is necessary when selecting systems of coordinate functions.

In carrying out the operation of variation, using Green's formula

$$\iint a\Delta b ds = \iint b\Delta a ds + \int (ab'_n - a'_n b)\, dl$$

and formulas

$$\left.\begin{aligned} &\iint a\Delta_k b ds = \iint b\Delta_k a ds + \int\left[\left(\frac{1}{R_l} b'_n + \frac{1}{R_{nl}} \times \right.\right.\\ &\left.\left. \times b'_l\right) a - \left(\frac{1}{R_l} a'_n + \frac{1}{R_{nl}} a'_l\right) b\right] dl;\\ &\iint aL_2(b, c)\, ds = \iint L_2(a, b)\, c ds + \int [(b''_{ll} c'_n -\\ &- b''_{nl} c'_l)\, a - (a'_n b''_{ll} - a'_l b''_{nl})\, c]\, dl;\\ &aL_1(\psi, \eta) + bL_2(\psi, \eta) = a\Delta\psi\Delta\eta - (a - b)\, L_2(\psi, \eta). \end{aligned}\right\} \qquad (2.27)$$

in which

$$b''_{nn} = \frac{\partial^2 b}{\partial n^2}; \quad b''_{nl} = \frac{\partial^2 b}{\partial n \partial l} - \varkappa \frac{\partial b}{\partial l}; \quad b''_{ll} = \frac{\partial^2 b}{\partial l^2} + \varkappa \frac{\partial b}{\partial n} \qquad (2.28)$$

(n and l - normal and tangent to the edge, κ - geodetic curvature of the edge), from equation (2.24) by virtue of the arbitrariness of variation δw and $\delta\phi$, we find the system

$$\begin{aligned} &\Delta(A_{ww}\Delta w) - L_2(A_{ww} - B_{ww}, w) - \Delta_k\varphi - \Delta(A_{w\varphi}\Delta\varphi) -\\ &- L_2(B_{w\varphi} - A_{w\varphi} + w_1, \varphi) = Z_u - \Delta M_t^0;\\ &\Delta(A_{\varphi\varphi}\Delta\varphi) - L_2(A_{\varphi\varphi} - B_{\varphi\varphi}, \varphi) + \Delta_k w + \Delta(A_{w\varphi}\Delta w) +\\ &+ L_2\left(B_{w\varphi} - A_{w\varphi} + \frac{w_2}{2}, w\right) = -\Delta T_t^0. \end{aligned} \qquad (2.29)$$

where $Z_u = Z + \left(\frac{1}{R'_1} + \frac{1}{R'_2}\right) U - F(w_1)$. From formulas (2.26) it is evident that $A_{ww} - B_{ww} = D_{20} - \frac{D^2_{10}}{D_{00}}$, $A_{\varphi\varphi} = B_{\varphi\varphi} = \frac{1}{D_{00}}$, $B_{w\varphi} - A_{w\varphi} = \frac{D_{10}}{D_{00}}$.

The system of differential equations of the fourth order (2.29) is a generalization of the system of equations (equilibrium and compatibility) of the technical theory of shells of V. Z. Vlasov [18] for the case of a heterogeneous material, taking into account the physical and geometric nonlinearity.

Equations of the physically linear theory do not differ from system (2 29) in form, but in this case $A_{\psi\eta}$ and $B_{\psi\eta}$ are the assigned functions of the coordinates and they do not depend on state of strain at the point, i.e., on the unknown functions w and ϕ.

Differential equations of the geometrically linear problem are obtained from system (2.29), if we assume that $w_1 = w_0$, $w_2 = 2w_0$. In this case clearly apparent is the role of relation D_{10}/D_{00}, which characterizes the degree of the asymmetry of distribution of properties of a material with respect to thickness and is equivalent to the supplementary chamber. The presence of asymmetry leads to the fact that even in the case of the linear problem of the curvature of the plate ($\Delta_k w \equiv 0$, $\Delta_k \phi \equiv 0$, $w_0 \equiv 0$) equations (2.29) are not divided. The effect of asymmetry is also felt in a decrease in the flexural rigidity (see formulas for A_{ww}, B_{ww}) and in the correction of temperature terms.

In the particular case of h = const, $\nu = \frac{1}{2}$, U = T = 0, upon the transition to the conjugate coordinates from (2.29), there follow the equations of N. F. Ershov [38] for a shell which is uniform before deformation. Equations of the geometrically linear problem (which consider the physical nonlinearity), which I. A. Tsurpal and N. A. Shul'ga [116] obtained, also follow from system (2.29).

Let us write the resolving equations for a shell uniform in the thickness ($E = E(\alpha, \beta)$, $\nu = \nu(\alpha, \beta)$) of a physically linear material. From formulas (2 8) and (2.26) in this case there follows $D_{10} = D_{11} = 0$, $D \equiv A_{ww} = \dfrac{Eh^3}{12(1-\nu^2)}$, $B_{ww} = \nu D$, $A_{w\phi} = B_{w\phi} = 0$. $B \equiv A_{\phi\phi} = \dfrac{1}{Eh}$, $B_{\phi\phi} = -\nu B$. Substitution of these expressions into system (2.29) gives

$$\Delta(D\Delta w) + L_2[(1-\nu)D, w] - \Delta_k\varphi - L_2(w_1, \varphi) = Z_u - \Delta M_t^0,$$

$$\Delta\left(\frac{1}{B}\Delta\varphi\right) - L_2\left(\frac{1+\nu}{B}, \varphi\right) + \Delta_k w + \frac{1}{2}L_2(w_2, w) = -\Delta T_t^0. \qquad (2.30)$$

Let us note that the operator $F(w_1)$ which enters into Z_u is absent from the analogous equations derived by S. Lukashevich [126] by means of an examination of the equilibrium of the element of the shell and the condition of consistency of the deformations. However, in the geometrically nonlinear case this operator cannot be disregarded; it reflects the effect of body forces in the middle surface on the equilibrium in direction γ.

By substituting approximately the metrics of the surface of the metric plane and assuming ν = const, from (2.30) let us arrive at equations (35) of source [61]. In the second of equations (35) [61] there are no projections of components of the tangential load on the vertical line

$$X\frac{\partial w}{\partial x} + Y\frac{\partial w}{\partial y}.$$

Coefficient 2 is dropped before the terms

$$\frac{\partial D}{\partial x}\cdot\frac{\partial\theta}{\partial x} + \frac{\partial D}{\partial y}\cdot\frac{\partial\theta}{\partial y},$$

which appear in the development of expression ΔM_t^0.

These examples again show the known advantage of the method of obtaining differential equations by the variational means.

5. Boundary Conditions.

In the variation of equation (2.24) it is clarified that the main boundary conditions for function w are

$$w/\Gamma = w^0; \quad w_n'/\Gamma = (w^0)_n'. \qquad (2.31)$$

where w^0 and $(w^0)_n'$ are the assigned functions. If these functions are not assigned, then by virtue of the arbitrariness of variations δw and $\delta w_n'$ on the contour, we find that the natural boundary conditions for the deflection

$$M_n = M_n^0; \quad Q_n = \frac{\partial M_n}{\partial n} + 2\frac{\partial M_{nt}}{\partial t} + \varkappa (M_n - M_t) = Q_n^0, \qquad (2.32)$$

where

$$\left.\begin{aligned} M_n &= -A_{ww} w_{nn}'' - B_{ww} w_{tt}'' + B_{w\varphi}\varphi_{tt}'' + A_{w\varphi}\varphi_{nn}'' - M_n^0; \\ M_t &= -B_{ww} w_{nn}'' - A_{ww} w_{tt}'' + A_{w\varphi}\varphi_{tt}'' + B_{w\varphi}\varphi_{nn}'' - M_t^0; \\ M_{nt} &= -(A_{ww} - B_{ww}) w_{nt}'' - (A_{w\varphi} - B_{w\varphi})\varphi_{nt}''. \end{aligned}\right| \qquad (2.33)$$

The second derivatives of w and ϕ should be calculated according to formulas (2.28), and in this case the partial derivatives along the normal and tangent to the edge are expressed as derivatives in terms of α and β by means of the usual formulas of rotation of the coordinate system.

The main boundary conditions for the function of stresses ϕ are conditions (2.22). If the edge of the shell is free in middle surface, $T_n^0 = T_l^0 = 0$ and expressions (2.22) are satisfied when

$$\varphi/\Gamma = \varphi_n'/\Gamma = 0. \qquad (2.34)$$

With functions $\delta\phi$ and $\delta\phi_n'$ arbitrary on the edge of the shell from equation (2.24) there ensue the natural conditions for ϕ:

$$M_n^* = 0; \quad Q_n^* = \frac{1}{R_t}\cdot\frac{\partial w}{\partial n} + \frac{1}{2}\varkappa\frac{\partial w}{\partial n}\cdot\frac{\partial w_1}{\partial n}. \qquad (2.35)$$

expressions for M_n^* and Q_n^* are obtained from formulas (2.32) and (2.33) by substitution according to the scheme $w \rightleftarrows \phi$, $A_{ww} \rightarrow B_{\phi\phi}$, $B_{ww} \rightarrow A_{\phi\phi}$, $B_{w\phi} \rightleftarrows A_{w\phi}$, $M_t^0 \rightarrow T_t^0$. Conditions (2.35) are geometric and should be fulfilled on the section of the edge of the shell attached in the middle surface. The value $- M_n^*$ is equal to the deformation of the edge in a direction tangent to it (see (2.10)). The second condition is equivalent to the requirement of the absence of bending of the edge in a plane tangent to the middle surface.

Thus, the main boundary conditions (2.31) and (2.34) are expressed as functions and their first-order derivatives along the normal to the contour of the shell. Into natural conditions (2.32) and (2.35) enter the second-order and third-order derivatives of the unknown functions.

Functions w and ϕ, subordinate to the main conditions and satisfying the variational equation (2.24), are the solution of the system of differential equations (2.29) subordinate to natural conditions. Such a position takes place as a result of the fact that the system (2.29) and conditions (2.32) and (2.35) ensue from the equation (2.24).

Natural boundary conditions have a comparatively complex nature; however, the use of a variational method of the solution makes it possible previously not to subordinate the coordinate functions to these conditions. This considerably facilitates the selection of systems of coordinate functions. Of course, if it appears possible to subordinate the coordinate system to all boundary conditions, this will contribute to the acceleration of convergence.

From formulas (2.33) and (2.35) it is evident that the natural boundary conditions for functions w and ϕ are connected. They are divided only in the case of the symmetry of properties of the material relative to the middle surface ($A_{w\phi} = B_{w\phi} = 0$).

CHAPTER III

METHOD OF THE SOLUTION OF GEOMETRICALLY AND PHYSICALLY NONLINEAR PROBLEMS

1. The Means of the Solution of Nonlinear Problems of the Theory of Heterogeneous Shells

The complexity of the variational equation (2.24) and the system of differential equations (2.29) equivalent to it of the nonlinear technical theory of heterogeneous shells makes, apparently, the obtaining of exact solutions to the problems in general impossible. Moreover, even the problems in which the unknown functions depend only on one coordinate, and the material is uniform and obeys Hooke's law, are difficult for analytical solution due to the geometric nonlinearity. The physical nonlinearity complicates equations considerably more than does the geometric nonlinearity.

It should be noted that the account of the heterogeneity of the material (or variability of the thickness) increases the difficulties of the solution of specific problems. However, in linear problems this difficulty is basically of a technical nature, and although their overcoming requires definite stresses and affects the selection of the method, nevertheless, in nonlinear cases the presence of heterogeneity complicates the problem to considerably greater degree.

The general idea which is used in all numerical methods of the solution consists in the approximate transition from the system with an infinite number of degrees of freedom to the system with their finite number. In this case the variational equation (or the system of differential equations), by one method or the other, is reduced to the system of nonlinear finite equations.

Without dwelling in detail on each of the known methods of information, let us note the presence of two groups of methods.

The methods of the first group are based on the system of the resolving differential equations and the boundary conditions corresponding to the problem. These are the classical methods of net-point, straight lines, collocations and the variants. The indicated methods are widely used for the solution of linear problems of the theory of uniform shells. In the case of heterogeneous shells, their use is also possible. However, the presence in differential equations (2.29) of second-order derivatives of functions $A_{\psi n}$ and $B_{\psi n}$ requires (to obtain the acceptable accuracy) an increase in the quantity of discrete unknowns, terms of a series, etc. This leads to an increase in the number of nonlinear finite equations. The latter substantially limits the possibilities of the implementation of calculations and leads to a considerable increase in machine time. It is necessary to consider also that the quantity of coefficients of the quadratic part (connected with the geometric nonlinearity) of finite equations is proportional to n^3 (n is the number of discrete unknowns), which also makes an increase in the dimension of this system undesirable.

Since the solved problems with the simultaneous account of the nonlinearity of two types and heterogeneity are very few, the total characteristic of the methods of the first group still cannot be given, and the deficiencies noted above cannot serve as a basis for the failure of their use. In selecting the method of the solution to the problem, generally the researcher's personal experience is of great significance.

In methods of the second group variational equations, directly (methods of Ritz and Bubnov-Galerkin) or in conjunction with differential equations, are used. Most widely used in a nonlinear theory (especially in problems of flexible shells) is the solution of the problem in two stages - in the first the differential equation of the consistency of deformations is used, and in the second - the variational method of Bubnov-Galerkin. An analysis of the possibilities of methods of Ritz and Bubnov-Galerkin in connection with the linear problems of the curvature of the plates of variable thickness, carried out in source [113], showed that among these methods, in the case of heterogeneous shells, preference should be given to the Ritz method. Since the technical difficulties caused by the heterogeneity become especially important in the nonlinear problems, and in the Ritz method the heterogeneity is considered most simply, let us give preference namely to it.

The Ritz method is used for equations which have the form of the equality to zero of a variation of the functional. Although equation (2.24) is written in another form, the use of the method of variable stiffness constants will make it possible to reduce the problem to the sequence of problems of steady state of functionals.

2. Variational Method.

The proposed method of the solution of nonlinear problems of the technical theory of heterogeneous shells is based on the combined use of methods of the variable elasticity parameters of I. A. Birger [10], Ritz and Newton-Kantorovich.

In accordance with the method of variable elasticity parameters, we will seek the solution to the physically nonlinear problem by means of successive approximations, assuming that on each step (external) of the iterative process the material of the shell possesses linear but heterogeneous properties. In this case functions E and ν are considered assigned and dependent directly on coordinates α, β, and γ.

To fulfill one iteration means to solve the geometrically nonlinear problem of the heterogeneous shell from a physically linear material. Having obtained the strain distribution as a result of the solution of this simpler problem, it is possible to compute at each point the volume strain ε_1 and deformation intensity ε_0 and determine the new refined values of functions[1] E and ν. The process is continued until two consecutive states of strain coincide with the assigned accuracy.

The variational equation, which describes the behavior of the flexible heterogeneous shell, is obtained from equation (2.24), if in it we replace operator R by δR^0, where

$$R^0(\psi, \eta) = A_{\psi\eta}L_1(\psi, \eta) + B_{\psi\eta}L_2(\psi, \eta), \quad (3.1)$$

and remove the sign of the variation from under the sign of the first integral in (2.24). This becomes possible, since when $E = E(\alpha, \beta, \gamma)$ and $\nu(\alpha, \beta, \gamma)$ functions $A_{\psi\eta}$ and $B_{\psi\eta}$ do not depend on the unknowns w and ϕ. The left side of equation (2.24) then acquires the form of the variation from the functional. We solve the obtained geometrically nonlinear problem by the Ritz method.

[1]We consider that there are formulas, tables or graphs on which (by knowing the coordinates of the point and ε_0, and ε_1 in it) it is possible to find $E(\alpha, \beta, \gamma, \varepsilon_0, \varepsilon_1)$ and $\nu(\alpha, \beta, \gamma, \varepsilon_0, \varepsilon_1)$.

Assuming

$$w = \sum_{i=1}^{n} x_i w_i(\alpha, \beta); \quad \varphi = \sum_{i=1}^{m} y_i \varphi_i(\alpha, \beta), \tag{3.2}$$

where x_i and y_i are the unknown parameters, and w_i and ϕ_i - the assigned coordinate functions, from conditions of steady state of the functional

$$\frac{\partial W}{\partial x_i} = 0, \ i = 1, 2, \ldots, n; \quad \frac{\partial W}{\partial y_i} = 0, \ i = 1, 2, \ldots, m$$

we obtain the system of quadratic algebraic equations (here it is assumed that the potential of applied forces in the middle surface U = 0; Z = q(α, β), $M_n^0 = Q_n^0 = 0$):

$$\begin{gathered}
\iint \Big\{ \sum_{j=1}^{n} x_j R^0(w_i, w_j) - \sum_{j=1}^{m} y_j [R^0(w_i, \varphi_j) + (\Delta_k \varphi_j + \\
+ L_2(w_0, \varphi_j)) w_i] - \sum_{j=1}^{n} \sum_{s=1}^{m} x_j y_s w_i L_2(w_j, \varphi_s) + \\
+ M_i^0 \Delta w_i - q w_i \Big\} ds = 0, \ i = 1, 2, \ldots, n; \\
\iint \Big\{ - \sum_{j=1}^{m} y_j R^0(\varphi_i, \varphi_j) - \sum_{j=1}^{n} x_j [R^0(w_j, \varphi_i) + (\Delta_k \varphi_i + \\
+ L_2(w_0, \varphi_i)) w_j] - \frac{1}{2} \sum_{j=1}^{n} \sum_{s=1}^{m} x_j x_s w_j L_2(w_s, \varphi_i) - \\
- T_i^0 \Delta \varphi_i \Big\} ds = 0, \ i = 1, 2, \ldots, m.
\end{gathered} \tag{3.3}$$

With the derivation of the first group of equations, it is taken into account that by virtue of formula (2.27)

$$\iint w_i L_2(w_j, \varphi_s) ds = \iint w_j L_2(w_i, \varphi_s) ds;$$

the contour integral disappears if on the edge of shell w = 0. We consider this condition to be fulfilled.

System (3.3) is solved by the Newton-Kantorovich method (internal iterative process), and their theory is presented in detail in a book [53].

The presence and rate of convergence of the iterative processes depend substantially on how closely the initial approximation is to the solution. Therefore, if in the nonlinear problem there is a parameter to each value of which corresponds one solution, then it is advantageous to seek not one solution for the assigned value of the parameter but a sequence of solutions for a certain range of values of this parameter. The search of the series of solutions for a number of monotonically varying values of the parameter makes it possible to use the information obtained on previous steps in the determination of the zero approximation necessary for the beginning of the iterative process with the following value of the parameter. Such a method makes it possible to decrease considerably the expenditures of machine time, and frequently - generally make the process converging. As a parameter it is possible to take the value of the load.

One of the methods which satisfies the indicated requirement is the method of differentiation with respect to the parameter proposed by V. S. Kiriya [59] and D. F. Davidenko [37]. This method is developed in connection with the theory of shells in works of I. I. Vorovich and his students [24, 40]. The method consists in the replacement of the system of the finite nonlinear equations by Cauchy problem for the system of ordinary differential equations in terms of the parameter. If the solution for a certain value of the parameter is known, then the numerical integration makes it possible to find the sequence of the solutions for the monotonically varying values of the parameter. The indicated method is successfully used in cases of nonlinear shells. A certain shortcoming in it is the possibility of the accumulation of an error in the process of integration.

In V. I. Feodos'yev's method [111] the transition to the Cauchy problem is fulfilled naturally - by the replacement of the static problem on the behavior of the shell by the quasi-dynamic. Introducing the inertia and dispersive terms and considering the load of a slowly changing time function, one of the approximation methods from the system of partial differential equations obtains with the problem the initial conditions for ordinary differential equations with respect to time.

M. S. Kornishin and Kh. M. Mushtari [62] proposed solving the system of nonlinear equations with the parameter

$$f_i(z_1, z_2, \ldots, z_n, p) = 0, \; i = 1, 2, \ldots, N$$

so that the solution Z_k, obtained for parameter p_k, is taken as the zero approximation Z^0_{k+1} for $p_{k+1} = p_k + \Delta p$. M. S. Kornishin [63], proposed using for determining the initial approximation the extrapolation with respect to formulas $Z^0_{k+1} = 2Z_k - Z_{k-1}$, $Z^0_{k+1} = 3Z_k - 3Z_{k-1} + Z_{k-2}$ when $z^0_{i,k+1} = (2z_{i,k} - z_{i,k-1})\frac{z_{i,K}}{z^{(0)}_{i,k}}$.

The advantage of such an approach consists in the absence of the accumulation of an error with a change in the parameter, since for each new value of it the solution is calculated by iterative means.

In the proposed method namely this method is used: Newton-Kantorovich method in conjunction with the extrapolation of the initial approximations. Before citing the formulas used, let us discuss the selection of the parameter.

As a parameter let us introduce the relative deflection of the shell $\zeta = w(\alpha_0, \beta_0)/h_0$ at a certain characteristic point α_0, β_0 (for circular plates and slightly curved shells of rotation - in the center) [48]. Such a selection of the parameter pursues two goals. It is known that if there occurs a knock of the flexible slightly curved shell (with a not very large ratio of the rise H to the thickness h_0), then function $\zeta(q_0)$ is ambiguous, and $q_0(\zeta)$ is single-valued (q_0 - the load parameter). Therefore, namely, quantity ζ is conveniently selected as a parameter, since the entire curve $q_0(\zeta)$ can be obtained with a monotonic charge in ζ. However, the following is more important. At the maximum points at which the load q_0 takes the upper or lower critical value, the determinant of the matrix (system of the linear algebraic equations of the Newton-Kantorovich method relative to correlations Δz_i), comprised of second-order partial derivatives of functional W in terms of x_i and y_i, vanishes [117]. At such points the process of iterations according to the Newton-Kantorovich method does not converge. But precisely these points are frequently of special interest for the researcher. Introducing as a parameter the value

$$\zeta = \frac{1}{h_0} \sum_{i=1}^{n} x_i \omega_i(\alpha_0, \beta_0), \tag{3.4}$$

we supplement system (3.3) with equation (3.4), and we consider q_0 to be unknown. The determinant of the Newton-Kantorovich matrix for thus expanded system at the maximum points does not vanish.

For components of the vector Z of the solution to this system, let us introduce notations $z_i = x_i$; $z_i = y_i$; $z_i = q_0$ respectively when $i = 1, 2, \ldots, n$; $i = n + 1, n + 2, \ldots, n + m$ and $i = n + m + 1$. Then the formula of the Newton-Kantorovich method take the form

$$Z^{(s+1)} = Z^{(s)} - A^{-1} F(Z^{(s)}), \tag{3.5}$$

where F is the column of discrepancies f_i of equations of system (3.3) and (3.4), and matrix A has components $\frac{\partial f_i}{\partial z_j}$.

We compute initial approximations by using the following system of extrapolaticnal formulas:

$$Z_1^{(0)} = Z_0;\ Z_2^{(0)} = 2Z_1 - Z_0;\ Z_3^{(0)} = 3Z_2 - 3Z_1 + Z_0; \tag{3.6}$$
$$Z_p^{(0)} = 4Z_{p-1} - 6Z_{p-2} + 4Z_{p-3} - Z_{p-4},\ p = 4, 5, \ldots,$$

whereupon $Z_p \equiv Z(\zeta_p)$, $\zeta_p = \zeta_0 + p\Delta\zeta$. These formulas provide for the gradual increase in the power of the extrapolational polynomial (from zero to the third power in the beginning of the process) and subsequent extrapolation by a polynomial of the third power.

It should be noted that formulas (3.6) are used not after the next (geometrically nonlinear with the assigned heterogeneity) problem of internal iterative cycle is solved, but upon the transition to the next value of the leading parameter ζ_p, i.e., after the solution of the complete (physically and geometrically nonlinear) problem for ζ_{p-1}.

In the solution of the physically nonlinear, but geometrically linear, problems, the internal iterative cycle is degenerated, since the system (3.3) and (3.4) becomes linear and the Newton-Kantorovich method gives its solution for one iteration. In this case the proposed process realizes one of the possible ways of the combination of methods of variable elasticity parameters and variation. These methods are distinguished by the form of the variational equation. For the first time the combined use of these methods was proposed by L. M. Kachanov [56], who used functionals of the theory of plasticity based on principles of possible displacements and the minimum of supplementary work of the body.

The convergence of the method of variable elasticity parameters has thus far not been strictly demonstrated; however, experience shows that usually the process converges.

3. Coordinate Functions

In order to obtain the solution to the problem of the steady state of the functional W, it is necessary to assign coordinate functions w_i and ϕ_i. From the theory of the Ritz method [80] it is known that the coordinate system should be subordinate to some conditions.

First, the coordinate functions should belong to the region of definition of the functional. This means that functions w_i and ϕ_i should be subordinate to the main boundary conditions (2.31) and (2.34); they also should be continuous together with the first-order derivatives and have the integrated squares of second-order derivatives. It is not difficult to fulfill all these requirements, since the main boundary conditions for w and ϕ are simple, and the class of functions which ensures the existence of the integrals in terms of the region in equation (2.24) is very wide.

In the second place, coordinate functions should be linearly independent at any n and m. It is possible to check the linear independence of the coordinate elements (if it is not obvious), having composed the Gram determinant - if it is not equal to zero, then this requirement is fulfilled.

The third condition - the completeness of the coordinate system in a certain linear separable metric space - will be considered in the solution of the specific problems to be fulfilled.

Usually when using variational methods in the theory of plates and shells, accepted as coordinate functions are trigonometric, hyperbola-trigonometric and exponential polynomials. The experience of the solution of linear problems [46] showed that under conditions of the conducting of calculations by the computer, preference should be given to the exponential polynomials. This is explained by the simplicity of programming and the small expenditure of time for the calculation of values of these functions and their derivatives in the nodes. The rate of convergence of the Ritz process according to number of parameters n and m when using exponential polynomials is not less but, in certain cases, more than that with other types of coordinate functions.

We will assume that in the nonlinear case these properties are retained. In this work the problems are solved with coordinate functions of precisely such a class.

If the unknown values w and ϕ are functions of two coordinates α and β, it is possible to assume that

$$w_i = \lambda_i(\alpha)\chi_i(\beta); \quad \varphi_i = \lambda_i^0(\alpha)\chi_i^0(\beta),$$

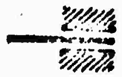
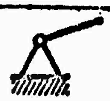
as is done, for example, in [46], and assign λ_i, χ_i, λ_i^0, and χ_i^0 in the form of exponential polynomials which satisfy the necessary conditions at the ends of the interval.

In problems on the axisymmetric deformation of circular plates and shells of rotation, functions w and ϕ depend on one coordinate. In the subsequent chapters namely such problems are examined, and it is considered that the shell is closed at the vertex, while accepted as coordinate α is the radius in plane r.

The set of the functions $w_i(\rho)$ and $\phi_j(\rho)$ when i, j = 1, 2, ... is reduced in Table 1 ($\rho = r/a$, where a is a radius of the shell in the plane). All these functions are subordinate to the symmetry condition relative to the center. Functions $w_i(\rho)$ satisfy conditions $w_i(1) = w_i'(1) = 0$ with the attachment of the edge and $w_i(1) = 0$ with a free support. Functions $\phi_j(\rho)$ in the case of rigid fastening in the middle surface (natural boundary conditions) are subordinate to no conditions on the edge; with the sliding attachment the edge is free from stresses $T_n(1) = T_1(1) = 0$. These conditions are fulfilled when $\phi_j(1) = \phi_j'(1) = 0$, and functions $\phi_j = (1 - \rho^2)^{j+1}$ satisfy them.

Coordinate functions of such type are used by many authors, beginning from the first works on the geometrically nonlinear theory of shells and until recently. The convenience in them consists of the simplicity because of which it is possible to obtain precise formulas (3.12) for coefficients of the nonlinear part of the system (3.3). Although the orthonormalized polynomials lead to the greater stability of the Ritz systems with respect to errors for calculations, the simplicity of functions of Table 1 makes them more preferable where a certain deterioration of feature of the Ritz system is not reflected on results of the calculation.

Table 1.

$\phi_j(\rho)$	$w_j(\rho)$	
	$(1-\rho^2)^{j+1}$	$(1-\rho^2)^j$
ρ^{2j} $(1-\rho^2)^{j+1}$	 	 

4. Algorithm

Taking into account the need for obtaining solutions in high approximations, let us construct the algorithm which we use for performing the calculations by computer.

Examined in subsequent chapters is the axisymmetric deformation of slightly curved shells of rotation closed in the vertex and circular plates (Fig. 5) under the action of the load $Z = q(r)$ distributed normal to the middle surface and temperature field $T(r, \gamma)$. Therefore, it is accepted that $U = M_n^0 = Q_n^0 = 0$. Furthermore, the shells are considered slightly curved so much that they cannot be presented in the form of plates with initial chamber ($\Delta_k w = \Delta_k \phi = 0$), and the metrics of the surface can be replaced by the metrics of the plane. Such an approach is widely used in literature.

In polar coordinates with axial symmetry we have $w = w(r)$, $\phi = \phi(r)$, $\alpha = r$, $\beta = \theta$, $ds = 2\pi r\,dr$; and the operators take the form

$$\left.\begin{aligned} \Delta &= \frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr};\\ L_1(\psi, \eta) &= \frac{d^2\psi}{dr^2}\cdot\frac{d^2\eta}{dr^2} + \frac{1}{r}\cdot\frac{d\psi}{dr}\cdot\frac{1}{r}\cdot\frac{d\eta}{dr};\\ L_2(\psi, \eta) &= \frac{d^2\psi}{dr^2}\cdot\frac{1}{r}\cdot\frac{d\eta}{dr} + \frac{1}{r}\cdot\frac{d\psi}{dr}\cdot\frac{d^2\eta}{dr^2}. \end{aligned}\right\} \tag{3.7}$$

We consider further all the functions α, β, and γ as functions only of the radius r and z-coordinate γ. We assign the form of the middle surface of the shell by the initial chamber

$$w_0 = -h_0 k(1 - c_1\rho - c_2\rho^2), \quad k = \frac{H}{h_0}. \tag{3.8}$$

Here H is the height of the shell above the plane. When $c_1 = 1$ and $c_2 = 0$ we obtain a cone; $c_1 = 0$ and $c_2 = 1$ gives a surface similar to a sphere; value $k = 0$ corresponds to the plate.

Let us introduce the relative values

$$\left.\begin{array}{c}
\bar{w}=\frac{w}{h_0};\ \bar{x}_i=\frac{x_i}{h_0};\ \bar{\varphi}=\frac{\varphi}{G_0h_0^3};\ \bar{y}_i=\frac{y_i}{G_0h_0^3};\ \bar{h}=\frac{h(\varrho)}{h_0};\\
h_0=h(0);\\
w_0^*=\frac{w_0}{h_0};\ \bar{w}_0=\frac{w_0^*}{k};\\
\bar{q}=\frac{q(\varrho)}{G_0}\left(\frac{a}{h_0}\right)^4;\quad \bar{q}_0=\bar{q}(0);\ \bar{\sigma}=\frac{\sigma}{G_0}\left(\frac{a}{h_0}\right)^2;\ \bar{\varepsilon}=\varepsilon\left(\frac{a}{h_0}\right)^2;\\
\bar{T}_i^0=\frac{T_i^0}{G_0t_0T_0h_0};\quad \bar{M}_i^0=\frac{M_i^0}{G_0t_0T_0h_0^2};\quad \bar{T}_0=t_0T_0\left(\frac{a}{h_0}\right)^2;\\
\begin{pmatrix}\bar{A}_{ww}\\ \bar{B}_{ww}\end{pmatrix}=\frac{1}{G_0h_0^3}\begin{pmatrix}A_{ww}\\ B_{ww}\end{pmatrix};\quad \begin{pmatrix}\bar{A}_{w\varphi}\\ \bar{B}_{w\varphi}\end{pmatrix}=\frac{1}{h_0}\begin{pmatrix}A_{w\varphi}\\ B_{w\varphi}\end{pmatrix};\\
\begin{pmatrix}\bar{A}_{\varphi\varphi}\\ \bar{B}_{\varphi\varphi}\end{pmatrix}=G_0h_0\begin{pmatrix}A_{\varphi\varphi}\\ B_{\varphi\varphi}\end{pmatrix}.
\end{array}\right| \qquad (3.9)$$

Here G_0, t_0, and T_0 are the characteristic values of the shear modulus in the undeformed state, the coefficient of linear expansion and temperature.

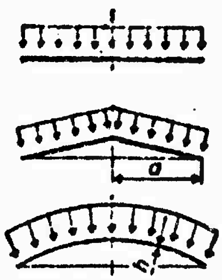


Fig. 5.

Replacing in operators (3.7) r by aρ and substituting formulas (3.9) into equations (3.3), after division by $2\pi G_0 h_0^5/a^4$ and deletion of the line above the relative values, we reduce the system (3.3) to the dimensionless form

$$\int_0^1\Big\{\sum_{j=1}^{n}x_jR^0(w_i,w_j)-\sum_{j=1}^{m}y_j[R^0(w_i,\varphi_j)+kw_iL_2(w_0,\varphi_j)]-$$
$$-\sum_{j=1}^{n}\sum_{s=1}^{m}x_jy_sw_iL_2(w_j,\varphi_s)+T_0M_i^0\Delta w_i-q_0qw_i\Big\}\varrho d\varrho=0, \qquad (3.10)$$
$$i=1,2,\ldots,n;$$

$$\int_0^1 \Big\{ -\sum_{j=1}^{m} y_j R^0(\varphi_i, \varphi_j) - \sum_{j=1}^{n} x_j [R^0(\varpi_j, \varphi_i) + k\varpi_j L_2(\varpi_0, \varphi_i)] -$$

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{s=1}^{n} x_j x_s \varpi_j L_2(\varpi_s, \varphi_i) - T_0 T_i^0 \Delta\varphi_i \Big\} \varrho d\varrho = 0;$$

$$i = 1, 2, \ldots, m.$$

We consider further that in operators (3.7), instead of dr and r, there are $d\rho$ and ρ.

The system of equations (3.10) describes the axisymmetric deformation of flexible slightly curved heterogeneous shells and is used in the internal iterative process in the solution of physically nonlinear problems.

The coefficients of the linear part are computed by a machine approximately - integration over ρ is made through a 12-nodal Gauss formula. Values of derivatives of coordinate functions are determined from precise formulas. Numerical integration is caused by the fact that functions $A_{\psi\eta}$, $B_{\psi\eta}$, M_t^0, and T_t^0 have a complex nature. The matrix of the linear part of system (3.10) is symmetric, and therefore only its right upper triangle is computed.

Coefficients g_{ijs} and g_{jsi} with quadratic terms $x_i y_s$ and $x_i x_s$ do not contain physical parameters of the material and are calculated from precise formulas. When using coordinate functions given in Table 1, these coefficients can be presented in the form of the sum of integrals of the form

$$I(c, d) = \int_0^1 \varrho^c (1 - \varrho^2)^d d\varrho.$$

Let us transform I(c, d) with the aid of the apparatus of Eulerian functions [121]:

$$I(c,d)=\frac{1}{2}B\left(\frac{c+1}{2},d+1\right)=\frac{1}{2}\frac{\Gamma\left(\frac{c+1}{2}\right)\Gamma(d+1)}{\Gamma\left(\frac{c+1}{2}+d+1\right)}=$$

$$=\frac{(2d)!!\,(c-1)!!}{(c+2d+1)!!}. \qquad (3.11)$$

Relation (3.11) makes it possible to obtain easily coefficients $g_{ijs} = -\int_0^1 w_i L_2(w_j, \phi_s)\rho\, d\rho$ for varied conditions of the attachment of the shell:

motionless fastening

$$g_{ijs}=4(i+1)(j+1)s\frac{s!}{(i+j+1)\ldots(i+j+s+1)},$$

fixed hinge

$$g_{ijs}=4ijs\frac{(i+j-2)!\,s!}{(i+j+s-1)!},$$

mobile fastening

$$g_{ijs}=-\frac{4(i+1)(j+1)(s+1)}{(i+j+s+1)(i+j+s+2)},$$

mobile hinge

$$g_{ijs}=-\frac{4ij(s+1)}{(i+j+s-1)(i+j+s)}. \qquad (3.12)$$

From formulas (3.12) it is evident that $g_{ijs} = g_{jis}$. By comparing the formula for g_{ijs} with the coefficient at $x_j x_s$ of the second group of equations (3.10) $g_{jsi} = -\frac{1}{2}\int w_j L_2(w_s, \phi_i)\rho\, d\rho$ we find $g_{jsi} = \frac{1}{2}g_{ijs}$. The noted symmetry makes it possible to calculate and, what is more important, to store in the immediate-access memory only $\frac{1}{2}(n^2 + n)m$ instead of $2n^2m$ coefficients of the quadratic terms. The three-dimensional matrix (its upper right half) with elements g_{ijs} is calculated in the beginning of the solution to the problem only one time, since it does not depend on the physical properties of the material and therefore remains constant during the execution of the external iterative process.

Derivatives of discrepancies $\frac{\partial f_i}{\partial z_j}$ of equations (3.10), which enter into the matrix A of the Newton-Kantorovich method (3.5), are computed accurately and incidentally with the calculation of the discrepancies themselves. This becomes possible because of the representation of the first group of equations (3.10) in the following form:

$$f_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{s=1}^{m} b_{is} y_s + \sum_{j=1}^{n} \sum_{s=1}^{m} g_{ijs} x_j y_s + q_0 c_i + T_0 d_i = 0 =$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left(a_{ij} + \frac{\partial f_i}{\partial x_j}\right) x_j + \frac{1}{2} \sum_{s=1}^{m} \left(b_{is} + \frac{\partial f_i}{\partial y_s}\right) y_s + \frac{\partial f_i}{\partial q_0} q_0 + T_0 d_i;$$

$$\frac{\partial f_i}{\partial x_j} = a_{ij} + \sum_{s=1}^{m} g_{ijs} y_s,$$

$$i = 1, 2, \ldots, n.$$

The second group of equations is similarly converted.

Actually instead of formula (3.5), which requires matrix inversion, we use another one:

$$A(Z^{(s-1)} - Z^{(s)}) = -F(Z^{(s)}). \qquad (3.13)$$

The system of algebraic equations (3.13) relative to corrections ΔZ is solved by the Gauss method with the selection of main element [107]. This method possesses high resistance to errors of calculations.

In the solution of the physically and geometrically nonlinear problem on each step of the external iterative process, the coefficients of the matrix of the linear part of the system (3.10) are computed again, since they include new values of functions $A_{\psi\eta}$, and $B_{\psi\eta}$ (the methodology for the calculation of these functions is given in Chapter VI). For the economy of machine time, in the calculation of values of operators L_1 and L_2 in nodes ρ_j of the quadrature Gauss formula, before the beginning of the iterations, $\frac{1}{\varrho}\frac{\partial \omega_i}{\partial \varrho}$, $\frac{\partial^2 \omega_i}{\partial \varrho^2}$, $\frac{1}{\varrho}\frac{\partial \varphi_i}{\partial \varrho}$, $\frac{\partial^2 \varphi_i}{\partial \varrho^2}$ are calculated and stored.

In the case of a material subordinate to the Hooke's law, the coefficients of the linear part of system (3.10) are computed only one time.

5. Flowchart

Enlarged flowchart of the major portion of the program developed for the computer "Ural-2" is given on Fig. 6.

Block 1 sends to the working cells the initial value of the leading parameter ζ_0 and vector of the initial $Z^{(0)}$. Block 2 sends vector Z to the group of cells Z*, and this is necessary for the organization of the dual iterative process. Block 3 calculates in 12 nodes of function $A_{\psi\eta}$, $B_{\psi\eta}$ and computes the coefficients of the linear part of the system (3.10). In block 4 the calculation of the matrix and right side of the system (3.13), its solution and the calculation of elements $Z^{(s)} = Z^{(s-1)} + \Delta Z$ are achieved. The result is sent to the Z cells. If the relative disagreement of all components of vectors $Z^{(s)}$ and $Z^{(s-1)}$ is less than that permissible, then we pass from block 6 to block 7.

The internal iterative process, which solves the heterogeneous geometrically nonlinear problem with fixed $A_{\psi\eta}$, $B_{\psi\eta}$ (block 4-6), is shown by a dashed line. If the problem is geometrically linear, block 6 is bypassed, since the solution is obtained for one iteration.

Block 8 checks if the necessary accuracy in the external iterative process (blocks 2-8) is reached. In physically linear problems this block is bypassed. After the process of successive approximations converged, block 9 calculates and prints the

Flowchart

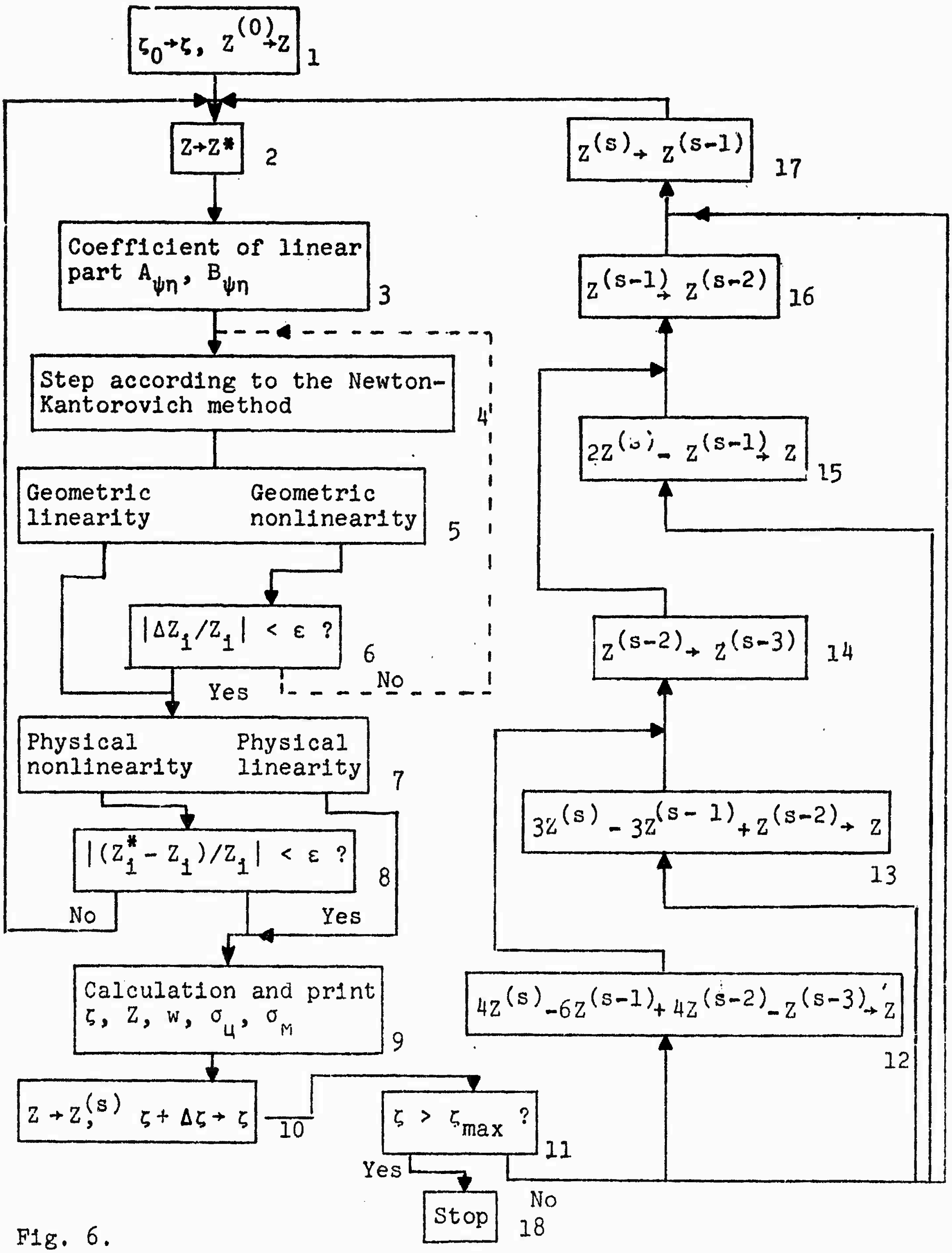


Fig. 6.

value ζ for which the problem is solved, the vector of the Ritz parameters, and the distribution of the deflection, flexural and diaphragm stresses.

Blocks 10-17 organized the transition to the following value ζ, and the circuit 11-17 with the aid of sendings and calculations according to formulas (3.6) fulfills the growth of the degree of the extrapolating polynomial (from 0 to 3).

Before the part of the program which corresponds to the flowchart begins to operate (see Fig. 6), another preparatory part fulfills the adjustment of the entire program for the quantity of coordinate functions n, m and auxiliary calculations, which can be done only once for the search of a series of solutions in the range $\zeta_0 \leq \zeta \leq \zeta_{max}$.

As initial information the program uses numerical data on the type and height of the shell (c_1, c_2, k), distribution of the thickness, transverse load, temperature field and on physical properties (data for the calculation of E and ν with respect to ε_0 and ε_1).

The method of attachment of the shell when using the Ritz method is wholly determined by conditions to which coordinate functions are subordinate. Introduced into the machine are coefficients of all the polynomials indicated in Table 1 (up to the 16th power) and the code information which make it possible to automatically take the necessary polynomials in the necessary sequence. The code information is a series of pseudoinstructions constructed so that in the first two octal positions (assigned on the computer "Ural-2" [2] for the number of operation the power of the polynomial is written, and in four positions (assigned for the address of the cell of the storage on ferrites) the address of the coefficient of the polynomial of the zero power of the argument is indicated. Thus, for a change in conditions of the attachment of the shell, it suffices to introduce into the memory of the machine only the other code information concerning the coordinate functions. This is one of the advantages of the Ritz method in the implementation of it on the computer.

CHAPTER IV

FLEXIBLE UNIFORM PLATES AND SLIGHTLY CURVED SHELLS OF ROTATION

1. Fundamental Principles

Examined in this chapter are the geometrically nonlinear problems of the bending of shells and plates made of a uniform ($E = E_0$, $\nu = \nu_0$, t = const) material subordinate to Hooke's law. The thickness of the shell is considered to be constant and equal to h_0 so that the relative thickness $\bar{h} = 1$. External loads are the evenly distributed pressure and, in individual problems, temperature field $T(r)$, which varies along the radius.

Taking the indicated conditions into account, and also expressions (2.8), (2.26) and (3.9), from equation (2.24) we obtain the variational equation of the problem in dimensionless form

$$\delta W = \delta \int_0^1 \left\{ \frac{1}{24(1-\nu_0^2)} [(\Delta w)^2 - (1-\nu_0) L_2(w, w)] - \frac{1}{2} \times \right.$$
$$\times \left[(\Delta \varphi)^2 - (1+\nu_0) L_2(\varphi, \varphi) \right] - \left[\frac{1}{2} L_2(w + 2k w_0, \varphi) + q_0 \right] \times$$
$$\left. \times\, w - T_0 T_i^0 \Delta \varphi \right\} \rho d\rho = 0, \qquad (4.1)$$

where q_0 and T_0 are the parameters of the load and temperature field. Function M_t^0 is equal to zero identically, since the temperature along the thickness is considered to be constant.

Let us write the formulas for the bending and torsional moments. From relations (2.33) it follows that

$$M_r = -D\left(w''_{rr} + \frac{\nu_0}{r} w'_r\right);$$

$$M_\theta = -D\left(\nu_0 w''_{rr} + \frac{1}{r} w'_r\right); \; M_{r\theta} = 0.$$

Hence flexural radial stress

$$\sigma_{ru} = \frac{6M_r}{h^2} = -E_0\left(\frac{h_0}{a}\right)^2 \frac{1}{2(1-\nu_0^2)}\left(\bar{w}''_{\varrho\varrho} + \frac{\nu_0}{\varrho}\bar{w}'_\varrho\right).$$

Bending stresses referred to $E_0(h_0/a)^2$, taking into account formulas (3.2), will be

$$\bar{\sigma}_{ru} = -\frac{1}{2(1-\nu_0^2)} \sum_{i=1}^{n} x_i \left(\bar{w}''_{\varrho\varrho} + \frac{\nu_0}{\varrho}\bar{w}'_\varrho\right)_i;$$
$$\bar{\sigma}_{\theta u} = -\frac{1}{2(1-\nu_0^2)} \sum_{i=1}^{n} x_i \left(\nu_0 \bar{w}''_{\varrho\varrho} + \frac{1}{\varrho}\bar{w}'_\varrho\right)_i. \qquad (4.2)$$

We will obtain diaphragm stresses by the division of stresses (2.11) by h_0:

$$\bar{\sigma}_{ru} = \sum_{i=1}^{m} y_i \frac{1}{\varrho}\bar{\varphi}'_{i\varrho}; \quad \bar{\sigma}_{\theta u} = \sum_{i=1}^{m} y_i \bar{\varphi}''_{i\varrho\varrho}; \qquad (4.3)$$

they are also referred to $E_0(h_0/a)^2$.

Let us recall that in view of the axial symmetry, after transition to the relative values in equation (4.1), operators

$$\Delta = \frac{d^2}{d\varrho^2} + \frac{1}{\varrho}\cdot\frac{d}{d\varrho}; \quad L_2(\psi, \eta) = \frac{d^2\psi}{d\varrho^2}\cdot\frac{1}{\varrho}\cdot\frac{d\eta}{d\varrho} + \frac{1}{\varrho}\cdot\frac{d\psi}{d\varrho}\cdot\frac{d^2\eta}{d\varrho^2}. \qquad (4.4)$$

In the solution to equation (4.1) according to the Ritz method, as algebraic equations relative to x_i, y_i it is possible to take the system (3.10). The dual iterative process, described in Sections 4 and 5 of Chapter III, is degenerated in the problems in question here into the single process of iterations of the solution to the system (3.10).

In the examples given below $\nu_0 = 0.3$.

2. The Nonlinear Bending of Circular Plates

Let us consider the bending of circular plates with rigid and sliding attachment caused by an evenly distributed load. The corresponding equation is obtained from relations for a slightly curved shell at the relative height of k = 0.

By introducing the coordinate functions from the first column of Table 1 into the calculation, in choosing $n = m = 4$, $\Delta\zeta = 0.5$ and beginning the calculations from $\zeta_0 = 0$, $Z^{(0)} = 0$, we obtain on the computer "Ural-2" functions $q_0(\zeta)$ shown on Fig. 7. The iterations ceased in the fulfilment of inequality $\left|\frac{z_i^{(s)} - z_i^{(s-1)}}{z_i^{(s)}}\right| \leq$ $\leq 10^{-5}$. Results of the calculation in the third approximation ($n = m = 3$) on the graphs merge with the appropriate curves of the fourth approximation, which indicates the good convergence according to the number of Ritz parameters. The straight lines correspond to the linear solution.

The data given (for $\zeta \leq 3.5$) are not new. The solution to the indicated problem in the first approximation is stated in a book [19]; the series method in the second approximation was used by Nadai [128]; by the method of collocation and net-point method this problem was solved by M. S. Kornishin [63]. The most precise results belong, apparently, to I. I. Vorovich and V. F. Zipalova [24], who used the Bubnov-Galerkin method.

Table 2 gives $q_0 = \frac{q}{E_0}\left(\frac{a}{h}\right)^4$ for the series ζ obtained in the article [24] and in this work. The divergence does not exceed unity in the fourth significant digit.

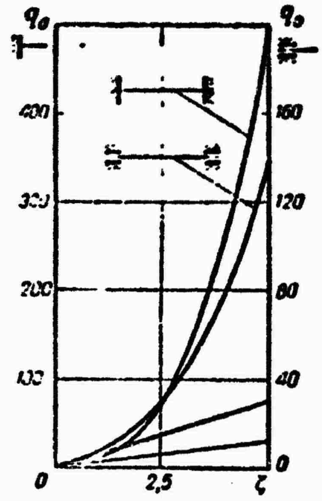


Fig. 7.

The convergence of the stresses is somewhat worse than that of values q_0. On Fig. 8a the solid line shows the curves of the greatest radial bending stresses when n = m = 4 and the dashed lines - when n = m = 3. Bending stresses of the fourth approximation differ when r = a and ζ = 5 from $\bar{\sigma}_{r\ \text{и}}$ of the fifth approximation by 1%. Let us note that the gradient $\bar{\sigma}_{r\ \text{и}}$ in the region of attachment (r ≈ a) is considerably more than that within the plate, and therefore the value $\bar{\sigma}_{r\ \text{и}}(a)$ is especially sensitive to the accuracy of the solution. The dependence of the greatest diaphragm radial stresses on the relative deflection in the center is given on Fig. 8b. Because of the use of formulas (3.6) with the assigned $\Delta\zeta$ the process coverges for two iterations.

Table 2.

ξ	From source [24]	n = m = 4
0.5	3.333	3.333
1.0	9.147	9.147
1.5	20.183	20.188
2.0	39.437	39.447
2.5	69.904	69.914

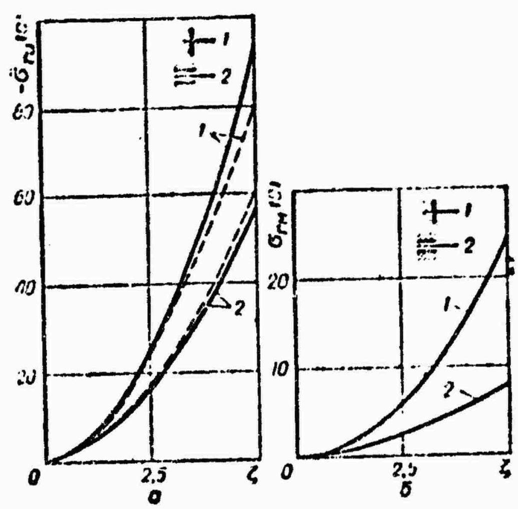


Fig. 8.

3. Bending and Stability in a Large Spherical Panel

The problem of the knocking of a slightly curved spherical shell under the action of a distributed load is classical for the geometrically nonlinear theory. The first works in this direction belong to V. I. Feodos'yev [109, 110]. Subsequently, the behavior of such a shell was investigated in the works of a number of American scientists and also I. I. Vorovich and V. F. Zipalova, M. S. Kornishin and other authors.

The inclusion into the book of the material of this section is explained not by the newness of the results given below but by the need for their comparison with those known in order to check the accuracy of the method.

The middle surface of the spherical cupola is assigned with the aid of formula (3.8), assuming in it that $c_1 = 0$, and $c_2 = 1$. We take the attachment in the form of a rigid seal. In accordance with Table 1, in this case

$$w_i = (1-\varrho^2)^{i+1};\ \varphi_j = \varrho^{2j},\ i, j = 1, 2, \ldots$$

The calculation is fulfilled when $n = m = 4$.

Figure 9 shows curves $q_0(\zeta)$ for $k = 4$ and $k = 5$. Calculations are made when $\Delta\zeta = 0.2$. The dependence $q_0(\zeta)$ for $k = 4$ was obtained earlier [24]. Table 3 compares values of q_0 from work [24] and our data. All values of q_0 are computed in the fourth approximation. The greatest disagreement takes place in the unstable region of curve ($\zeta = 2$). In the region of the upper q_B and lower q_H critical values, the divergence does not exceed 0.3%. Data of M. S. Kornishin and F. S. Isanbayeva [64] obtained by the finite-difference method are very close to these results. In this work the table is given for the leading parameter q_0 and not for ζ.

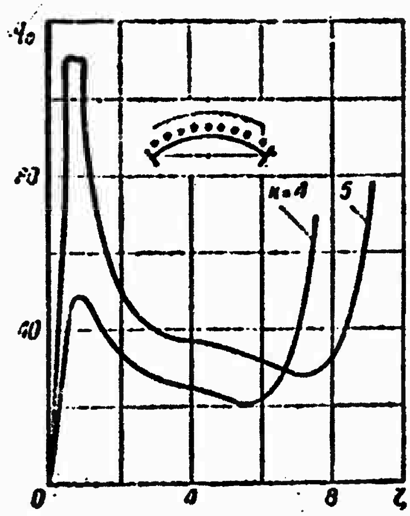


Fig. 9.

Table 3.

ζ	From source [24]	n = m = 4	ε, %
0.4	35.06	35.08	0.06
1.0	48.36	48.36	0
2.0	34.29	33.69	-1.78
5.6	21.04	21.10	0.28
6.8	32.35	32.45	0.31

Curves $q_0(\zeta)$ when k = 5 in the range $0 \leq \zeta \leq 1.5$, given on Fig. 10, are obtained when $\Delta\zeta = 0.02$. Shown there are shapes of deflection at the characteristic points. Values of the upper q_B and lower q_H critical loads coincide with data of the article [136] in which on the graphs, instead of the critical value of load $q_* \sqrt{3(1 - \nu^2)}q_*/8k^2$ is plotted, and instead of k - the value $4\sqrt{3(1 - \nu^2)}k$. Curves $q_0(\zeta)$ with the shape characteristic for k = 5 were obtained previously V. I. Feodos'yev [111].

Let us also give the distribution curves of radial stresses in the upper ($\zeta = 0.6$) and lower ($\zeta = 7.2$) critical states when k = 5 (Fig. 11); the solid lines are $\bar{\sigma}_{r\,H}$; the dashed lines are

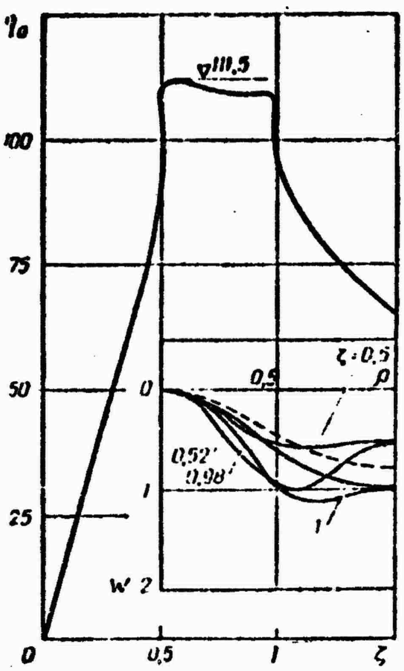


Fig. 10.

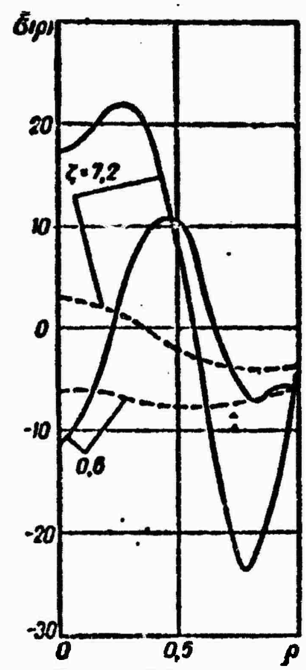


Fig. 11.

$\bar{\sigma}_{r\,M}$. In the upper critical state there occur compressive diaphragm stresses almost uniform on the radius, which contribute to the loss of stability in general and to a knock. In the lower critical state (after the knock) the level $\bar{\sigma}_{r\,M}$ noticeably decreased, but the bending stresses increased. It is interesting that value $\bar{\sigma}_{r\,И}(a)$, i.e., in sealing, is not greatest. An analysis of the distribution of stresses shows that for the correct determination q_B an obligatory account of the fact that the shell is not zero-moment is necessary.

The calculations of cupolas fulfilled according to the proposed method with other conditions of attachment also confirmed that the method gives accurate and not less accurate results than do other methods.

4. Conical Slightly Curved Shell

We obtain the initial chamber which assigns the shape of the middle surface of conical panel, assuming in formula (3.8) that $c_1 = 1$ and $c_2 = 0$.

Figures 12 and 13 show curves $q_0(\zeta)$ calculated in the fourth approximation [8] at the height of the shells k, equal to 1, 3, and 5, with constant thickness and with rigid and sliding attachment of the edge, respectively. The characteristic points obtained in source [30] by the Bubnov-Galerkin method (first approximation) are shown by small circles on Fig. 12. When $k > 3$ the divergence rapidly increases. The value of the upper critical load of the shell with height $k = 5$ coincides with that given in the article [115].

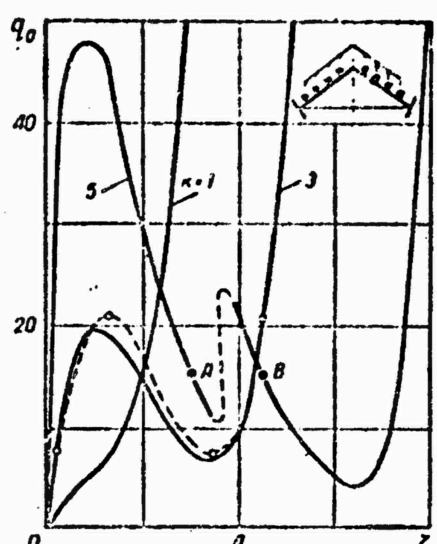


Fig. 12.

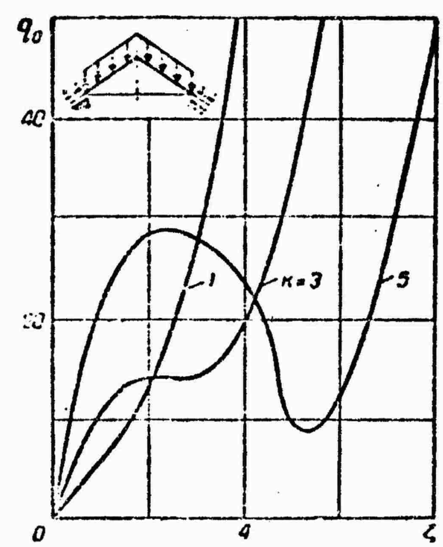


Fig. 13.

In the case of the rigidly attached shell with height $k = 5$ curve $q_0(\zeta)$ is obtained in the form of two parts: $\zeta < 3.5$ and $\zeta \geq 3.5$. In the method used for the solution of the problem in certain ζ, knowledge of solutions with previous values of deflection in the center is not required. Therefore, transition from one segment of the curve of states of equilibrium to another is achieved simply by an assignment of the new initial approximation for the solution of the Ritz system with the parameter ζ at which the process of iterations, connected with the motion along one of the parts of the curve, ceases to converge. The necessary initial approximation can be established by analyzing the nature of vector Z for values k, at which the discontinuity is absent.

In actuality the curve $q_0(\zeta)$ has no discontinuity. With a monotonic increase ζ (for example, from $\zeta = 0$) in certain cases (usually at relatively high values of k) the process of iterations of Newton-Kantorovich, in the solution of expanded system (3.3) and (3.4), ceases to converge. This occurs at the place where the derivative $\frac{dq_0}{d\zeta}$ becomes equal to infinity and the curve sharply changes direction.

When it is necessary to obtain the curve in the region of point $q = q^*$, $\zeta = \zeta^*$ with a vertical tangent, it is possible, without substituting the leading parameter ζ with another, to begin in the following way. Let us assume that at the next value $\zeta_p = \zeta_{p-1} + \Delta\zeta$ the process did not converge for a certain assigned number of iterations. Since the initial approximation of vector $Z_p^{(0)}$ for $\zeta = \zeta_p$ was calculated from solutions for four previous values of ζ (3.6), and there is no solution for $\zeta = \zeta_p$, then $Z_p^{(0)}$ corresponds to the point when $\zeta = \zeta_{p-1}$ on the part of the curve $q_0(\zeta)$ which lies on the other side (above when $\left.\frac{dq_0}{d\zeta}\right|_{\zeta=\zeta_{p-1}} > 0$ or below - otherwise) of point $q = q^*$. After repeating the process of iterations with the initial approximation $Z_p^{(0)}$ when $\zeta = \zeta_{p-1}$, we find this solution for the further motion along the curve $q_0(\zeta)$ and change the sign of $\Delta\zeta$ to the opposite.

The described algorithm is programmed simply and makes it possible to automate the control of the leading parameter ζ in order to obtain the entire curve, following its course.

With the bending of the rigidly attached shell with height k = 3 a smooth sagging of the center section occurs (Fig. 14). The shell with height k = 5 sags first at the periphery so that its shape (in the state which corresponds to the upper critical load) noticeably differs from the shape of the shell when k = 3. However, with a load equal to the lower critical load, the shapes of the deformed middle surface of these shells differ little. A sharp change in shapes of the shells when k = 5, upon the transition from one critical state to another, is accompanied by the appearance of a "discontinuity" of curve $q_0(\zeta)$. The difference between these shapes is especailly clearly visible on graphs $\zeta = 3$ and $\zeta = 4.5$, obtained when $q_0 \approx 15.5$ and corresponding to points A and B (Fig. 12).

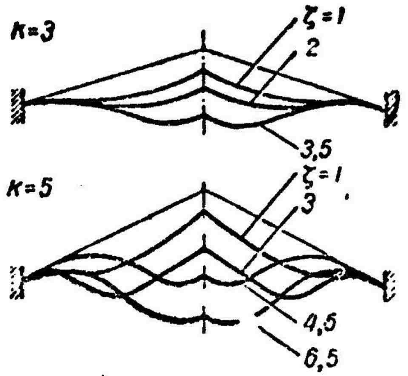


Fig. 14.

Figure 15 gives the dependence of critical values of the load on the height of the shell. When k < 1.8 the shell is stable in general. The solid line shows the result of the calculation in the fourth approximation, the dashed line - in the second, and dot-dash line - in the first according to the article [30]. The fifth approximation in this figure coincides with the fourth.

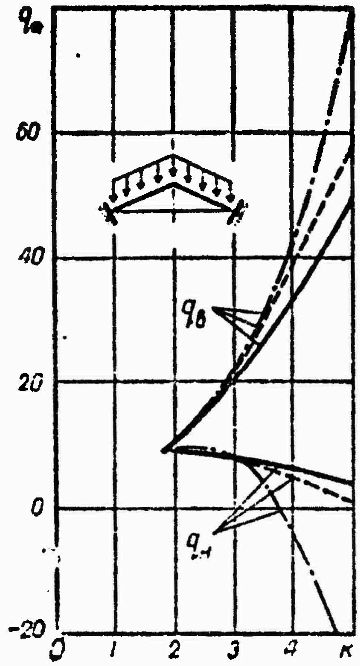


Fig. 15.

5. Plates and Shells in a Temperature Field Loaded by Pressure

The supercritical behavior of flexible plates and slightly curved shells loaded by pressure is of great interest for practice. However, the obtaining of the sufficiently exact solutions of this nonlinear problem requires the overcoming of a number of difficulties. Usually in the calculations we are limited to the first or second approximation, fulfilling the calculations manually, which cannot satisfy the practice either in accuracy or in the labor expense of the process of the obtaining of results.

The loss of the stability of plates in the temperature field is the subject of many works in which the linear uniform problem of eigenvalues - critical temperatures, is solved. If the temperature exceeds critical, the plate bulges and takes the form of a slightly curved shell but does not lose the abibity to resist a transverse load. Subsequently, depending on the direction of the bulge, a knock or monotonic increase in the deflection with an increase in the load is possible. Since in the supercritical state of deflection of the plate is of the order of the thickness, the problem becomes geometrically nonlinear.

Let us consider the circular plate strictly fastened on the edge stressed by the evenly distributed pressure. We consider the temperature field to be constant in thickness, and we assign it one of the following functions: $T_t^0 = 1 - \rho^2$, $T_t^0 = \frac{1}{2}$, $T_t^0 = \rho^2$ subordinate to the condition of normalization $\int_0^1 T_t^0(\varrho)\varrho d\varrho = \frac{1}{4}$. Such functions are selected in order to model most simply the decrease, constancy and increase in the temperature along the radius.

Figure 16 gives curves $q_0(\zeta)$ with uniform heating and $T_t^0 = \frac{1}{2}$ (function $q_0(\zeta)$ - odd; the figure depicts its branch for $\zeta \geq 0$). Curves of the third and fourth approximations merge. The dashed line shows the result of the calculation when $n = m = 2$ at $T_0 = 5$.

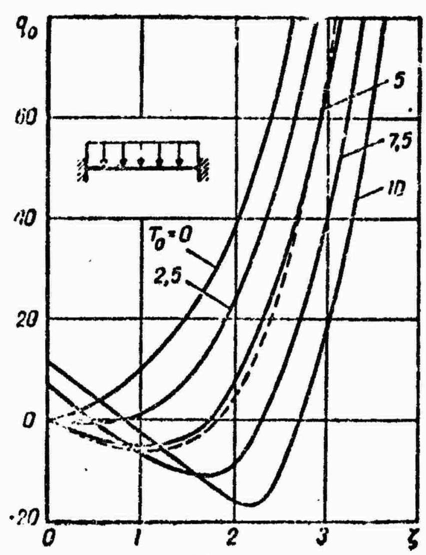


Fig. 16.

With an increase in the parameter of temperature T_0 the value $\left.\frac{dq}{d\zeta}\right|_{\zeta=0}$ decreases and becomes equal to zero at the critical value $T_0 = T_* = \frac{u_1^2}{6(1+\nu_0)}$, where $u_1 = 3.832$ is the first root of the Bessel function $J_1(u)$. When $\nu_0 = 0.3$, $T_* = 1.883$. The parameter T_* is determined in the solution of the linear problem and corresponds to the loss in stability in particular. A further increase in T_0 leads to an appearance and increase in absolute value of the critical load; the point of the curve $q_0 = 0$, which previously coincided with the beginning of the coordinates, is displaced along the axis ζ to the right.

Shapes of the curved surface for a number of values ζ (when $T_0 = 10$, $T_t^0 = \frac{1}{2}$) are shown on Fig. 17. Point $\zeta = 0$, as can be seen from Fig. 16, belongs to the section of curve $q_0(\zeta)$, which corresponds to the unsteady states of equilibrium.

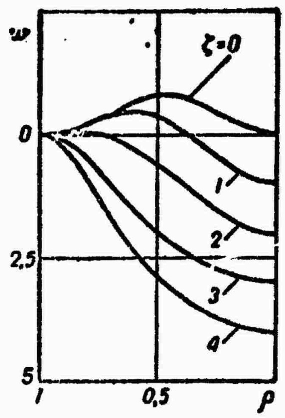


Fig. 17.

The behavior of the plate stressed by pressure when $T_0 > T_*$ can be traced on Fig. 18. When $q_0 = 0$ the plate has two steady states of equilibrium at points A and B with flexural shapes, which are a mirror image with respect to each other. The peculiarity of the position in comparison with the slightly curved shells consists in the fact that the plate does not have a lower critical load. Actually, in moving from A to the right and from B to the left, we arrive at equal in absolute value upper critical loads. With an increase in q_0 (in absolute value) a jump into new stable positions of equilibrium (A' and B') occurs.

Since with an increase in temperature the plate equiprobably bulges both in the direction of the γ axis and opposite it, the entire graph $q_0(\zeta)$ possesses a central symmetry relative to the beginning of the coordinates. The stresses and strains in the middle plane coincide at points with equal (in absolute value) values ζ; bending stresses and deflection have opposite signs at these points. When $\zeta = 0$ there are two mirror reflected unsteady states of equilibrium.

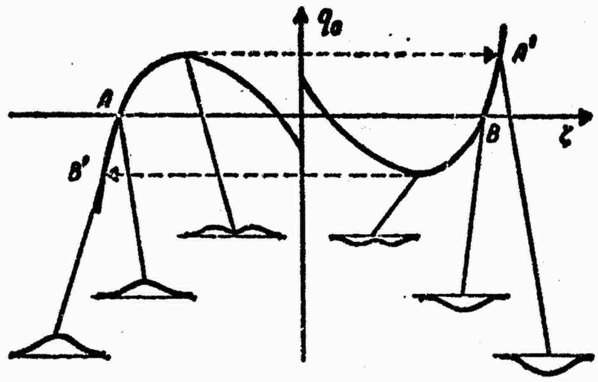


Fig. 18.

The indicated properties ensue from equation (4.1). In the case of the plate (k = 0) a simultaneous change in the sign at $x_j(x_s)$ and q_0 do not change these equations. The solution for y_j remains the previous one. The dependence of the flexural radial stresses in the sealing on the deflection in the center is given on Fig. 19.

We show one of the possible ways of the solution of geometrically nonlinear problems in the example [51] by the bending of the strictly attached conical shell stressed by an evenly distributed pressure and placed in a temperature field.

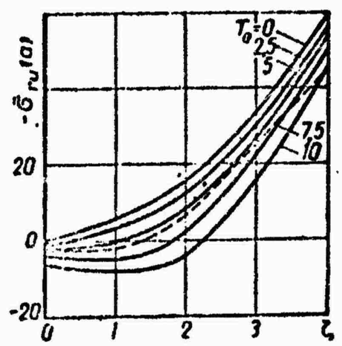


Fig. 19.

In the second approximation we take $w = (1 - \rho^2)^2 x_1 + (1 - \rho^2)^3 x_2$, $\phi = \rho^2 y_1 + \rho^4 y_2$. Substituting these formulas into (4.1) and performing integration, we obtain

$$W = c\left(\frac{16}{3}x_1^2 + 8x_1x_2 + \frac{24}{5}x_2^2\right) - \left(\frac{7}{5}y_1^2 + \frac{28}{5}y_1y_2 + \frac{164}{15}y_2^2\right) -$$

$$- k\left[\left(\frac{16}{15}y_1 + \frac{32}{35}y_2\right)x_1 + \left(\frac{32}{35}y_1 + \frac{64}{105}y_2\right)x_2\right] +$$

$$
-\left(\frac{2}{3}y_1+\frac{8}{15}y_2\right)x_1^2+\left(\frac{6}{5}y_1+\frac{4}{5}y_2\right)x_1x_2+\left(\frac{3}{5}y_1+\frac{12}{35}y_2\right)x_2^2-
$$

$$
-\left[\left(\frac{1}{6}x_1+\frac{1}{8}x_2\right)q_0+(y_1+py_2)T_0\right],\quad c=\frac{1}{12(1-\nu^2)}. \tag{4.5}
$$

The coefficient p is equal to 4/3; 2; and 8/3 for $T_t^0 = 1 - \rho^2$; 1/2; ρ^2, respectively.

Conditions of the stationarity of function (4.5) give the system of nonlinear algebraic equations

$$
\frac{\partial W}{\partial z_i}=0,\quad i=1,2,3,4. \tag{4.6}
$$

We supplement it by equation

$$
f_5\equiv x_1+x_2-\zeta=0\,; \tag{4.7}
$$

we assume further ζ to be assigned and q_0 to be the fifth unknown. To calculate the discrepancies f_i on each step of the process of the iterations of the solution of system (4.6) and (4.7), let us use the finite-difference formula

$$
f_i=\frac{\partial W}{\partial z_i}=\frac{1}{2\delta}\left[W(z_1,\ldots,z_i+\delta,\ldots,z_4,q_0,T_0)-\right.
$$

$$
\left.-W(z_1,\ldots,z_i-\delta,\ldots,z_4,q_0,T_0)\right],\quad i=1,2,3,4. \tag{4.8}
$$

The formula in central differences (4.8) is accurate for polynomials up to the second power inclusively, and, consequently, it is accurate for function (4.5).

The indicated method makes it possible to avoid the actual construction of nonlinear algebraic equations. In order to solve the problem, it is sufficient only to compile a program of the calculation of function W. The accuracy of the calculations in the solution of the nonlinear system $f_i = 0$, $i = 1, 2, \ldots, N$ depends on the accuracy of the calculation f_i. The values of derivatives of f_i, necessary for the method of Newton-Kantorovich et. al. can contain the error whose value will affect only the rate of the convergence of the process of the iterations. However, in the work with functions of form (4.5) the calculation of derivatives of f_i by means of centralized formulas will also lead to almost precise results, since the f_i are polynomials of not more than the second power.

The solution to system (4.6) and (4.7) was carried out on the computer "Ural-2" by the method of gradient descent of V. A. Matveyev [77] at the following values of the parameters: $T_0 = 0; 5; k = 0, 1, 2, 3, 4, 5$. Step δ is accepted equal to 0.01. Here the f_i are obtained with eight precise significant digits. The use of $\delta << 0.01$ lowers the accuracy as a result of the emergence of small differences.

The system of equations was solved with the assigned T_0 and k for the relative deflection ζ, which varies from 0 to 8 and from 0 to -2 with the step 0.2 and -0.1, respectively. The initial value of the vector Z with components x_1, x_2, y_1, y_2 and q_0 when $k = 0$, and $\zeta = 0$ is accepted as $Z^{(0)}(0; 0) = 0$. The vector $Z^{(0)}(0; 0.2) = Z(0; 0)$. Further it was considered that $Z^{(0)}(k; \zeta_{i+1}) = 2Z(k; \zeta_i) - Z(k; \zeta_{i-1})$. Upon transition to the next value of k the zero approximation is $Z^{(0)}(K_{i+1}, 0) = Z(k_i, 0)$. The process of iterations was finished with $\max |f_i| \leq 10^{-3}$.

Figure 20 gives graphs of $q_0(\zeta)$, obtained when $T_t^0 - = \frac{1}{2}$ [sic] and $T_0 = 5$. From the curve of states of equilibrium it is possible to trace the behavior of the shell. Let us assume that ζ is beginning with a large negative value. Load $q_0 < 0$ will also increase and at a certain moment will becone equal to zero. The state of strain of the shell in this case is determined only by the temperature field. The corresponding value of ζ shows how much higher the conical panel became due to a temperature expansion. With a further increase in ζ the load reaches the upper critical level, then the section of the unsteady states of equilibrium proceeds. The lower critical load in this example is close to zero. A further increase in ζ requires a rapid increase in q_0 – the knocked shell is now expanded under the effect of pressure.

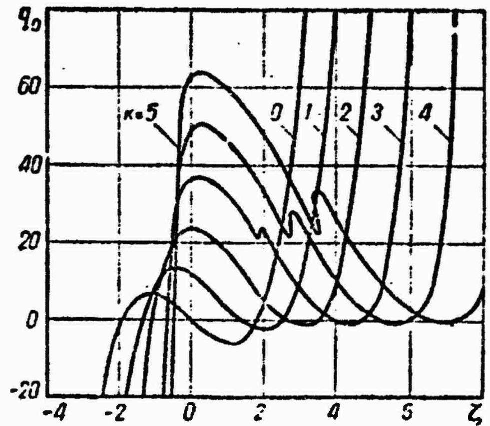


Fig. 20.

The dependence of critical pressures q_* on the nature of the distribution of temperature (Fig. 21) shows that the form of function $T_t^0(\rho)$ substantially affects values of q_*. With an increase in values of k, the connection of q_B^* with T_t^0 decreases. On Fig. 21 curves 1 correspond to $T_t^0 = 1 - \rho^2$, curves 2 – $T_t^0 = \frac{1}{2}$, and curves 3 – $T_t^0 = \rho^2$. In more detail let us give results [52] for $T_t^0 = 1 - \rho^2$. Figure 22 shows curves of $q_*(k)$ for a number of values of parameter T_0. Curve of the change in the deflection in the center of critical states are given on Fig. 23, and flexural forms for k = 5 – on Fig. 24. As a rule, the flexural radial stresses in the center when $q_* = q_B$ are greatest. The dependence of these stresses on k when $q_* = q_B$ is shown on Fig. 25, and when $q_* = q_B$ – on Fig. 26.

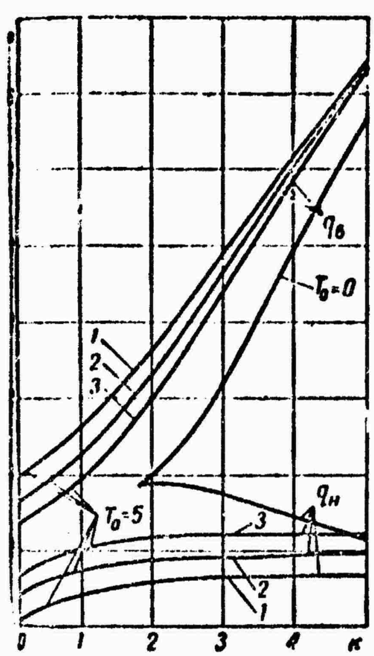


Fig. 21.

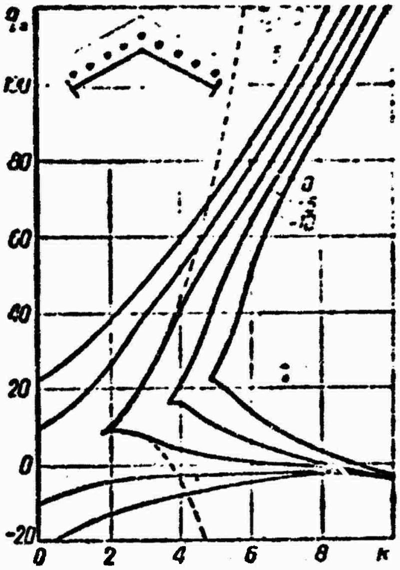


Fig. 22.

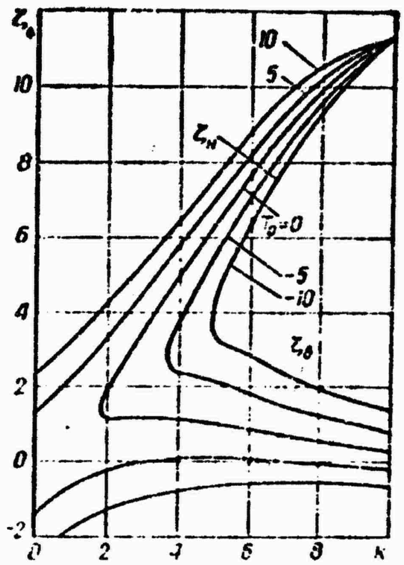


Fig. 23.

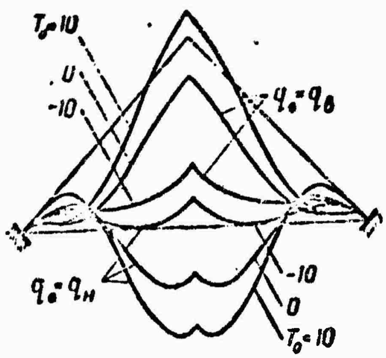


Fig. 24.

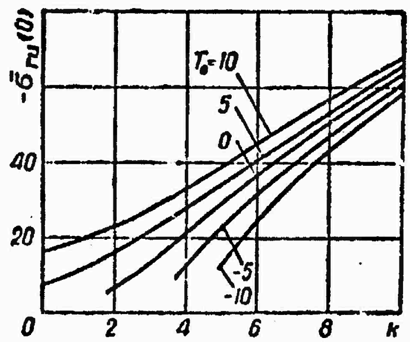


Fig. 25.

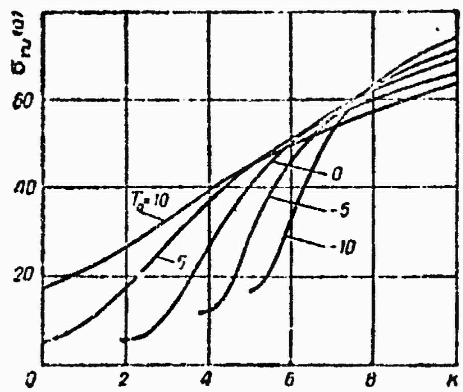


Fig. 26.

6. The Direct Determination of Critical States of Flexible Slightly Curved Shells

One of the main purposes in the study of nonlinear elastic systems is the search for critical points which separate the regions of steady and unsteady states of equilibrium. In the case of flexible slightly curved shells such points are usually the limiting points at which the upper or lower critical state is reached. The method of the direct determination of critical states which, unlike the usual method, does not require the plotting of curves of positions of equilibrium proposed by A. A. Kurdyumov [66].

In this work the system of equations whose solutions describe the critical states is compiled on the basis of the theory given above. As an example of the use of the method, problems of stability in large conical and spherical panels, which are under effect of pressure with four basic kinds of boundary conditions, are solved.

Let us consider conservative elastic nonlinear system [50] with n degrees of freedom. The variation of total energy of such a system is a function of the generalized coordinates z_i and parameter of the load q_0:

$$\delta U = \delta U(z_1, z_2, \ldots, z_n, q_0).$$

Equilibrium states (point of stationarity of the function U) are determined by equations

$$f_i = \frac{\partial U}{\partial z_i} = 0, \quad i = 1, 2, \ldots, n. \qquad (4.9)$$

Each assigned value of parameter q_0, by virtue of the nonlinearity of system (4.9), corresponds to $m \geq 1$ solutions $\{z_i^{(k)}\}$. The dependences

$$z_i^{(k)} = z_i^{(k)}(q_0),\ i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m \qquad (4.10)$$

represent parametrical equations of curves of states of equilibrium.

It is known [96, 117] that at points of bifurcation (branching off of the curves) and the limiting points in which $\frac{dz_i}{dq_0} = \infty$, the Jacobian functional determinant is equal to zero:

$$F(z_1, z_2, \ldots, z_n, q_0) \equiv \begin{vmatrix} \frac{\partial^2 U}{\partial z_1^2} & \frac{\partial^2 U}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2 U}{\partial z_1 \partial z_n} \\ \frac{\partial^2 U}{\partial z_2 \partial z_1} & \frac{\partial^2 U}{\partial z_2^2} & \cdots & \frac{\partial^2 U}{\partial z_2 \partial z_n} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2 U}{\partial z_n \partial z_1} & \frac{\partial^2 U}{\partial z_n \partial z_2} & \cdots & \frac{\partial^2 U}{\partial z_n^2} \end{vmatrix} = 0. \qquad (4.11)$$

Formula (4.11) expresses the condition of ambiguity of the solution of the linear algebraic system relative to $\frac{dz_i}{dq_0}$. This system can be constructed by substituting relations (4.10) into equation (4.9) and differentiating the obtained identities with respect to q_0. Formula (4.11) shows also that at the critical points the second variation in the total energy of the elastic system is equal to zero.

In the linear problems U there is a uniform quadratic function z_i, and $F(q_0) = |A - q_0 B| = 0$, where A and B are numerical matrices. Methods of the solution of such equations worked out in detail.

In general when system (4.9) is nonlinear, for determining the critical parameters, we usually plot curves of equilibrium states, and the points of bifurcation and limiting points are found on the graph. This way requires performing of very laborious calculations, since for the search of each point of each of the curves it is necessary to solve the nonlinear problem.

The following method is more advisable. We will consider the parameter of the load to be unknown and will form the system which includes equations (4.9) and (4.11):

$$\frac{\partial U}{\partial z_i}=0,\ i=1,2,\ldots,n;\ F(z_1,z_2,\ldots,z_{n+1})=0, \qquad (4.12)$$

where $z_{n+1} = q_0$. The solutions to problem (4.12) are the critical equilibrium states, since the first n of the equations determine the equilibrium states, and the latter is fulfilled only at critical points.

The configuration of the elastic system in the critical state is described by a set of generalized coordinates $z_i = z_{i*}$, and the load - by the value of parameter $q_0 = q_*$.

In order to explain if the obtained value q_* is equal to the upper (lower) critical load, one should establish if the quadratic form $Q = \frac{1}{2}\sum_{i,j=1}^{n+1}\frac{\partial^2\Psi}{\partial z_i \partial z_j}dz_i dz_j$ of function $\Psi=q_0+\sum_{k=1}^{n}\lambda_k f_k$ is negatively (positively) determined.

Actually, the search of critical values q_* can be considered as the problem to the extremum of function $p(z_1, z_2, \ldots, z_n, q_0) \equiv q_0$ under conditions (4.9). The solution to this problem, according to the method of Lagrange's indefinite factors, coincides with the absolute extremum of function $\Psi(z_1, z_2, \ldots, z_n, q_0, \lambda_1, \lambda_2, \ldots, \lambda_n)$.

The factors λ_k necessary for calculation of derivatives $\frac{d^2\Psi}{\partial z_i \partial z_j}$ in the critical point can be found from linear algebraic system $\frac{\partial\Psi}{\partial z_i} = 0$, $i = 1, 2, \ldots, n + 1$ relative to λ_k. The first n of the equations of this system are uniform but have a significant solution, since their determinant coincides with F and at the critical points is equal to zero.

Let us note that the system of equations which determines the positions of equilibrium should not have the form (4.9), i.e., it should directly ensue from the condition of stationarity of the total energy. These equations can be obtained by other ways - by the Bubnov-Galerkin method, the net point method, etc.

In the theory of slightly curved shells, for the construction of these equations, it is convenient to proceed from the expression for a variation in total energy (2.24). Therefore, as system (4.9) we use equations (3.10) and will solve the combined system of n + m + 1 equations (3.10) and (4.11) with unknowns x_i ($i = 1,2, \ldots, n$), y_i ($i = 1,2, \ldots, m$), and q_0. This system includes parameter k (relative height of the shell). To seek the initial approximations, we use the extrapolation according to formulas (3.6), where each of the solutions corresponds to one of the monotonically varying values k. As the initial approximation for the first value k, we take the critical state obtained for this k by the usual means.

We use the Newton-Kantorovich method. We compute first-order derivatives of the determinant F in terms of z_i, necessary for the construction of the matrix of the system of algebraic equations of relative discrepancies, according to the formula of central differences. The fulfilment of one iteration requires, with such a method, the calculation of $2(n + m) + 1$ value of a determinant of the n + m-th order. The derivative $\frac{\partial F}{\partial q_0} = 0$, since F does not directly depend on q_0. For the economy of machine time, the determinants are calculated from the square root method [107]. This is possible, since matrix $\frac{\partial^2 U}{\partial z_i \partial z_j}$ ($i, j = 1, 2, \ldots, n + m$) is symmetric.

Figure 27 shows[1] the dependences obtained by the given method of critical loads of slightly curved spherical cupolas on the geometric parameter of the shell. Plotted along the axis of the abscissas is the parameter of height $\lambda = 4\sqrt{3(1-\nu^2)} \times \frac{H}{h} \approx 6.6$ k; along the axis of the ordinates - the value q^0 equal to q_* referred to the critical pressure of a complete spherical shell of radius $R = a^2/2H$ according to the linear theory $q^0 = \frac{\sqrt{3(1-\nu^2)}q_*}{8k^2}$. Curves 1 and 2 denote the results of the calculation of the upper critical loads of a rigidly attached panel in the fourth and fifth approximations, respectively. In the region $\lambda < 60$ data of the fifth and sixth approximations differ by not more than 2%. Values of q^0 in this region are 5% different from those given in source [137]. At $\lambda < 25$ values q^0_B and q^0_H coincide with the data of source [24].

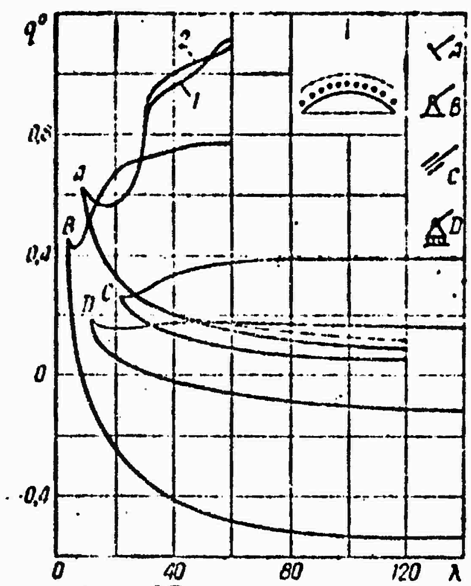


Fig. 27.

Curves q^0_B for boundary conditions B, C and D are given in the fourth approximation. Lower critical loads q^0_H for all types of attachments are also given in the fourth approximation. For q^0_H curve A almost coincides with that obtained in source [137]; curves B, C and D are new.

Forms of deflection of the shell in the upper critical states differ by the complex wave formation (Fig. 28a, k = 10), while with the lower critical load the shape of the middle surface is simple (Fig. 28b, k = 10).

[1]The data given below are obtained in conjunction with that of L. M. Afanas'yeva, (News of the Accademy of Sciences of the USSR, Mechanics of a Solid body, 1969, 5).

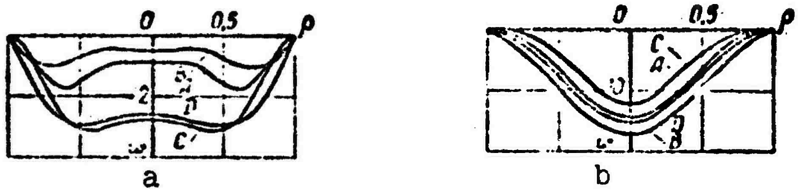


Fig. 28.

The completeness of the data shown on Fig. 27 becomes clearer in its comparison with Fig. 18.11 of the monograph [22].

Figure 29 shows q_B^0 and q_H^0 of conical shells with four kinds of boundary conditions. The convergence of the Ritz process was checked for all kinds of the attachments. All curves of q_B^0 are obtained in the fifth approximation, with the exception of case A, which when $\lambda > 60$ is calculated when $n = m = 6$. For this kind of boundary conditions the fifth and sixth approximations coincide when $\lambda < 60$.

Values of q_H^0, under all conditions of support, are given in the fifth approximation. Its results are merged on the graphs with results of the fourth approximation in the indicated range of values λ and are new.

The values obtained for the cone q_B^0 differ from those given in work [120] by not more than 5%. Let us say, however, the concept of the upper critical pressure used in L. I. Shkutin's work differs somewhat from our own.

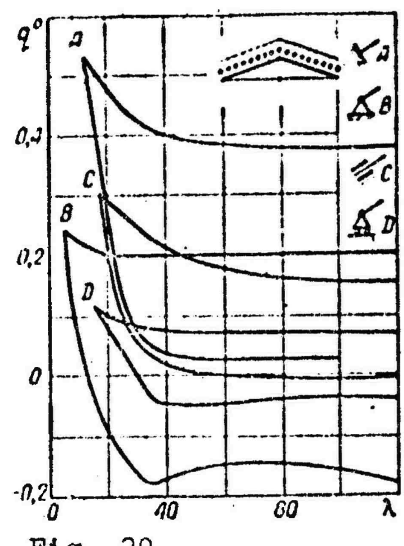
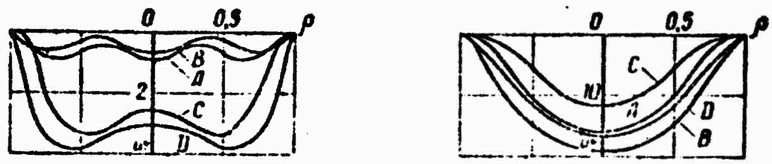


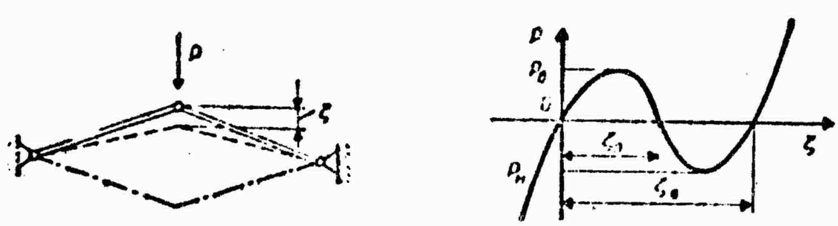
Fig. 29.

The forms of the deflections of the conical shell with a height of k = 10 are shown on Fig. 30.



a b
Fig. 30.

Let us note that with the hinged attachment of spherical and conical cupolas and with the sliding attachment of the conical panels, there are wide regions of λ in which $q_H^0 < 0$. The emergence of negative lower critical loads is easy to comprehend if we analyze the behavior of the simple rod system shown on Fig. 31a. The shape of curve $P(\zeta)$ is given on Fig. 31b. The point $\zeta = \zeta_0$ corresponds to the unstable horizontal position of the rods. When $\zeta = \zeta_*$ the form of system is a mirror reflection of the initial one. Function $P(\zeta - \zeta_0)$ is odd (is polar symmetric relative to point $\zeta = \zeta_0$). The upper critical load P_B is equal to the absolute value of the negative lower load P_H.



a b
Fig. 31.

The characteristic of this rod system is the absence of strains and stresses at the origin of the coordinates and at point $\zeta = \zeta_*$ after a knock. In the case of slightly curved shells (even with the free support of the edge) in a knocked state strains and stresses are not equal to zero, and the shape of the shells is not a precise mirror reflection of the initial. In this way the absence of the polar symmetry of curves $q_0(\zeta)$ relative to any point on the horizontal axis is caused. However, the lower critical load with not very small λ, nevertheless, remains negative, although its absolute value becomes less than q_B. The qualitative analysis of different forms of the symmetry of curves $q_0(\zeta)$ was done by D. I. Shil'krut [119].

CHAPTER V

FLEXIBLE HETEROGENEOUS PLATES AND SHELLS

1. Variation Equation and Basic Formulas

The study of uniform slightly curved shells and plates in a geometrically nonlinear setting is the subject of a large number of works. However, the effect of heterogeneities on the behavior of flexible shells still remains almost completely unstudied. This is explained, apparently, only by the considerable difficulties in the solution of similar problems, since their practical importance is doubtless.

With the presentation of the theory and method proposed in this work, it was stressed that they are convenient for the solution of problems in which one must take the heterogeneity into account. Actually, in one of the advantages of method it is easy to be convinced, after noting that the variational equation upon transition from a uniform to a heterogeneous case barely changes, the algorithm does not become more complex.

In this chapter let us consider two types of heterogeneities. The first - the variability of thickness (with uniform properties of the material) - can be conditionally classified under this section, since in an algorithmic sense the introduction of function $\bar{h} = \bar{h}(\rho)$ leads to those same consequences as that of the appearance of functions $E = E(\rho)$; $\nu = \nu(\rho)$: coefficients $A_{\psi\eta}$ and $B_{\psi\eta}$ become functions of the radius. We assign the second form of heterogeneity, considering Young's modulus to be the function ρ as a result of the dependence of it on the temperature field unevenly distributed along the radius. We assume that on z-coordinate the temperature does not change, and therefore E is the function only of the radius and $M_t^0 = 0$. We disregard the dependence of Poisson's ratios and linear expansion on temperature, assuming $\nu = \nu_0$, $t = t_0$. We consider the distributed load to be uniform, so that $\bar{q} = 1$.

The variational equation for slightly curved shells of rotation with the indicated heterogeneities are obtained from equation (2.24). After transition to the dimensionless form, we find

$$\delta W = \delta \int_0^1 \Big\{ \frac{\bar{E}\bar{h}^3}{24(1-\nu_0^2)} [(\Delta w)^2 - (1-\nu_0) L_2(w, w)] - \frac{1}{2\bar{E}\bar{h}} [(\Delta\varphi)^2 - (1+\nu_0) L_2(\varphi, \varphi)] - \Big[\frac{1}{2} L_2(w + 2kw_0, \varphi) + q_0 \Big] w - T_0 T_t^0 \Delta\varphi \Big\} \rho \, d\rho = 0. \quad (5.1)$$

Here $\bar{E}(\rho) = E(\rho)/E_0$, whereupon E_0 is Young's modulus at a temperature of 20°C, $\bar{h}(\rho) = h(\rho)/h_0$; the lines above the remaining relative values are dropped. In their obtaining we consider that in formulas (3.9) everywhere, instead of G_0, there is E_0.

We will obtain the resolving system of quadratic algebraic equations from (3.10), using formulas for $A_{\psi\eta}$ and $B_{\psi\eta}$ used for the derivation of equations (2.30), and the latter from relations (2.27)

$$\int_0^1 \left\{ \sum_{j=1}^{n} x_j \frac{\bar{E}\bar{h}^3}{12(1-\nu_0^2)} [\Delta w_i \Delta w_j - (1-\nu_0) L_2(w_i, w_j)] - \right.$$
$$- k \sum_{j=1}^{m} y_j w_i L_2(w_0, \varphi_j) - \sum_{j=1}^{n} \sum_{s=1}^{m} x_j y_s w_i L_2(w_j, \varphi_s) -$$
$$\left. - q_0 w_i \right\} \varrho d\varrho = 0, \; i = 1, 2, \ldots, n;$$
$$\int_0^1 \left\{ - \sum_{j=1}^{m} y_j \frac{1}{\bar{E}\bar{h}} [\Delta\varphi_i \Delta\varphi_j - (1+\nu_0) L_2(\varphi_i, \varphi_j)] - \right. \qquad (5.2)$$
$$- k \sum_{j=1}^{n} x_j w_j L_2(w_0, \varphi_i) - \frac{1}{2} \sum_{j=1}^{n} \sum_{s=1}^{n} x_j x_s w_j L_2(w_s, \varphi_i) -$$
$$\left. - T_0 T_i^0 \Delta\varphi_i \right\} \varrho d\varrho = 0, \; i = 1, 2, \ldots, m.$$

To these equations, as usually, we connect equation (3.4). System (5.2) can also be obtained by equating derivatives of the integral in (5.1) to zero.

Formulas for the flexural and membrane stresses have the form:

$$\left.\begin{aligned} \bar{\sigma}_{ra} &= -\frac{\bar{E}\bar{h}}{2(1-\nu_0^2)} \sum_{i=1}^{n} x_i \left(\bar{w}''_{\varrho\varrho} + \frac{\nu_0}{\varrho} \bar{w}'_{\varrho}\right)_i; \\ \bar{\sigma}_{\theta a} &= -\frac{\bar{E}\bar{h}}{2(1-\nu_0^2)} \sum_{i=1}^{n} x_i \left(\nu_0 \bar{w}''_{\varrho\varrho} + \frac{1}{\varrho} \bar{w}'_{\varrho}\right)_i; \\ \bar{\sigma}_{rm} &= \frac{1}{\bar{h}} \sum_{i=1}^{m} y_i \frac{1}{\varrho} \bar{\varphi}'_{i\varrho}; \quad \bar{\sigma}_{\theta m} = \frac{1}{\bar{h}} \sum_{i=1}^{m} y_i \bar{\varphi}''_{i\varrho\varrho}. \end{aligned}\right\} \qquad (5.3)$$

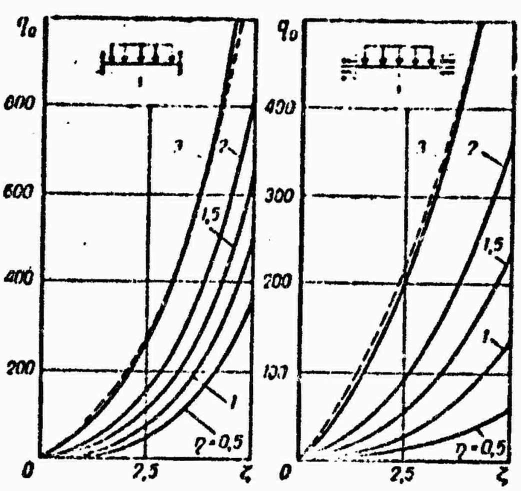
By comparing equations (4.1) and (5.1), we see that for the transition from the uniform case to the heterogeneous, it was required only to introduce under the integral the functions $\bar{E}(\rho)$ and $\bar{h}(\rho)$. The differential equations (2.30) corresponding to this problem contain the second derivatives of $\bar{E}$ and $\bar{h}$. This in general makes the analytical solution of the second of equations (2.30) impossible, and the usual scheme of the solution of the geometrically nonlinear problems is destroyed. In the case of the use of numerical integration of differential equations, the presence of derivatives of hardnesses introduces difficulties. The method used in this work is free of the indicated deficiencies.

2. Circular Plates of Variable Thickness

Let us give the results of calculating the bending of plates with rigid and sliding attachments of the edge under the action of a uniform load. We assign the distribution of thickness by the function

$$\bar{h} = 1 + (\eta - 1)\varrho; \quad \eta = h(1)/h_0. \qquad (5.4)$$

Figure 32 shows curves $q_0(\zeta)$ for the different ratios of thickness near the edge and in the center. The solid line denotes results of the fourth approximation; curves for $n = m = 3$ merge with the appropriate curves $n = m = 4$; the dashed lines are the second approximation.



a b
Fig. 32.

The value of parameter η has a great effect on the rigidity of the plate. It is interesting that an increase in η leads to a decrease in the curvature of curves $q_0(\zeta)$. Thus, when $\eta = 3$ on Fig. 32b curve $q_0(\zeta)$ differs little from a straight line in the range $0 < \zeta < 4$. Although the deflection in the center reaches the value 4h, i.e., the problem is substantially geometrically nonlinear, the dependence $q_0(\zeta)$ resembles a linear dependence. This is explained by the redistribution of the stresses caused by a change in functions $\bar{h}(\rho)$.

The flexural radial stresses of a rigidly attached plate when $\zeta = 4$ are shown on Fig. 33. With an increase in η they increase in the center section of the plate and decrease somewhat on the periphery.

Greatest (with respect to ρ) flexural and membrane stresses are given on Figs. 34 and 35. The solid lines correspond to anechoic sealing and dashed lines - the sliding. By points Fig. 34 gives the results of the third approximation for plates with dead-end sealing at values of η equal to 1 and 3. With an increase in η the convergence of the stresses is improved. The bending stresses when $\eta = 3$, which correspond to the second and third approximations, coincide on Fig. 34. Membrane stresses are established more slowly than the flexural are: when $\eta = 3$ (Fig. 35) the values obtained in the second approximation (crosses) are distant from data of the third approximation (points).

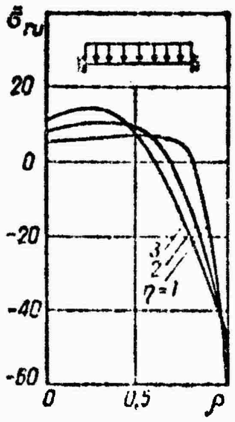


Fig. 33.

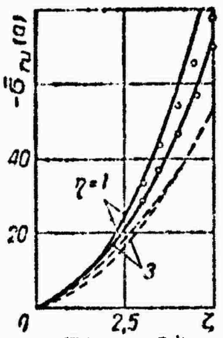


Fig. 34.

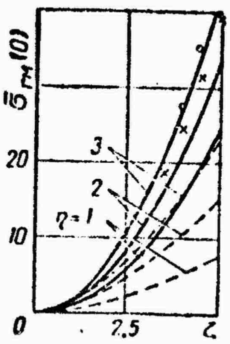


Fig. 35.

3. Flexible Spherical Panel of Variable Thickness

Let us investigate the effect of the variability of thickness on the behavior of shells with rigid and sliding attachments of the edge [48] with the deflection commensurable with the thickness. We take the distribution of thickness to be linear (5.4); we assign the form of the meridian of the shell by the relative chamber $w_0 = -(1 - \rho^2)$. Let us give results of the calculation by the stressed pressure of a panel with height $k = 5$.

Figure 36 shows by solid lines curves $q_0(\zeta)$ of a shell with the sliding attachment of the edge when $n = m = 4$, dashed lines - results of the third approximation, and the dot and dashed lines (for the shell of constant thickness) - the second approximation. A change in parameter η leads, as is evident, not only to quantitative changes in value in the upper and lower critical pressures, but to important qualitative consequences. Precisely, with an increase in η the difference between q_B and q_H rapidly decreases, and when $\eta \geq 2$ the phenomenon of knock generally disappears - the shell becomes stable in general.

In the case of the rigid fixing of the edge (Fig. 37) the effect of the value η is also great. With a change in η from 1.25 to 1.5 the deflection in the center when $q_0 = q_H$ decreases from 6.8 to 3.8. The relative change in the lower critical pressure in this case is approximately 31%, and that of the upper - approximately 7%. The tendency toward the elimination of knock with an increase in η here is also clearly visible, but the shell becomes stable when $\eta > 2$.

For shells with the sliding attachment Fig. 38 shows the dependence of flexural radial strains near the edge from deflection in the center when $k = 5$. A dashed curve is obtained when $\eta = 1$ and $n = m = 3$. The distribution $\sigma_{r\,H}(\rho)$ of a rigidly attached shell for a number of values ζ when $k = 5$ and $\eta = 1.5$ is given on Fig. 39. In the steady-states of equilibrium before and after the knock the edge stresses are large in absolute value.

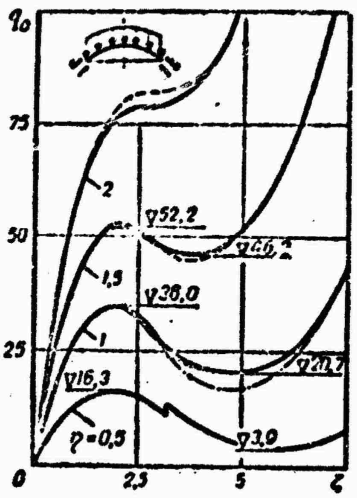


Fig. 36.

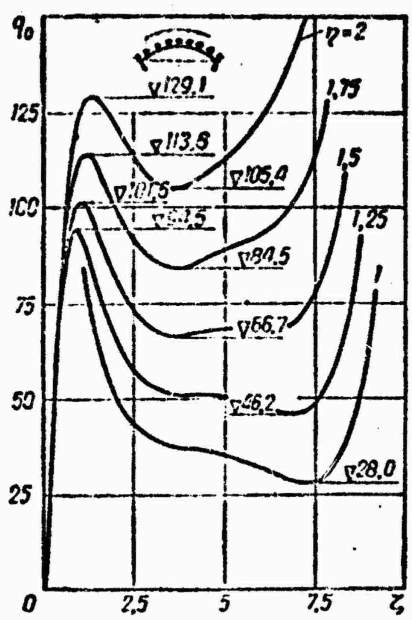


Fig. 37.

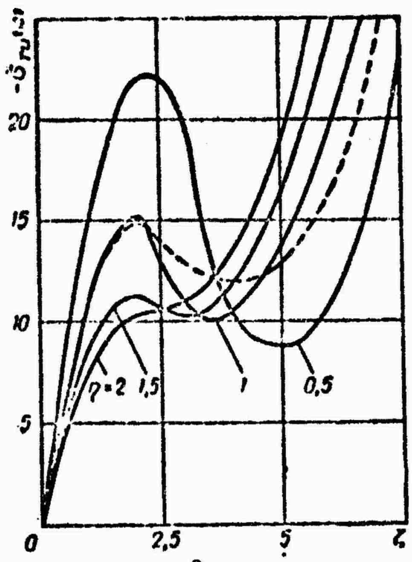


Fig. 38.

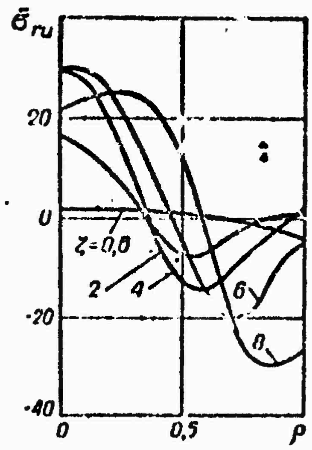


Fig. 39.

4. Conical Slightly Curved Shell of Variable Thickness

In a linear setting such shells are well studied in the works of A. D. Kovalenko and his students (see, for example, [60]).

However, the effect of the variability of thickness on stability in large flexible conical panels has not been investigated.

Let us consider results of the solution [8] of the system of equations (5.2) when $w_0' = -(1-\rho)$ and the distribution of thickness $\ddot{h}$ (5.4) for conical shells with the rigid and sliding attachments of the edge. We consider that $\bar{E} = 1$ and $T_0 = 0$.

The graphs of functions $q_0(\zeta)$ for height $k = 5$ and a number of values of parameter η are given on Figs. 40 and 41. Solid lines refer to $n = m = 4$ and dashed lines - to $n = m = 3$. Results of the calculations in the fifth and fourth approximations merge everywhere, with the exception of the region of unsteady states of equilibrium.

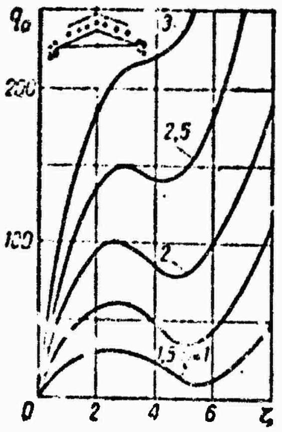


Fig. 40.

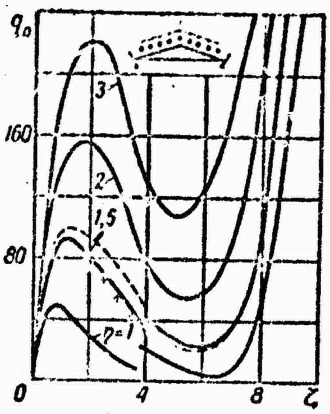


Fig. 41.

Just as in the case of the spherical panel with a sliding attachment, an increase in η conducts to a more rapid increase in the lower critical loads than it does in the upper ones. When η somewhat exceeds the value 2.5, the shell becomes stable in general. In the case of the rigid fixing of the edge (Fig. 41) this phenomena is expressed less sharply as compared with the spherical shell. Here when $\eta = 3$ the difference between q_B and q_H is still great.

The effect of the ratio of thicknesses η on the critical loads is very considerable. Thus, with a change in η from 1 to 1.5 (50%) the value q_B on Fig. 41 increases by approximately 100%, and that of q_H - by approximately 400%. Thus, it is possible to assume that the less deviations of thickness of the conical panel from the constant will lead to noticeable changes in its stability.

The distribution of flexural radial stresses before and after the knock of a rigidly attached shell is given for $\eta = 1.5$ and $k = 5$ on Fig. 42. The dashed line corresponds to the third approximation. Membrane stresses for this shell are given on Fig. 43, where the dashed line denotes the fifth approximation. After the knock compression stresses are decreased, and when $\zeta > 8$ only tensile stresses are present. Membrane stresses are less flexural, but they converge more slowly.

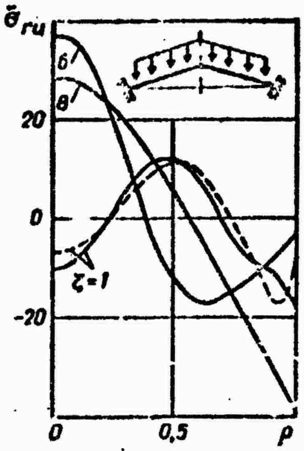


Fig. 42.

5. Geometrically Nonlinear Problems with Heterogeneous Material

Let us now consider the effect of the heterogeneity of mechanical properties of the material, caused by the dependence of Young's modulus on the temperature, on the behavior of flexible plates and slightly curved shells. In the first approximation, such problems are examined in work [28]. Let

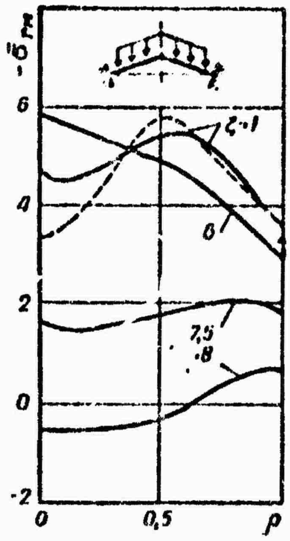


Fig. 43.

us assume that the material is steel of the brand St. 3, for which over a wide range of temperatures it is possible to take with sufficient practical accuracy

$$\bar{E} = 1 - \lambda T_t^0(\rho). \qquad (5.5)$$

here $\lambda = \frac{\beta T(0)}{E_0}$, whereupon coefficient $\beta = 1000$ kg/(cm^2·deg) [9]. When $T_0 = 500$°C and $E_0 = 2 \cdot 10^6$ kg/cm^2 we have $\lambda \approx 0.25$.

Results of the calculation of a rigidly attached and evenly stressed plate when $n = m = 4$ and $T_0 = 5$ are shown on Figs. 44 and 45. The solid lines correspond to $\bar{E} = 1$, dashed lines - to dependence (5.5), curves 1 - $T_t^0 = \rho^2$, curves 2 - $T_t^0 = 1 - \rho^2$, and curves 3 (Fig. 44) - $T_t^0 = \frac{1}{2}$.

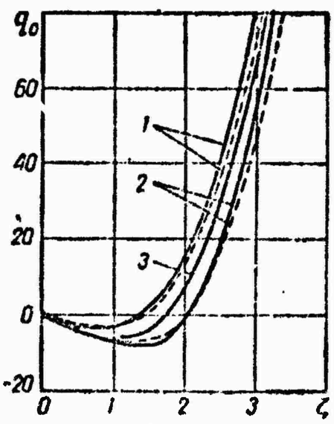


Fig. 44.

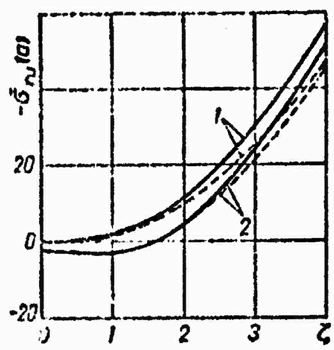


Fig. 45.

In a subcritical state the distinctions between these curves are insignificant. However, when $\zeta > 2$ the divergence noticeably increases. If the temperature in the center is less than that on the periphery ($T_t^0 = \rho^2$), then the account of the dependence of Young's modulus on the temperature leads to a reduction in q_0 (with the same deflection) by approximately 11%. The distribution of $T_t^0 = 1 - \rho^2$ (curves 2) leads to a decrease in the load by approximately 19%. The flexural radial stresses in the sealing are decreased, respectively, by approximately 20 and 11%.

Appearing especially noticeable is the dependence of Young's modulus on the temperature with bending by the transverse load of slightly curved shells. As an example let us give results of the calculation of a rigidly attached spherical shell with height k = 3 when $T_0 = 5$ (Fig. 46). The solid curves are computed taking into account formula (5.5) when $\lambda = 0.25$ and the dashed curves - when $\lambda = 0$, and dot-dashed curves corresponds to the unheated shell, whereupon curves 1 correspond to $T_t^0 = \rho^2$, curves 2 - $T_t^0 = 1 - \rho^2$, and curves 3 - $T_t^0 = \frac{1}{2}$. It is evident that although the value q_H changes comparatively little, the upper critical load depends both on the nature of the distribution of the temperature and on whether or not the temperature effect on mechanical properties is taken into account. When $T_t^0 = 1 - \rho^2$ the difference in values q_B when $\lambda = 0$ and when $\lambda = 0.25$ is 24%, and the value itself $q_B(\lambda = 0.25; T_0 = 5)$ is two times more than it is when $T_0 = 0$.

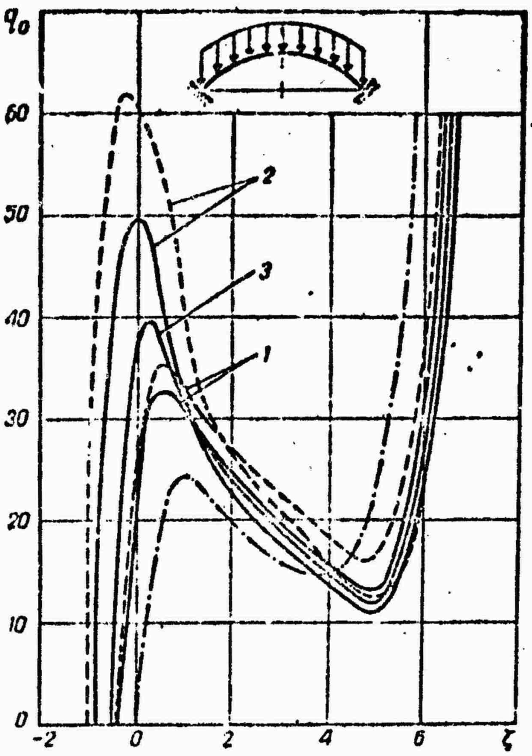


Fig. 46.

The conical slightly curved shell with sliding attachment of the edge can also serve as an example. Figure 47 by a solid line shows the dependence of $q_0(\zeta)$ for $\lambda = 0.25$ and dashed line - for $\lambda = 0$. It is interesting that when $T_t^0 = \rho^2$ (curves 1) the deflection in the center when $q_0 = 0$ is positive (the shell sags downward), and with $T_t^0 = 1 - \rho^2$ (curves 2) the defelction is negative (the shell stands higher). This is connected with the fact that in the first case the temperature expansion is more on the periphery.

With the sliding attachment of the edge, the account of the dependence of Young's modulus on the temperature is reflected in the behavior of the shell less noticeably than that with rigid attachment. The shell is stable in general if the temperature is distributed according to the law $T_t^0 = 1 - \rho^2$ and is unstable when $T_t^0 = \rho^2$.

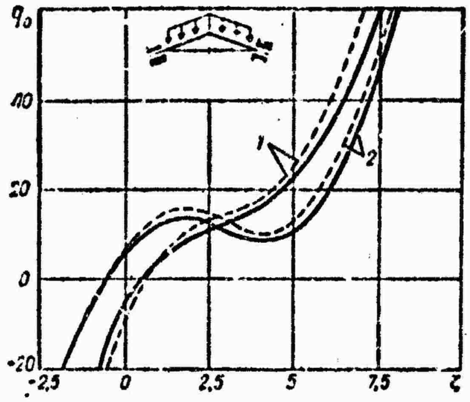


Fig. 47.

CHAPTER VI

ELASTIC-NONLINEAR PLATES AND SLIGHTLY CURVED SHELLS OF ROTATION

1. Fundamental Principles

Geometrically linear problems are examined in this chapter.

In the derivation of the variational (2.24) and resolving algebraic (3.10) equations, it was assumed that dependences $E = E(r, \gamma, \varepsilon_0, \varepsilon_i)$ and $\nu = \nu(r, \gamma, \varepsilon_0, \varepsilon_i)$ were assigned. In order to define them concretely, let us use the Hencky theory of small elasto-plastic deformations [42]. It is known that if at not one of the points of a body does unloading appear, then this theory coincides with the theory of an elastic physically nonlinear body. The cases of unloading, i.e., a decrease in ε_i upon the transition from one step of loading to another, will be noted. Accepting that the material possesses nonlinearly elastic properties, we subordinate unloading to those same laws as the process of loading.

The relationships of the theory of small elasto-plastic deformations are based on the assumption that at each point of the body the path of loading is a straight line. In axisymmetric problems this means that in the plane ε_r and ε_θ

the points which correspond to different values of the load parameter should lie on the straight ray which emerges from the origin of the coordinates. This limitation is sufficiently rigid. Even in the construction of this theory, it was considered that it will give acceptable results also for paths of loading close to straight lines.

Later in the works of B. Budyanskiy [122] and Yu. N. Rabotnov [94], under the assumption of the singularity of the surface of loading, it was proved that Hencky's deformation theory does not contradict the postulate of plasticity of Drucker [124] (it is physically consistent) in the deviation of the trajectories of loading from straight lines. Yu. N. Shevchenko [118] demonstrated an analogous theorem for the nonsingular surface of loading in the nonisothermal process, taking into account the dependence of the matrix of coefficients of elasticity on the temperature and plastic deformations. In order to be convinced of the correctness of results obtained from this theory, one should check (at each point of the body and at all stages of deformation) to see if the vector of preloading lies in the region limited by the cone, the permissible solution of which can be computed.

Since [89] shows that in the case of an ideal elasto-plastic diagram $\sigma_i(\varepsilon_i)$ without strain hardening, when using the condition of plasticity of Mises, the criterion of B. Budyanskiy of the applicability of the deformation theory is fullfilled. Because of these works, the region of the theoretically substantiated use of the deformation theory was substantially expanded, although the most complete checking can be given only by a comparison of results of the calculation with the experimental data.

In the theory of small elasto-plastic deformations, the relationships between the stresses and strains have the form of (2.4); however, values of E and ν depend on state of strain at the point. The modulus of expansion and the coefficient of lateral compression are connected with the shear moduli G and volumetric strain K by formulas

$$E=\frac{9KG}{3K+G};\quad \nu=\frac{1}{2}\,\frac{3K-2G}{3K+G}. \tag{6.1}$$

In this theory it is assumed that K does not depend on the state of strain at the point, so that in the case of the body heterogeneous before deformation K = K(r, γ); for the body which is uniform before deformation $K = K_0$ = const.

The shear modulus in the deformation theory is determined by the formula

$$G=\frac{1}{3}\cdot\frac{\sigma_i(r,\gamma)}{\varepsilon_i(r,\gamma)} \tag{6.2}$$

and is called the intersecting modulus. The deformation intensity with axial symmetry has the form

$$\varepsilon_i=\frac{2}{3}\sqrt{\left[1+\frac{\nu}{(1-\nu)^2}\right](\varepsilon_r+\varepsilon_\theta)^2-3\varepsilon_r\varepsilon_\theta}. \tag{6.3}$$

In the derivation of this formula the normal deformation in direction γ is expressed by ε_r and ε_θ from the condition in the equality to zero of the stress σ_γ; the shear $\varepsilon_\rho\theta$ is equal to zero with respect to symmetry. It is important to note that the value ν here is not constant, but it depends on accordance with formulas (6.2) and (6.1) on the state of strain at the point.

The mechanical properties of the material are determined by functions $\sigma(\varepsilon_i, r, \gamma)$ and K(r, γ).

2. Approximation of the Connection Between the Intensities of Stresses and Deformations.

Let us consider some methods of the representation of the dependence of the intensity of stresses on ε_i.

Ideally the elasto-plastic body (Fig. 48a) is characterized by formulas

$$\sigma_i = 3G_0\varepsilon_i \text{ when } \varepsilon_i < \varepsilon_{is}; \qquad (6.4)$$
$$\sigma_i = \sigma_s \text{ when } \varepsilon_i \geqslant \varepsilon_{is}.$$

The relationship $\sigma_i = \sigma_s$ is the Mises criterion of plasticity. From formulas (6.2) and (6.4) it follows that

$$G = G_0 \text{ when } \varepsilon_i < \varepsilon_{is}; \qquad (6.5)$$
$$G = \frac{1}{3}\cdot\frac{\sigma_s}{\varepsilon_i} \text{ when } \varepsilon_i \geqslant \varepsilon_{is}.$$

The elasto-plastic body with linear strain hardening (Fig. 48b) is usually assigned by formulas

$$\sigma_i = 3G_0\varepsilon_i \text{ when } \varepsilon_i < \varepsilon_{is};$$
$$\sigma_i = 3G_0\varepsilon_{is} + 3G_1(\varepsilon_i - \varepsilon_{is}) \text{ when } \varepsilon_i \geqslant \varepsilon_{is}. \qquad (6.6)$$

where G_1 is the modulus of strain hardening; $3G_0\varepsilon_{is} = \sigma_s$. When $G_1 = 0$, from (6.6) we obtain expressions (6.4). From formulas (6.6) we find the intersecting shear modulus

$$G = G_0 \text{ when } \varepsilon_i < \varepsilon_{is};$$
$$G = G_0 + G_1\left(1 - \frac{\varepsilon_{is}}{\varepsilon_i}\right) \text{ when } \varepsilon_i \geqslant \varepsilon_{is}. \qquad (6.7)$$

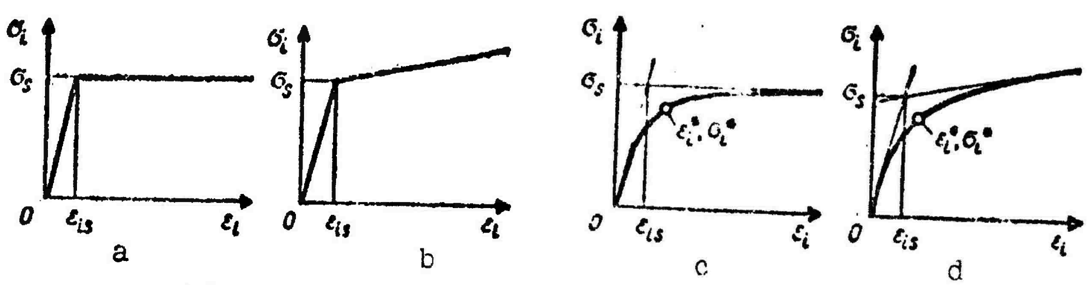


Fig. 48.

The nonlinear diagram (Fig. 48c) for pure aluminum [131] is described well by the dependence

$$\sigma_i = \sigma_s\left[1 - \exp\left(-\frac{\varepsilon_i}{\varepsilon_{is}}\right)\right]. \qquad (6.8)$$

in which $\varepsilon_{is} = \sigma_s/3G_0$ is the conditional intensity of yield strains. Hence the intersecting modulus is

$$G = G_0\frac{\varepsilon_{is}}{\varepsilon_i}\left[1 - \exp\left(-\frac{\varepsilon_i}{\varepsilon_{is}}\right)\right]. \qquad (6.9)$$

With the approximation of dependences of the type shown on Fig. 48c, the formula

$$\sigma_i = \sigma_s\left\{1 - \exp\left[-\frac{\varepsilon_i}{\varepsilon_{is}}\left(1 + b\frac{\varepsilon_i}{\varepsilon_{is}}\right)\right]\right\}. \qquad (6.10)$$

can be useful. The determination of coefficient b from the condition that the curve passes through a certain selected point (obtained with the aid of the experiment of the graph) with coordinates ε_i^*, and σ_i^*, namely,

$$b = -\frac{\varepsilon_{is}}{\varepsilon_i^*}\left[1 + \frac{\varepsilon_{is}}{\varepsilon_i^*}\ln\left(1 - \frac{\sigma_i^*}{\sigma_s}\right)\right].$$

makes it possible to consider more accurately the rate of convergence of σ_i to σ_s with an increase in ε_i.

The generalization of formula (6.10), in the case of strain hardening (Fig. 48d), is the dependence

$$\sigma_i = (b + c\varepsilon_i)\{1 - \exp[-d\varepsilon_i(1 + e\varepsilon_i)]\}. \qquad (6.11)$$

in which $b = 3\varepsilon_{is}(G_0 - G_1)$, $c = 3G_1$, $d = G_0/\varepsilon_{is}(G_0 - G_1)$. Constant e is calculated from the condition that curve (6.11) would pass through assigned point ε_i^*, σ_i^*:

$$e = -\frac{1}{\varepsilon_i^*}\left\{1 + \frac{\varepsilon_{is}}{\varepsilon_i^*}\left(1 - \frac{G_1}{G_0}\right)\times\right.$$

$$\left.\times \ln\left[1 - \frac{\sigma_i^*}{\sigma_s}\cdot\frac{1}{1 - \frac{G_1}{G_0} + \frac{G_1}{G_0}\cdot\frac{\varepsilon_i^*}{\varepsilon_{is}}}\right]\right\}.$$

If the body is nonhomogeneous before deformation, then in formulas (6.4)-(6.11) one should consider that $G_0 = G_0(r, \gamma)$ (elastic heterogeneity). In the case of plastic heterogeneity the functions of the radius and z-coordinate will be σ_s, and ϵ_{is}, the modulus of strain hardening G_1, and coefficients b, c, d, and e.

Exponential and the polynomial approximations of the dependence $\sigma_i(\epsilon_i)$ give good results in the narrow region ϵ_i, and therefore they are barely suitable for the solution of problems with development zones of plasticity.

In the given formulas the intersecting shear modulus is calculated from the value of the deformation intensity and dependence $\sigma_i(\epsilon_i)$. Such a method (unlike determination of G from value σ_i at the point) leads to the best convergence of the iterative processes [55].

3. Features of the Algorithm

When using the method of variable parameters of elasticity on each step of external iterative process, one should compute the values of functions $A_{\psi\eta}(\rho)$ and $B_{\psi\eta}(\rho)$ at each point ρ_k (let us recall that the calculation of integrals in terms of the radius is fulfilled in the program according to 12-nodal Gaussian formula so that we have 12 calculated positions).

The indicated process itself is iterative and consists in the following. According to vector Z of the Ritz parameters and formulas (3.2), (2.3), (2.10) and (2.1), utilizing the initial approximation for $A_{\psi\eta}$ and $B_{\psi\eta}$ at point γ_ν the values ϵ_r and ϵ_θ. Then according to formula (6.3) and the initial approximation

$\nu(r, \gamma)$, we find the deformation intensity. Further we compute the shear modulus (6.2) and values E and ν (6.1). In this case the obtained value ν will differ from the initial one. In order to reduce disagreement, the new value ν is substituted into formula (6.3), the refined value is determined ε_i and E and ν are again determined. This (internal) process of iterations is finished upon achieving the assigned accuracy, and $E(r_k, \gamma_p)$, $\nu(r_k, \gamma_p)$ obtained in summation are stored.

After repeating these calculations for all γ_p, by integration over γ we determine the values of functions D_{bl} (2.8), and according to formulas (2.26) we find the values $A_{\psi\eta}(r_k)$ and $B_{\psi\eta}(r_k)$. However, these values differ from the initial ones. They are again used for the calculation of ε_r and ε_θ and the process is repeated until the assigned accuracy will be achieved.

Integration over γ if fulfilled by a 6-nodal Gaussian formula (precise for polynomials of up to 11 degrees inclusively).

A flowchart of the calculation of $A_{\psi\eta}$ and $B_{\psi\eta}$ is given on Fig. 49.

In the solution of the physically nonlinear problem for the first value of the leading parameter, as an initial approximation of functions ν, $A_{\psi\eta}$ and $B_{\psi\eta}$ we take their values for a linearly elastic material. Further we use the values $\nu(r, \gamma)$, $A_{\psi\eta}(r)$ and $B_{\psi\eta}(r)$, obtained for a certain ζ_i, as the initial approximation for the solution to the problem with ζ_{i+1}. An experiment showed that with such a selection of the initial approximations the process of the determination of ν, $A_{\psi\eta}$ and $B_{\psi\eta}$ converges on a one-two iteration with the permissible relative error equal to 10^{-4}.

Taking into account the geometric linearity, we obtain the variational equation from (2.24) by replacement of $w_2 = w + 2w_0$ by $w_2 = 2w_0$. The algebraic equations (3.10) become linear, and the internal iterative process of the general scheme of calculation (paragraph 5, Chapter III) is degenerated. The external process of iterations, which corresponds to the method of variable parameters of elasticity, with the permissible error of Ritz's parameters of 10^{-4} converges usually in one - three iterations depending on the step $\Delta\zeta$.

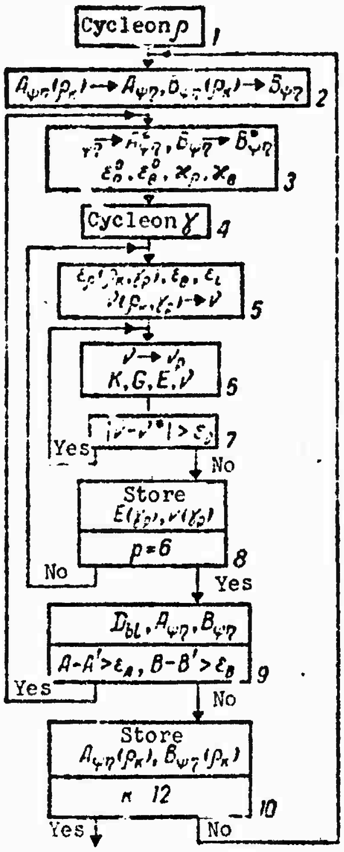


Fig. 49.

It follows, however, to select correctly the initial approximation for the first value ζ_0. If this value is large, and the greatly developed plastic regions will correspond to it, then the process of iterations will converge very slowly or not converge at all. Therefore, it is desirable to choose ζ_0 in order that in the shell there would not appear plasticity or (with the nonlinear dependences of Fig. 48c and d) the values ε_i are small.

The results of the calculations given in this chapter can be used only when $\zeta < 0.25$; with great deflections the effect of the geometric nonlinearity becomes noticeable.

In Chapters VI and VII we assign the relative values by formulas (3.9), so that unlike Chapters IV and V the parameter of the load is $q_0 = \frac{q}{G_0}(\frac{a}{h_0})^4$, and the stresses are referred to

$G_0(h_0/a)^2$. Further we will use the given deformation intensities $\mu = \epsilon_1(a/h_0)^2$ and the yield deformations $\mu_s = \epsilon_{1s}(a/h_0)^2$.

4. The Elasto-Plastic Bending of Circular Plates

The problem indicated in the title of the section is usually used for the testing of different methods of calculation. However, in spite of the relative simplicity (if applied forces in middle surface are equal to zero and geometric nonlinearity is not considered, then $\phi \equiv 0$), an exact solution is absent.

One of the first approximate solutions to problems of elasto-plastic bending of freely supported circular and annular plates was given by V. V. Sokolovskiy [100]. In A. A. Ilyushin's book [42] different methods are proposed, and examples and approximation formulas are given. Many problems of the theory of circular plates were solved by A. S. Grigor'yev. He performed detailed studies of the effect of strain hardening, compressibility of material, plastic heterogeneity, and supporting power [32-34]. The most precise results of recent time belong, apparently, to Okhashi and Kamiya [131].

The method of the solution given in Chapter III, in the case of small deflections of the plate, is somewhat simplified.

Assuming in the system of equations (3.10) that $k = 0$, rejecting the m of equations of the second group and connecting equation (3.4), we obtain n + 1 linear algebraic equations, which make it possible to fulfill one iteration of the method of variable parameters of elasticity.

As the first example let us consider the hinged supported plate under the action of an evenly distributed load. Let $a/h_0 = 12.5$, the material of the plate (pure/aluminum) has $\sigma_s = 1023$ bar, $G_0 = 0.3483 \times 10^6$ bar, and its properties are assigned by formulas (6.8) and (6.9). The parameter μ_s in this case is equal to 0.1530. Source [131] shows that the dependence (6.8) clearly coincides with the experimental curve $\sigma_i(\varepsilon_i)$. An analysis of data of this work makes it possible to consider that $K = K_0$ and take $K_0 = 2G_0$.

On Fig. 50 the letters HΦ (nonlinear physically) mark the result [131] of the calculation according to the method [130] based on the separation of the elasto-plastic and elastic regions, each of which is described by the individual system of differential equations. These equations, as the authors of source [130] note, are very bulky. Used for the solution are the method of initial parameters, the numerical integration of systems of differential equations (Cauchy problem) and the dual process of iterations.

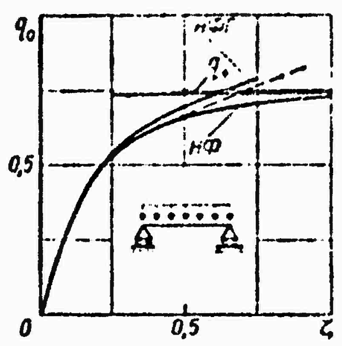


Fig. 50.

The dependence $q_0(\zeta)$, calculated when $n = 4$, merges on Fig. 50 with curve HΦ. An increase in n does not change the result. The horizontal line indicates the maximum load $q_+ = 0.748$ which is obtained [36] on the assumption that the material is rigid-plastic and incompressible. It is evident that the curve $q_0(\zeta)$ asymptotically tends to this straight line. It follows to refer to values q_0 for $\zeta > 0.25$ with caution, since in this region the process of deformation affects the geometric nonlinearity.

Figure 51 shows results of the calculation when $n = 5$ of freely supported and rigidly fastened circular evenly stressed plates made from an ideally elasto-plastic material ($\nu_0 = 0.3$).

Plotted along the axis of the abscissas is the value $\bar{\zeta} = \zeta/\sqrt{3\mu_s}$, and at the axis of the ordinates - $q_0 = q_0/\sqrt{3\mu_s}$. In these coordinates the curves which correspond to calculation of only the physical nonlinearity are obtained unique and do not depend on μ_s. According to data of source [36], the maximum loads $\bar{q}_+$ for two indicated conditions of the attachment are equal to 2.82 and 5.44. The distribution of zones of plasticity for some values of ζ is shown on Fig. 52.

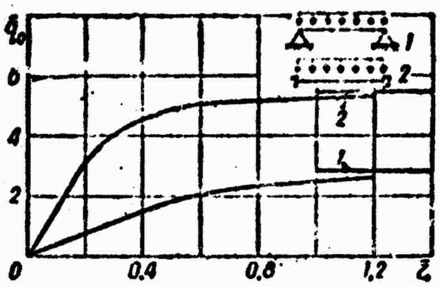


Fig. 51.

The effect of the law of the change in Poisson's ratio on the dependence $\bar{q}_0(\zeta)$ can be traced on Fig. 53, on which shown are results of the calculation (n = 4) of the bending of a freely supported circular plate by a uniform load. Curve 1 corresponds to $\nu = 0.5$, and curve 2 to $\nu = 0.3$ when $\mu < \mu_s$ and $\nu = 0.5$ when $\mu \geq \mu_s$. The curve obtained from the proposed method with complete calculation of the compressibility of the material coincides with curve 2. The assumption about the incompressibility of the material in the elastic and plastic range ($\nu = 0.5$) leads to very approximate results in the elasto-plastic state of the plate, but with the greatly developed plasticity (when almost in the entire volume of the material ν is close to 0.5) the error in the determination of the load for the same value of deflection is small. However, at the identical

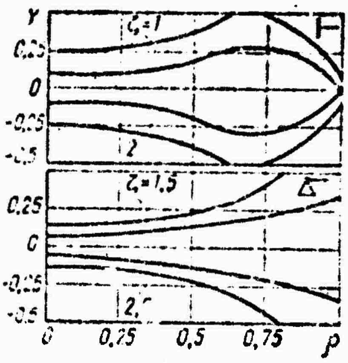


Fig. 52.

value of the load, the deflection differs. So, if $q_0 = 0.945q_+$, then the deflection in the center calculated with $\nu = 0.5$ is obtained 35% greater. The widespread method of the approximate account of compressibility, with which outside the dependence on ε_i in a plastic state $\nu = 0.5$ is accepted, gives the curve $\bar{q}_0(\bar{\zeta})$, which barely differs from the precise one. Therefore, when such an assumption noticeably simplifies the calculation, it is possible to use it.

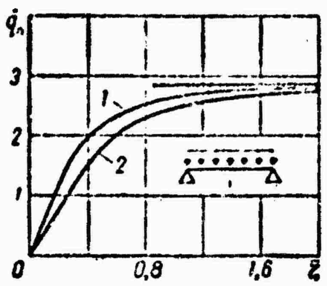


Fig. 53.

If it is necessary to obtain a detailed picture of the state of strain, all the same one should not allow the disruptive behavior of Poisson's ratio - one of the two physical parameters which characterize the state of strain at the point. One should also take into account that the effect of ν is explained in only one example. The propagation of conclusions to other cases, without sufficient substantiation, can lead to errors. In the first approximation, an analysis of the effect of different methods of the account of compressibility is given in source [70].

5. The Plastically Heterogeneous Circular Freely Supported Plate

Investigated in source [34] is the bending of the plate whose material steel 60, after the oil quenching, became plastically heterogeneous in thickness:

$$\sigma_s = \sigma_{s0}(1 + 16|\gamma|^3); \quad -0.5 \leqslant \gamma \leqslant 0.5. \qquad (6.12)$$

Here σ_{s0} is the value of the yield point in the middle plane. Tresca's flow condition, the ideal elasto-plastic diagram and the assumption about the incompressibility of the material both in an elastic and plastic range are used. The calculations are made by the method of numerical integration of systems of two nonlinear differential equations.

Let us solve this problem by the variational method, and unlike source [34] let us completely take the compressibility into account. Let us present Tresca's condition of plasticity in the form

$$\max_i |\gamma_i| = \gamma_s.$$

Here γ_i is one of the main shears

$$\gamma_1 = \frac{1}{2}(\varepsilon_\theta - \varepsilon_\gamma),\quad \gamma_2 = \frac{1}{2}(\varepsilon_\gamma - \varepsilon_r),\quad \gamma_3 = \frac{1}{2}(\varepsilon_r - \varepsilon_\theta), \tag{6.13}$$

value $\gamma_s = G_0\tau_s$, and the deformation in the direction γ

$$\varepsilon_\gamma = -\frac{\nu}{1-\nu}(\varepsilon_r + \varepsilon_\theta)$$

is determined from the condition $\sigma_\gamma = 0$. We consider that $K \equiv K_0$; we compute the shear modulus according to formulas

$$G = G_0 \text{ when } \max_i |\gamma_i| < \gamma_s;$$

$$G = G_0 \frac{\gamma_s}{\max_i |\gamma_i|} \text{ when } \max_i |\gamma_i| \geq \gamma_s. \tag{6.14}$$

The relation $G_0 = \frac{\tau_s}{\gamma_s} = \frac{\sigma_s}{3\varepsilon_{is}}$, taking into account the fact that with Tresca's criterion $\sigma_s = 2\tau_s$, gives the relation $\gamma_s = \frac{3}{2}\varepsilon_{si}$ [100]. Thus, the expression (6.12) corresponds to

$$\gamma_s \left(\frac{w}{h_0}\right)^* = \frac{3}{2}\mu_s(1 - 16|\gamma_i|^3). \tag{6.15}$$

Formulas (6.13)-(6.15) are used in the calculation; it is accepted that $\mu_s = 1$; $\nu_0 = 0.3$; $n = 4$.

On Fig. 54 the solid line shows the dependence $\bar{q}_0(\bar{\zeta})$ obtained by the variational method; the dashed line corresponds to the data of source [34]. The precise value of the maximum load with plastic heterogeneity (6.15) is $\frac{8}{7}\bar{q}_+$ of the uniform plate. This coefficient is taken from formula (38) of source [34]; after the replacement of z by 2γ it is determined by expression $8\int_0^{1/2}(1+16\gamma^5)\gamma d\gamma$.

The value of maximum load calculated by the variational method consists of 3.0 and only exceeds the precise value 2.970 by 1%.

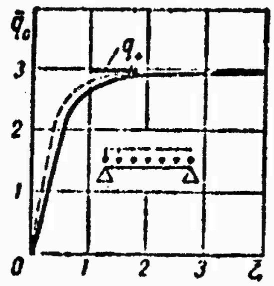


Fig. 54.

Let us recall that the maximum load of the uniform freely supported plate when using Tresca's condition of plasticity [100]

$$q^* = \frac{q_+}{\sigma_{s0}}\left(\frac{a}{h_0}\right)^2 = \frac{6}{4}.$$

Value

$$q_0 = \frac{q_+}{G_0}\left(\frac{a}{h_0}\right)^4 = 3\frac{q_+}{\sigma_{s0}}\left(\frac{a}{h_0}\right)^2 \varepsilon_{is}\left(\frac{a}{h_0}\right)^2 = 3\mu_s q^*,$$

so that when $\mu_s = 1$ the maximum load

$$\bar{q}_+ = q_0/\sqrt{3}\mu_s = 3\sqrt{3}/2 = 2,598.$$

For transition to the heterogeneous plate, this value should be multiplied by 8/7, and this gives $\bar{q}_+ = 2.970$.

Let us note that the complete account of compressibility in an elastic and plastic range, simply achieved according to the proposed method, makes it possible to obtain a considerably more precise dependence $\bar{q}_0(\bar{\zeta})$. The difference between values q_0 (with the same deflection), obtained by taking into account the compressibility and without taking it into account, decreases in proportion to the approach of load to the maximum.

It should also be noted that the quantity of precalculated joints taken in the program for a computer, according to thickness (six abscissas of the formula for the numerical Gaussian integration) proved to be sufficient in order to obtain a high accuracy of the value of maximum load, although the plastic heterogeneity was changed with respect to γ very sharply. The use of the Gaussian formula in this case proved to be especially successful, because its nodes are concentrated at the edges of the interval of the argument where function (6.15) is changed most rapidly.

6. Elasto-Plastic Bending of Slightly Curved Spherical Shells

If relation $a/h \approx 20-30$ and $\varepsilon_{1s} \approx 2 \cdot 10^{-3}$ (aluminum alloys. steel), the value of the parameter is obtained close to unity. Calculations of slightly curved shells show that the plasticity then appears with small deflections $\zeta < 0.25$, and therefore a study can be made in geometrically linear setting. The greatest load q_+ which is withstood by the shell is determined in this case by the loss of the supporting power caused by the rapid propagation of plastic zones; the value of function $q(\zeta)$, with an increase in ζ asymptotically tends to the horizontal line $q = q_+$.

One should emphasize that values q_+ obtained for elasto-plastic slightly curved shells, in a geometrically linear setting, have a conditional nature. An account of the effect of geometrically nonlinear terms (Chapter VII) shows that the behavior of the plastically deformed shell even with very small deflections depends substantially on the load direction. If it is directed from the center of curvature, then soon after the development of plastic ranges there begins the phenomenon of geometric strain hardening - the load increases together with the deflection, although the entire material has already been found in a plastic state. In the case of the load direction to the center of curvature, a knock occurs, and q_B proves to be less in value than q_+. Thus, these solutions to the problems, only physically nonlinear, make it possible to explain the behavior of the slightly curved shells in the stage of the accumulation of plastic deformations, but they do not make it possible to judge the type of the curve $q(\zeta)$ after plastic regions became vast. Therefore, the use in this section of the term "supporting power" is caused by tradition rather than the essence of the phenomenon.

Let us consider results of the calculation of spherical cupolas with rigid and free motionless and movable support. We take the relative height k = 3, the material to be ideally elasto-plastic (Fig. 48a), and $\nu_0 = 0.3$.

The supporting power of a shell depends on the boundary conditions. With a motionless free support and rigid fixing, $\bar{q}_+$ is equal to 19.0 and 18.8, respectively. The hinge and sealing movable in the middle surface cause values of $\bar{q}_+$ equal to 4.5 and 8.5. By comparing curves $\bar{q}_0(\bar{\zeta})$ for shells and plates (Figs. 55 and 51), it is possible to note that the transition from the elasto-plastic state to the exhaustion of the supporting power in the plates occurs more smoothly than that of the shells. This is especially vividly evident in the cases of free support. Then in the figure the straight line of elastic equilibrium states soon after the appearance of plasticity sharply turns and becomes horizontal, having achieved the extreme value of the load.

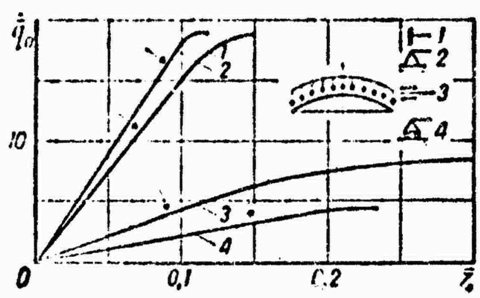


Fig. 55.

Sources [35, 36] show that the exhaustion of the supporting power of the freely supported circular plates takes place with the propagation of zones of plasticity for the entire upper ($-\frac{1}{2} \leq \leq \gamma \leq 0$) and entire lower ($0 \leq \gamma \leq \frac{1}{2}$) part of the plate with respect to thickness.

With the bending of slightly curved shells, the propagation of zones of plasticity in the limiting state onto the entire volume of material is possible only with a very small indicator of lift of the shell. The complexity of the stress-strain state of the more slightly curved shells, which is connected with the interaction of bending and deformation in middle surface, leads to a more whimsical distribution of the zones of the plasticity. For spherical cupolas with k = 3, these zones in limiting states are shown on Fig. 56. Given there are forms of function $\frac{w(\theta)}{h}$ reduced to identical deflection in the center.

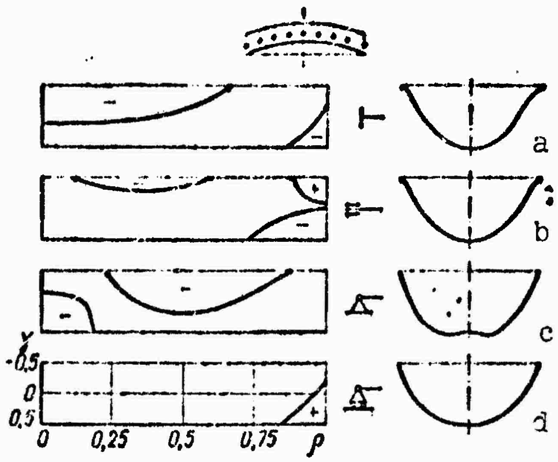


Fig. 56.

In the first case (Fig. 56a) the reason for the exhaustion of the supporting power is the vast plastic zone in the central upper part of the plate and the zone near the sealing; they are caused by the large radial compressive stresses. In the second case (Fig. 56b) the limiting state is connected with the bending in the sealing, where the plastic hinge was formed. With a fixed hinge (Fig. 56c) plasticity is caused by compressive stresses, while with a movable hinge (Fig. 56d) - by powerful extension below near the support.

Let us note that in the cases considered from the onset of plasticity up to the limiting state, at not one point did there appear unloading, and therefore this calculation, made according to the elastic-nonlinear theory, coincides with the calculation according to the deformation theory of plasticity.

CHAPTER VII

FLEXIBLE ELASTIC-NONLINEAR PLATES AND SHELLS

1. Basic Concepts and Dependences

The degree of effect of the physical and geometric nonlinearity on the stress-strain state of thin slightly curved shells is determined by the value of the given intensity of yield deformation μ_s, which in a composite manner characterizes properties of the material and relative thickness of the shell. With small μ_s (ε_{1s} or a/h are small) the plasticity begins with such insignificant deflections that the problem with high accuracy can be considered geometrically linear. If μ_s is great, then even with great deflections the plasticity does not appear, and calculations can be conducted by taking into account only geometric nonlinearity. These two limiting cases are examined in Chapters IV-VI.

There is great scientific and practical interest in the examination of plates and shells for intermediate values of the parameter μ_s, when the behavior of the object to an equal degree is affected by the physical and geometric nonlinearity. It is important, furthermore, more accurately to determine which values μ_s can be considered "small" and "large," depending on the form, conditions of attachment of the shell and the load which acts on it. Recently problems of such type have received even greater attention; however, specific numerical results of accuracy acceptable for practice are obtained little.

Using the variational method described in Chapters II and III, let us consider twice the nonlinear problems of bending, stability and supporting power of circular plates and slightly curved spherical cupolas under varied conditions of attachment. We proceed from the variational equation (2.24), which in accordance with the general algorithm by the method of variable parameters of elasticity we reduce to the sequence of systems of quadratic algebraic equations (3.10) and (3.4). We solve the problems on the computer "Ural-2" by the dual iterative process described in Chapter III.

Below everywhere, except the cases especially stipulated, we assign the physical nonlinearity by dependences (6.4) and (6.5), considering the material to be ideally elasto-plastic, and we take the Poisson coefficient before deformation equal to $\nu_0 = 0.3$. In all cases the load is considered to be transverse evenly distributed, although the program makes it possible to perform calculations for other forms of loading.

2. The Bending of Circular Plates with Varied Conditions of Attachment

The examination of the combined effect of the physical and geometric nonlinearity on the bending of circular plates was initiated to source [129], where by the finite-difference method a freely supported plate under the action of a concentrated force in the center is examined. In sources [1, 69] in the first approximation, problems of the action of an evenly distributed load are solved, and the principle of possible displacements and the assumption about the incompressibility of the material are used. The method, based on the separation of zones with the characteristic distribution of the plastic regions each of which is described by a separate system of differential equations, is proposed in source [130].

Let us consider a plate with a sliding hinged support of the edge. Let $a/h_0 = 12.5$, the material of the plate (pure aluminum) has $\sigma_s = 1023$ bar, $G_0 = 0.3483 \cdot 10^6$ bar and its properties are assigned by the dependence (6.8), $\mu_s = 0.1530$, $K_0 = 2G_0$. The calculation of this plate, without the effect of geometric nonlinearity, is made in Section 4 of Chapter VI.

On Fig. 50 the dashed line shows an experimental curve obtained in source [131]. Dependence $q_0(\zeta)$, calculated when $n = 4$, coincides with this curve. Line with the mark HФГ (nonlinear physically and geometrically) is obtained when $n = m = 5$. As is evident, calculation of the geometric nonlinearity made it possible to bring together the theoretical and experimental results, especially when $\zeta = 0.5$.

A plate of the same dimensions but made of soft steel, with $\sigma_s = 3110$ bar, $G_0 = 0.7848 \cdot 10^6$ bar and $\nu_0 = 0.28$, is calculated in source [89] with the diagram of Fig. 48a (authors of source [89] reported that the calculations were made with $\sigma_s = 31.7$ kg/mm^2 and not with the value $\sigma_s = 34.5$ kg/mm^2, as indicated in their article). Results of source [89] coincide with the obtained variational method. In the subsequent cases it is accepted that the deformation of the edge of the plate in the middle plane is absent.

Figure 57 shows curves plotted according to results of the calculation when $n = m = 4$ of freely supported plates with different parameter value μ_s. The letter Л marks the line of the linear solution. The dashed line separates the region in which the plate remains elastic from the region of elasto-plastic deformations. The arrows show the points at which unloading appears; the continuation of the curves for large $\bar{\zeta}$ is plotted for an elastic-nonlinear body.

Similar curves obtained when n = m = 5 for rigidly attached plates are given on Fig. 58. The curve μ_s = 0.2255 corresponds to data of source [90], and it coincides with that calculated in the variational method.

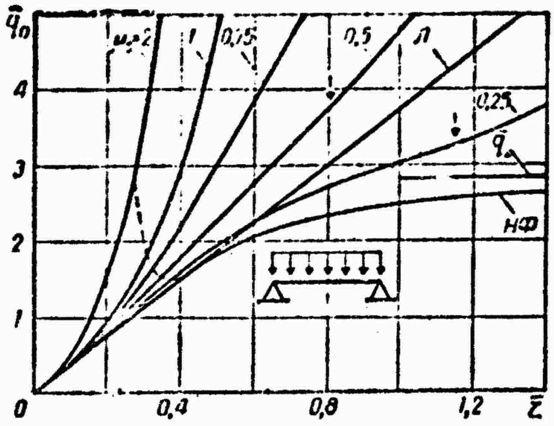


Fig. 57.

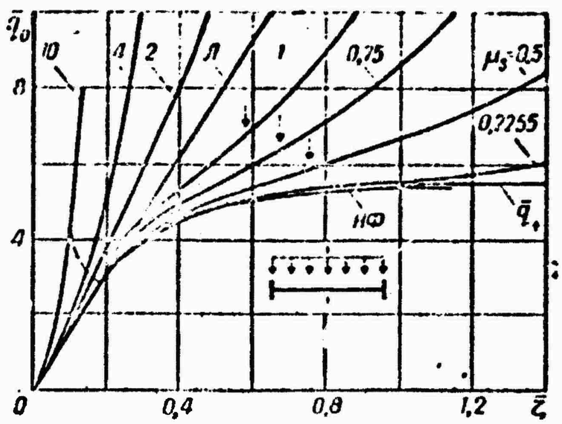


Fig. 58.

It is interesting that it is possible to select such a value μ_s for each of Figs. 57 and 58 at which the dependence $\bar{q}_0(\bar{\zeta})$ will hardly differ from the straight line of the linear solution.

According to results of the analysis of these curves it is possible to establish which values of μ_s should be considered large or small. In the case of the free support already when μ_s = 0.25, it is necessary to consider the geometric nonlinearity, and when μ_s > 2 it is possible to consider only it. In the case of rigid attachment the lower boundary can be left the same, but the possibility of not considering the physical nonlinearity appears only when μ_s > 10.

The distribution of radial stresses referred to $G_0(h/a)^2$ on planes $\gamma = \pm 1/2$ of a rigidly attached plate when $\zeta = 0.25$ (dot-dash lines), $\zeta = 0.5$ (dashed lines) and $\zeta = 1$ (solid lines) and when $\mu_s = 1$, is given on Fig. 59. The maximum value of the stresses in the center of the plate (with developed plasticity) $\bar{\sigma}_r(0) = \bar{\sigma}_\theta(0) = 3\mu_s$, in sealing $\sigma_r(1) = 2\sqrt{3}\mu_s$, $\bar{\sigma}_\theta(1) = \sqrt{3}\mu_s$.

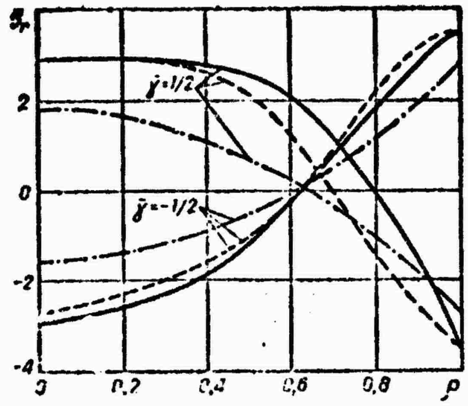


Fig. 59.

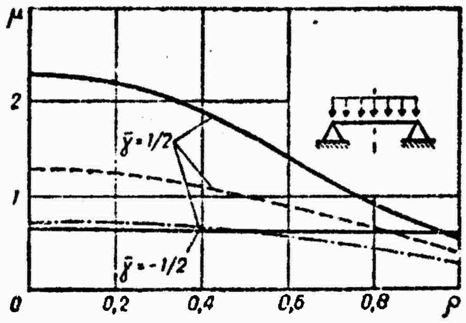


Fig. 60.

Figure 60 shows the dependence $\mu(\rho)$ of the given deformation intensity of a freely supported plate also when $\mu_s = 1$. The dot-dash, dashed and solid lines correspond to values ζ equal to 0.4, 0.6 and 0.8. With an increase in ζ on the lower ($\gamma = 1/2$) extended surface of the plate the plasticity rapidly develops, and when $\zeta = 0.8$ the plastic region occupies the region $0 < \rho < 0.76$.

The given data show that over a wide range of values of μ_s it is completely necessary to consider the effect of membrane forces and deformations in the middle plane (appearing in uniform plates as a result of the geometric nonlinearity) on the work with elasto-plastic deformations.

3. Bending and Stability in Large Elasto-Plastic Slightly Curved Shells

The account of geometric nonlinearity with the calculation of shells whose material is subordinated to Hook's law makes it possible to determine the critical loads connected with the stability in general and to explain the effect of the deflections commensurable with the thickness on bending. However, as is shown in Chapter I, shells with thickness $h/R > 1/300$ usually lose stability in the presence of plastic deformations. This leads to the need for solving geometrically and physically nonlinear problems.

Furthermore, the use of concepts of a plastic-rigid material with small deflections is connected with the assumption that the shell does not lose stability up to the exhaustion of the supporting power [108, 127]. The calculation of flexible plastic-rigid slightly curved shells[1] leads to the conclusion that such shells already with zero deflection are unstable in general. The assumption mentioned becomes unnecessary if we forego the plastic-rigid model and consider not only the plastic but also elastic deformations. This way gives also the possibility of discovering that shells made from real materials with a linearly elastic zone on the diagram $\sigma_i - \varepsilon_i$ are stable in general up to the load at which the plastic regions occupy almost the whole section.

The joint account of the geometric and physical nonlinearity makes it possible to study the bending and find the critical loads

[1]Shlabiy, O. N., Large deflections of a plastic-rigid slightly curved spherical shell. Data of summer school on the problem "Physically and Geometrically Nonlinear Problems of the Theory of Plates and Shells," Part II. Tartu, 1966.

of flexible shells in the presence of elasto-plastic deformations, depending on the value of the given intensity of yield deformations.

Let us consider the results of calculations of slightly curved spherical cupolas of different height with the usual four forms of attachment, an evenly distributed load, an ideal elasto-plastic diagram (Fig. 48a) and $\nu_0 = 0.3$. We assign the parameter μ_s in the range 0.1-10. The quantity of Ritz's parameters is accepted at n = m = 4, since a further increase in n and m (for the indicated μ_s and k = 1-5) did not change the solutions.

Figures 61 and 62 show curves of states of equilibria of shells with rigid and sliding attachment of the edge. The straight arrows denote the onset of plasticity. The wavy arrow on Fig. 62 indicates the value ζ at which the unloading appeared.

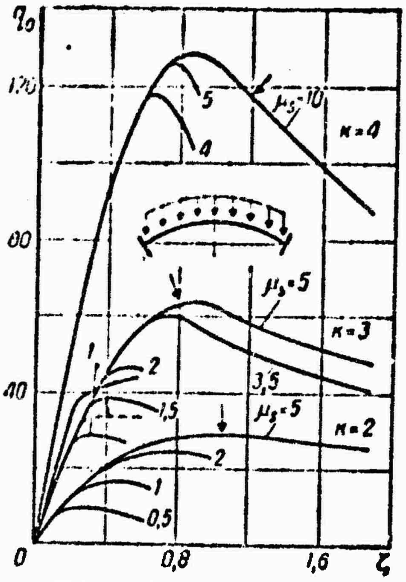


Fig. 61.

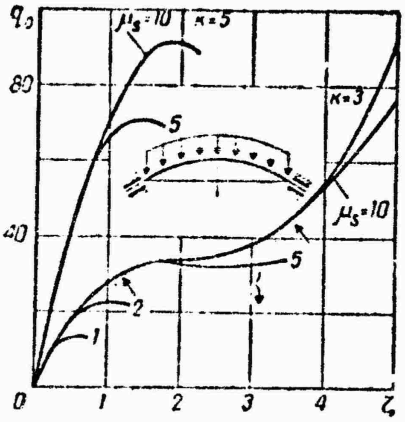


Fig. 62.

An analysis of curves $q_0(\zeta)$ with different μ_s leads to a number of conclusions. First of all, let us note that each height of the shell corresponds to a definite parameter value $\mu_s = \mu_s^0$ at which the plasticity appears for the first time in the critical state $q = q_s$. With μ_s equal to that indicated or larger, the upper critical load is completely determined by the shape of the shell, its attachment, and the form of the load and does not depend on μ_s, i.e., on the plastic properties of the material. It would be interesting to consider approximately the mentioned boundary value μ_s^0. On the basis of the formula for the breaking stress of a complete spherical shell $\sigma_{\kappa p} = 0.6Eh/R$ and accepting $R = a^2/2H$ and $\sigma_{\kappa p} = \sigma_s$, we obtain $\mu_s^0 \approx \frac{\sigma_s}{E}(\frac{a}{h})^2 = 1.2k$. The values of μ_s^0, which can be determined by the graphs, are somewhat higher than those calculated from this formula, but the nature of the dependence $\mu_s^0(k)$ is close to the estimate found.

With a decrease in μ_s the deflection at which there appears plasticity also decreases. Therefore, the less μ_s, then with less deflections the plastic regions become more developed, in consequence of which the value q_B decreases.

Let us examine the three curves $q_0(\zeta)$ when $k = 3$ and $\mu_s = 1$ given on Fig. 61. The dashed curve is obtained from a physically nonlinear theory not allowing for the geometric nonlinearity. Soon after the development of plastic deformations value q_0 ceases to change and becomes equal to the maximum load in connection with the exhaustion of the supporting power. However, an account of the geometric nonlinearity (lower curve) shows that in actuality the elasto-plastic shell reaches not the maximum load but the upper critical load and makes a knock - the sign $dq_0/d\zeta$ at a certain point of the curve $q_0(\zeta)$ is changed and the steady states of equilibrium are changed to unsteady. The zones of plasticity in the critical state are shown on Fig. 63.

Calculations show that even with very small μ_s (for example, $\mu_s = 0.1$), when the plastic regions prove to be greatly developed already with $\zeta \approx 0.05$, the geometric nonlinearity produces a decrease in value q_0 after a certain greatest value of $q = q_B$. Let us emphasize that this occurs with deflections $\zeta < 0.25$, when usually the effect of geometric nonlinearity is disregarded. Hence, there follows the important conclusion: the slightly curved shells loaded in the direction to the center of curvature in the development of plastic deformations even in the region of of very small deflections make a knock; the asymptotic tendency of q_0 to the maximum load (dashed curve on Fig. 61) is not realized. However, one should clearly imagine that with loads $q_0 < q_B$ the elasto-plastic shell is stable, since this region corresponds to $dq_0/d\zeta > 0$.

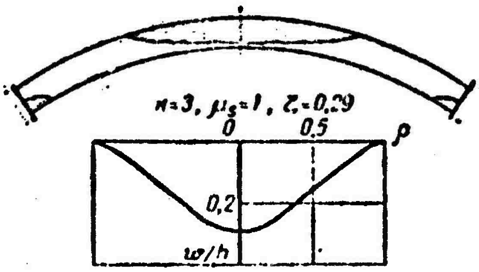


Fig. 63

The upper curve $\mu_s = 1$ and $k = 3$ on Fig. 61 corresponds to the calculation of the shell under the action of the load $q_0 < 0$ – from the center of curvature. Values of ζ for it should be taken with the "minus" sign. After the development of plasticity when $\zeta = 0.2$ the curve sharply turns and asymptotically tends to an inclined straight line of zero-moment solution for a plastic-rigid shell. An increase in the load with an increase in deflection is connected here with the "geometric strain hardening."

Figure 62 shows that the shell with height $k = 3$ and sliding attachment is stable in general if the value μ_s is sufficiently great. At smaller values of μ_s curves $q_0(\zeta)$ have a maximum – a knock appears.

Dependences $q_B(\mu_s)$ are shown on Figs. 64 and 65. They are obtained as a result of an analysis of curves $q_0(\zeta)$ for different

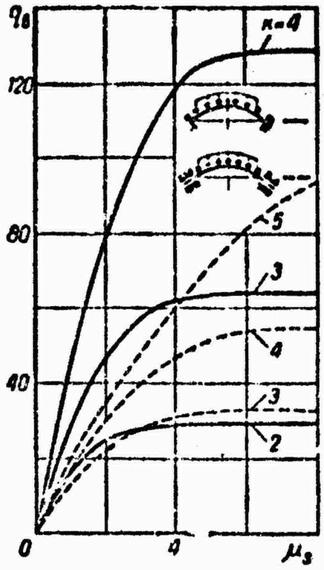


Fig. 64.

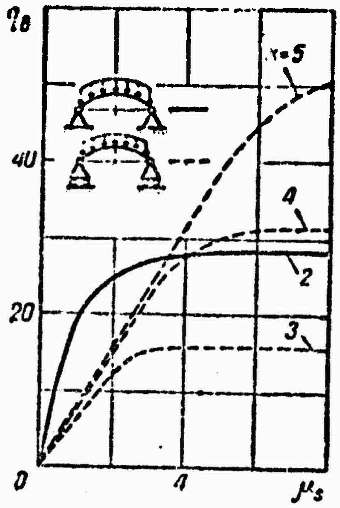


Fig. 65.

forms of the attachment and different values of k and μ_s. On horizontal sections curves of value $q_в$ coincide with values of the upper critical loads known for linearly elastic shells. By knowing the values k, μ_s and the type of attachment, according to Figs. 64. and 65 it is possible to find the greatest load, taking into account the elasto-plasticity and nonlinear effect of the deflections commensurable with the thickness.

4. The Effect of Neutron Irradiation on Stability in a Large Spherical Panel

With the irradiation of metal by rapid neutron flux, a change in the number of physical and mechanical properties occurs [68]. To the greatest degree the effect of irradiation shows up in the value of the yield point σ_s, which increases with an increase in the obtained dose, reaching a certain limit. The dependence of σ_s on the total flow is nonlinear. This dependence for silicon-carbide steel A-212V is given on Fig. 3 of source [43].

If the flow is directed perpendicular to the flat surface of the material, then the dependence of the total flow from the z-coordinate can be taken [43] in the form

$$N = N_0 e^{-ch\left(\frac{1}{2} \pm \gamma\right)}, \quad -\frac{1}{2} < \gamma < \frac{1}{2}. \tag{7.1}$$

Let us use formula (7.1) for calculating the slightly curved shells [49]. Assuming that ch = 2, $N_0 = 4 \cdot 10^{19}$ nvt, $(a/h)^2 = 1000$, $G_0 = 0.8 \cdot 10^6$ bar and $\nu_0 = 0.3$, let us find that the yield point is described well by the formula

$$\sigma_s(\gamma) = (5000 \mp 1800\gamma)\,\text{bar}, \tag{7.2}$$

and the normalized intensity of deformations at the moment of the onset of the plasticity - by formula

$$\mu_s = 2.08 \mp 0.75\gamma. \tag{7.3}$$

In formulas (7.1)-(7.3) the upper and lower signs of the terms which contain γ are related to cases of irradiation on the side of convexity and concavity, respectively. The nonirradiated material has $\sigma_s = 3375$ bar and $\mu_s = 1.405$. Let us emphasize that the linearity of formula (7.3) does not create any important simplifications on which the possibility of the realization of the calculation depends. It is necessary only to assign values of σ_s in six nodes of the Gaussian formula of integration with respect to γ, so that the analytic form of concentration $\sigma_s(\gamma)$ has no vital importance.

Thus, in the statement in question the problem of the effect of irradiation by fast neutron flux is reduced to the calculation of plastically heterogeneous (in thickness) bending of the shell made from an ideal elasto-plastic material.

Let us investigate the effect of irradiation and its direction on the behavior of a slightly curved spherical shell k = 3 high whose edge is rigid or attached in a sliding manner. Figure 66 gives curves $q_0(\zeta)$ of irradiated (flow direction shown by the wavy arrow) and nonirradiated shells ($n = m = 4$). The appearance of plasticity is noted by dots and unloading - by crosses. The dot-dash line shows results of the calculation of flexible linearly elastic shells.

With the rigid attachment of edge the irradiation on the side of convexity increases the upper critical pressure q_B by 37%, and on the side of concavity it increases it by 20% in comparison with the non-irradiated shell. The direction of irradiation with the sliding attachment virtually does not affect the dependence $q_0(\zeta)$. Value q_B increases in this case by approximately 26%.

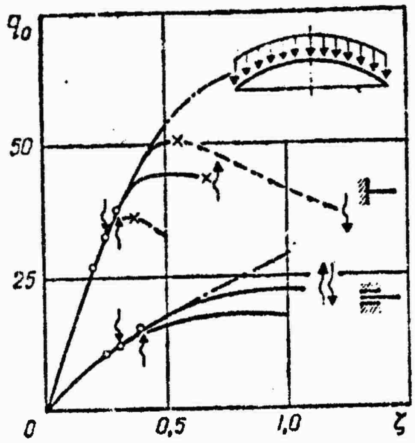


Fig. 66.

The distribution of plastic zones of shells with different attachment in states close to the critical is shown on Fig. 67. It is evident that the regions of plasticity are developed more near the surface opposite to the irradiated surface, which corresponds to the nature of the change in the yield point with respect to thickness.

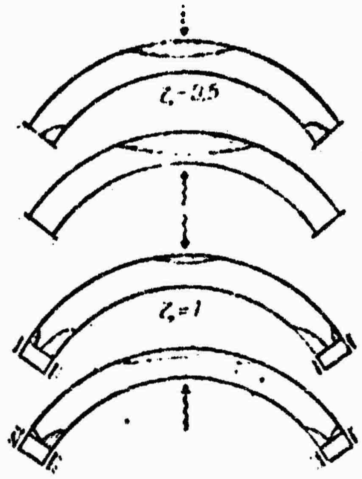


Fig. 67.

Figure 68 gives the distribution $\bar{\sigma}_r$ in the center for different values of ζ of the rigidly attached shell. The solid lines correspond to the irradiation on the side of concavity, dashed lines - on the side of convexity; the dot-dash lines are the nonirradiated shell; $\breve{\sigma}_s(\gamma)$ are shown by dots.

Irradiation noticeably affects the stability and stressed state of slightly curved shells and can be used as the production process which improves the properties of design. If the radiation dose is sufficiently large, then the yield point will become constant according to thickness (the shell will be plastically uniform) but greater than that of the nonirradiated shell. The calculation of such a shell can be made when μ_s = const.

The proposed methodology makes it possible to study also the effect of other changes in the mechanical properties produced by irradiation, for example, elastic heterogeneity $G_0 = G_0(\gamma)$, $\nu_0 = \nu(\gamma)$.

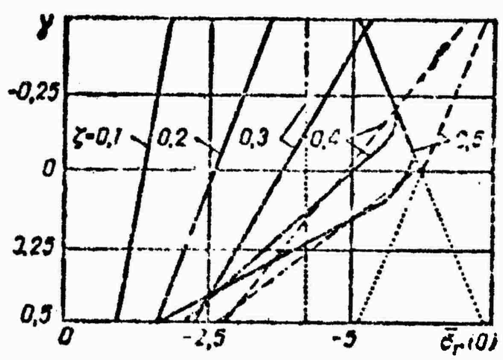


Fig. 68

CHAPTER VIII

DYNAMIC BEHAVIOR OF FLEXIBLE SLIGHTLY CURVED SHELLS

Variational equations of the mixed type are a convenient apparatus for the solution not only of static, but also dynamic problems. The inclusion of this chapter[1] into the book has the purpose of giving an example of such an approach. Examined here is the geometrically nonlinear problem for slightly curved shells made from an isotropic material subordinate to Hooke's law. We consider the load to be evenly distributed and normal to the middle surface but time-dependent. The effect of the force of inertia in directions tangent to the middle surface and the inertia of rotation of the right section are disregarded. The theory is stated for shells with an arbitrary form of the middle surface and the algorithm and concrete calculations are given for the axisymmetric deformation of slightly curved shells of rotation.

1. Brief Survey of Literature

We find the systematic presentation of the theory and results of a study of the stability of plates and shells with dynamic load in the books of A. S. Vol'mir [22] and V. V. Bolotin [12, 13].

[1]The material is obtained by the author in conjunction with L. A. Starosel'skiy.

Let us consider only some methods of the solution of problems of the dynamic behavior of shells, which can be described by the nonhomogeneous differential equations

$$D\Delta\Delta w - \Delta_k\varphi - L_2(w + w_0, \varphi) + \frac{h\gamma}{g}(\ddot{w} + \varepsilon\dot{w}) = q(\alpha, \beta, t);$$
$$\frac{1}{Eh}\Delta\Delta\varphi + \Delta_k w - \frac{1}{2}L_2(w + 2w_0, \varphi) = 0. \qquad (8.1)$$

System (8.1) is obtained from equations (2.29) if we consider constants E, ν, and h and in accordance with the d'Alembert principle take into account the force of inertia and attenuation along the normal to the middle surface. Here γ is the specific gravity of the material, q - the acceleration of gravity, ε - the attenuation factor, t - time, and the dots above w indicate the time differentiation.

The most widespread method of the approximate transition from the problem with the boundary and initial conditions (8.1) to the Cauchy problem consists of the use of the generalized procedure of P. F. Papkovich. In searching for the deflection in the form of

$$w = \sum_{i=1}^{n} x_i(t)\, w_i(\alpha, \beta) \qquad (8.2)$$

and accurately integrating the second equation of system (8.1), we substitute the expression obtained for ϕ and formula (8.2) into the first equation, which further according to the Bubnov-Galerkin method leads to the Cauchy problem relative to functions $x_1(t)$. Such a method (usually with one term of a series (8.2)) is used in works of A. S. Vol'mir [22], V. V. Bolotin and G. A. Boychenko [14] and P. M. Ogibalov [86]. The clicking of the cylindrical panel, under the action of impact acoustic loading, is examined in a one-term approximation in source [31].

The ternary representation of the deflection and function of forces with the use of the Bubnov-Galerkin method directly to system (8.1) permitted V. I. Feodos'yev [111] to solve more accurately the problem of the stability of a slightly curved spherical shell under the action of the load $q = \lambda t$. The mathematical foundation of the use of the Bubnov-Galerkin method in the nonlinear theory of slightly curved shell was given by I. I. Vorovich [23].

Simitsis [98] in a one-term and binomial approximation investigated the knock of a slightly curved spherical cupola under the effect of pressure applied suddenly and also in the form of an ideal pulse. The slightly curved conical shell, under the action of a pulsating load, is considered as the single-degree-of-freedom system of Fulton [114]. The behavior of a slightly curved spherical shell, under the action of a load periodic in time, is studied in source [41]. Used as the coordinates were the Bessel functions, the periodic solution of the Cauchy problem was sought. The finite-difference method for the solution on each step with respect to time of the nonlinear boundary value problem was used by A. Yu. Birkgan and A. S. Vol'mir [11] for calculating the reaction of a slightly curved panel square in design for the linear (with respect to time) load. A similar procedure was used by Archer and Leng [6] during a study of the action of the square pressure pulse on a slightly curved sphere. Even with a large step on coordinates of middle surface, such an approach requires noticeable expenditures of machine time.

One of the possible ways of the solution of dynamic problems of the theory of flexible slightly curved shells is the use of the variational equation of the mixed type and the transition to the system of ordinary differential equations with respect to time by the method which is similar to Ritz's procedure. The advantage of such a method consists in the simplicity of the account of heterogeneities of the material and also the algorithm of the construction and solution on the computer of equations of the problem.

2. Variational Equation and Transition to the Cauchy Problem

We proceed from the principle of possible displacements, the fundamental relation of which has the form

$$-\delta(U_{и}+U_{c})+\iint R\delta w\,ds=0. \tag{8.3}$$

The first component is the virtual work of elastic forces which act in the material of the shell. The second term corresponds to the virtual work of the load and also of forces of inertia (applied in accordance with the d'Alembert principle) and is dissipative:

$$R=q-\frac{h\gamma}{g}(\ddot{w}+\varepsilon\dot{w}). \tag{8.4}$$

The relation (8.3), taking into account (2.24) and (8.4), reduces to the variational equation

$$\begin{aligned}\delta\iint\Big\{&\frac{D}{2}\left[(\Delta w)^2-(1-\nu)L_2(w,w)\right]-\Big[\Delta_k\varphi+L_2\Big(\frac{1}{2}w+\\&w_0,\varphi\Big)\Big]w-\frac{1}{2Eh}\left[(\Delta\varphi)^2-(1+\nu)L_2(\varphi,\varphi)\right]\Big\}ds-\\&-\iint\left[q-\frac{h\gamma}{g}(\ddot{w}+\varepsilon\dot{w})\right]\delta w\,ds=0.\end{aligned} \tag{8.5}$$

For the solution of equation (8.5), in which w and ϕ are the independently varied unknown functions, it is not possible to use directly Ritz's procedure, since this equation does not have the form of equality to zero of the variation from the functions. Therefore, let us use the method (let us call it the method of generalized coordinates) which was used by G. A. Marchenko in the investigation of dissipative problems of the vibrations of plates.[1]

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[1]News of Colleges. Aviation Technology, 1966, 3.

Let us decompose the deflection and function of forces with respect to two complete systems of the linearly independent functions, subordinate to the main boundary conditions, and we will be limited to the finite sums

$$w = \sum_{i=1}^{n} x_i(t)\,w_i(\alpha, \beta); \qquad q = \sum_{i=1}^{m} y_i(t)\,\varphi_i(\alpha, \beta). \tag{8.6}$$

The coefficients $x_i(t)$ and $y_i(t)$ are unknown functions of time. Substituting expressions (8.6) into equation (8.5), performing the operation of variation and equating to zero the coefficients of δx_i and δy_i, we will obtain the system of ordinary differential equations relative to functions $x_i(t)$ and $y_i(t)$

$$\left.\begin{aligned} A_{ik}(\ddot{x}_k + \bar{\varepsilon}\dot{x}_k) + B_{ik}x_k + C_{ip}y_p + D_{ikp}x_k y_p = Q_i q_0; \\ C_{pi}x_i - E_{pj}y_j + \frac{1}{2}D_{pkl}x_k x_l = 0, \\ i, k = 1, 2, \ldots, n; \quad p, j = 1, 2, \ldots, m. \end{aligned}\right\} \tag{8.7}$$

Let us introduce the dimensionless quantities $t = \bar{t}_\tau$, $\varepsilon = \bar{\varepsilon}/\tau$, where $\tau = \frac{a}{h_0} = \sqrt{\frac{a^2\gamma}{Eg}}$, and the parameter of the load $q_0(t) = q(t)a^4/Eh_0^4$.

In the case of the axisymmetric deformation of a slightly curved shell of rotation with a thickness of $h = h_0(1 + c\rho)$, the matrices of coefficients of system (8.7) have the form

$$\begin{aligned} A_{ik} &= \int_0^1 (1 + c\rho)\,w_i w_k \rho\,d\rho; \\ B_{ik} &= \frac{1}{12(1-\nu^2)} \int_0^1 (1 + c\rho)^3 [\Delta w_i \Delta w_k - (1-\nu) L_2(w_i, w_k)]\,\rho\,d\rho; \\ C_{ip} &= -\int_0^1 [\Delta_k \varphi_p + L_2(w_0, \varphi_p)]\,w_i \rho\,d\rho; \end{aligned}$$

$$D_{ikp}=-\int_0^1 w_i L_2(w_k, \varphi_p)\rho d\rho; \quad Q_i=\int_0^1 w_i \rho d\rho;$$

$$E_{jp}=-\int_0^1 \frac{1}{1+c\rho}[\Delta\varphi_j\Delta\varphi_p-(1+\nu)L_2(\varphi_j,\varphi_p)]\rho d\rho. \tag{8.8}$$

Solving the second equation of system (8.7) relative to y_j, we obtain

$$y_j=\left[E_{jp}^{-1}C_{ps}+\frac{1}{2}(E_{jp}^{-1}D_{ips}x_i)_s\right]x_s. \tag{8.9}$$

Multiplying by A_{ik}^{-1} the first equation of system (8.7), and designating $\dot{x}_i = r_i$ we arrive at the Cauchy problem

$$\begin{aligned}\dot{r}_i&=-\bar{\varepsilon}r_i+[A_{ik}^{-1}C_{kj}+(A_{ik}^{-1}D_{ksj}x_s)_j]y_j-A_{ik}^{-1}B_{ks}x_s-q_0(\bar{t})A_{ik}^{-1}Q_k,\\ \dot{x}_i&=r_i, \quad i,k,s=1,2,\ldots,n; \quad p,j=1,2,\ldots,m\end{aligned} \tag{8.10}$$

for the nonlinear system of equations of the first order. The conversion carried out is possible, since matrices A_{ik}^{-1} and E_{jp}^{-1} exist if the coordinate functions are linearly independent.

We will integrate system (8.10) with the initial conditions

$$x_i=0, \ \dot{x}_i=0 \ \text{when} \ t=0.$$

3. Some Formulas

We will obtain precise formulas for the matrix elements of system (8.10). In turning to the problem of the behavior of a slightly curved spherical cupola, let us consider the latter as a plate ($\Delta k\phi \equiv 0$) with an initial chamber $\bar{w}_0 = -k(1 - \rho^2)$, $k = H/h_0$ and use the coordinate functions given in Table 1. It is easy to note that in this case each of the formulas (8.8) can be represented by the sum of the integrals of the form

$$I(x,y)=\int_0^1 \rho^x(1-\rho^2)^y d\rho=\frac{(2y)!!\,(x-1)!!}{(x+2y+1)!!}. \tag{8.11}$$

Using the relation (8.11) and without discussing the calculations, let us give the final results for each of the four types of attachments.

1. The fixed attachment:

$$
\left.
\begin{aligned}
&A_{ik}^{(1)} = \frac{1}{6+2i+2k} + c\,\frac{(4+2i+2k)!!}{(7+2i+2k)!!},\\
&B_{ik}^{(1)} = \frac{4(i+1)(k+1)}{3(1-\nu^2)}\left\{\frac{1}{i+k+1}\left(\frac{ik}{(i+k)(i+k-1)} + \right.\right.\\
&\quad + \frac{3c^2}{2(i+k+2)}\left[\frac{6ik}{(i+k)(i+k-1)} - \frac{1+\nu}{2}\right]\Bigg) + \\
&\quad + \frac{3c(2i+2k-4)!!}{(2i+2k+3)!!}\left[15ik-(1+\nu)(i+k)(i+k-1)\right]\Bigg\},\\
&C_{ip}^{(1)} = -4\,\frac{H}{h_0}\,p\,\frac{(1+i)!\,p!}{(i+p+1)!}; \quad Q_i = \frac{1}{2(i+2)};\\
&E_{jp}^{(1)} = 4jp\left[(j+p-1)(1+\nu)-2jp\right]\times\\
&\quad \times\left(\frac{1}{j+p-1} - \frac{c}{j+p-1/2} + \frac{c^2}{j+p}\right).
\end{aligned}
\right| \qquad (8.12)
$$

2. The movable attachment:

$$
\left.
\begin{aligned}
&A_{ik}^{(2)} = A_{ik}^{(1)}; \quad B_{ik}^{(2)} = B_{ik}^{(1)};\\
&C_{ip}^{(2)} = 4\,\frac{H}{h_0}(i+1)(p+1)\,\frac{(i+p)!}{(i+p+2)!}; \quad Q_i^{(2)} = Q_i^{(1)};\\
&E_{jp}^{(2)} = -16(j+1)(p+1)\left\{\frac{pj}{(j+p+1)(j+p)(j+p-1)}\right.\\
&\quad + \frac{c^2}{2(j+p+1)(j+p+2)}\left[\frac{6jp}{(j+p)(j+p-1)} - \frac{1-\nu}{2}\right] -\\
&\quad \left. - c\,\frac{(2j+2p-4)!!}{(2j+2p+3)!!}\left[15jp-(1-\nu)(j+p)(j+p-1)\right]\right\}.
\end{aligned}
\right| \qquad (8.13)
$$

Matrices D_{ikp} of these two forms of attachment are equal to g_{ijs} (first and third formulas (3.12)) with the replacement of j and s by k and p, respectively.

3. The fixed hinge:

$$B_{ik}^{(3)} = \begin{cases} \dfrac{2 + c(4 + 3c)}{6(1-\nu)}, & k = i = 1, \\ -\dfrac{c(1 + 5c)}{15(1-\nu)}, & i = 1,\ k = 2;\ i = 2,\ k = 1; \\ B^{(1)}_{i-1,k-1}; \end{cases} \qquad A_{ik}^{(3)} = A^{(1)}_{i-1,k-1};$$

$$C_{ip}^{(3)} = C^{(1)}_{i-1,p}; \quad D_{ikp}^{(3)} = D^{(1)}_{i-1,k-1,p}; \quad Q_i^{(3)} = Q^{(1)}_{i-1}; \quad E_{ip}^{(3)} = E_{ip}^{(1)}. \qquad (8.14)$$

4. The mobile hinge:

$$A_{ik}^{(4)} = A^{(1)}_{i-1,k-1}; \quad B_{ik}^{(4)} = B_{ik}^{(3)}; \quad C_{ip}^{(4)} = C^{(2)}_{i-1,p};$$

$$D_{ikp}^{(4)} = D^{(2)}_{i-1,k-1,p}; \quad Q_i^{(4)} = Q^{(2)}_{i-1}; \quad E_{ip}^{(4)} = E_{ip}^{(2)}. \qquad (8.15)$$

In the derivation of formulas (8.12)-(8.15) there are used the approximate relations $(1 + c\rho)^3 \approx 1 + 3c\rho + 3c^2\rho^2$, $(1 + c\rho)^{-1} \approx \approx 1 - c\rho + c^2\rho^2$, which when $c \leq 0.2$ give an error which does not exceed 1%.

To study the action of pulse loads, it is useful to place formula $q_0(t)$ which, without the inclusion of machine logical operations, would describe a graph of the type shown on Fig. 69.

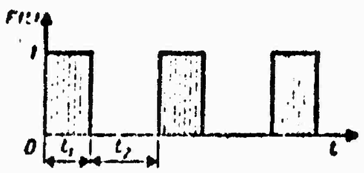

Fig. 69.

The function which has the values

$$F(t) = \begin{cases} 1, & t \in [0, t_1) \\ 0, & t \in [t_1, t_1 + t_2], \end{cases} \qquad (8.16)$$

is represented by formula

$$F(t) = 1 - E\left[\frac{t + t_2 E(t/t_1)}{t_1 + t_2 E(t/t_1)}\right], \qquad (8.17)$$

in which E(x) is the whole part of the number.

Let us show that the introduced function possesses properties of (8.16). When $t \in [0, t_1]$ this is simple to check by substitution. The second condition in (8.16) will be satisfied if with any

t_1 and t_2 the expression included in the outer brackets is less than two. We have

$$\frac{t + t_2 E(t/t_1)}{t_1 + t_2 E(t/t_1)} = 1 + \frac{t - t_1}{t_1 + t_2 E(t/t_1)} < 1 +$$
$$+ \frac{t - t_1}{t_1 + t_2} < 1 + \frac{t_2}{t_1 + t_2} < 2.$$

The behavior of function F(t) outside the interval $[0, t_1 + t_2]$ does not interest us. The pulses which are repeated with the period $t_1 + t_2$ can be obtained by making the replacement of the variable according to formula

$$t = z - (t_1 + t_2) E\left(\frac{z}{t_1 + t_2}\right). \qquad (8.18)$$

It is obvious that for any $z \in [0, \infty]$ there will be $t \in [0, t_1 + + t_2]$. In this case, taking into account (8.18), function $q_0(t)$ coincides with the right side of (8.17).

In choosing t_2 sufficiently large, according to formula (8.17) it is possible to describe the single pulse.

The construction of system (8.9) and its integration by the Adams method are programmed for the computer "Ural-2."

4. The Dynamic Behavior of a Slightly Curved Spherical Panel

Let us check the rate of the convergence of the method according to the number of degrees of freedom on a rigidly attached shell stressed by a square pulse. Figure 70 gives the dependences of deflection in the center on the time parameter; $k = 4$, $\varepsilon = 3$. Even with $n = m = 3$ acceptable results are obtained; the data obtained with four and five terms of a series for deflection and the function of forces coincide on the figure.

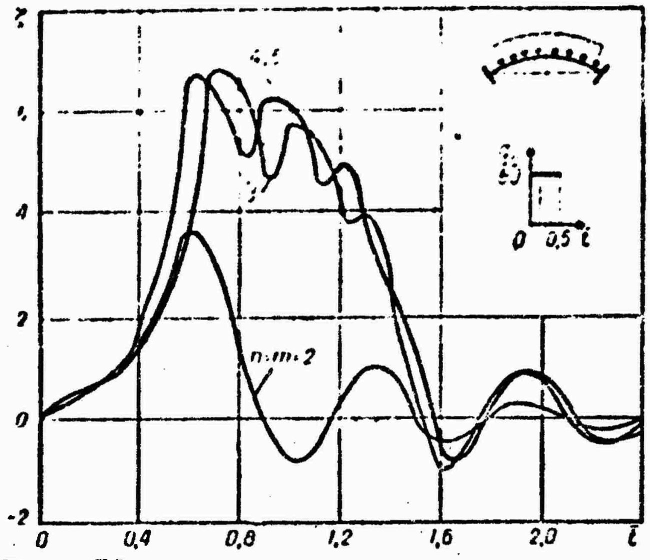

Fig. 70.

The effect of conditions of attachment on the behavior of the shell with the relative height equal to four thicknesses with the action of the pulse of 60 × 0.9 is shown on Fig. 71 ($n = m = 4$, $\varepsilon = 3$). The shell, fastened by the scheme "fixed hinge," after a knock vibrates near the inverted position. With other methods of support, after unstressing there occurs the discharge to the initial state and damping oscillations near it.

Unlike the problems of statics, where the concepts about the upper and lower critical loads were established, in dynamic problems there is no such certainty. Since it is not clear which deflections or which state of the shell in time can be considered corresponding to the knocking, let us conditionally assume that the shell knocks if the deflection in the center ζ reaches a value large in relative height k. The pulse with such amplitude and duration, which lead to the knocking of the shell, is called critical.

On Fig. 72 ($k = 4$, $n = m = 4$) the pulse 45 × 2 is not critical for the rigidly attached shell. The amplitude change per unit leads to knocking, and the shell is reversed ($\zeta_{max} \approx 2k$). A similar pattern can be observed by varying not the amplitude but a pulse duration.

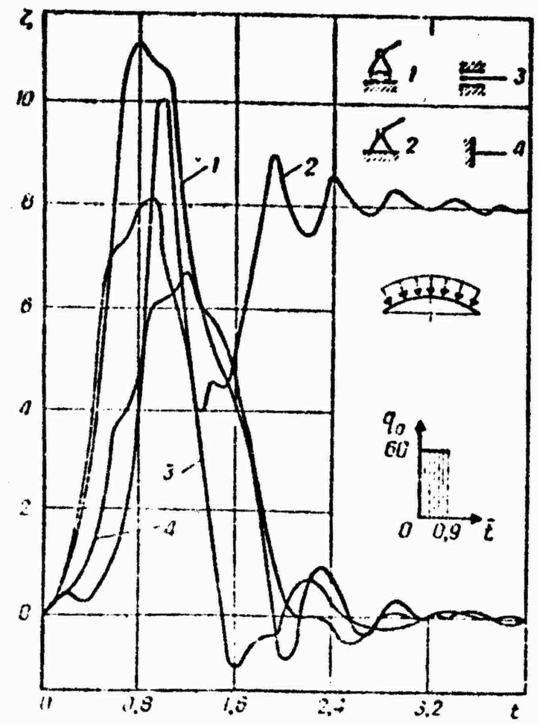

Fig. 71.

The dependence of amplitude on duration of the critical pulse for the rigidly attached cupola with height k = 3.8 when ε = 0 and n = m = 5 is given on Fig. 73 by a solid line. The dashed curve is the dependence obtained in source [123], and the dot-dash curve represents data of Archer and Leng [6], who took into account forces of inertia in a direction tangent to the middle surface. The stress distribution of this shell in a knocked state is shown on Fig. 74, where k = 3.8; ε = 3; n = m = 4; ζ = 7.55. Figure 74 shows that the bending stresses predominate over the membrane stresses over the entire length of the radius.

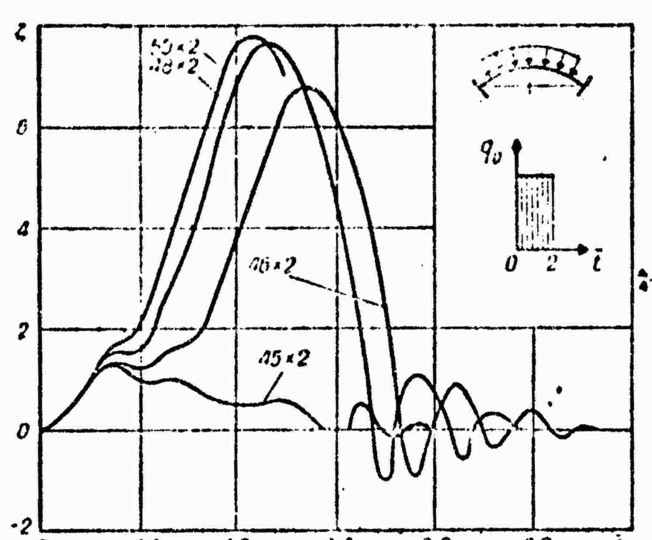

Fig. 72.

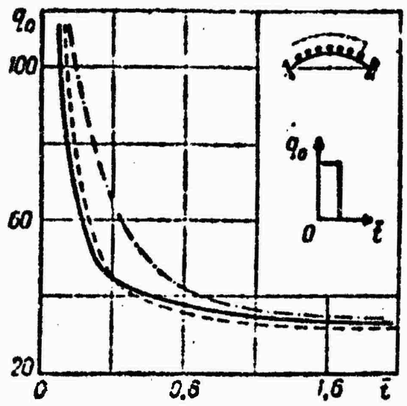

Fig. 73.

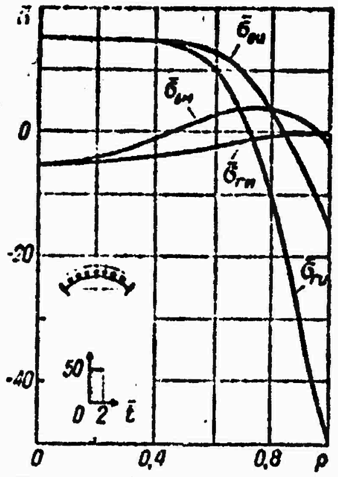

Fig. 74.

The effect of the form of the attachment on dimensions of the critical pulse can be traced on Fig. 75 ($k = 4$, $n = m = 4$, $\varepsilon = 3$). It is interesting to note that the phenomenon of knocking is especially affected by the boundary conditions superimposed on the function of forces (mobility of immobility of the edge in the middle surface). At the same time the subordination of deflection to the condition of attachment or free support affects less the amplitude of the critical pulse.

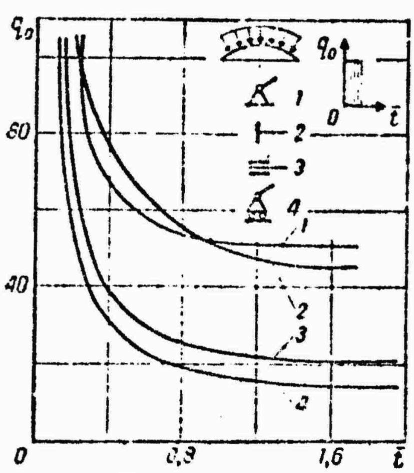

Fig. 75.

By comparing curves for the fixed attachments and hinge, it is possible to see that in the range $q_0 > 100$ and $q_0 < 53$ the hinged attached shell possesses greater supporting power.

Let us consider the behavior of the shell stressed according to the law $q_0 = \lambda\bar{t}$ with different proportionality factors λ when $k = 4$ and $n = m = 3$. With an increase in λ the load at which the shell knocks increases (Fig. 76).

As an example let us also give results of the calculating of slightly curved spherical shells of linear-variable thickness when $\lambda = 1$, $k = 4$, $\varepsilon = 3$, $n = m = 3$. A shell with thickening from the center to the periphery ($c = 0.2$) and vice versa is calculated. The critical load for a shell with thickening in the center is somewhat higher (Fig. 77). Let us note that the curves $c = 0$ of a shell of constant thickness on Fig. 77 and $\lambda = 1$ on Fig. 76 coincide with the curve obtained in source [130].

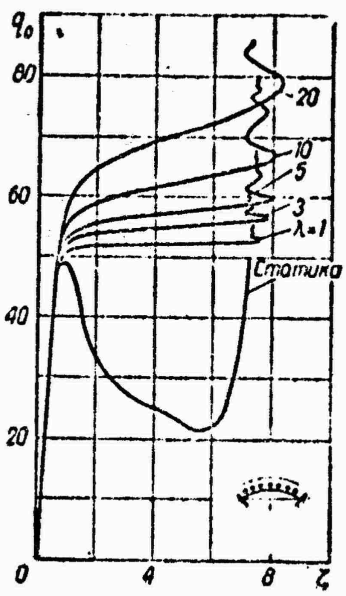

Fig. 76.

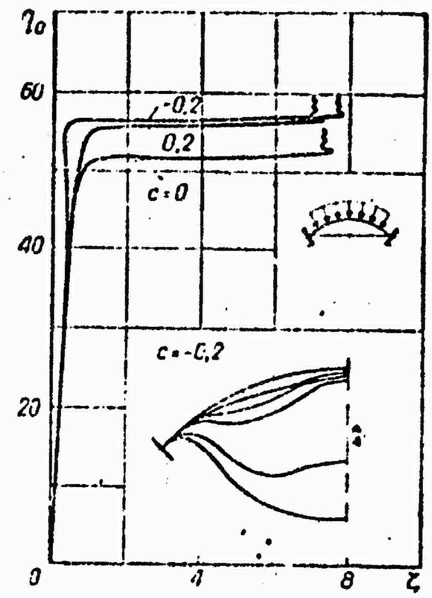

Fig. 77.

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