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IMPROVED CONVEXITY CUTS FOR LATTICE POINT PROBLEMS

Fred Glover, et al

Texas University

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IMPROVED CONVEXITY CUTS FOR LATTICE POINT PROBLEMS

by

Freu Glover*

Darwin Klingman**

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*Professor of Management Science, University of Colorado, Boulder.

**Associate Professor of Operations Research and Computer Science, University of Texas, Austin.

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

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The University of Texas

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13. ABSTRACT

The generalized lattice point (GLP) problem provides a formulation that accommodates a variety of discrete alternative problems. In this paper we show how to substantially strengthen the convexity cuts for the GLP problem. The new cuts are based on the identification of "synthesized" lattice point conditions to replace those that ordinarily define the cut. The synthesized conditions give an alternative set of hyperplanes that enlarge the convex set, thus allowing the cut to be shifted deeper into the solution space. A convenient feature of the strengthened cuts is the existence of linking relationships by which they may be constructively generated from the original cuts. Geometric examples are given in the last section to show how the new cuts improve upon those previously proposed for the GLP problem.

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Abstract

The generalized lattice point (GLP) problem provides a formulation that accommodates a variety of discrete alternative problems. In this paper we show how to substantially strengthen the convexity cuts for the GLP problem. The new cuts are based on the identification of "synthesized" lattice point conditions to replace those that ordinarily define the cut. The synthesized conditions give an alternative set of hyperplanes that enlarge the convex set, thus allowing the cut to be shifted deeper into the solution space. A convenient feature of the strengthened cuts is the existence of linking relationships by which they may be constructively generated from the original cuts. Geometric examples are given in the last section to show how the new cuts improve upon those previously proposed for the GLP problem.

1. INTRODUCTION

The generalized lattice point (GLP) problem provides a formulation that accommodates a variety of discrete alternative problems. Significantly, it accomplishes this without the necessity of introducing integer variables and their associated constraints. The advantage of such a formulation lies not only in the reduced problem size it affords, but also in the fact that solution methods for the GLP problem do not have to wrestle with "extraneous structure" created by standard tricks for formulating such problems as integer programs. A class of cutting planes that apply directly to the GLP problem formulation utilizing the convexity cut framework was first developed in [2]. Extensions of these to the related structures of "facet problems," have also been given in [3].

In this paper we show how to substantially strengthen the convexity cuts for the GLP problem. The new cuts are based on the identification of "synthesized" lattice point conditions to replace those that ordinarily define the cut. The synthesized conditions give an alternative set of hyperplanes that enlarge the circumscribing convex set, thus allowing the cut to be shifted deeper into the solution space.

A convenient feature of the strengthened cuts is the existence of linking relationships by which they may be constructively generated from the original cuts. Geometric examples are given in the last section to show how the new cuts improve upon those previously proposed for the GLP problem.

2. THE GLP PROBLEM

The GLP problem is an ordinary linear program subject to an additional "lattice point condition." This condition stipulates that all admissible solutions must lie on an (m-q) dimensional face of some convex polytope defined by a subset of the constraints of the linear program, where m is the full dimension of the polytope and q is a constant between 0 and m. Stated in the language of basic solutions, feasibility for the GLP problem requires that at least q slack variables associated with a specific set of constraints be excluded from the linear programming basis. Thus, the problem may be expressed

$$\begin{aligned} &\text{Maximize} && cx \\ &\text{Subject to} && Ax \leq b \\ & && Dx \leq d \\ & && \text{and at least q slack variables for } Dx \leq d \text{ are nonbasic.} \end{aligned}$$

The vector of slack variables for $Dx \leq d$ may be written $u = d - Dx$, and hence the lattice point condition requires q of the components u_i of u to be nonbasic. (Nonnegativity conditions such as $x \geq 0$ may be assumed to be absorbed into the inequalities $Ax \leq b$ and $Dx \leq d$.)

A variety of problems can be expressed in this form, including the 0 - 1 mixed integer programming problem (see [2,3]) and all problems that can be formulated in terms of the latter. For some of these problems, however, the GLP formulation is more direct, suggesting that an approach that is specially designed for this problem may have advantages over methods designed for other formulations. In particular, it is shown in [3] that 0 - 1 mixed integer formulations of certain problems give rise to a "structural redundancy" -- requiring an undetermined number of cuts to move -- which is avoided by the more direct formulation.

A consequence of the requirement that q of u_i variables be nonbasic is that q of these variables receive a 0 value. For many problems (including the 0 - 1 problem) these conditions are equivalent, and we will focus here on this simple version of the lattice point condition. Our results apply as well to the "nonbasic" version by the procedures indicated in [2].

3. GLP CUTS

A central result for the GLP problem states that one can select any set of

$m - q + 1$ of the u_i variables and determine the convexity cut relative to the convex region given by $u_i \geq 0$ for these selected variables. Here m is the total number of u_i variables. (The rule for generating the cut is to extend each edge associated with the current LP tableau until it intersects the boundary of the convex region, and then pass the cut hyperplane through the endpoints of these edges. See [2] for detailed illustrations of how this may be done. The geometric examples of Section 4 also provide a pictorial analog.) One can always find such a set of u_i variables all of which are positive whenever fewer than q of the u_i are zero. The positivity restriction on the values of these $m - q + 1$ variables assures that the cut is not "degenerate" relative to the current LP extreme point.

To determine strengthened cuts, we seek new sets of $m - q + 1$ variables, concatenated out of the original components of u, which have the requisite properties to define suitable convex sets.

The result that identifies these new variables and specifies how they give rise to alternative convexity cuts is the following.

Theorem 1. Partition the index set $\{1,\ldots,m\}$ for the u_i variables into $m - q + 1$ nonempty sets P_j, $j = 1,\ldots,m - q + 1$. For each of these sets P_j, define a variable

$$z_j = \sum_{i \in P_j} \lambda_i u_i$$

where the parameters λ_i are chosen so that each z_j is positive in the current basic solution. Then a legitimate convexity cut for the GLP problem is given by the convex set consisting of the intersection of the half spaces $z_j \geq 0$, $j=1,\ldots,m - q + 1$.

Proof: Drawing on the results of [1,2], the validity of the theorem is easily established. It is only reqiired to demonstrate that at least one of the variables z_j must be 0 in every feasible solution to the GLP problem. Thus, suppose to the contrary that there exists a feasible solution to the GLP problem in which all of the z_j are nonzero. This implies that at least one u_i in each set P_j is nonzero, and hence a total of at least $m - q + 1$ of the u_i variables are nonzero. But this is impossible by the requirement that q of the u_i are 0 in every feasible solution.

Note that this foregoing result specializes to the earlier result for the GLP problem by imposing the restriction that only one λ_i is nonzero for each P. (Thus $z_j \geq 0$ corresponds to $\lambda_i u_i \geq 0$, or equivalently $u_i \geq 0$; for each of $m-q+1$ variables.) We now indicate how the theorem can be used to generate improved cuts.

Remark 1: For each set P_j there must be at least one u_i $(i \in P_j)$ that is nonzero in the current basic solution in order for the z_j all to be positive. Moreover, any collection of $m - q + 1$ positive u_i variables can be taken to be the "starting" (or representative) variables for the sets P_j. Given a choice for these starting variables, the remainder of the sets P_j can be constructed by successively assigning each of the remaining $q - 1$ variables to them. This permits the cuts from Theorem 1 to be generated by stepwise modification of the earlier cuts (i.e., their corresponding convex regions).

Remark 2: Assume the basic u_i variables are the first to be assigned to the sets P_j. (The starting variable for each set will of course always be basic.) Then the constructive process of Remark 1 can be simplified for each of the nonbasic u_i variables to determine the "best" set P_j to which it should be assigned. Moreover, it is possible to identify the deepest cut from such an assignment without bothering to specify the corresponding λ_i values. In particular, suppose that "incomplete" z_j variables have been defined relative to the partially constructed P_j sets (consisting of the basic u_i) so that the halfspaces $z_j \geq 0$ already implicitly define a cut. To deepen this cut, assign each nonbasic u_i to the set for the incomplete z_j variable that first goes to 0 as a result of extending the edge corresponding to u_i. (This

z_j variable is identified by the same ratio test used to identify the variable to leave the basis in the primal simplex method.) The best λ_i value will produce a new z_j variable with a 0 coefficient in its "defining equation" (constructed from the defining equations of the u_i variables in the current LP tableau), and thus the edge corresponding to the nonbasic u_i variable will no longer be blocked by this z_j variable. Thus, the completed cut is constructed simply by permitting each edge corresponding to a nonbasic u_i variable to bypass the first z_j that blocks it. (This will always yield a deeper cut unless more than one z_j qualifies as the "first" blocking variable.)

<u>Remark 3</u>: Basic u_i variables may be added to the P_j sets (given a selection of $m - q + 1$ starting variables, as indicated in Remark 1) by similarly considering incomplete z_j variables that currently block particular edges. To understand how this may be accomplished, note that a linear combination of an incomplete z_j with a u_i creates a new z_j variable z_j' whose associated hyperplane $z_j' = 0$ is rotated through the intersection of the hyperplanes $z_j = 0$ and $u_i = 0$. Provided the hyperplane $z_j = 0$ does not block more than one of the edges, it is always possible to specify a z_j' which allows the edge blocked by z_j to be extended an additional distance without restricting the extensions of other edges. (This is a simple matter algebraically, accomplished by examining ratios.) In the absence of "blocking ties," such a procedure results in a strictly deeper cut. (Linear combinations of basic u_i variables may alternatively be selected, if desired, to allow certain edges to be extended still more deeply at the expense of curtailing other edge extensions.)

<u>Remark 4</u>: A good strategy for assigning a basic u_i variable to an incomplete set P_j is to select a variable z_j, when possible, which currently blocks a given edge and which would continue to block that edge if $u_i \geq 0$ were added to the set of half spaces defining the cut. This means that the edge intersects $u_i = 0$ after intersect-

ing $z_j = 0$, tending to increase the depth to which a linear combination of u_i and z_j will allow the given edge to penetrate before interfering with the extensions of other edges.

More complex strategies can be employed for generating additionally strengthened cuts by Theorem 1. For example, basic u_i variables whose defining equations have a number of oppositely signed coefficients in the LP tableau may be used to "offset" each other in the context of Remark 3, and thus may be paired instead of considered independently to permit deeper edge extensions. Similarly, the effect of the final strengthening of the nonbasic u_i variables may be anticipated when determining linear combinations of basic variables. Indeed, having determined an assignment of the u_i variables to the sets P_j, "optimal" values of the parameters λ_i (e.g., those which maximize a weighted sum of the depths of the edge extensions) may be generated by solving a simple linear program. Such refinements, however, are not required to produce cuts that are stronger than the original GLP cuts since the procedures of the preceding remarks will typically suffice, sometimes dramatically so.

4. ILLUSTRATIONS

We give two examples, each in two dimensions, to illustrate the new cuts and their relationship to the old ones. The first is

$$\begin{aligned} \text{Minimize} \quad & x_1 + x_2 \\ \text{Subject to:} \quad & x_1 + 2x_2 \le 8 \\ & x_1 - 2x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

where the lattice point conditions require at least 2 of u_1, u_2 and u_3 to be 0, defining $u_1 = 8 - x_1 - 2x_2$, $u_2 = 2 - x_1 + 2x_2$ and $u_3 = x_1$ (that is u_1, u_2 and u_3 are the slack variables for the first three inequalities of the problem).

The constraints of this problem are depicted in Figure 1, with the hyperplanes corresponding to $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$ denoted by (1), (2) and (3) respectively. Thus the feasible region of the problem excluding the lattice point conditions is the quadrilateral bounded by (1), (2), (3) and the x_1 axis (i.e., the hyperplane $x_2 = 0$). The optimal LP vertex is the origin. The GLP cut of [2] requires selecting 2 ($= m - q + 1$, where $m = 3$ and $q = 2$) of the positive u_i variables to define the convex set. Only u_1 and u_2 are positive at the origin, and thus the convex region is uniquely given as the intersection of the half spaces corresponding to (1) and (2). Extending the edges from the origin to the boundary of the convex set gives the GLP cut indicated by the dotted line.

The new cut for this problem may be determined by reference to Remark 2, since u_1 and u_2 must be the starting variables for the sets P_1 and P_2, completing the assignment of all basic u_i, leaving u_3 which is nonbasic at the LP solution. Thus, the edge corresponding to this variable (which is in this case the x_2 axis) bypasses the first hyperplane it intersects ($u_2 = 0$) and the resulting cut is precisely the hyperplane (1) itself (i.e., its associated half space).

The second of the two example problems is the same as the first with the additional constraint

$$x_2 \leq 2 \quad .$$

The variable u_3 is redefined to be the slack for this new constraint (i.e., $u_3 = 2-x_2$). The geometric analog of this problem appears in Figure 2. The feasible LP region is bounded by (1), (2), (3) (which correspond to $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ -- for the new u_3) and by the two coordinate axes. Once again the origin is the optimal LP solution, and the GLP cut of [2] specifies the selection of 2 positive u_i variables to determine the convex set. This time, however, there are three possible choices since all three of the u_i are positive at the origin, and any pair is a legitimate selection. The two best choices are the pairs u_1, u_2 and u_1, u_3 giving rise to the cuts indicated by the dotted lines. (The third choice, u_2, u_3 gives the cut that goes through the intersections of (2) and (3) with the coordinate axes, and hence is clearly dominated by the other cuts.)

The new cut for this problem may be determined by Remark 3, since all of the u_i are basic. The three choices for variables to define the GLP cuts of [2] provide the choices for the "starting" variables of the sets P_1 and P_2. To derive the new cut we take $P_1 = \{1\}$ and $P_2 = \{2\}$, corresponding to the selection of the starting variables u_1 and u_2 (which thus become the initial z_1 and z_2). The basic variable u_3 will be added to one of these sets in the manner indicated by Remark 4. Extending the edges from the LP vertex, we note that the blocking hyperplane (2) ($z_2 = 0$) is intersected before (3) along the x_1 edge, and thus we assign u_3 to the same set as u_2, producing the completed sets $P_1 = \{1\}$ and $P_2 = \{2,3\}$. Linear combinations of u_2 and u_3 (to create the new z_2) give the hyperplanes passing through the intersection of (2) and (3). In this fashion the first z_2 hyperplane ($u_2 = 0$) may be "rotated aside" by the addition of u_3 to P_2 to produce a hyperplane that is not intersected by either of the edges until after their intersection with (1). Correspondingly, it is easy to specify a linear combination that removes z_2 from the category of a blocking variable for the x_1 edge without making it into a blocking variable for the x_2 edge. (The reader may verify this by reference to the LP tableau for this example problem.) As a consequence both edges may be extended to their intersections with (1), and the new cut is given by (1) itself. As in the previous example, this is the strongest cut possible.

It is interesting to note that if the pair u_1, u_3 were selected to be the starting variables, then the same cut would still be obtained. (That is, the strategy of Remark 4 would still generate the same sets P_1 and P_2 and the same hyperplanes $z_1 = 0$ and $z_2=0$ as before -- with the components of P_2 selected in reverse order.) Moreover, if the "worst" pair u_2, u_3 were selected, then the new cuts that result by associating u_1 with either of these variables are still stronger than the best of the original GLP cuts.

The increased strength of the new cuts, as clearly illustrated by the preceding examples, suggests the desirability of using them in place of those of [2] in applications.

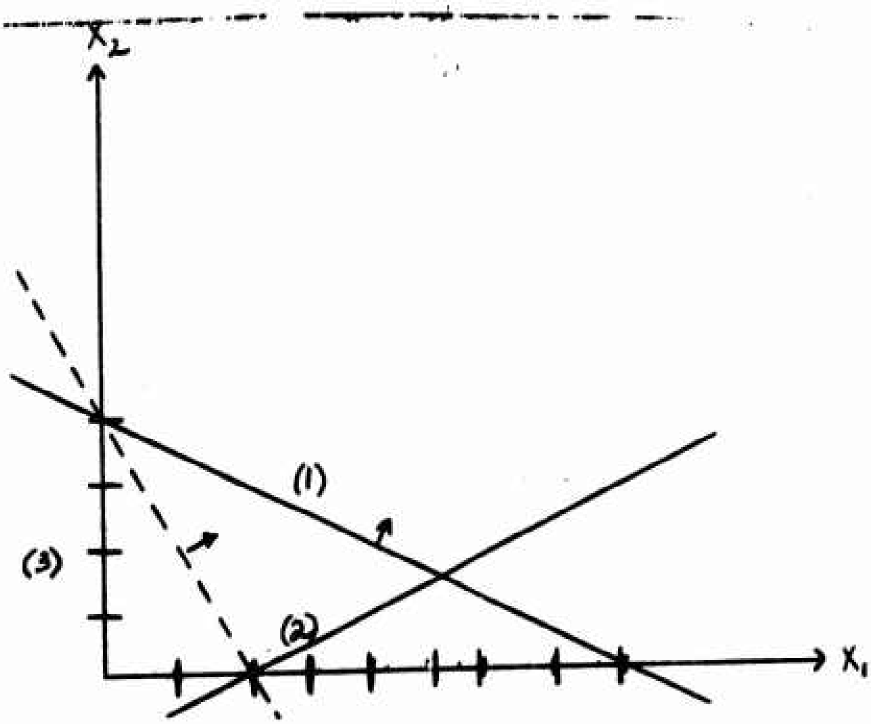

Figure 1:
Old cut given by dotted line.
New cut coincides with hyperplane (1).

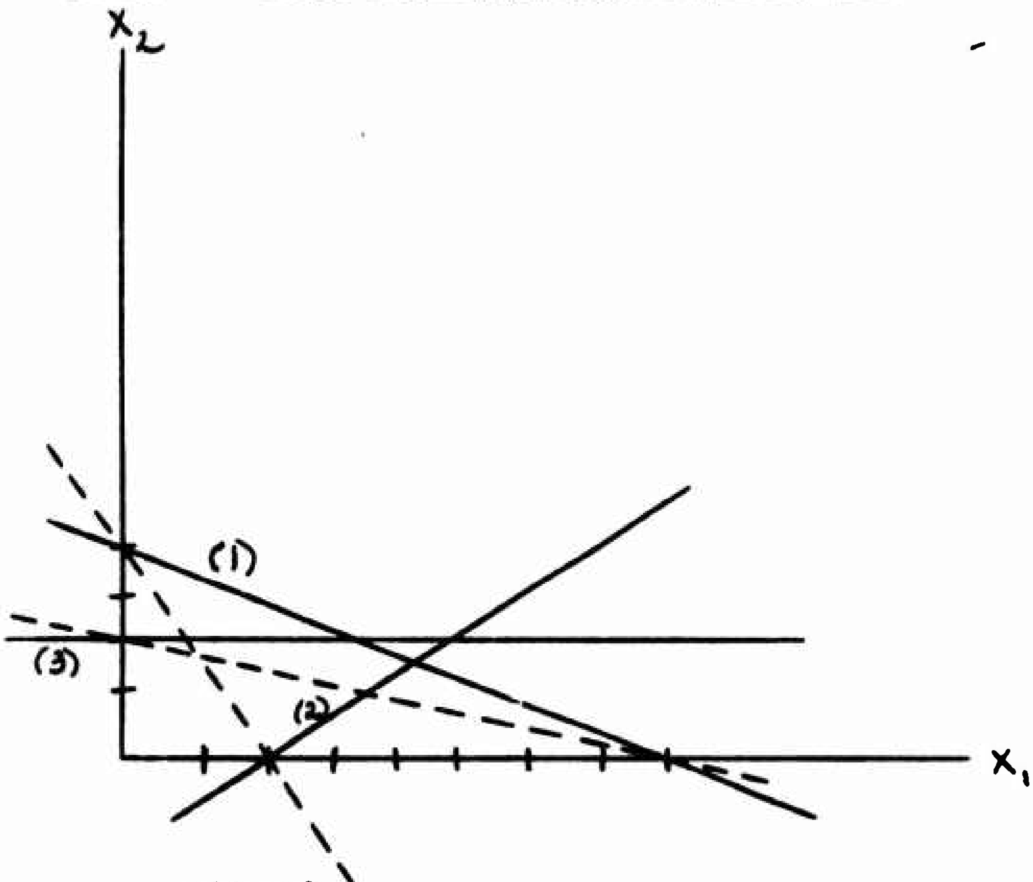

Figure 2:
Two best old cuts given by dotted lines.
New cut coincides with hyperplane (1).

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