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TOLERANCE LIMITS FOR THE MAXWELL DISTRIBUTION WITH EMPHASIS ON THE SEP. (SPHERICAL ERROR PROBABLE)

Marlin A. Thomas, et al

Naval Weapons Laboratory
Dahlgren, Virginia

June 1973

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Tolerance limits are formulated for the Maxwell distribution, and a table of tolerance limit factors for the upper tolerance bound is provided as a function of P (percent of the population below the bound), γ (confidence level), and n (sample size). Values of P, γ, and n considered are P = .50, .75, .90, .95, .99; γ = .75, .90, .95, .99; n = 2(1)25(5)100(10)200(50)300(100)1000,∞.

While the formulation is sufficiently general to be of use to anyone who deals with Maxwell data, examples are restricted to the area of weapon systems analysis. It is in this area that the analyst is often confronted with the problem of estimating the radius of a sphere which will include 100P% of the future burst points from an air burst weapon. Under the assumption that the distribution of burst points about the target center is trivariate normal with common standard deviation σ in all three directions, this development will enable him to attach a confidence statement to the percent of the population encompassed. In particular, it will enable him to determine, on the basis of test firings, the radius of a sphere which will encompass at least 100P% of the future burst points with 100γ% confidence.

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FOREWORD

The work covered in this Technical Report was performed in the Mathematical Statistics and Systems Simulation Branch (KCM), Operations Research Division, Warfare Analysis Department. The date of completion was 4 May 1973.

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Released by:

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ABSTRACT

Tolerance limits are formulated for the Maxwell distribution, and a table of tolerance limit factors for the upper tolerance bound is provided as a function of P (percent of the population below the bound), γ (confidence level), and n (sample size). Values of P, γ, and n considered are P = .50, .75, .90, .95, .99; γ = .75, .90, .95, .99; n = 2(1)25(5)100(10)200(50)300(100)1000,∞.

While the formulation is sufficiently general to be of use to anyone who deals with Maxwell data, examples are restricted to the area of weapon systems analysis. It is in this area that the analyst is often confronted with the problem of estimating the radius of a sphere which will include 100P% of the future burst points from an air burst weapon. Under the assumption that the distribution of burst points about the target center is trivariate normal with common standard deviation σ in all three directions, this development will enable him to attach a confidence statement to the percent of the population encompassed. In particular, it will enable him to determine, on the basis of test firings, the radius of a sphere which will encompass at least 100P% of the future burst points with 100γ% confidence.

CONTENTS

INTRODUCTION

A measure of dispersion often applied to air burst weapons is the SEP (Spherical Probable Error). This parameter is the radius of a mean-centered sphere which includes 50% of the trivariate probability, or in terms of a particular weapon, it is the radius of a sphere centered on the target center within which 50% of the rounds will burst. Its bivariate counterpart for ground burst weapons is, of course, the CEP which is the radius of a mean-centered circle which includes 50% of the bivariate probability. Estimation of the SEP involves firing a sample of n independent rounds at a target arbitrarily placed at the center of the Cartesian coordinate system. The radial burst distances of these rounds from the target center, denoted by $\{r_i\}_{i=1}^{n}$, are then recorded and used to compute an estimate of SEP, say $\hat{SEP}$, which is taken as the radius of a sphere within which 50% of the future rounds from this weapon will burst.

$\hat{SEP}$ is only a point estimate of SEP and, as such, it will vary from sample to sample. To provide a measure of precision concerning the percent of the population encompassed, the probability that a sphere of radius $\hat{SEP}$ will include at least 50% of the future rounds is considered. Since this probability is found to be quite low, an alternative procedure is suggested through the development of tolerance limits for the Maxwell distribution. This alternative procedure enables one to increase this probability (or confidence) to more reasonable levels not only for 50% of the trivariate probability but also for 75%, 90%, 95%, and 99%. The development is an extension of Thomas et al. (1973) for the CEP and is sufficiently general to be useful to those other than the weapon systems analyst.

DERIVATION OF THE MAXWELL DISTRIBUTION

The use of the SEP as a measure of dispersion requires the assumption that the burst points of a weapon about the target center are distributed according to the uncorrelated trivariate normal distribution with common variance σ^2 in all three directions (sometimes referred to as the spherical normal distribution). If one lets the trivariate random variable (X,Y,Z) designate the miss distance of a burst point from the target center (arbitrarily placed at the origin) in the x, y, and z directions respectively, this density is given by

$$f(x,y,z) = (2\pi\sigma^2)^{-3/2} e^{-(x^2+y^2+z^2)/2\sigma^2}, \quad -\infty < x,y,z < \infty . \tag{1}$$

Consider now the distribution of the radial burst distance from the target center, i.e., the distribution of $R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$. Transforming to spherical coordinates by letting

$$X = R \sin \phi \cos \theta$$

$$Y = R \sin \phi \sin \theta$$

$$Z = R \cos \phi ,$$

it is easy to show that the density of the trivariate random variable (R,ϕ,θ) is

$$g(r,\phi,\theta) = (2\pi\sigma^2)^{-3/2} e^{-r^2/2\sigma^2} r^2 \sin \phi \tag{2}$$

for $r > 0$, $0 < \theta < 2\pi$, and $0 < \phi < \pi$. The marginal density of R is now obtained by integrating $g(r,\phi,\theta)$ over the entire range of both θ and ϕ. It turns out that the density of R is

$$h(r) = (2/\pi)^{\frac{1}{2}} (r^2/\sigma^3) e^{-r^2/2\sigma^2} , \qquad r > 0 . \tag{3}$$

This density is commonly referred to as the Maxwell density and represents the distribution of the radial burst distances from the target center under the spherical normality assumption. (See, for example, Lindgren (1968).) Its counterpart for ground burst weapons is referred to as the Rayleigh distribution and is the distribution of the radial miss distances in the ground plane.

Using the density in (3), one can establish the relationship between SEP and σ by solving

$$P(R < SEP) = \int_0^{SEP} h(r)dr = .5 \tag{4}$$

for SEP in terms of σ. Since $h(r)$ is not integrable in closed form, numerical procedures can be employed to show that SEP = 1.5382σ. Hence, if the population standard deviation σ were known for a given weapon-to-target range, a sphere of radius 1.5382σ centered on the target center would encompass 50% of the trivariate probability or 50% of the future burst points from this weapon under similar conditions. (In the bivariate analogy for ground burst weapons, it is well known that CEP = 1.1774σ.)

Unfortunately, σ and hence SEP are never known and must be estimated from test firings. To estimate σ, n rounds are fired at a target at a given range and the distances of the burst points from the target center are recorded. If these distances are designated as r_i, it is easy to show that the maximum likelihood estimate for σ is given by

$$\hat{\sigma} = \{\sum_{i=1}^{n} r_i^2/3n\}^{\frac{1}{2}} \tag{5}$$

and thus the maximum likelihood estimate for SEP is given by

$$\hat{SEP} = 1.5382\hat{\sigma}. \tag{6}$$

Suppose now that one estimates SEP with $\hat{SEP}$ as given in (6) above. One cannot state that a sphere of radius $\hat{SEP}$ will encompass 50% of the future rounds from this weapon (under similar conditions) with certainty since $\hat{SEP}$ is a continuous random variable which varies from sample to sample. However, as a measure of precision, one could consider the probability that a sphere of radius $\hat{SEP}$ encompasses <u>at least</u> 50% of the trivariate probability (at least 50% of the future burst points). This probability will be explored in the next section.

PROBABILISTIC DEVELOPMENT

This section concerns the development of an expression for the probability that a sphere of radius $\hat{SEP} = 1.5382\hat{\sigma}$ encompassing at least 50% of the trivariate probability However, to keep the development general, this will be accomplish. by obtaining an expression for the probability that a sphere of radius $k\hat{\sigma}$ encompasses at least 100P% of the trivariate probability. The mathematical expression for this probability is

$$\text{Prob}\ \{\int_0^{k\hat{\sigma}} (2/\pi)^{\frac{1}{2}}\ (r^2/\sigma^3)\ e^{-r^2/2\sigma^2}\ dr \geq P\} = \gamma \tag{7}$$

where γ is the quantity sought. Letting $y = r^2/\sigma^2$ in the integral in (7), one obtains

$$\text{Prob}\ \{\int_0^{k^2\hat{\sigma}^2/\sigma^2} (2\pi)^{-\frac{1}{2}}\ y^{\frac{1}{2}}\ e^{-y/2}\ dy \geq P\} = \gamma\ . \tag{8}$$

The integrand in (8) is recognized as a chi-square density with three degrees of freedom. Hence (8) can be written as

$$\text{Prob}\ \{F(\frac{k^2\hat{\sigma}^2}{\sigma^2}) \geq P\} = \gamma \tag{9}$$

where F denotes the cumulative distribution function for the chi-square with three degrees of freedom. Since F is a one-to-one function, (9) can be expressed by

$$\text{Prob}\ \{\frac{k^2\hat{\sigma}^2}{\sigma^2} \geq F^{-1}\ (P)\} = \gamma \tag{10}$$

or

$$\text{Prob}\ \{\hat{\sigma}^2 \geq \frac{\sigma^2 F^{-1}(P)}{k^2}\} = \gamma\ . \tag{11}$$

Note that F^{-1} (P) is the 100P percentage point of the chi-square density with three degrees of freedom. Tabular values of F^{-1} (P) are available from various sources, most notably from Biometrika Tables for Statisticians, Vol. I where they are presented to six significant digits for a wide range of values of P. For P = .50, the value of P under present consideration, $F^{-1}(.50) = 2.36597$.

Using the maximum likelihood estimator for $\hat{\sigma}$ as given in (5), it is easily shown that the density of $W = \hat{\sigma}^2$ is given by

$$f(w) = \frac{(3n)^{3n/2}\ w^{3n/2\ -1}\ e^{-3nw/2\sigma^2}}{\Gamma(3n/2)\ 2^{3n/2}\ \sigma^{3n}}\ , \qquad w > 0\ . \tag{12}$$

Therefore, equation (11) can be expressed by

$$\int_0^v f(w)\ dw = 1 - \gamma \tag{13}$$

where f(w) is given in (12) and $v = \sigma^2 F^{-1}(P)/k^2$. It appears from (13) that γ, the probability under question, is a function of the unknown parameter σ. A simple transformation reveals that it is not. Letting $z = w/\sigma^2$ so that $w = \sigma^2 z$ and $dw = \sigma^2 dz$, equation (13) becomes

$$\int_0^{v'} \frac{(3n)^{3n/2}\ z^{3n/2\ -1}\ e^{-3nz}}{\Gamma(3n/2)\ 2^{3n/2}}\ dz = 1 - \gamma \tag{14}$$

where $v' = F^{-1}(P)/k^2$. Equation (14) is free of the unknown parameter σ so that the probability γ is a function of only P (% of the population), k (multiplying constant for $\hat{\sigma}$), and n (sample size).

Recall now the question posed at the end of the last section, namely, with what probability (or confidence) can one state that a sphere of radius $\hat{SEP} = 1.5382\hat{\sigma}$ will encompass at least 50% of the trivariate probability. This probability can be obtained from equation (14) by setting $k = 1.5382$, $P = .50$, and solving for γ for various values of n. This equation was solved with k and P set as above for sample sizes of $n = 2(1)25(5)100(10)200(50)300(100)1000,\infty$. The results are set out in Table 1 and reveal that this probability or confidence is quite low, i.e., less than .50 unless one has an infinite sample size (tantamount to having complete knowledge about the unknown parameter σ). To increase this confidence to more reasonable levels, it is clear that one must increase the multiplying constant above 1.5382. This will be discussed next through the development of tolerance limits for the Maxwell distribution.

UPPER TOLERANCE BOUND FOR THE MAXWELL DISTRIBUTION

Making confidence or probability statements concerning the percent of the population which lies below an estimate of the SEP involves the concept of an upper tolerance bound. In the more general sense, an upper tolerance bound, $U(P,\gamma)$, is a point defined such that at least 100P% of the population lies below it with 100γ% confidence. (See, for example, Bowker and Lieberman (1972), Proschan (1953), or Thomas, et al. (1973).). $U(P,\gamma)$ is constructed as a function of the estimate(s) of unknown population parameter(s) based on a random sample from the population in question. For the case at hand in which one is sampling from a trivariate normal distribution with common variance, there is only one unknown parameter, i.e., σ, and the upper tolerance bound will be formulated as a function of the estimate of this parameter. (Since only the radial distance between burst point and target is under consideration, this can also be viewed as sampling from the Maxwell distribution with parameter σ. See equation (3).)

The maximum likelihood estimator for σ as given in (5) can be shown to be a sufficient estimator for σ, so that the upper tolerance bound should be of the form $U(P,\gamma) = k(P,\gamma,n)\hat{\sigma}$. The constant $k(P,\gamma,n)$ (tolerance limit factor) is to be determined such that one is 100γ% confident that at least 100P% of the population lies below $U(P,\gamma)$. While a value of $P = .50$ is of primary concern here, other values could well be of interest so the general notation will be used. Deleting the arguments for notational simplicity, $k(P,\gamma,n)$ is sought such that

$$\text{Prob}\left\{\int_0^{k\hat{\sigma}} (2/\pi)^{\frac{1}{2}} (r^2/\sigma^3)\, e^{-r^2/2\sigma^2}\, dr \geq P\right\} = \gamma . \tag{15}$$

This is, of course, precisely equation (7) of the last section which reduced to equation (14). While equation (14) was originally derived for the purpose of evaluating γ for fixed values of P, k, and n, it can also be used to evaluate k for fixed values of P, γ, and n. The value of k so obtained would be an exact tolerance limit factor which,

Table 1

PROBABILITY THAT AT LEAST 50% OF POPULATION LIES WITHIN $\hat{SEP}$

n	γ	n	γ
2	.4232	60	.4860
3	.4373	65	.4865
4	.4457	70	.4870
5	.4514	75	.4875
6	.4556	80	.4879
7	.4589	85	.4882
8	.4616	90	.4886
9	.4638	95	.4889
10	.4656	100	.4891
11	.4672	110	.4896
12	.4686	120	.4901
13	.4699	130	.4905
14	.4710	140	.4908
15	.4720	150	.4911
16	.4728	160	.4914
17	.4737	170	.4917
18	.4744	180	.4919
19	.4751	190	.4921
20	.4757	200	.4923
21	.4763	250	.4931
22	.4768	300	.4937
23	.4774	400	.4946
24	.4778	500	.4951
25	.4783	600	.4956
30	.4802	700	.4959
35	.4816	800	.4962
40	.4828	900	.4964
45	.4838	1000	.4966
50	.4846	∞	1.0000
55	.4854		

when multiplied times $\hat{\sigma}$, would provide the upper tolerance bound $U(P,\gamma)$. As aforementioned, the interpretation of $U(P,\gamma)$ is that one is $100\gamma\%$ confident that at least 100P% of the population lies within a sphere of radius $U(P,\gamma)$. For a particular weapon at a specified gun-to-target range, this means that one is $100\gamma\%$ confident that at least 100P% of the future rounds from this weapon, fired under similar conditions, will burst within a sphere of radius $U(P,\gamma)$ centered on the target center.

To utilize the above concept, it is necessary for one to have a table of tolerance limit factors at his disposal for reasonable values of P, γ, and n. Hence, equation (14) was solved for k for P = .50, .75, .90, .95, .99; γ = .75, .90, .95, .99; and n = 2(1) 25(5)100(10)200(50)300(100)1000,∞. The solutions were obtained using Simpson's integration rule and successive binary cuts beginning with an appropriate starting value for the upper limit v'. The computations were performed on the CDC 6700 at the Naval Weapons Laboratory. Tolerances were set to provide an accuracy in k of four decimal digits; the values of k are set out in Table 2 for the above listed values of P, γ, and n.

As an example of employing this procedure, suppose eight rounds are fired at a target to obtain the radius of a sphere about the target center within which at least 50% of the future rounds (under similar conditions) from this weapon will burst with 95% confidence. The radial miss distances from the target center are shown below. All measurements are in feet.

r
201.92
52.98
210.10
120.43
48.17
96.40
85.84
104.71

It can be verified that $\sum_{i=1}^{8} r_i^2 = 132,169.9603$ and hence that

$$\hat{\sigma} = 74.21$$

and

$$S\hat{E}P = 1.5382\hat{\sigma} = 114.15$$

Table 2

TOLERANCE LIMIT FACTORS FOR THE MAXWELL DENSITY

	γ = .75					γ = .90				
n \ P	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
2	2.0271	2.6712	3.295[illegible]	3.6841	4.4389	2.5378	3.3441	4.1251	4.6122	5.5571
3	1.8999	2.5036	3.0883	3.4530	4.1604	2.2602	2.9784	3.6739	4.1077	4.9493
4	1.8343	2.4171	2.9816	3.3336	4.0166	2.1222	2.7966	3.4497	3.8570	4.6472
5	1.7932	2.3630	2.9148	3.2590	3.9267	2.0378	2.6852	3.3123	3.7034	4.4622
6	1.7647	2.3254	2.8685	3.2072	3.8643	1.9798	2.6089	3.2182	3.5982	4.3354
7	1.7435	2.2975	2.8341	3.1687	3.8179	1.9372	2.5527	3.1489	3.5207	4.2420
8	1.7271	2.2758	2.8073	3.1388	3.7818	1.9043	2.5094	3.0954	3.4609	4.1699
9	1.7138	2.2584	2.7858	3.1147	3.7528	1.8779	2.4746	3.0526	3.4130	4.1122
10	1.7029	2.2439	2.7680	3.0948	3.7289	1.8563	2.4461	3.0173	3.3736	4.0648
11	1.6936	2.2318	2.7530	3.0780	3.7086	1.8381	2.4221	2.9878	3.3405	4.0249
12	1.6857	2.2214	2.7402	3.0637	3.6914	1.8225	2.4016	2.9625	3.3122	3.9909
13	1.6789	2.2123	2.7290	3.0512	3.6763	1.8090	2.3838	2.9405	3.2877	3.9613
14	1.6728	2.2044	2.7192	3.0402	3.6631	1.7972	2.3682	2.9213	3.2662	3.9354
15	1.6675	2.1973	2.7105	3.0305	3.6514	1.7867	2.3544	2.9043	3.2472	3.9125
16	1.6627	2.1910	2.7027	3.0218	3.6409	1.7774	2.3421	2.8891	3.2302	3.8920
17	1.6584	2.1853	2.6957	3.0140	3.6314	1.7690	2.3310	2.8754	3.2149	3.8736
18	1.6545	2.1802	2.6893	3.0068	3.6229	1.7613	2.3210	2.8630	3.2011	3.8569
19	1.6509	2.1754	2.6835	3.0003	3.6150	1.7544	2.3118	2.8517	3.1884	3.8417
20	1.6476	2.1711	2.6782	2.9944	3.6079	1.7480	2.3034	2.8414	3.1769	3.8277
21	1.6446	2.1671	2.6733	2.9889	3.6013	1.7422	2.2957	2.8319	3.1662	3.8149
22	1.6418	2.1635	2.6687	2.9838	3.5952	1.7367	2.2886	2.8230	3.1564	3.8030
23	1.6392	2.1600	2.6645	2.9791	3.5895	1.7317	2.2820	2.8149	3.1472	3.7920
24	1.6368	2.1569	2.6606	2.9747	3.5842	1.7270	2.2760	2.8073	3.1387	3.7818
25	1.6345	2.1539	2.6569	2.9706	3.5792	1.7227	2.2700	2.8002	3.1308	3.7722
30	1.6251	2.1415	2.6416	2.9535	3.5587	1.7045	2.2461	2.7707	3.0978	3.7325
35	1.6180	2.1321	2.6300	2.9405	3.5430	1.6907	2.2279	2.7482	3.0727	3.7022
40	1.6123	3.1246	2.6208	2.9302	3.5305	1.6798	2.2135	2.7304	3.0528	3.6783
45	1.6076	2.1185	2.6132	2.9218	3.5204	1.6708	2.2017	2.7159	3.0365	3.6587
50	1.6038	2.1133	2.6069	2.9147	3.5118	1.6633	2.1918	2.7037	3.0230	3.6423

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE MAXWELL DENSITY

P / n	$\gamma = .75$: .50	.75	.90	.95	.99	$\gamma = .90$: .50	.75	.90	.95	.99
55	1.6004	2.1090	2.6015	2.9086	3.5046	1.6570	2.1834	2.6934	3.0114	3.6284
60	1.5976	2.1052	2.5968	2.9034	3.4983	1.6514	2.1762	2.6844	3.0014	3.6163
65	1.5950	2.1018	2.5927	2.8988	3.4928	1.6466	2.1698	2.6766	2.9926	3.6057
70	1.5928	2.0989	2.5891	2.8948	3.4879	1.6424	2.1642	2.6696	2.9848	3.5964
75	1.5908	2.0963	2.5859	2.8912	3.4835	1.6386	2.1592	2.6634	2.9779	3.5880
80	1.5890	2.0939	2.5830	2.8879	3.4796	1.6351	2.1546	2.6578	2.9717	3.5805
85	1.5874	2.0918	2.5803	2.8850	3.4760	1.6320	2.1506	2.6528	2.9660	3.5737
90	1.5859	2.0898	2.5779	2.8823	3.4728	1.6292	2.1468	2.6482	2.9609	3.5675
95	1.5846	2.0880	2.5757	2.8798	3.4698	1.6266	2.1434	2.6440	2.9562	3.5618
100	1.5833	2.0864	2.5737	2.8775	3.4671	1.6242	2.1402	2.6401	2.9518	3.5566
110	1.5811	2.0835	2.5700	2.8735	3.4622	1.6199	2.1346	2.6332	2.9441	3.5472
120	1.5792	2.0809	2.5669	2.8700	3.4580	1.6162	2.1298	2.6271	2.9373	3.5391
130	1.5774	2.0787	2.5641	2.8669	3.4542	1.6130	2.1255	2.6219	2.9314	3.5320
140	1.5760	2.0767	2.5617	2.8641	3.4510	1.6101	2.1217	2.6172	2.9262	3.5257
150	1.5746	2.0749	2.5595	2.8617	3.4480	1.6075	2.1183	2.6130	2.9215	3.5201
160	1.5734	2.0733	2.5575	2.8595	3.4453	1.6052	2.1152	2.6092	2.9173	3.5150
170	1.5723	2.0719	2.5557	2.8575	3.4429	1.6031	2.1125	2.6058	2.9135	3.5104
180	1.5713	2.0705	2.5541	2.8556	3.4407	1.6012	2.1099	2.6027	2.9100	3.5062
190	1.5704	2.0693	2.5526	2.8540	3.4387	1.5994	2.1076	2.5998	2.9068	3.5023
200	1.5695	2.0682	2.5512	2.8524	3.4368	1.5978	2.1055	2.5972	2.9038	3.4988
250	1.5661	2.0636	2.5456	2.8462	3.4293	1.5912	2.0968	2.5865	2.8919	3.4844
300	1.5635	2.0603	2.5415	2.8416	3.4238	1.5864	2.0905	2.5787	2.8832	3.4739
400	1.5600	2.0557	2.5358	2.8352	3.4161	1.5798	2.0817	2.5679	2.8710	3.4593
500	1.5577	2.0526	2.5320	2.8309	3.4109	1.5752	2.0757	2.5605	2.8628	3.4494
600	1.5560	2.0503	2.5291	2.8277	3.4071	1.5719	2.0714	2.5551	2.8568	3.4421
700	1.5546	2.0485	2.5269	2.8251	3.4041	1.5694	2.0680	2.5510	2.8522	3.4365
800	1.5535	2.0471	2.5252	2.8233	3.4018	1.5673	2.0653	2.5476	2.8484	3.4320
900	1.5526	2.0459	2.5237	2.8217	3.3998	1.5656	2.0630	2.5448	2.8453	3.4282
1000	1.5518	2.0449	2.5225	2.8203	3.3981	1.5642	2.0611	2.5425	2.8427	3.4251
∞	1.5382	2.0269	2.5003	2.7955	3.3682	1.5382	2.0269	2.5003	2.7955	3.3682

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE MAXWELL DENSITY

	γ = .95					γ = .99				
P / n	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
2	2.9463	3.8825	4.7892	5.3547	6.4517	4.0344	5.3163	6.5579	7.3322	8.8344
3	2.5305	3.3346	4.1134	4.5990	5.5413	3.1934	4.2081	5.1908	5.8037	6.9928
4	2.3309	3.0715	3.7888	4.2361	5.1040	2.8199	3.7158	4.5836	5.1248	6.1748
5	2.2108	2.9133	3.5937	4.0180	4.8412	2.6050	3.4328	4.2344	4.7344	5.7044
6	2.1296	2.8062	3.4616	3.8703	4.6633	2.4[illegible]48	3.2467	4.0050	4.4778	5.3952
7	2.0704	2.7282	3.3654	3.7627	4.5336	2.3630	3.1139	3.8411	4.2946	5.1745
8	2.0250	2.6684	3.2915	3.6802	4.4342	2.2870	3.0137	3.7175	4.1564	5.0080
9	1.9888	2.6207	3.2327	3.6144	4.3549	2.2271	2.9347	3.6201	4.0475	4.8767
10	1.9591	2.5816	3.1845	3.5605	4.2900	2.1787	2.8709	3.5414	3.9596	4.7708
11	1.9344	2.5490	3.1443	3.5156	4.2358	2.1386	2.8180	3.4762	3.8866	4.6829
12	1.9132	2.5212	3.1100	3.4772	4.1895	2.1044	2.7731	3.4208	3.8246	4.6082
13	1.8950	2.4971	3.0803	3.4440	4.1496	2.0752	2.7346	3.3732	3.7715	4.5442
14	1.8790	2.4761	3.0544	3.4150	4.1146	2.0498	2.7011	3.3319	3.7253	4.4885
15	1.8649	2.4575	3.0314	3.3894	4.0838	2.0275	2.6717	3.2956	3.6847	4.4397
16	1.8523	2.4409	3.0110	3.3665	4.0562	2.0076	2.6455	3.2634	3.6487	4.3962
17	1.8410	2.4260	2.9926	3.3459	4.0314	1.9898	2.6221	3.2344	3.6163	4.3572
18	1.8308	2.4125	2.9760	3.3274	4.0090	1.9738	2.6009	3.2084	3.5872	4.3221
19	1.8215	2.4003	2.9609	3.3105	3.9887	1.9592	2.5818	3.1847	3.5607	4.2903
20	1.8130	2.3891	2.9470	3.2950	3.9700	1.9460	2.5643	3.1632	3.5367	4.2613
21	1.8052	2.3788	2.9343	3.2808	3.9529	1.9339	2.5484	3.1435	3.5147	4.2348
22	1.7980	2.3692	2.9225	3.2676	3.9371	1.9227	2.5336	3.1253	3.4944	4.2103
23	1.7913	2.3604	2.9117	3.2554	3.9224	1.9123	2.5199	3.1084	3.4754	4.1875
24	1.7850	2.3522	2.9015	3.2441	3.9088	1.9028	2.5073	3.0929	3.4581	4.1666
25	1.7792	2.3445	2.8921	3.2336	3.8960	1.8938	2.4956	3.0784	3.4418	4.1470
30	1.7551	2.3128	2.8529	3.1898	3.8433	1.8569	2.4469	3.0183	3.3747	4.0661
35	1.7368	2.2887	2.8232	3.1565	3.8032	1.8292	2.4104	2.9733	3.3244	4.0054
40	1.7224	2.2696	2.7997	3.1304	3.7716	1.8073	2.3816	2.9378	3.2846	3.9576
45	1.7106	2.2541	2.7805	3.1088	3.7458	1.7896	2.3582	2.9089	3.2524	3.9187
50	1.7008	2.2412	2.7646	3.0910	3.7242	1.7748	2.3388	2.8850	3.2256	3.8865

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE MAXWELL DENSITY

	γ = .95					γ = .99				
n \ P	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
55	1.6924	2.2301	2.7510	3.0758	3.7059	1.7623	2.3223	2.8646	3.2028	3.8590
60	1.6852	2.2206	2.7392	3.0626	3.6901	1.7515	2.3080	2.8470	3.1832	3.8354
65	1.6788	2.2122	2.7289	3.0511	3.6762	1.7421	2.2956	2.8318	3.1661	3.8148
70	1.6732	2.2049	2.7198	3.0410	3.6640	1.7338	2.2848	2.8183	3.1511	3.7967
75	1.6682	2.1983	2.7117	3.0319	3.6531	1.7264	2.2750	2.8063	3.1376	3.7804
80	1.6638	2.1924	2.7044	3.0237	3.6432	1.7198	2.2662	2.7955	3.1255	3.7659
85	1.6597	2.1870	2.6978	3.0164	3.6343	1.7137	2.2583	2.7857	3.1146	3.7527
90	1.6560	2.1822	2.6918	3.0096	3.6262	1.7083	2.2511	2.7768	3.1046	3.7407
95	1.6526	2.1777	2.6863	3.0034	3.6188	1.7033	2.2445	2.7687	3.0956	3.7298
100	1.6495	2.1736	2.6812	2.9978	3.6120	1.6987	2.2384	2.7612	3.0872	3.7198
110	1.6439	2.1663	2.6722	2.9877	3.5998	1.6906	2.2277	2.7480	3.0724	3.7019
120	1.6391	2.1599	2.6643	2.9789	3.5892	1.6835	2.2184	2.7365	3.0596	3.6865
130	1.6349	2.1543	2.6475	2.9712	3.5800	1.6773	2.2102	2.7264	3.0483	3.6729
140	1.6311	2.1494	2.6514	2.9644	3.5718	1.6718	2.2030	2.7175	3.0384	3.6609
150	1.6278	2.1450	2.6459	2.9584	3.5644	1.6670	2.1966	2.7096	3.0295	3.6502
160	1.6248	2.1410	2.6410	2.9529	3.5578	1.6625	2.1908	2.7024	3.0215	3.6406
170	1.6220	2.1374	2.6366	2.9479	3.5518	1.6586	2.1855	2.6960	3.0143	3.6313
180	1.6195	2.1341	2.6325	2.9434	3.5464	1.6549	2.1808	2.6901	3.0077	3.6239
190	1.6172	2.1311	2.6288	2.9392	3.5414	1.6516	2.1764	2.6846	3.0016	3.6166
200	1.6151	2.1283	2.6254	2.9354	3.5367	1.6485	2.1723	2.6796	2.9960	3.6098
250	1.6066	2.1171	2.6116	2.9199	3.5181	1.6362	2.1560	2.6596	2.9736	3.5828
300	1.6004	2.1089	2.6014	2.9086	3.5045	1.6272	2.1442	2.6449	2.9572	3.5631
400	1.5918	2.0975	2.5874	2.8929	3.4856	1.6147	2.1277	2.6246	2.9345	3.5358
500	1.5859	2.0898	2.5779	2.8822	3.4728	1.6063	2.1166	2.6109	2.9192	3.5173
600	1.5816	2.0842	2.5709	2.8744	3.4634	1.6001	2.1085	2.6009	2.9080	3.5038
700	1.5783	2.0798	2.5655	2.8684	3.4561	1.5953	2.1022	2.5932	2.8994	3.4934
800	1.5756	2.0763	2.5612	2.8636	3.4503	1.5915	2.0972	2.5870	2.8924	3.4850
900	1.5734	2.0734	2.5576	2.8596	3.4455	1.5884	2.0931	2.5819	2.8868	3.4782
1000	1.5716	2.0710	2.5546	2.8562	3.4414	1.5857	2.0896	2.5776	2.8819	3.4724
∞	1.5382	2.0269	2.5003	2.7955	3.3682	1.5382	2.0269	2.5003	2.7955	3.3682

From Table 1, it is seen that one is only 46% confident that a sphere of radius $\hat{SEP}$ = 114.15 will include at least 50% of the future bursts from this weapon. To increase the confidence, or probability to 95% as specified, one refers to Table 2 under $P = .50$, $\gamma = .95$, and $n = 8$ to find the tolerance limit factor k (.50, .95, 8) = 2.0250 (vice 1.5382). This is then multiplied times $\hat{\sigma}$ to obtain $U(.50, .95) = k(.50, .95, 8) \hat{\sigma} = (2.0250)(74.21) = 150.28$. Hence, a sphere of radius 150.28 feet about the target center will include at least 50% of the future bursts from this weapon under similar conditions with 95% confidence. Should the experimenter want to increase the confidence to 99%, he would refer to Table 2 under $P = .50$, $\gamma = .99$, $n = 8$ to find k(.50, .99, 8) = 2.2870. This, when multiplied times $\hat{\sigma}$, provides U(.50, .99) = 169.72 feet. Finally, should one want the radius of a sphere which would include 95% of the bursts from this weapon (vice 50%) under similar conditions with 95% confidence, he simply refers to Table 2 to find k(.95, .95, 8) = 3.6802. This is then multiplied times $\hat{\sigma}$ to obtain U(.95, .95) = 273.11 feet.

CONCLUSIONS

The tolerance limit factor, $k(P,\gamma,n)$, derived in this report can also be expressed as a function of chi-square percentage points. Referring to equation (11), it can be shown that $3n\hat{\sigma}^2/\sigma^2$ is distributed according to the chi-square distribution with 3n degrees of freedom. Hence, one could write equation (11) as

$$\mathrm{Prob}\left\{\frac{3n\hat{\sigma}^2}{\sigma^2} \geq \frac{3nF^{-1}(P)}{k^2}\right\} = \gamma \tag{16}$$

where, as before, the arguments of k have been deleted. In order for equation (16) to be satisfied, one must have $3nF^{-1}(P)/k^2 = \chi^2_{3n,1-\gamma}$ or

$$k = \left\{\frac{3nF^{-1}(P)}{\chi^2_{3n,1-\gamma}}\right\}^{\frac{1}{2}} \tag{17}$$

where $\chi^2_{3n,1-\gamma}$ is the $100(1-\gamma)$ percentage point of a chi-square density with 3n degrees of freedom. Utilizing this notation, $F^{-1}(P)$ can be expressed as $\chi^2_{3,P}$ so that $k(P,\gamma,n)$ can be written as

$$k = \left\{\frac{3n\chi^2_{3,P}}{\chi^2_{3n,1-\gamma}}\right\}^{\frac{1}{2}} . \tag{18}$$

Hence, should one need a value of k not tabulated in this report, it can be obtained by referring to a table of chi-square percentage points and computing k as in (18) above.

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