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DEFENSE APPLICATIONS OF MATHEMATICAL PROGRAMS WITH OPTIMIZATION PROBLEMS IN THE CONSTRAINTS

Jerome Bracken
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13. ABSTRACT

Theory, computation, and an example of mathematical programming models with optimization problems in the constraints have been discussed in a previous paper [1]. A computer program for solving mathematical programming models with nonlinear programs in the constraints has been presented in a subsequent paper [2]. A procedure for transforming mathematical programs with two-sided optimization problems in the constraints into mathematical programs with nonlinear programs in the constraints, enabling solution by the computer program of [2], has been given in [3].

The present paper formulates models of defense problems which are convex programs having the mathematical properties treated in the previous papers. The models include several strategic forces planning models and two general purpose forces planning models.

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ABSTRACT

Theory, computation, and an example of mathematical programming models with optimization problems in the constraints have been discussed in a previous paper [1]. A computer program for solving mathematical programming models with nonlinear programs in the constraints has been presented in a subsequent paper [2]. A procedure for transforming mathematical programs with two-sided optimization problems in the constraints into mathematical programs with nonlinear programs in the constraints, enabling solution by the computer program of [2], has been given in [3].

The present paper formulates models of defense problems which are convex programs having the mathematical properties treated in the previous papers. The models include several strategic forces planning models and two general purpose forces planning models.

I.   INTRODUCTION

Reference [1] presents theory, interpretations and an example of mathematical programs with optimization problems in the constraints. The first result there deals with   Problem (A):   find vectors $x = (x_1, \ldots, x_n)$ and $v^i = (v_1^i, \ldots, v_{k_i}^i)$, for $i = 1, \ldots, m$, to[1]

$$\text{minimize } f(x)$$
$$x \in X$$

subject to

$$h_i(x) = \min_{v^i \in V^i} g^i(x, v^i) \geq 0 \ , \qquad i = 1, \ldots, m \quad .$$

If $f(x)$ is a convex function of x on a convex set X and $g^i(x, v^i)$ is concave in x on X for every $v^i \in V^i$ and for $i = 1, \ldots, m$, then the mathematical program is a convex program. In order to computationally solve this problem it is desirable that $g^i(x, v^i)$ be convex in v^i and the set V^i be convex. In Reference [1] an example is presented and solved in which $g^i(x, v^i)$ is concave in x , and $\min_{v^i \in V^i} g^i(x, v^i)$ is a linear programming problem in v^i.

The second result in Reference [1] deals with Problem (B):   find vectors $x = (x_1, \ldots, x_n)$, $v^i = (v_1^i, \ldots, v_{k_i}^i)$, and $u^i = (u_1^i, \ldots, u_{\ell_i}^i)$, for $i = 1, \ldots, m$, to

---

1. All indicated maximums and minimums are assumed to exist.

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$$\text{minimize } f(x)$$
$$x \in X$$

subject to

$$\bar{h}_i(x) = \max_{u^i \in U^i(x)} \min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0 , \quad i = 1, \ldots, m .$$

If $f(x)$ is a convex function of x on a convex set X, $g^i(x, u^i, v^i)$ is concave in (x, u^i) for every $v^i \in V^i$, and $U^i(x)$ is a convex set for $x \in X$, for $i = 1, \ldots, m$, then the mathematical program is a convex program. In order to computationally solve the problem it is also desirable that $g^i(x, u^i, v^i)$ be convex in v^i and the set V^i be convex.

Reference [2] presents a computer program for solving Problem (A) where the constraints $\min_{v^i \in V^i} g^i(x, v^i) \geq 0$, $i = 1, \ldots, m$ contain non-linear programs.

Reference [3] shows how Problem (B) can be solved by the computer program of Reference [2].

The purpose of this paper is to present optimization models having the structure of the above mathematical programs for some defense planning problems. Each of the problems that follows is formulated in terms of its major planning variables. Specific functional forms are assumed which have the general properties possessed by the physical process being modeled. These functional forms have parameters, the values of which could be determined from empirical data or detailed quantitative descriptions of the physical processes. This paper concentrates on the problem formulation. All of the models considered are convex programs.

II. GENERAL CHARACTERISTICS OF THE MODELS

Consider Problem (A). The constraint set for the mathematical program is equivalent to

$$\left\{ x \in X: \ g^i(x, v^i) \geq 0 \text{ for all } v^i \in V^i, \ i = 1, \ldots, m \right\}.$$

This formulation renders a ready interpretation of the problem. The "outside optimizer" chooses forces x such that each of the m effectiveness requirements in the constraint set can be met at a minimal value of the objective function, no matter what feasible choice of $v^1, \ldots, v^m$ the "inside optimizer" makes. An alternative interpretation is that of the outside optimizer choosing x, followed by (or simultaneous with) the inside optimizer's choice to minimize the effect of x.

The first military model having the structure of Problem (A) is that of an outside optimizer choosing forces x with minimum cost $f(x)$ which can achieve effectiveness objectives $r_1, \ldots, r_m$ in the face of optimal allocations of $v^i \in V$ ($i = 1, \ldots, m$) by one inside optimizer in an attempt to minimize the effects. The model is to choose x and $v^1, \ldots, v^m$ to

$$\text{minimize } f(x)$$

subject to

$$\min_{v^i \in V} g^i(x, v^i) \geq r_i, \quad i = 1, \ldots, m.$$

The outside optimizer can achieve his objectives against any of the various uses of one inside optimizer's resources V.

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The second military model having the structure of Problem (A)
is that of an outside optimizer choosing forces x with minimum cost $f(x)$
which can achieve any one of the objectives $r_1, \ldots, r_m$ in the face
of optimal allocations of $v^i \in V^i$ ($i = 1, \ldots, m$) by m inside
optimizers. The model is to choose x and $v^1, \ldots, v^m$ to

$$\text{minimize } f(x)$$

subject to

$$\min_{v^i \in V^i} g^i(x, v^i) \geq r_i , \qquad i = 1, \ldots, m .$$

The outside optimizer chooses x, and m inside optimizers choose
$v^1, \ldots, v^m$. The outside optimizer will be capable of achieving his
objective in any one conflict in which he engages. One example might
be the outside optimizer planning to fight and win any one conventional
war but not more than one simultaneously.

The third military model having the structure of Problem (A) is
that of an outside optimizer choosing forces $x^1, \ldots, x^m$ with minimum
cost $f(x^1, \ldots, x^m)$ which can achieve all of the objectives $r_1, \ldots, r_m$
in the face of optimal allocations of $v^i \in V^i$ ($i = 1, \ldots, m$). The model
is to choose $x^1, \ldots, x^m$ and $v^1, \ldots, v^m$ to

$$\text{minimize } f(x^1, \ldots, x^m)$$

subject to

$$\min_{v^i \in V^i} g^i(x^i, v^i) \geq r_i , \qquad i = 1, \ldots, m .$$

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There are two interpretations of this problem. There may be one inside optimizer who divides his resources into $V^1, \ldots, V^m$, or there may be m inside optimizers with resources $V^1, \ldots, V^m$. In either case, the outside optimizer can simultaneously achieve all of the objectives.

Consider Problem (B). The constraint set for the mathematical program is equivalent to

$$\left\{ x \in X: \min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0 \text{ for some } u^i \in U^i(x) , i = 1, \ldots, m \right\} .$$

Again, this formulation provides a ready military interpretation. The outside optimizer chooses forces x . For this choice, there is a feasible set of alternative uses of these forces, in achieving an effectiveness objective i , namely $U^i(x)$. Thus the outside optimizer chooses a force level x and an optimal and feasible use of these forces for each of the objectives, namely $u^i \in U^i(x)$ (i = 1, ..., m). The inside optimizer makes his feasible choices $v^i \in V^i$ (i = 1, ..., m) to minimize the effects of the outside optimizer's choices.

It is shown in [3] that Problem (B) is equivalent to choosing x and $u^1, \ldots, u^m$ to

$$\text{minimize } f(x)$$

subject to

$$x \in X ,$$
$$u^i \in U^i(x) , \quad i = 1, \ldots, m ,$$
$$\min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0 , \quad i = 1, \ldots, m .$$

Variants of Problem (B) analogous to those discussed above for Problem (A) can also be treated.

III. STRATEGIC OFFENSE OR DEFENSE FORCE STRUCTURE OPTIMIZATION

This section presents an offense and a defense force structure model. Each is a mathematical program with nonlinear programs in the constraints. Both are convex programs. A passive defense model is presented in detail.

In the strategic offense model, the outside optimizer chooses minimum-cost offensive forces capable of achieving specified destruction of various resources by type, despite allocation of specified defensive forces of the inside optimizer to minimize the destruction.

In the strategic defense model, the outside optimizer chooses minimum-cost defensive forces capable of assuring specified survival of various resources by type, despite the optimal allocation of specified offensive forces of the inside optimizer to minimize the surviving resources.

Definitions

Let the indexes of locations be $i = 1, \ldots, p$, of offensive weapon types be $j = 1, \ldots, q$, and of defensive resource types be $k = 1, \ldots, r$. Let the index of target types be $\ell = 1, \ldots, s$.
Define

y_{ij} = offensive weapons of type j targeted to location i,

z_{ik} = defensive resources of type k assigned to location i,

α_{ij}^{ℓ} = parameter associated with destruction of target type ℓ in location i by offensive weapons of type j,

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β_{ik}^{ℓ} = parameter associated with defense of target type ℓ in location i by defensive resources of type k,

W_i^{ℓ} = value of target type ℓ at location i

$f(y)$ = cost of providing offensive weapons $y = (y_{ij})$ (i = 1, ..., p; j = 1, ..., q),

$g(z)$ = cost of providing defensive resources $z = (z_{ik})$ (i = 1, ..., p; k = 1, ..., r).

Effectiveness Function for Offense Optimization

The measure of effectiveness for offense optimization is destroyed value of type ℓ . One function which is concave in y_{ij} and convex in z_{ik} is

$$\sum_i W_i^{\ell} \exp\left(-\sum_k \beta_{ik}^{\ell} z_{ik}\right)\left[1 - \exp\left(-\sum_j \alpha_{ij}^{\ell} y_{ij}\right)\right] .$$

In the strategic offense models the outside optimizer chooses y_{ij} and the inside optimizer chooses z_{ik}, so the function is appropriately behaved to yield a convex program.

Effectiveness Function for Defense Optimization

The measure of effectiveness for defense optimization is surviving value of type ℓ . One function which is concave in z_{ik} and convex in y_{ij} is

$$\sum_i W_i^{\ell} \left\{1 - \exp\left(-\sum_k \beta_{ik}^{\ell} z_{ik}\right)\left[1 - \exp\left(-\sum_j \alpha_{ij}^{\ell} y_{ij}\right)\right]\right\} .$$

In the strategic defense models the outside optimizer chooses z_{ik} and

the inside optimizer chooses y_{ij}, so the function is appropriately behaved to yield a convex program.

Force Structure Optimization Models

The strategic offense optimization model is to choose y_{ij} $(i = 1, \ldots, p; j = 1, \ldots, q)$ and $z_{ik}(i = 1, \ldots, p; j = 1, \ldots, r)$ to

$$\text{minimize } f(y)$$

subject to

$$\left[\begin{array}{l} \min\limits_{z} \left\{ \sum\limits_{i} W_i^{\ell} \exp(- \sum\limits_{k} \beta_{ik}^{\ell} z_{ik})[1 - \exp(- \sum\limits_{j} \alpha_{ij}^{\ell} y_{ij})] \right\} \\ \sum\limits_{i} z_{ik} \leq Z_k \end{array} \right] \geq \overline{D}^{\ell}$$

where

$\overline{D}^{\ell}$ = specified destroyed value to be assured by outside optimizer,

Z_k = defensive forces of type k available to inside optimizer.

The strategic defense optimization model is to choose z_{ik} $(i = 1, \ldots, p; k = 1, \ldots, r)$ and $y_{ij}(i = 1, \ldots, p; j = 1, \ldots, q)$ to

$$\text{minimize } g(z)$$

subject to

$$\left[\begin{array}{l} \min\limits_{y} \left\{ \sum\limits_{i} W_i^{\ell} \left\{ 1 - \exp(- \sum\limits_{k} \beta_{ik}^{\ell} z_{ik})[1 - \exp(- \sum\limits_{j} \alpha_{ij}^{\ell} y_{ij})] \right\} \right\} \\ \sum\limits_{i} y_{ij} \leq Y_j \end{array} \right] \geq \overline{S}^{\ell}$$

where

$\overline{S}^{\ell}$ = specified surviving value to be assured by outside optimizer,

Y_j = offensive forces of type j available to inside optimizer.

Properties of Models

The two models are convex programming problems. They contain convex programming problems in the constraints. Thus the procedure of Reference [2] can be used for their solution.

Example: Passive Defense Model

The outside optimizer's problem is to provide hardening and/or evacuation capabilities for population and hardening and/or dispersion capabilities for industry. Specified survival levels must be achieved in the face of a nuclear attack by the inside optimizer. These passive defense measures are to be supplied at minimum cost to protect against attacks on either population or industry. Let $\ell = 1$ denote population and $\ell = 2$ denote industry.

Define

y_{ij}^{ℓ} = offensive weapons of type j targeted to location i when attack is against target type ℓ,

Y_j = offensive weapons of type j available,

z_{i1}^{ℓ} = hardness of target type ℓ at location i,

z_{i2}^{ℓ} = evacuation capability of population or dispersal capability for industry from location i,

α_{ij}^{ℓ} = scaling factor for damage by offensive weapon of type j on target type ℓ in location i,

β_{i1}^{ℓ} = hardness parameter for target type ℓ at location i,

β_{i2}^{ℓ} = target evacuation or dispersion parameter at location i,

W_i^{ℓ} = value of the target type ℓ at location i.

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Let the surviving value of target type ℓ at location i. for offensive allocations y_{ij} (j = 1, ..., q) be given by

$$W_i^\ell \left\{ 1 - \exp\left(-\beta_{i1}^\ell z_{i1}^\ell - \beta_{i2}^\ell z_{i2}^\ell \right)\left[1 - \exp\left(-\sum_{j=1}^{q} \alpha_{ij}^\ell y_{ij}^\ell \right) \right] \right\} .$$

The expression $\exp(-\beta_{i1}^\ell z_{i1}^\ell - \beta_{i2}^\ell z_{i2}^\ell)$ gives the fraction of the target that is susceptible to attack. Thus the effect of increasing z_{i1}^ℓ and/or z_{i2}^ℓ is to remove a portion of the target. If $z_{i2}^1 = 0$, then none of the population is evacuated and if $z_{i2}^2 = 0$, then none of the industry is dispersed. The variables z_{i1}^ℓ may be bounded from below, say by $\bar{z}_{i1}^\ell$, to represent the natural hardness of the population and disperson of industry.

The expression $\left[1 - \exp\left(-\sum_{j=1}^{q} \alpha_{ij}^\ell y_{ij}^\ell \right) \right]$ gives the fraction of the target which is destroyed by the attack y_{ij}^ℓ (j = 1, ..., q). The sum $\sum_{j=1}^{q} \alpha_{ij}^\ell y_{ij}^\ell$ provides a measure of the joint effects of various types of weapons. The parameters α_{ij}^ℓ scale weapons of different yields to an equivalent number of a standard weapon.

A cost function for hardening the targets might be

$$c(z_{i1}^1 - \bar{z}_{i1}^1)^d ,$$

where c > 0, $z_{i1}^1 \geq \bar{z}_{i1}^1$, and d $\geq$ 1. Assume further that the cost of evacuation is linear, namely, $e_i z_{i2}^1$. A similar cost function for industrial protection is assumed.

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The overall problem of providing defensive forces at minimum cost to achieve weighted population survival r_1 and weighted industry survival r_2 is to choose z_{i1}^1 ($i = 1, \ldots, p$), z_{i2}^1 ($i = 1, \ldots, p$), z_{i1}^2 ($i = 1, \ldots, p$), z_{i2}^2 ($i = 1, \ldots, p$), y_{ij}^1 ($i = 1, \ldots, p$; $j = 1, \ldots, q$), y_{ij}^2 ($i = 1, \ldots, p$; $j = 1, \ldots, q$) to

$$\text{minimize} \quad \sum_{i=1}^{p} c^1(z_{i1}^1 - \bar{z}_{i1}^1)^{d^1} + \sum_{i=1}^{p} e_i^1 z_{i2}^1$$

$$+ \sum_{i=1}^{p} c^2(z_{i1}^2 - \bar{z}_{i1}^2)^{d^2} + \sum_{i=1}^{p} e_i^2 z_{i2}^2$$

subject to

$$\left[\underset{\substack{y_{ij}^1}}{\text{minimum}} \; \sum_{i=1}^{p} w_i^1 \left\{ 1 - \exp\left(- \beta_{i1}^1 z_{i1}^1 - \beta_{i2}^1 z_{i2}^1\right) \left[1 - \exp\left(- \sum_{j=1}^{q} \alpha_{ij}^1 y_{ij}^1\right)\right] \right\} \right. $$
$$\left. \sum_{i=1}^{p} y_{ij}^1 \leq Y_j, \quad j = 1, \ldots, q \right] \geq r_1 \; ,$$

$$\left[\underset{\substack{y_{ij}^2}}{\text{minimum}} \; \sum_{i=1}^{p} w_i^2 \left\{ 1 - \exp\left(- \beta_{i1}^2 z_{i1}^2 - \beta_{i2}^2 z_{i2}^2\right) \left[1 - \exp\left(- \sum_{j=1}^{q} \alpha_{ij}^2 y_{ij}^2\right)\right] \right\} \right. $$
$$\left. \sum_{i=1}^{p} y_{ij}^2 \leq Y_j, \quad j = 1, \ldots, q \right] \geq r_2 \; ,$$

$$z_{i1}^1 \geq \bar{z}_{i1}^1 \; , \quad i = 1, \ldots, p$$

$$z_{i1}^2 \geq \bar{z}_{i1}^2 \; , \quad i = 1, \ldots, p \qquad .$$

IV. STRATEGIC BOMBER FORCE STRUCTURE AND BASING OPTIMIZATION

This model is an extension of the model presented in Reference [4]. The problem is to provide a minimum-cost force structure and basing of strategic bombers which will enable the outside optimizer to achieve specified survivability in the face of a submarine-launched ballistic missile (SLBM) threat. The outside optimizer chooses bomber forces and basing, and the inside optimizer chooses allocations of submarines to launch areas and targeting of SLBMs to bases to minimize surviving bombers.

Let $i = 1, \ldots, p$ index the missile launch area, $j = 1, \ldots, q$ index the bomber bases and $t = 0, 1, \ldots, r$ index time. Define

$$x_j = \text{number of bombers assigned to base } j \text{ ,}$$

$$f(x_1, \ldots, x_q) = \text{cost of providing bombers } x_1, \ldots, x_q,$$

$$g_j^t(x_j) = \text{number of bombers on base } j \text{ at time } t \text{ when}$$
$$\text{the number at } t = 0 \text{ is } x_j \text{ ,}$$

$$v_{ij}^t = \text{number of missiles launched from area } i \text{ which}$$
$$\text{are targeted to arrive at base } j \text{ at time } t \text{ ,}$$

$$p_{ij}^t = \text{probability that a missile launched from}$$
$$\text{area } i \text{ hits base } j \text{ at time } t \text{ ,}$$

$$t_{ij} = \text{missile flight time from launch area } i \text{ to}$$
$$\text{base } j \text{ ,}$$

$$M_i = \text{number of missiles at launch area } i \text{ ,}$$

$$M = \text{total number of missiles available.}$$

If the objective of the attacker is to minimize the number of bombers surviving, his problem is to choose v_{ij}^t and M_i to minimize

$$\sum_{j=1}^{q} \sum_{t=0}^{r} g_j^t(x_j) \prod_{i=1}^{p} (1 - p_{ij}^t)^{v_{ij}^t} \prod_{s=0}^{t-1} (1 - p_{ij}^s)^{v_{ij}^s}$$

subject to

$$\sum_{j=1}^{q} v_{ij}^{t+t_{ij}} \leq M_i , \quad i = 1, \ldots, p, \; t = 0, \ldots, r ,$$

$$\sum_{i=1}^{p} M_i \leq M .$$

Let S be the required number of surviving bombers after absorbing the optimal attack. The overall problem is to choose $x_1, \ldots, x_q$, v_{ij}^t ($i = 1, \ldots, p$; $j = 1, \ldots, q$; $t = 0, \ldots, r$), and M_i ($i = 1, \ldots, p$) to

$$\text{minimize } f(x_1, \ldots, x_q)$$

subject to

$$\left[\; \underset{v_{ij}^t, \, M_i}{\text{minimum}} \; \sum_{j=1}^{q} \sum_{t=0}^{r} g_j^t(x_j) \prod_{i=1}^{p} (1 - p_{ij}^t)^{v_{ij}^t} \prod_{s=0}^{t-1} (1 - p_{ij}^s)^{v_{ij}^s} \right.$$

$$\sum_{j=1}^{q} v_{ij}^{t+t_{ij}} \leq M_i , \quad i = 1, \ldots, p, \; t = 0, \ldots, r, \; \bigg] \geq S .$$

$$\left. \sum_{i=1}^{p} M_i \leq M \right.$$

The optimization problem in the constraint is a convex nonseparable nonlinear program. A detailed example of a special case of the inside mathematical program is given in Reference [4]. If the functions $g_j^t(x_j)$ are concave in x_j, then the constraints satisfy the conditions necessary for a convex program. Finally, if $f(x_1, \ldots, x_q)$ is convex, the overall problem is a convex program.

V. STRATEGIC DEFENSE OPTIMIZATION TO ACHIEVE SPECIFIED POST-ATTACK PRODUCTION CAPABILITIES

This model is described in more detail in Reference [5]. It is summarized here.

One of the planning considerations for defense against a nuclear attack is the assurance of sufficient surviving economic capacity to support the surviving population. The present model contains an aggregate representation of the nation-wide production base. The effect of strategic defenses, consisting of both active defenses (for example, anti-ballistic missiles) and passive defenses (for example, population shelters), against offensive weapons is modeled. The defender chooses a minimum-cost mix of active and passive components which ensures that specified post-attack production capacities will survive after an optimized attack.

The country is partitioned into $j = 1, \ldots, n$ geographic regions, allowing consideration of varying population and production-base densities. The general model allows $i = 1, \ldots, m$ different economic sectors in each geographic region. Each economic sector is characterized by a Cobb-Douglas production function. Also, $k = 1, \ldots, p$ denotes the different types of defensive resources and $\ell = 1, \ldots, q$ the different types of offensive weapons.

Define

x_{jk} = number of defensive resources of type k assigned to region j ,

$$v^i_{j\ell} = \text{number of offensive resources of type } \ell \text{ targeted}$$

on region j in an attack on economic sector i ,

$$V_\ell = \text{number of offensive weapons of type } \ell .$$

The post-attack production function (in terms of value added) in economic sector i in region j is assumed to be

$$H_{ij} \left[1 - e^{-\sum_{k=1}^{p} a^H_{ijk} x_{jk}} \left(1 - e^{-\sum_{\ell=1}^{q} b^H_{ij\ell} v^i_{j\ell}} \right) \right]$$

$$\cdot K^{\alpha_{ij}}_{ij} \left[1 - e^{-\sum_{k=1}^{p} a^K_{ijk} x_{jk}} \left(1 - e^{-\sum_{\ell=1}^{q} b^K_{ij\ell} v^i_{j\ell}} \right) \right]^{\alpha_{ij}}$$

$$\cdot L^{\beta_{ij}}_{ij} \left[1 - e^{-\sum_{k=1}^{p} a^L_{ijk} x_{jk}} \left(1 - e^{-\sum_{\ell=1}^{p} b^L_{ij\ell} v^i_{j\ell}} \right) \right]^{\beta_{ij}}$$

If $v^i_{j\ell} = 0$, $\ell = 1, \ldots, q$, then the production function has the standard Cobb-Douglas form

$$H_{ij} K^{\alpha_{ij}}_{ij} L^{\beta_{ij}}_{ij} \quad ,$$

where H_{ij} represents the technological efficiency, K_{ij} is the capital base, and L_{ij} is the labor base. The exponents α_{ij} and β_{ij} are the elasticities of value added with respect to capital and labor, respectively.

The expressions of the form

$$1 - e^{-\sum\limits_{k=1}^{p} a_{ijk}x_{jk}} \left(1 - e^{-\sum\limits_{\ell=1}^{q} b_{ij\ell}v_{j\ell}^{i}} \right)$$

modify the efficiency, the capital base, and the labor base as a function of the offense and defense allocations. The expression is convex in $v_{j\ell}^{i}$, concave in x_{jk}, and has the asymptotic properties expected for the physical processes being modeled. The parameters a_{ijk} and $b_{ij\ell}$ can be estimated from detailed analyses.

The aggregate production for the whole country is the sum of the production across geographic regions.

The cost of defensive resources is taken to be $\sum\limits_{k=1}^{p} \sum\limits_{j=1}^{n} c_{jk}x_{jk}$, where c_{jk} is the unit cost of defensive resource k in region j .

The requirements for surviving post-attack production capacity are given by r_i(i = 1, ..., m).

The overall model is to choose x_{jk} (j = 1, ..., n; k = 1, ..., p) and $v_{j\ell}^{i}$ (i = 1, ..., m; j = 1, ..., n; ℓ = 1, ..., q) to

$$\text{minimize} \sum\limits_{k=1}^{p} \sum\limits_{j=1}^{n} c_{jk}x_{jk}$$

subject to

$$
\left[
\begin{array}{l}
\displaystyle\operatorname*{minimum}_{v_{j\ell}^i} \ \sum_{j=1}^{n} H_{ij}\left[1 - e^{-\sum\limits_{k=1}^{p} a_{ijk}^{H} x_{jk}\left(1 - e^{-\sum\limits_{\ell=1}^{q} b_{ij\ell}^{H} v_{j\ell}^{i}}\right)}\right] \\[20pt]
\cdot K_{ij}^{\alpha_{ij}}\left[1 - e^{-\sum\limits_{k=1}^{p} a_{ijk}^{K} x_{jk}\left(1 - e^{-\sum\limits_{\ell=1}^{q} b_{ij\ell}^{K} v_{j\ell}^{i}}\right)}\right]^{\alpha_{ij}} \\[20pt]
\cdot L_{ij}^{\beta_{ij}}\left[1 - e^{-\sum\limits_{k=1}^{p} a_{ijk}^{L} x_{jk}\left(1 - e^{-\sum\limits_{\ell=1}^{q} b_{ij\ell}^{L} v_{j\ell}^{i}}\right)}\right]^{\beta_{ij}} \\[20pt]
\displaystyle\sum_{j=1}^{n} v_{j\ell}^{i} \leq V_{\ell} \ , \qquad \ell = 1, \ldots, q
\end{array}
\right] \geq r_i \ , \quad i=1,\ldots,m.
$$

The overall model is a convex program. The inside optimization problems are convex nonseparable nonlinear programs.

VI. OPTIMIZATION OF WEAPON MIX AND
TARGETING FOR ATTRITION PROCESSES

A conventional warfare model is formulated for the attrition of
a heterogeneous mix of weapons. The problem for the outside optimizer
is to choose a minimum-cost mix of weapons and their optimal targeting
patterns against the optimal targeting of the weapons of the inside
optimizer. The force size must be sufficient to allow the outside
optimizer to achieve a specified differential value of weapons at
the end of the engagement.

Let the index $i = 1, \ldots, m$ denote weapon type for the outside
optimizer and $j = 1, \ldots, n$ denote weapon type for the inside
optimizer. Define

x_i = initial number of outside optimizer's weapons of type i ,

y_j = initial number of inside optimizer's weapons of type j ,

u_{ij} = number of outside optimizer's weapons of type i assigned
to fire on inside optimizer's weapons of type j ,

v_{ji} = number of inside optimizer's weapons of type j assigned
to fire on outside optimizer's weapons of type i ,

b_{ij} = effectiveness of outside optimizer's weapon type i on
inside optimizer's weapon type j ,

r_{ji} = effectiveness of inside optimizer's weapon type j on outside
optimizer's weapon type i .

In classical Lanchester theory the instantaneous attrition to
a weapon by type i from an opposing weapon of type j is given by

$$\dot{x}_i = - r_{ji} y_j$$

or

$$\dot{x}_i = - r_{ji} x_i y_j \, ,$$

the first equation being the "square" law and the second the "linear" law. More generally,

$$\dot{x}_i = - r_{ji} x_i^{2-p} y_j \, ,$$

where $1 \le p \le 2$, gives the "p"-law, with the square and linear laws being special cases.

The heterogeneous Lanchester attrition considers more than one weapon type for each opponent. The heterogeneous analog to the p-law is

$$\dot{x}_i = - x_i^{2-p_i} \sum_{j=1}^{n} r_{ji} v_{ji} \, ,$$

where $1 \le p_i \le 2$, $i = 1, \ldots, m$.

In conventional Lanchester analysis the parameters b_{ij} (and r_{ji}) are instantaneous attrition rates. For purposes of the model here it is useful to consider these parameters to be the kill rates over some positive length of time. Thus, b_{ij} might be the average effectiveness of a weapon type i firing at a weapon type j for a day, a week, or longer. The number of surviving weapons of type i for the outside optimizer at the end of the length of time is

$$x_i - x_i^{2-p_i} \sum_{j=1}^{n} r_{ji} v_{ji} \, .$$

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Similarly the surviving weapons of type j for the inside optimizer
are

$$y_j - y_j^{2-q_j} \sum_{i=1}^{m} b_{ij} u_{ij} \,,$$

where $1 \le q_j \le 2$, $j = 1, \ldots, m$.

If β_i and ρ_j are the values of the surviving weapons for the
outside optimizer and for the inside optimizer, respectively, then
the model for finding minimum-cost forces subject to the difference
in value of surviving weapons being at least r is to choose
$x = (x_1, \ldots, x_m)$, $u = (u_{ij})$, and $v = (v_{ji})(1 = 1, \ldots, m;$
$j = 1, \ldots, n)$ to

$$\text{minimize } f(x)$$
$$x$$

subject to

$$\left[\max_{u \in U(x)} \min_{v \in V} \left\{ \sum_{i=1}^{m} \beta_i \left[x_i - x_i^{2-p_i} \sum_{j=1}^{n} r_{ji} v_{ji} \right] - \sum_{j=1}^{n} \rho_j \left[y_j - y_j^{2-q_j} \sum_{i=1}^{m} b_{ij} u_{ij} \right] \right\} \right] \ge r,$$

where

$$U(x) = \left\{ u_{ij} \ge 0: \quad \sum_{j=1}^{n} u_{ij} \le x_i, \; i = 1, \ldots, m \right\},$$

$$V = \left\{ v_{ji} \ge 0: \quad \sum_{i=1}^{m} v_{ji} \le y_j, \; j = 1, \ldots, n \right\},$$

and where $f(x)$ is the cost of forces x.

Applying the result in Reference [3], the above optimization model is equivalent to the following:

$$\text{minimize } f(x)$$
$$x, u$$

subject to

$$\sum_{j=1}^{n} u_{ij} \leq x_i , \quad i = 1, \ldots, m$$

$$u_{ij} \geq 0 , \quad i = 1, \ldots, m; \ j = 1, \ldots, n$$

$$\left[\min_{v} \left\{ \sum_{i=1}^{m} \beta_i \left[x_i - x_i^{2-p_i} \sum_{j=1}^{n} r_{ji} v_{ji} \right] - \sum_{j=1}^{n} \rho_j \left[y_j - y_j^{2-q_j} b_{ij} u_{ij} \right] \right\} \quad \sum_{i=1}^{m} v_{ji} \leq y_j , \quad j = 1, \ldots, n \quad v_{ji} \geq 0 , \quad i = 1, \ldots, m; \ j = 1, \ldots, n \right] \geq r .$$

The inside mathematical program is a linear program. Further, the inside objective function is concave in (x, u). Thus, if $f(x)$ is convex, the program is convex. There is a possibility that an "overkill" of forces will occur if the attrition parameters are not small relative to the number of weapons. The numerical solution of the problem should be examined for this eventuality.

VII. OPTIMIZATION OF AIRCRAFT DEPLOYMENT AND SORTIE ALLOCATION

The purpose of this model is to choose optimal Blue aircraft deployments to theater, and Blue and Red optimal sortie allocations, assuring that Blue achieves specified differences of cumulative ground and air firepower at the end of a war. The war consists of two periods. Red aircraft deployments are fixed. The sortie allocations are to combat air support (CAS) and air-base attack (ABA).

Let $t = 1, 2$ index time, and

x_t = number of Blue aircraft deployed to theater for use in period t

y_t = number of Red aircraft deployed to theater for use in period t

u_1, u_2 = number of Blue aircraft assigned to CAS and ABA, respectively, in period 1

v_1, v_2 = number of Red aircraft assigned to CAS and ABA, respectively, in period 1

α = kill parameter for Blue ABA killing Red aircraft

β = kill parameter for Red ABA killing Blue aircraft

f = firepower per Blue CAS sortie

g = firepower per Red CAS sortie

c_t = cost per Blue aircraft introduced at time t

S_t = required cumulative firepower difference by time t.

Destruction of aircraft is a function of the number of opposing ABA aircraft. The attrition of Blue aircraft is given by

$$x(1 - e^{-\beta v_2}) ,$$

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and similarly for Red. Both sides allocate all of their aircraft
to CAS in the second period, since aircraft allocated to ABA would
have no effect due to the war having ended.

The two-period optimization model is to choose x_1, x_2, u_1, u_2,
v_1, v_2 to

$$\underset{x_1, x_2}{\text{minimize}} \ c_1 x_1 + c_2 x_2$$

subject to

$$\underset{u_1 + u_2 \leq x_1}{\max} \left[\underset{v_1 + v_2 \leq y_1}{\min} \left\{ f[u_1 + (x_1 + x_2 - x_1(1 - e^{-\beta v_2})] \right. \right. $$
$$\left. \left. - g[v_1 + (y_1 + y_2 - y_1(1 - e^{-\alpha u_2}))] \right\} \right] \geq S_2 \ .$$

Applying the result of Reference [3] the problem can be written as
to choose x_1, x_2, u_1, u_2, v_1, v_2 to

$$\underset{x_1, x_2, u_1, u_2}{\text{minimize}} \ c_1 x_1 + c_2 x_2$$

subject to

$$\underset{v_1, v_2}{\min} \left[\left\{ f[u_1 + (x_1 + x_2 - x_1(1 - e^{-\beta v_2})] \right. \right.$$
$$\left. - g[v_1 + (y_1 + y_2 - y_1(1 - e^{-\alpha u_2})] \right\} \right] \geq S_2$$
$$\left. v_1 + v_2 \leq y_1 \right.$$
$$u_1 + u_2 \leq x_1$$

The overall program is convex, and the inside program is also convex.

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