AD-763 383

EQUIVALENCE OF GENERALIZED NETWORK AND GENERALIZED TRANSPORTATION PROBLEMS

Fred Glover, et al

Texas University

Prepared for:

Office of Naval Research

January 1973

CENTER FOR CYBERNETIC STUDIES

The University of Texas
Austin, Texas 78712

15

Research Report

CS 125

EQUIVALENCE OF GENERALIZED NETWORK AND GENERALIZED TRANSPORTATION PROBLEMS

by

Fred Glover

D. Klingman

Al Napier

January 1973

This research was partly supported by a grant from the Farah Foundation and by ONR Contracts N00014-67-A-0126-0008 and N00014-67-A-0126-0009 with the Center for Cybernetic Studies, The University of Texas.

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

Business-Economics Building, 512

The University of Texas

Austin, Texas 78712

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for Cybernetic Studies University of Texas at Austin	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Equivalence of Generalized Network and Generalized Transportation Problems

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

F. Glover A. Napier
D. Klingman

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1973	12	7

8a. CONTRACT OR GRANT NO. NR-047-021 b. PROJECT NO. N00014-67-A-0126-0008 c. d. N00014-67-A-0126-0009	9a. ORIGINATOR'S REPORT NUMBER(S) Center for Cybernetic Studies Research Report No. 125 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research (Code 434) Washington, D.C.

13. ABSTRACT

This paper shows that any generalized network problem can be transformed into a generalized transportation problem. Our approach extends earlier procedures for transforming pure network problems to ordinary transportation problems. Also we show that start and solution algorithms developed for certain classes of generalized network problems can be applied to any generalized network problem.

//

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified

Security Classification

Unlcassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Generalized Networks						
Generalized Transportation Problems						
Networks						
Linear Programming						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0102-014-6800

iii

Unclassified

Security Classification

A-31409

Abstract

This paper shows that any generalized network problem can be transformed into a generalized transportation problem. Our approach extends earlier procedures for transforming pure network problems to ordinary transportation problems. Also we show that start and solution algorithms developed for certain classes of generalized network problems can be applied to any generalized network problem.

1. Introduction

The purpose of this paper is to show that any generalized network problem can be transformed into a generalized transportation problem. Our approach constitutes an extension of the procedures [6,7] for transforming ordinary ("pure") network problems into pure transportation problems. Since it has been shown [4] that any generalized network whose incidence matrix does not have full row rank is equivalent to a pure network problem, our results imply that such generalized networks are equivalent to ordinary transportation problems.

Our proposed transformation makes it possible to extend the range of application of those procedures [2,3] which have been developed for generalized network problems whose arc multipliers (arc amplication coefficients) are between 0 and 1 permitting them to be applied to any generalized network. (Note that it is not possible to develop a linear procedure for scaling an arbitrary generalized network problem to yield all arc multipliers in such a range.) This extension follows from the fact that any generalized transportation problem can be made to assume the desired form by simply scaling each column of the coefficient matrix to contain a 1 in its origin row and then dividing each destination row by the largest coefficient in the row. One computational aspect of these observations is that the procedure in [3] can be used to yield a dual feasible solution for any generalized network problem.

2. Transformation

For our purposes, a generalized network will be defined to consist of n nodes or junction points that are connected pairwise by a collection of m directed arcs (links). It is not necessary for all pairs of nodes to be joined.

For each arc (i,j) in the network, we also define the following items:

1. x_{ij} is the flow from node i to node j.
2. c_{ij} is the cost of sending a single unit of flow from node i to node j.
3. p_{ij} is the amplication (or attenuation) coefficient (or multiplier) on the flow from node i to node j; i.e., if y units of flow leave node i then $p_{ij}y$ units enter node j.

Letting N denote the set of arcs, a generalized network problem may be stated as:

<u>Problem I</u>

$$\text{Minimize} \quad \sum_{(i,j)\epsilon N} c_{ij}x_{ij} \tag{1.1}$$

subject to:

$$-\sum_{(i,j)\epsilon N} x_{ij} + \sum_{(j,i)\epsilon N} p_{ji}x_{ji} = d_i, \; i = 1,2,\ldots,n \tag{1.2}$$

$$x_{ij} \geq 0, \; (i,j)\epsilon N \; , \; p_{ij} > 0 \tag{1.3}$$

where d_i is the amount of supply (demand) at node i, where the supply (demand) at node i is denoted by a negative (positive) d_i.

Each node in a network can be classified as a source, sink, or transshipment node. A source node only has arcs emanating from it, while a sink node only has arcs entering it. A transshipment node has arcs both entering it and emanating from it. (Note: a transshipment node may have a supply or a demand.) In this formulation, we assume for simplicity that there are n_s sources, n_t transshipment nodes, n_d sinks and that these nodes are numbered in order from 1 to n. That is, the sources are numbered from 1 to n_s, the transshipment nodes from $n_s + 1$ to $n_s + n_t$ and the sinks from $n_s + n_t + 1$ to $n_s + n_t + n_d$.

Our main result is that the following generalized transportation is equivalent to Problem I:

Problem II

$$\text{minimize} \quad \sum_{(i,j+n_s)\epsilon N} c'_{ij}x'_{ij} + \sum_{i=1}^{n_t} c'_{i+n_s,i}y_{i+n_s,i} \qquad (2.1)$$

subject to:

$$\sum_{(i,j+n_s)\epsilon N} x'_{ij} = a_i, \; i = 1,\ldots,n_s \qquad (2.2)$$

$$\sum_{(i,j+n_s)\epsilon N} x'_{ij}+y_{i,i-n_s} = a_i, \; i = n_s + 1,\ldots,n_s + n_t \qquad (2.3)$$

$$\sum_{(i,j+n_s)\epsilon N} p'_{ij}x'_{ij} + y_{j+n_s,j} = b_j, \; j = 1,\ldots,n_t \qquad (2.4)$$

$$\sum_{(i,j+n_s)\epsilon N} p'_{ij}x'_{ij} = b_j, \; j = n_t + 1,\ldots,n_t + n_d \qquad (2.5)$$

$$x'_{ij} \geq 0 \quad , \; (i,j+n_s)\epsilon N \qquad (2.6)$$

$$y_{j+n_s,j} \geq 0 \quad , \; j = 1,\ldots,n_t \qquad (2.7)$$

where $c'_{ij} = c_{i,j+n_s}$, $p'_{ij} = p_{i,j+n_s}$ for $(i,j+n_s)\epsilon N$

$c'_{i+n_s,i} = 0$ for $i = 1,2,\ldots,n_t$

$a_i = -d_i$ for $i = 1,\ldots,n_s$

$a_i = -d_i + B$ for $i = n_s + 1,\ldots,n_s + n_t$

$b_j = B$ for $j = 1,\ldots,n_t$

$b_j = d_{j+n_s}$ for $j = n_t + 1,\ldots,n_t + n_d$

and B is a buffer to be specified later.

A succint way of describing the transformation procedure for obtaining Problem II from Problem I is the following:

1. Designate an origin for each source i of the network and let the supply value a_i of this origin i be the negative of the amount of supply d_i at source i. ("Supplies" are positive in transportation formulations.)

2. Designate a destination $j - n_s$ for each sink j of the network and let the demand value b_{j-n_s} of this destination be the amount of demand d_j at sink j.

3. For each transshipment node k designate an origin k and destination $k - n_s$. Let $a_k = -d_k + B$ and $b_{k-n_s} = B$. (B is a buffer stock that must be large enough to insure that all $y_{k,k-n_s}$ will be basic. A procedure for determining the appropriate value of B when all $p_{ij} \geq 1$ or $0 < p_{ij} \leq 1$ or $p_{ij} > 0$ is examined in a later section.)

4. For each arc $(i,j+n_s)$ of the generalized network introduce an arc (i,j) in the transportation problem with a cost c'_{ij} and multiplies p'_{ij} equal to the cost and multiplier associated with the original arc. In addition, for each transshipment node k, introduce an arc $(k,k-n_s)$ in the transportation problem with a cost $c'_{k,k-n_s}$ equal to zero and a multiplier $p'_{k,k-n_s}$ equal to one. Let $y_{k,k-n_s}$ denote the flow on this arc.

3. Equivalence

Theorem: Assume that $0 < p_{ij} \leq 1$, $(i,j)\epsilon N$ and let $B = -\sum_{\{i:d_i < 0\}} d_i$. The solution x_{ij}, $(i,j)\epsilon N$ is feasible (optimal) for Problem I if and only if the solution $x'_{i,j-n_s} = x_{ij}, (i,j)\epsilon N$ and $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s)\epsilon N} x'_{ij}$, $i = n_s+1,\ldots,n_s+n_t$ is feasible (optimal) for Problem II. (Furthermore, the theorem is valid if the objective is to maximize the functionals (1.1) and (2.1).)

Proof:

It is apparent that the functionals for Problems I and II will have the same value for the solution as indicated. Thus, to prove the theorem, it suffices simply to prove the feasibility assertion.

First assume that the solution x_{ij}, $(i,j)\epsilon N$ is feasible for Problem I. We will show that $x'_{i,j-n_s} = x_{ij}$, $(i,j)\epsilon N$ and $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s)\epsilon N} x'_{ij}$, $i=n_s+1,\dots,n_s+n_t$ is feasible for Problem II. By assumption $x'_{i,j-n_s}$, $(i,j)\epsilon N$ satisfies (2.6) and clearly by definition $y_{i,i-n_s}$ and $x'_{i,j-n_s}$ satisfy (2.3). Note that for $i = 1,\dots,n_s$, (1.2) is equal to $-\sum_{(i,j)\epsilon N} x_{ij} = d_i$. Since $x'_{i,j-n_s} = x_{ij}$ and x_{ij} is a feasible solution for Problem I, then $\sum_{(i,j)\epsilon N} x'_{i,j-n_s} = -d_i$ or

$\sum_{(i,j+n_s)\epsilon N} x'_{ij} = a_i$, $i=1,\dots,n_s$; thus (2.2) is satisfied. Similarly, for $i = n_s+n_t+1,\dots,n$ (1.2) is equal to $\sum_{(j,i)\epsilon N} p_{ji}x_{ji} = d_i$. From $x'_{i,j-n_s} = x_{ij}$ and the feasibility of x_{ij} for Problem I we have $\sum_{(j,i)\epsilon N} p_{ji}x'_{j,i-n_s} = d_i$ or

$\sum_{(j,i)\epsilon N} p'_{j,i-n_s}x'_{j,i-n_s} = d_i$ for $i = n_s + n_t + 1,\dots,n$. Setting $k = i-n_s$ and rewriting we obtain $\sum_{(j,k+n_s)\epsilon N} p'_{jk}x'_{jk} = b_k(=d_{k+n_s})$. Thus x'_{ij} satisfies (2.5).

The equality $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s)\epsilon N} x'_{ij}$, $i=n_s+1,\dots,n_s + n_t$ can be rewritten as $y_{i,i-n_s} = -d_i + B - \sum_{(i,j)\epsilon N} x_{ij}$, and thus we have from (1.2)

$$y_{i,i-n_s} + d_i - B + \sum_{(j,i)\epsilon N} p_{ji}x_{ji} = d_i;\ i = n_s + 1,\dots,n_s + n_t \text{ or}$$

$$\sum_{(j,i)\epsilon N} p_{ji}x_{ji} + y_{i,i-n_s} = B,\ i = n_s + 1,\dots,n_s + n_t. \text{ Setting } k = i-n_s \text{ yields}$$

$$\sum_{(j,k+n_s)\epsilon N} p_{j,k+n_s}x_{j,k+n_s} + y_{k+n_s,k} = B,\ k = 1,\dots,n_t. \text{ Since}$$

$p'_{ij} = p_{i,j+n_s}$, $x'_{ij} = x_{i,j+n_s}$ and $b_j = B$ for $j=1,\dots,n_t$ the solution x'_{ij} satisfies (2.4).

Because all of the $0 < p_{ij} \leq 1$, the flow out of any transshipment node i cannot be greater than the total supply (B) less $\max(d_i, 0)$. Thus $y_{i,i-n_s} = -d_i + B - \sum_{(i,j)\in N} x_{ij}$ is nonnegative for $i = n_s + 1, \ldots, n_s + n_t$. Therefore (2.7) is satisfied and this completes the first half of the proof.

Next, assume x'_{ij} is feasible for Problem II. We must now show that this solution is feasible for Problem I. It is immediately apparent that (1.3) is satisfied from (2.6); for $i = 1, \ldots, n_s$ (1.2) is satisfied from (2.2); and for $i = n_s + n_t + 1, \ldots, n$ (1.2) is satisfied from (2.5).

For $k = n_s+1, \ldots, n_s+n_t$, if equation k of (2.3) is subtracted from equation $k-n_s$ of (2.4) we obtain

$$-\sum_{(k,j+n_s)\in N} x'_{kj} - y_{k,k-n_s} + \sum_{(i,k)\in N} p'_{i,k-n_s} x'_{i,k-n_s} + y_{k,k-n_s} = b_{k-n_s} - a_k .$$

Since $b_{k-n_s} = B$ and $a_k = -d_k + B$ we have

$$-\sum_{(k,j)\in N} x_{kj} + \sum_{(i,k)\in N} p_{ik} x_{ik} = d_k, \quad k = n_s + 1, \ldots, n_s+n_t.$$ Thus (1.2) is satisfied for all nodes 1, 2, ..., n, and the proof is complete.

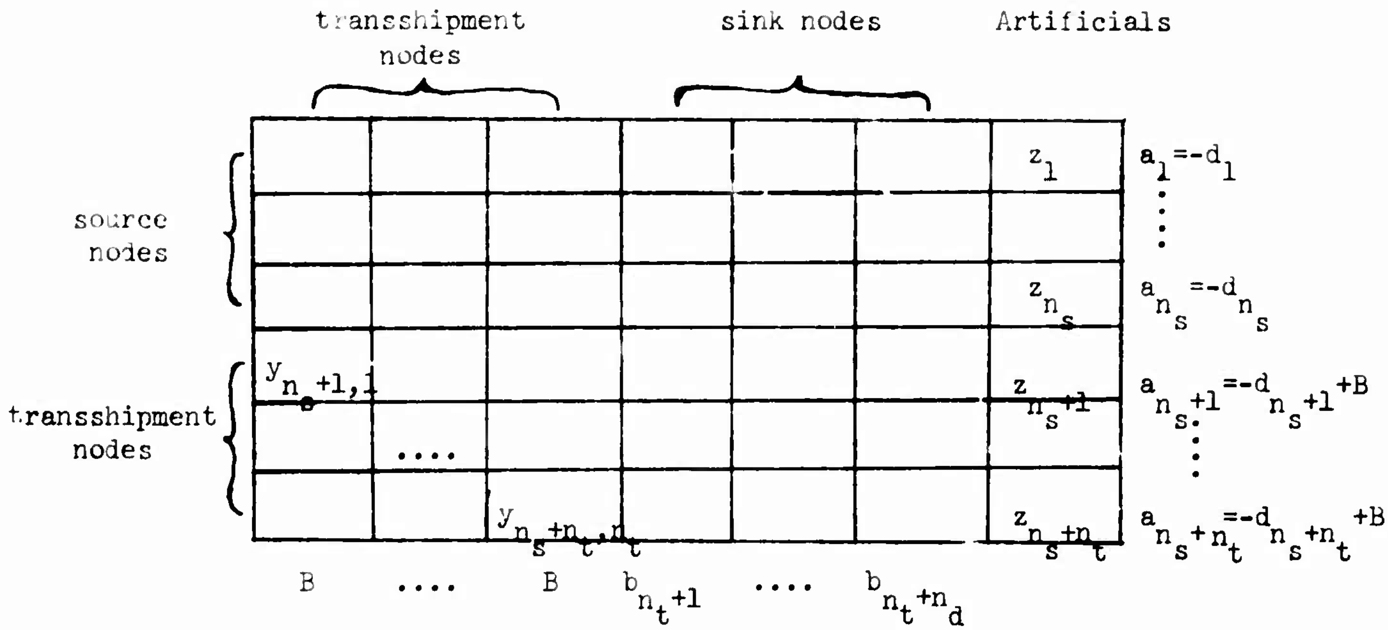
Corollary: Assume that $p_{ij} \geq 1$, $(i,j)\in N$ and let $B = \sum_{\{i:d_i > 0\}} d_i$. The solution x_{ij} is feasible (optimal) for Problem I if and only if the solution $x'_{i,j-n_s} = x_{ij}, (i,j)\in N$ and $y_{i,i-n_s} = a_i - \sum_{(i,j+n_s)\in N} x'_{ij}, i = n_s + 1, \ldots, n_s + n_t$ is feasible (optimal) for Problem II. (Furthermore, the corollary is valid if the objective is to maximize the functionals (1.1) and (2.1)).

Proof: From the proof of the theorem, if x'_{ij} is a feasible solution for Problem II then x_{ij} is a feasible solution for Problem I regardless of the values of B and p_{ij}. Similarly, the proof of the theorem establishes that if x_{ij} is a feasible solution to Problem I, then x'_{ij} is a feasible solution to Problem II if and only if $y_{i,i-n_s} \geq 0$. Thus it suffices simply to show that $y_{i,i-n_s} \geq 0$.

To do this, note that $p_{ij} \geq 1$ for all (i,j) implies that the flow out of any transshipment node i cannot be greater than the total demand(B)less $\max(0,d_i)$. Otherwise, the amplification of the flow leaving the node would render the solution infeasible since this amplified flow could not be absorbed by the demand. Thus $y_{i,i-n_s} = -d_i + B - \sum_{(i,j)\epsilon N} x_{ij}$ is nonnegative for B equal to the total demand and this completes the proof.

If some of the multipliers p_{ij} are less than one and others are greater than one, then the problem may contain "creator" and/or "destructor" loops (See Jewell [5].) In such a case the solution region may be unbounded since arbitrarily large amounts of flow may be created and later destroyed. Thus, it is not possible to derive a sufficiently large value for the buffer B without assuming the nonexistence of creator or destructor loops. From a computational standpoint, however, it is not necessary to know a sufficiently large buffer size a-priori since the buffer can be successively increased until either an optimal solution is found to the generalized network or the problem is determined to be infeasible or unbounded. Moreover, this manipulation of the buffer can be done without interrupting the ordinary calculations and without shifting from a primal method to a dual method. This may be seen as follows.

Set the buffer at some positive value (i.e., $B > 0$) and try to solve the problem using a special purpose primal approach. First pick an artificial primal feasible starting basis containing the $y_{i+n_s,i}$ variables. This can be done by considering the transportation tableau format for Problem II with a column of artificial variables z_j adjoined:



Set $y_{n_s+i,i}=B$, $z_{n_s+i}=|d_{n_s+i}|$, $i=1,\dots,n_t$, and subtract the artificial variable z_{n_s+i} from origin constraint n_s+i if $d_{n_s+i} < 0$; otherwise add z_{n_s+i}. In addition if $n_s \geq n_d$ set $x'_{i,n_t+i}= (1/p'_{i,n_t+i})\, b_{n_t+i}$, $i=1,\dots,n_d$ and set $z_i=|a_i-(1/p'_{i,n_t+i})b_{n_t+i}|$, $i=1,\dots,n_d$ and subtract z_i if $a_i-(1/p'_{i,n_t+i})b_{n_t+i} < 0$; otherwise add z_i Also set $z_i=a_i$, $i=n_d+1,\dots,n_s$ and add it. (Note if x'_{i,n_t+i}, $i=1,\dots,n_d$ does not exist then it is also an artificial variable and let $p'_{i,n_t+i}=1$.)

If $n_s \leq n_d$, set $x'_{i,n_t+i}=(1/p'_{i,n_t+i})b_{n_t+i}$, $i=1,2,\dots,n_s$ and $x'_{n_s,n_t+i} = (1/p'_{n_s,n_t+i})\, b_{n_t+i}$, $i=n_s+1,\dots,n_d$. Further set $z_i = |a_i - (1/p'_{i,n_t+i})\, b_{n_t+i}|$, $i=1,2,\dots,n_s-1$ and $z_{n_s} = |a_{n_s} - \sum_{i=n_s}^{n_d} (1/p'_{i,n_t+i})\, b_{n_t+i}|$. Add z_i to the i^{th} origin if $a_i -(1/p'_{i,n_t+i})b_{n_t+i} \geq 0$, $i = 1,2,\dots,n_s-1$; otherwise subtract z_i. Similarly add or subtract z_{n_s}.

This is a basic artificial feasible solution since the set of the first $n_s+2n_t+n_d$ unit vectors is a subset of the span of the vectors associated with the basic variables. This can be easily seen by observing that the vector associated with the variable z_i consist of the first n_s+n_t unit vectors and subtracting these unit vectors from the other vectors associated with the otherbasic vectors yield the remaining unit vectors.

Using this artificial primal basic feasible solution consider performing a

Phase I optimization (i.e., minimizing the sum of the artificials). Throughout this minimization the buffer can be manipulated in a manner which enables the $y_{i+n_s,i}$ variables to be kept basic. To see this first note that increasing the buffer B will increase only the basic $y_{i+n_s,i}$ variables. Consequently, during Phase I whenever a variable x'_{ij} would enter the basis in place of some variable $y_{n_s+i,i}$, it is possible to increase the buffer sufficiently to prevent such a replacement from occurring. This is a consequence of the fact that the basis representation of any candidate to enter the basis must have a positive coefficient associated with at least one artificial variable. Thus, at the termination of Phase I all of the $y_{n_s+i,i}$ variables will be basic. If any artificial variable is basic at a positive value then the generalized network problem is of course infeasible since increasing the buffer will only increase the $y_{n_s+i,i}$ variables and thus not affect the artificial variables. (Specifically, there exist no buffer values for which the generalized transportation problem is feasible, consequently, the generalized network problem is infeasible.)

After completing Phase I and pivoting all zero-valued artificials out of the basis, it is either possible in Phase II to continue to keep the $y_{n_s+i,i}$ variables basic by the same procedure of manipulating the buffer, or, the generalized network must be unbounded due to the fact that the incoming variable can be brought into the basis at an infinite amount by infinitely increasing the buffer. If the problem is not unbounded Phase II will terminate with a finite optimal solution to the generalized network since any increase in the buffer will not alter the solution value of the x'_{ij} to the corresponding transportation problem. (This illustrates that the constraints associated with the buffer act as "regularization constraints" as defined by Charnes [1].)

References

1. Charnes, A. and W. W. Cooper. Management Models and Industrial Applications of Linear Programming, VolumesI and II. New York: John Wiley and Sons, Inc. 1961.
2. Charnes, A. and W. M. Raike, "One-Pass Algorithms for Some Generalized Network Problems", Operations Research, 14,5 (1966), 914-924.
3. Glover, F., D. Klingman, and Al Napier, "Basic Dual Feasible Solution for a Class of Generalized Networks", Operations Research, 20,1 (1972), 126-137.
4. Glover, F. and D. Klingman, "On the Equivalence of Some Generalized Network and Pure Network Problems", to appear in Mathematical Programming.
5. Jewell, W. S., "Optimal Flow Through Networks with Gains", Operations Research, 10,4 (1962), 476-499.
6. Orden, Alex, "The Transhipment Problem", Management Science, 2 (1956), 276-285.
7. Wagner, H. M., Principles of Operations Research With Applications to Management Decisions, New Jersey: Prentice Hall, Inc., 1969.