NUMERICAL MODEL FOR MIXED REGION COLLAPSE IN A STRATIFIED FLUID

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Numerical Model for Mixed Region Collapse in a Stratified Fluid

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Introduction

The flow phenomena accompanying the collapse of a homogeneous fluid mass immersed in a stratified fluid have been the center of some attention over the past ten years. The fluid mechanical model has various applications in geophysics and in engineering but the flow phenomena are complex enough to have yielded only marginally to analysis.

The model is taken to be a cylindrically shaped homogeneous fluid mass immersed in a linearly stratified fluid. The cylindrical region is initially circular in shape and the density of the mixed fluid within is equal to the density of the exterior fluid at the axis of the cylinder. Since the surrounding fluid is assumed to be stably stratified, the fluid in the upper part of the mixed region is heavier, and that in the lower part lighter, than its surroundings. Thus, when released from rest, the region expands horizontally while collapsing vertically to the level of equilibrium of the homogeneous fluid. The motion of the mixed region occurs at the expense of the potential energy originally stored in the initial configuration. The distortion from the circular shape causes motions in the exterior fluid that eventually radiate the energy away in the form of internal gravity waves.

Previous studies aimed directly toward the collapse process have been mostly experimental (cf. Schooley and Stewart [1], Schooley [2], Wu [3]). There have been several numerical experiments (Wessel [4], Padmanabhan et al [5], Vasiliev et al [6]) but only one relevant analytical effort (Bell and Dugan [7]). There have been analytical contributions to certain aspects of the collapse process by Hartman and Lewis [8] and Schooley and Hughes [9] but these will be excluded from discussion because of severe limitations of the linearized equations of motion in the fully mixed case.

Numerical experiments aimed at clarifying the physical processes should be most enlightening but they have not been entirely successful. The physical model solved numerically by Padmanabhan et al [5] is an idealized one in which fluid motions exterior to the homogeneous

region are ignored. Thus, no energy can be transferred to the exterior fluid and the model could be an accurate one only for the initial stage of collapse. In the initial stage, though, those predictions are at variance with an analytical solution of the same model (cf. Bell and Dugan [7]) and it appears that the solution did not conserve energy. Wessel [4] solved the Navier-Stokes equations numerically in such a way that exterior fluid motions were included in the computation. Gross corroboration with the experiment of Wu [3] was obtained but the calculation was not accurate enough to attempt detailed predictions of energy balances. Finally, Vasiliev et al [6] have solved numerically a particular physical model that retains some aspects of the growth of a mixed region due to turbulence as well as the resulting collapse. However, no comparisons with other data or conclusions were made.

In summing up the previous contributions, there remain several outstanding questions about the mechanics of the collapse process. First, the results for the initial stage of collapse are contradictory. Bell and Dugan [7] discuss several analytical models for the initial stage that would appear to be mechanically convenient. However, in those models the homogeneous region is assumed to conserve its energy and this is a questionable hypothesis. The experiments of Wu [3] are not helpful in answering this question because the experimental method masked the initial stage, and the numerical predictions of Wessel [4] and Vasiliev et al [6] are not accurate enough. Second, there has been no analytical or numerical method that is adequate to predict the phenomena in the late stage of collapse. Last of all, the energetics of the motions and the resulting physical implications have not been explored.

This paper is written to answer these questions. The Navier-Stokes equations in the Boussinesq approximation are solved numerically in a large box. The fluid is assumed to have constantly increasing density with depth except for a circular region that is initially constrained to be homogeneous. The numerical method follows along lines laid out by Williams [10] and Piacsek and Williams [11] and it conserves the total energy in the box remarkably well. The source of energy for the fluid motions is the potential energy initially stored in the homogeneous region. The rate of energy transfer from the homogeneous region to the surrounding fluid is followed in detail and the result substantiates the speculation of Bell and Dugan [7] that the mixed region does not conserve its energy even in the early stage. Also, confirming another speculation of Bell and Dugan [7], the viscous stresses are shown to affect the solution only at a late stage of collapse.

Numerical Analysis

The Navier-Stokes equations are solved by an adaptation of a method proposed by Williams [10] and Piacsek and Williams [11] . The

equations of motion are

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial z}\right) + \alpha g T \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

where (u, w) are the (x, z) components of velocity, p is the pressure, T is the temperature, ν is the kinematic viscosity, and κ is the thermometric diffusivity. The operator D/Dt is the convective derivative $\partial/\partial t + \underline{v} \cdot \nabla$ and the density is related to the temperature by the relation $\rho = \rho_0[1 - \alpha(T - T_0)]$ where α is the thermal expansion coefficient. The first two equations represent the conservation of momentum within the Boussinesq approximation wherein density changes are assumed negligible in the acceleration terms. The third equation represents the condition of incompressibility and the last one the conservation of energy. These equations are to be solved subject to approximate boundary and initial conditions. The boundary conditions are taken to be no-stress conditions with the temperature (density) fixed at the top and bottom of a rectangular box. In detail, the boundary conditions are

$$u = 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0 \qquad \text{on } x = 0, b \qquad (2a)$$

$$\frac{\partial p}{\partial x} = 0$$

and

$$w = 0$$

$$\frac{\partial u}{\partial z} = 0 \qquad \text{on } z = 0, d \qquad (2b)$$

$$\frac{\partial T}{\partial z} = \alpha g T$$

where T is fixed on $z = 0$ and d, and b and d are the horizontal and vertical sizes of the box, respectively. The initial condition is specified as quiescent with horizontal, equally spaced isopycnals except for a circular region in which the fluid is totally mixed.

A Poisson equation for the pressure can be derived from equations (1) so that

$$\frac{1}{\rho_0}\nabla^2 p = -\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{G} \tag{3}$$

where Θ is the velocity divergence and $\mathbf{G}$ represents the nonlinear and viscous terms in the Navier-Stokes equations. The integral form of this equation is

$$\iint \nabla \cdot \left[\frac{1}{\rho_0}\nabla p + \frac{\partial \mathbf{u}}{\partial t} - \mathbf{G}\right] dV = 0$$

so that

$$\oint \left[\frac{1}{\rho_0}\nabla p + \frac{\partial \mathbf{u}}{\partial t} - \mathbf{G}\right] \cdot \mathbf{n}\, dS = 0 \tag{4}$$

has to be satisfied around the boundary. Since the no-stress boundary conditions are to be imposed, the boundary condition on equation (3) is of Neumann type and it is important that the finite-difference form of equation (3) satisfy the integral (4) on the boundaries.

In order to establish the finite-difference scheme for solving the above equations, the difference operator

$$\delta_x f = \frac{1}{\Delta x}\left[f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)\right]$$

and the sum operator

$$\overline{f}^{x} = \frac{1}{2}\left[f\left(x + \frac{\Delta x}{2}\right) + f\left(x - \frac{\Delta x}{2}\right)\right]$$

are defined in the notation of Grammeltvedt [12] . These operators form a linear commutative and distributive algebra for which various operator rules and identities can be constructed, such as

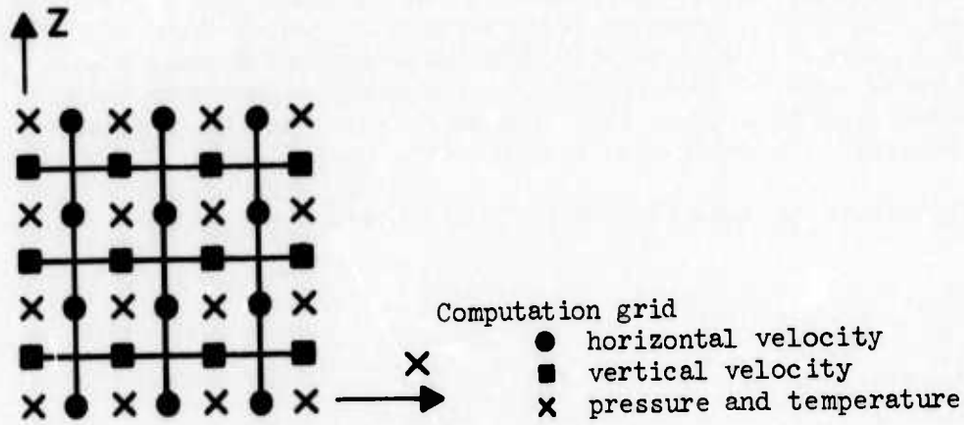
$$\delta_x\left(\overline{A}^{x}\overline{B}^{x}\right) = \overline{A}^{xx}\delta_x \overline{B}^{x} + \overline{B}^{xx}\delta_x \overline{A}^{x}.$$

A quadratically conservative scheme in the sense of Piacsek and Williams [11] can be constructed for the relevant equations as

$$\begin{aligned}
\delta_t \bar{u}^{t} &= -\delta_x\left(\bar{u}^{x}\bar{u}^{x}\right) - \delta_z\left(\bar{w}^{x}\bar{u}^{z}\right) + \tfrac{1}{2}u\left(\delta_x \bar{u}^{x} + \delta_z \bar{w}^{x}\right) \\
&\quad -\tfrac{1}{\rho_0}\delta_x p + \nu\left(\delta_{zz}u - \delta_{xz}w\right) \\
\delta_t \bar{w}^{t} &= -\delta_x\left(\bar{u}^{z}\bar{w}^{x}\right) - \delta_z\left(\bar{w}^{z}\bar{w}^{z}\right) + \tfrac{1}{2}w\left(\delta_x \bar{u}^{z} + \delta_z \bar{w}^{z}\right) \\
&\quad -\tfrac{1}{\rho_0}\delta_z p + \nu\left(\delta_{xx}w - \delta_{xz}u\right) + \alpha g T \\
\delta_t \bar{T}^{t} &= -\delta_x\left(u\bar{T}^{x}\right) - \delta_z\left(w\bar{T}^{z}\right) + \tfrac{1}{2}T\left(\delta_x u + \delta_z w\right) \\
&\quad + K\left(\delta_{xx}T + \delta_{zz}T\right),
\end{aligned} \tag{5}$$

where variables are defined on the staggered grid as shown in Figure 1.

Figure 1



The terms of the form

$$\tfrac{1}{2}u(\delta_x \bar{u}^x + \delta_z \bar{w}^x)$$

force the finite-difference scheme for the nonlinear terms to be quadratically conservative regardless of whether the divergence is zero or not. In other words, the quadratic conservation is algeberaic and is independent of the accuracy of the solution in the sense of how close the divergence is to being zero. This does introduce an error proportional to the divergence (which is not identically zero during the computation) into the integrals of linear quantities. However, these linear conservation integrals are not as necessary or as meaningful a requirement for computational stability as are the quadratic ones.

The nonlinear terms are evaluated at time t, thus constituting the "leap-frog" method (Richtmyer [13], p. 17) which has a time truncation error of $O(\Delta t^2)$ and a von Neumann condition of

$$\Delta t \leq \Delta x / \left[|u| + \frac{\Delta x}{\Delta z} |w| \right].$$

The viscous terms are lagged in time at time level t-Δt and are subject to the stability criteria of

$$\Delta t < (\Delta x)^2 / 8\nu$$

and the pressure and temperature terms are time centered (evaluated at time t).

Another instability that can appear is aliasing (Phillips [14]). This instability arises when waves that are too short to be resolved by a given set of grid points are misrepresented. Such instabilities

are controlled successfully by using schemes like this one that maintain the integral constraints of physical importance on the quadratic quantities u^2, w^2, and T^2. For a three-level time scheme like this one, computational modes are present due to the fact that a first order continuum equation in time has been raised to a second order difference equation in time. The computational modes are usually small and it takes several hundred time steps for serious deviations from the correct solution to arise. These computational modes are suppressed by periodically averaging over adjacent time steps.

The governing equation (3) in the finite-difference notation is

$$\delta_{xx} p + \delta_{zz} p = \rho_0 \left(S - \delta_t \bar{\mathcal{D}}^{t} \right)$$

$$= \rho_0 \left(S + \frac{1}{2\Delta t} \mathcal{D}^{t-\Delta t} \right) \qquad (6)$$

where

$$\mathcal{D} = \delta_x u + \delta_z w$$

and

$$S = \delta_x \left[-\delta_x (\bar{u}^x \bar{u}^x) - \delta_z (\bar{u}^z \bar{w}^x) + \tfrac{1}{2} u (\delta_x \bar{u}^x + \delta_z \bar{w}^x) \right]$$

$$+ \delta_z \left[-\delta_x (\bar{u}^z \bar{w}^x) - \delta_z (\bar{w}^z \bar{w}^z) + \tfrac{1}{2} w (\delta_x \bar{u}^z + \delta_z \bar{w}^z) \right]$$

$$+ \delta_z \alpha g T .$$

The Harlow and Welch [15] numerical strategem to drive the divergence, $\mathcal{D}$, to zero was applied. This is necessary because a degree of round-off error is inevitable and it leads to the creation of an artificial divergence. The computational strategem works as follows: since the divergence in the computation at a given time step $t-\Delta t$ is not exactly zero ($\mathcal{D}^{t-\Delta t} \neq 0$) but that of time $t+\Delta t$ ought to be zero ($\mathcal{D}^{t+\Delta t}=0$), $\partial_t \mathcal{D}^t$ is treated as equal to $-\mathcal{D}^{t-\Delta t}/2\Delta t$.

The actual solution of equation (6) with the predictors (5) is by partial Fourier reduction. The boundary conditions (2b) on the pressure are made homogeneous by a simple transformation and the pressure is then expanded in a Fourier cosine series in the vertical direction. The expansion and decomposition are accomplished by means of fast Fourier transforms. The equation that has to be solved then takes the form

$$(\delta_{xx} - \lambda_n) p_n = \rho_0 S_n \tag{7}$$

where n refers to the Fourier mode number, λ_n is the eigenvalue for the nth mode, and S_n is the transformed source function. This is a tridiagonal equation in x and it can be solved subject to the Neumann boundary conditions by means of a tridiagonal algorithm (Varga [16]). For the degenerate case n=0, the matrix is no longer diagonally dominant and the algorithm breaks down so the equation is solved as a marching problem in this case. The integral constraint (4) is satisfied by the finite-difference formalism within round-off so this degenerate case poses no problem.

The normal checks on the computation for consistency and convergence were made but, since the computation of Padmanabhan et al [5] apparently dissipated energy in a physical model where none should have been dissipated, it was felt especially important to prove that the present method did conserve energy at least approximately in the case of zero viscosity.

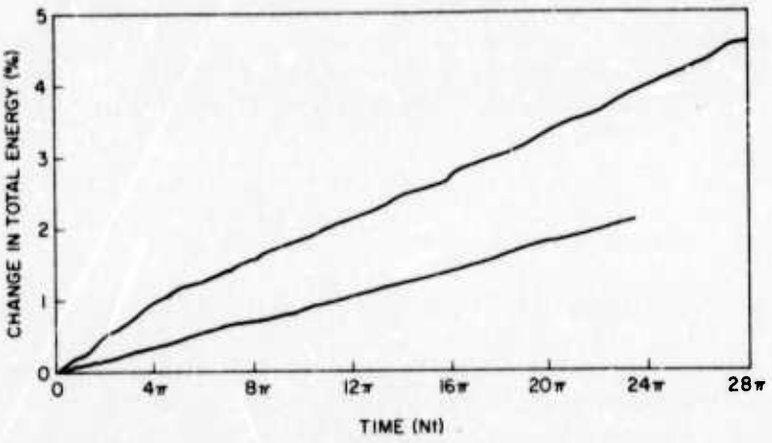


Figure 2

Total energy in computation box

In Figure 2, the total energy in the box is plotted versus time. The time axis is scaled by the Brunt-Vaisala frequency N where

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}$$

and $\bar{\rho}$ is the density profile of the exterior fluid. The energy is conserved to within 5% for 14 periods which is over 2500 time steps. This is remarkable considering that the initial condition data includes a discontinuity and that in such a long run the mixed region is highly elongated so that it virtually disappears into the grid spacing.

Results and discussion

The box size chosen for the computations is elongated so that the collapsing mixed region can expand unrestricted by the sidewall boundary. The initial mixed region is taken to be a quarter circle in a corner of the box since the problem exhibits both horizontal and vertical symmetry about the mixed region centerline. The distance to the top of the box is 4 radii and that to the side is 16 radii. The number of grid points was varied somewhat to check for convergence of the solution but the majority of the results are for 50X200 grid points.

Figure 3 shows a sequence of mixed region shapes taken at intervals of one-sixth of a Brunt-Vaisala period. The shape is obtained from particle tracers that are massless points pushed along by the velocity field. The short-time history of the width of the mixed region is shown in Figure 4 along with the theoretical prediction of Bell and Dugan [7] for the accelerative stage. Actually, the numerical result is comprised of two curves which bound the region size above and below; the uncertainty is entirely in the plotting routine utilized to display the results. That theory (as well as similar ones of Padmanabhan et al [5] and Mei [17]) essentially

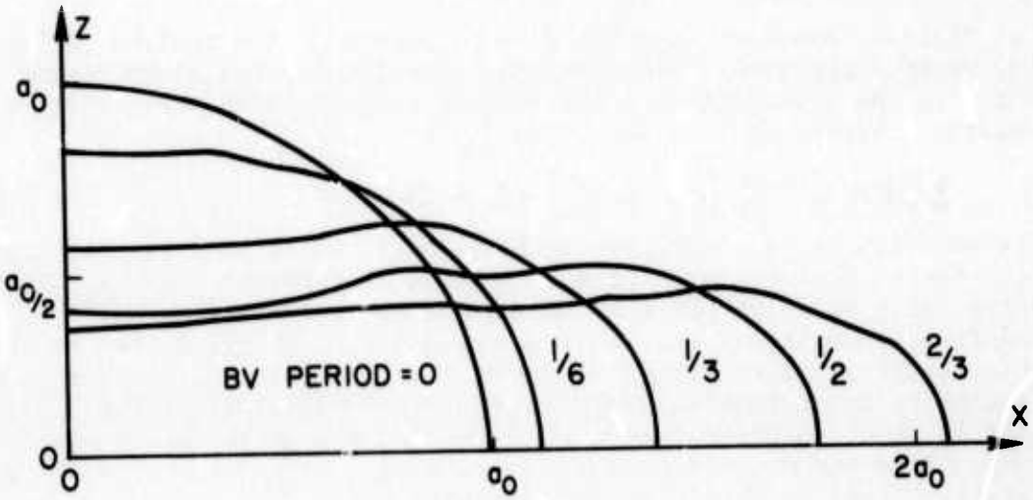


Figure 3
Mixed region shape

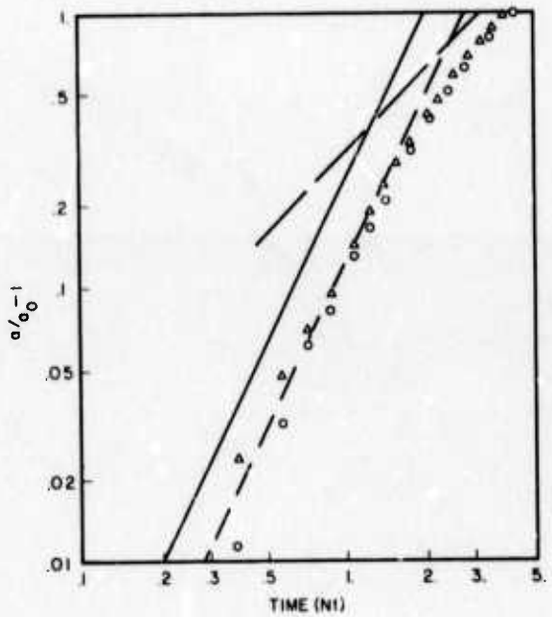


Figure 4
Mixed region width

assumed that the potential energy that was originally stored in the mixed region is converted to kinetic energy but no energy is transferred to the surrounding fluid. In other words, that theory neglects all motions in the surrounding fluid. This can only be an approximation since all changes of shape of the mixed region force accommodating motions in the surrounding fluid. It is plausible to extend that theory slightly by making the assumption that, for short times, all motions inside the mixed region cause comparable motions outside

the region. Assuming that the kinetic energy of the motions in the two regions are equal, the governing equation for the mixed region width in the accelerative stage that is comparable to the governing equation derived by Bell and Dugan [7] is

$$2(1+\alpha^{-4})\left(\frac{d\alpha}{d\tau}\right)^2 + \alpha^{-2} - 1 = 0. \tag{8}$$

The solution of this equation is identical to that of Bell and Dugan [7] except that the time axis is shifted by a factor of $2^{\frac{1}{2}}$. This curve falls right on the mean of the numerical results so this modified assumption is at least consistent with the numerical result. This result is interesting because of its physical implications. Evidently, the potential energy of the mixed region initially is lost to an equipartition of kinetic energy between the fluid inside and outside the mixed region and no potential energy is stored in the exterior fluid in this stage of collapse.

Figure 5 is another plot of the width of the mixed region versus time. The dots are the raw data from the tank experiments of Wu [3]

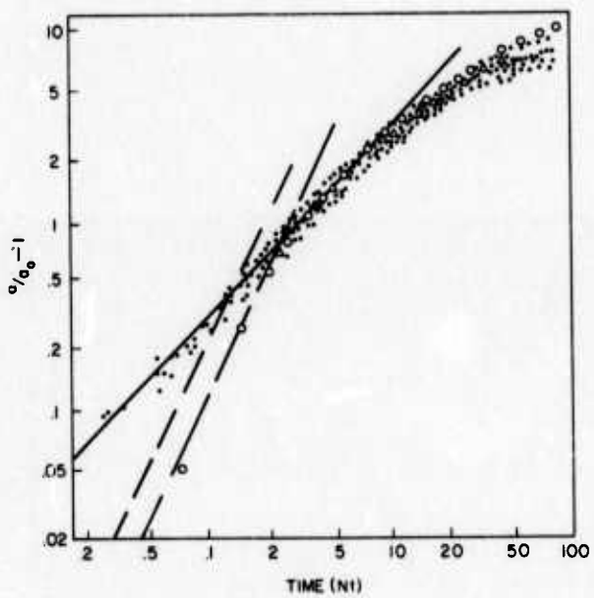


Figure 5
Mixed region width

and the solid line is the intermediate stage theory of Bell and Dugan [7] . The circles indicate the numerical prediction of the region width in the case of zero viscosity and the triangles indicate the prediction in which the molecular viscosity is on the scale used by Wu [3] . The agreement of the numerical results with the theory and the experiment is quite good. This result confirms the speculation of Bell and Dugan [7] that viscous effects are important only late in the collapse. It also sheds some doubt on the accuracy of the numerical methods used by Padmanabhan et al [5] since the inviscid

and viscous results differed much earlier in time in those predictions.

Some physical insight is gained by considering the energetics of the motions. From the mixed region shapes of the form shown in Figure 3, along with tabulations of energy densities, it is possible to determine the amount of potential and kinetic energy in the mixed region at any given time. Figure 6 is a plot of these energies versus

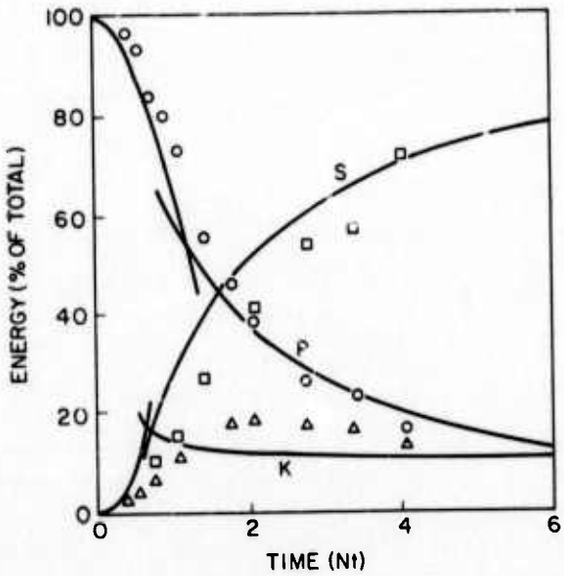


Figure 6
Energy balance of mixed region

P, potential energy, [7]
K, kinetic energy, [7]
S, residual energy, [7]
○, potential energy, numerical
△, kinetic energy, numerical
□, residual energy, numerical

time. The energies are normalized by the total potential energy originally stored in the mixed region. These computations are for zero viscosity (as shown above, viscous effects are not important in the early stage) so the energy transferred to the exterior fluid is just the total energy minus the sum of the potential and kinetic energies in the mixed region. The analytical predictions of the theories of Bell and Dugan [7] are also illustrated in Figure 6. It is remarkable that the results agree as well as they do because the analytical results for the energetics assumed an elliptical-shaped mixed region even for times for which the numerical results show the shape to be quite different from an ellipse. The numerical result confirms the analytical prediction that most of the energy has been transferred to the surrounding fluid by the end of one Brunt-Vaisala period and it leads to the conclusion that the energy transfer

mechanism operates quickly. In turn, this leads to the conclusion that flow models that assume energy conservation in the mixed region (including those of Mei [17] , Padmanabhan et al [5] , and the early time models of Bell and Dugan [7]) have very limited applicability to the real problem of mixed region collapse.

A plot of the potential and kinetic energy in the whole box versus time is also of interest and is shown in Figure 7. Although

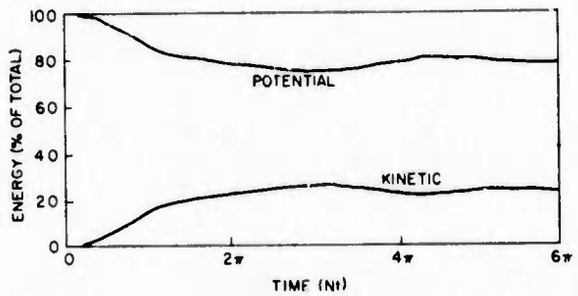


Figure 7
Energy balance of computation box

initially all of the energy is potential, the collapse transforms roughly one-quarter of it to kinetic energy. This prediction is consistent with the fact that there is an equipartition between kinetic and potential energy in small amplitude internal gravity waves. The time-averaged mean of the kinetic and potential energies is not the same but this discrepancy is attributed to the method of computing the potential energy. The zero level of potential energy is assumed to be that of the linearly stratified fluid outside the collapse region. However, in a box of finite size, the final equilibrium state is distorted from the linear stratification by the homogenous fluid lying on the mixed region axis. This distortion fully accounts for the discrepancy between the means of kinetic and potential energies so that equipartition is in fact achieved.

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