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TURBULENT DIFFUSION IN A STRATIFIED FLUID WITH APPLICATION TO THE OCEAN

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13. ABSTRACT

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Abstract

The turbulent diffusion of a passive scalar in a stratified fluid is studied using a simplified model for the calculations. A semi-empirical formula for the diffusion coefficient is obtained as a function of turbulent energy dissipation rate and scale of the diffusing patch. The formula is demonstrated to give values which are consistent with known experimental results. It is suggested that the diffusion coefficient at intermediate depths is an order of magnitude smaller than that observed in near surface waters. Using these values of the diffusion coefficients, calculations show that the mean concentration for a patch initially 100 meters across decreases by a factor of 10 in 2 hours near the surface, but in 8 hours at intermediate depths.

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1.0 Introduction

The turbulent diffusion of a passive scalar in a stratified fluid was studied using three models for turbulent diffusivity. The calculations are approximate in the sense that only diffusion in one horizontal direction is considered and that the effects of the turbulence are "modeled". However, the "model" diffusion equations are solved exactly, in some cases, or numerically with high accuracy in all other cases.

It is assumed that the quantity whose diffusion is being studied is a passive scalar. That is, the quantity undergoing diffusion can be, like temperature or concentration of a substance, specified by a scalar function of space and time and the dynamics of the flow are totally unaffected by the variation in concentration of the diffusing quantity. It is further assumed that, whatever the diffusing substance is, it does not undergo any chemical reaction with significant release or absorption of heat.

Finally it will be assumed that the fluid is stratified. In many areas of the ocean there is an appreciable variation of density with depth outside of a well mixed surface layer. It has been known for many years (see Sverdrup, et al 1942) that the presence of a stable stratification reduces the turbulent intensity and inhibits vertical transport of both scalar and vector properties. Okubo and Pritchard (1969) give measured values of the horizontal and vertical diffusion coefficients for the ocean. The horizontal diffusion coefficients are in the range of 10^3 to $5 \times 10^3\ cm^2/sec$ + while the vertical diffusion coefficients

+ Ozmidov (1959) quotes values ranging from 10^4 to $10^{12}\ cm^2/sec$ for the coefficient of horizontal diffusivity.

fall in the range 1 to 100 cm^2/sec. Okubo and Pritchard recommend, for rough estimates, that the coefficient of vertical diffusivity be taken as:

in the upper mixed layer	10 - 100 cm^2/sec
in a significant thermocline	0.01 - 1.0 cm^2/sec
in the deep layer	0.1 - 10.0 cm^2/sec
near bottom	1 - 10 cm^2/sec.

It is clear that, excluding the upper mixed layer, whichever of these values is taken for the coefficient of vertical diffusivity, the ratio of the coefficient of horizontal diffusivity to the coefficient of vertical diffusivity is greater than 100. It therefore seems reasonable to neglect the vertical diffusion and to consider only diffusion in the horizontal plane.

The source of the scalar is assumed to be a self-propelled body traveling through the ocean. The scalar is assumed to be released into the wake. The wake grows by entrainment and eventually collapses (see the model of Ko, 1971). If the scalar is uniformly distributed throughout the wake just behind the body, the model of the growth and collapse of the wake describes the spread of the scalar. That is, the scalar is uniformly (on average) distributed within the wake and the calculation of the wake boundary as a function of time gives the mean distribution of the scalar as a function of time. The more difficult question of the diffusion of the scalar from, say, a point source within the wake will be considered in a following report.

Ko's model shows that the turbulent intensity decreases rapidly with time after the beginning of collpase. When the turbulent intensity within the wake drops appreciably below the ambient turbulent intensity of the ocean, Ko's model can no longer describe the diffusion process and the turbulence of the ocean controls the diffusion process.

In the calculation reported here, the "equivalence principle" is used, i.e., the model is that of a "slice of the ocean" in a vertical plane perpendicular to the axis of the wake. Since, as discussed above, vertical diffusion can be neglected compared to horizontal diffusion, the problem can be reduced in this approximation to that of turbulent diffusion in one space dimension and time.

In fact, the problem is really a function of both spatial coordinates in the horizontal plane and of time. Within the approximation used in this report, for example, the meander of the wake due to large eddies cannot be described. This is a serious limitation. A possible experiment would involve sampling the wake along some path. If the sample were taken along a path parallel to the track of the body, the meanders of the wake could cause large variations in the observed concentration. The calculations reported here are of the average, over an ensemble, of the concentration. While the variation of the mean concentration as a function of time is important, it is not the whole story. The variation of the concentration with time in the horizontal plane is now being investigated and will be reported at a later date.

2.0 Formulation of the Problem

Let x be the spatial coordinate parallel to the free surface and perpendicular to the path of the self-propelled body, t be the time and S be the mean concentration of some passive scalar. The concentration $S(x,t)$ is then governed by a diffusion equation

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial S}{\partial x} \right) . \qquad (1)$$

It has been assumed that mean velocity of the fluid is zero. K is the effective diffusion coefficient. If the fluid is "at rest", i.e., there is no turbulence, then K is the molecular diffusivity, but if, as is usually the case, there is turbulence, then K is an eddy diffusivity.+

It is assumed that the concentration is uniform in a region of width l_0 at some initial time which is taken to be $t = 0$. As discussed in the introduction, this is some arbitrary time after the wake has collapsed and the intensity of the wake turbulence has dropped below the ambient turbulence intensity. Thus the initial condition is that

$$S = S_0 \ ; \ t = 0 \ , \ -l_0/2 \le x \le l_0/2 \ . \qquad (2)$$

+ See Appendix A for a detailed discussion of the derivation of equation (1) and the approximations used.

Without any loss of generality S_0 can be taken equal to 1.

Equation (1) can be put into dimensionless form by defining appropriate length and time scales. Let the length scale be l_0 and let $\mathcal{K}_0$ be a reference diffusivity (dimension (length)^2/time). The time scale, t_0, is then

$$t_0 = l_0^2/\mathcal{K}_0 . \qquad (3)$$

Define the dimensionless spatial coordinate

$$\xi = x/l_0 , \qquad (4)$$

and the dimensionless time coordinate

$$\eta = t/t_0 . \qquad (5)$$

Under this transformation, equation (1) becomes

$$\frac{\partial S}{\partial \eta} = \frac{\partial}{\partial \xi}\left(\mu \frac{\partial S}{\partial \xi}\right), \qquad (6)$$

where the dimensionless diffusion coefficient

$$\mu = \mathcal{K}/\mathcal{K}_0 . \qquad (7)$$

The initial condition is now,

$$S = 1 \; ; \; \eta = 0, \; -1/2 \le \xi \le 1/2. \qquad (8)$$

The problem is uniquely specified when $\mu(\xi, \eta)$ is given. Since a general solution to the turbulence problem is not available, μ must be approximated or, better, modeled. A common approximation is to use a constant value for the eddy diffusivity. This is, at best, a very rough approximation and, of course, is crucially dependent on the choice of a numerical value of μ. The value of μ not only varies from one location to another in the ocean but also varies, at a given location, with the scale of the diffusing patch. Because the constant eddy diffusivity model is so widely used, and because other, more appropriate diffusion models can sometimes be reduced, under a coordinate transformation, to an equation with a constant eddy diffusivity, this model will be treated first.

The variation of μ (or K) with patch size can be understood in terms of the variation of energy in the spectrum of turbulent eddies. Consider a patch of size λ. Turbulent eddies which are much larger than λ simply convect the patch without appreciably distorting or spreading it. Turbulent eddies which are much smaller than λ stir up the patch and provide the fine scale mixing. Of course the small eddies do provide enhanced mixing at the edges of the patch. However, it is the eddies with a scale approximately equal to λ that provide the major mixing. These eddies distort the patch and greatly increase the surface area of the patch and thus enhance the mixing due to the small eddies at the edges. Clearly the most important quantity is the rate of distortion of patch by the eddies of scale ℓ. But, the turbulent energy, $E(\vec{k})$, increases with decreasing $|\vec{k}|$, i.e., with increasing scale, at least up to the maximum scale of the turbulence. Therefore, the larger the patch, the more energy there is in the turbulent eddies of the same scale which distort the patch. Of course there is an upper limit in that the turbulent energy spectrum does not increase indefinitely with decreasing $|\vec{k}|$; there is a peak in the spectrum at the scales at which the turbulence is being generated. This is, however, unimportant in this problem. The peak in the spectrum lies at scales greater than tens of kilometers and the patches to be studied here have sizes in range 1 meter to 10 kilometers.

Richardson (1926) studied the turbulent dispersion of particles in the atmosphere and showed experimentally that

$$K = \tilde{\varepsilon}\,\ell^{4/3}, \qquad (9)$$

where $\tilde{\varepsilon}$ is a constant (dimensions $cm^{2/3}$/sec when ℓ is in cm) and ℓ is the separation of the particles. A number of other experiments have been performed on water surfaces, Richardson & Stommel (1948), Stommel & Marin (1949), Ozmidov (1959) and Gray & Pochapsky (1964), and have also shown agreement with equation (9). In a forthcoming report, Kuo (1972) will discuss a theoretical justification of the "4/3" law. In view of this experimental evidence and theoretical justification, it will be assumed that the best model of the turbulent diffusion coefficient is the "4/3" law (equation (9)).

The remaining conceptual problems are (1) what is the appropriate scale, i.e., what is λ and (2) what is the value of $\tilde{\epsilon}$? These questions will be discussed in detail below. First, however, a number of eddy diffusivity models will be studied.

3.0 Diffusivity Models

In this section, three models for the eddy diffusivity are examined, namely, a constant, a variable, and an integral scale value. The resulting mean concentration is calculated for each of these models in terms of a dimensionless time variable.

3.1 Constant Eddy Diffusivity Model

If the eddy diffusivity is constant,

$$\mathcal{K} = \mathcal{K}_0 \quad \text{and} \quad u = 1. \tag{10}$$

Equation (6) is then

$$\frac{\partial S}{\partial \eta} = \frac{\partial^2 S}{\partial \xi^2} , \tag{11}$$

with the initial condition given by (8).

The solution is easily found to be

$$S(\xi,\eta) = \frac{1}{2}\left\{ \operatorname{erf}\left[\frac{\xi + \frac{1}{2}}{\sqrt{4\eta}}\right] - \operatorname{erf}\left[\frac{\xi - \frac{1}{2}}{\sqrt{4\eta}}\right]\right\} , \tag{12}$$

with $\operatorname{erf}(z)$ defined, as usual,

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds . \tag{13}$$

The three quantities of interest are (1) the spatial distribution of the scalar at a fixed time (given by equation (12)); (2) the concentration at the center of the patch as a function of time

$$S(0,\eta) = \operatorname{erf}\left[\frac{1}{4\sqrt{\eta}}\right], \qquad (14)$$

and, (3) the "size" or "scale" of the patch.

The scale of the diffusing patch cannot be uniquely defined for $t \neq 0$, because $S(\xi,\eta) \neq 0$ for all ξ if $\eta \neq 0$. A reasonable, but arbitrary, definition of ℓ, the patch scale is

$$S(\tfrac{1}{2}\ell,\eta) = a^{*} S(0,\eta), \qquad (15)$$

where $0 < a^{*} < 1$. That is, $\ell/2$ is defined as the distance from the center of the patch at which the concentration has dropped to an arbitrary but fixed fraction, a^{*}, of the concentration at the center. For given a^{*}, equation (15) can be solved numerically for ℓ as a function of η.

Another possibility is to define the scale of the patch in terms of the variance of the distribution. Defining the variance, as usual, by

$$\sigma^{2}(\eta) \equiv \int_{-\infty}^{\infty} \xi^{2} S(\xi,\eta)\, d\xi, \qquad (16)$$

it is found that

$$\sigma^{2} = \frac{1}{12} + 2\eta. \qquad (17)$$

Then, if ℓ^2 is proportional to σ^{-2},

$$\ell/\ell_0 = \sqrt{\sigma^2/\sigma_0^2} = \sqrt{1+24\eta}, \qquad (18)$$

where the zero subscript refers to values at $t = 0$ ($\eta = 0$).

Equation (15), using equations (12) and (14), has been solved numerically for $a^* = 0.1$ and 0.25. The results, curves of ℓ/ℓ_0 as a function of t/t_0 are plotted in Figure (1), along with values of the scale defined in terms of the variance, i.e., ℓ/ℓ_0 from equation (18).

It can be seen from the results plotted in Figure (1) that there is little difference between the size of the patch as given by the different definitions of ℓ/ℓ_0. As expected, the largest size is obtained by defining ℓ as the 1/10 concentration point. The scale sizes defined by the 1/4 concentration point and the variance agree within about 10% and are about 25% smaller than the scale defined by the 1/10 concentration point. These scales are the same to within the accuracy of the approximations which are imbedded in the calculations. In particular, there is uncertainty in the choice of an eddy diffusion model and error in the use of an eddy diffusion to represent the turbulence in the first instance. Finally, the scale, as calculated here is an ensemble average and in any single realization the actual scale will differ from the ensemble average. In view of these uncertainties, it seems reasonable to use the variance definition of the scale as a reasonable approximation. The mean concentration is then just the reciprocal of ℓ/ℓ_0 i.e.

$$\langle S \rangle = \left[1+24(t/t_0)\right]^{-1/2}. \qquad (19)$$

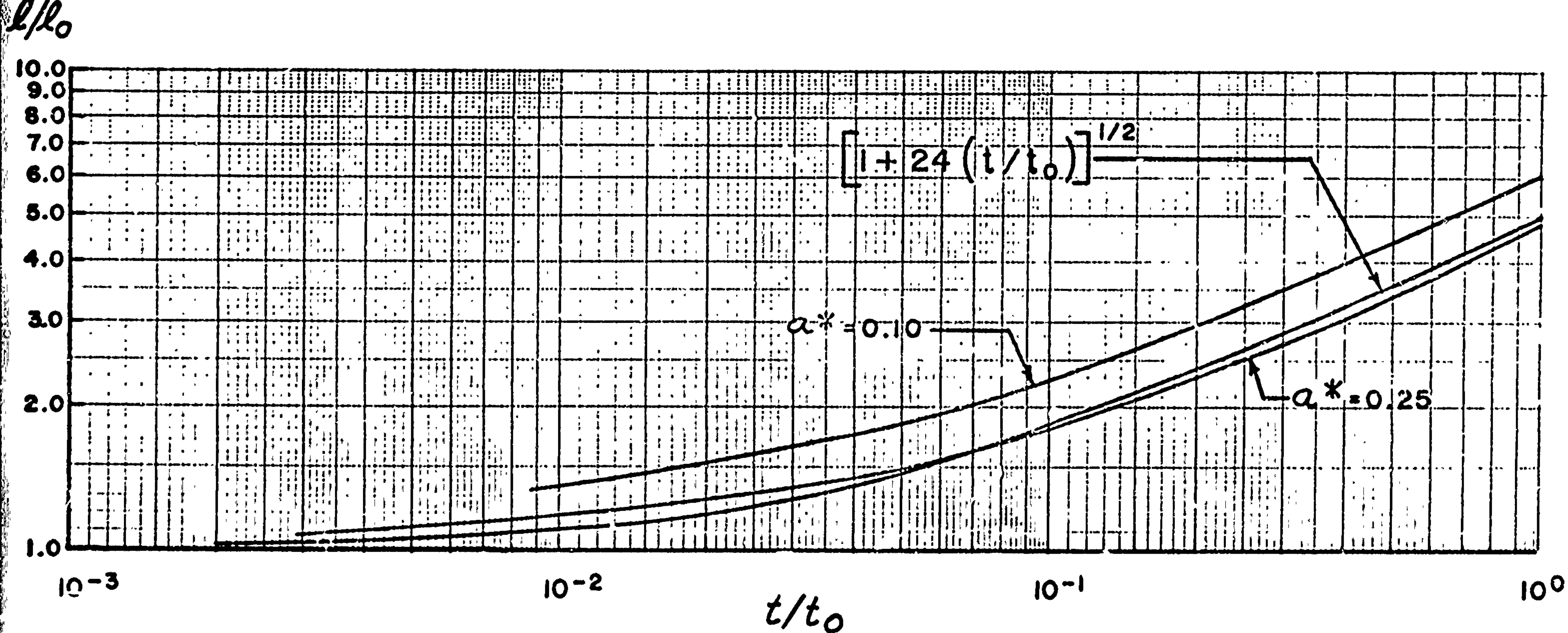


Figure 1. Variation of the dimensionless patch size (ℓ/ℓ_0) as a function of the dimensionless time (t/t_0). It has been assumed that the eddy diffusivity is a constant ($\mu = 1$). The different curves correspond to the different definitions of the patch size discussed in the text.

3.2 Variable Eddy Diffusivity Model

One possible choice for the scale to be used in the "4/3" law is to take

$$\ell = \chi . \tag{20}$$

This model is based on the idea that the scalar is diffusing away from the center of the patch and therefore the appropriate scale is the distance from the center of the patch. With this approximation

$$K = \tilde{\epsilon}\, \chi^{4/3} , \tag{21}$$

and

$$K_0 = \tilde{\epsilon}\, \ell_0^{4/3} . \tag{22}$$

Therefore

$$\mu = (\chi/\ell_0)^{4/3} = \xi^{4/3} , \tag{23}$$

and equation (6) becomes

$$\frac{\partial S}{\partial \eta} = \frac{\partial}{\partial \xi}\left(\xi^{4/3} \frac{\partial S}{\partial \xi} \right) . \tag{24}$$

It can be shown using elementary techniques, see Appendix B, that the solution to equation (24) which satisfies the initial condition (8), is

$$S(\xi,\eta) = \frac{1}{2}\left\{ \operatorname{erf}(z_2) - \operatorname{erf}(z_1) - \frac{2}{\sqrt{\pi}}\left[z_2 e^{-z_2^2} - z_1 e^{-z_1^2} \right] \right\}, \quad (25)$$

with

$$z_1 = 3\left(\xi - \tfrac{1}{2}\right)^{1/3} / \sqrt{4\eta} \quad , \quad (26)$$

$$z_2 = 3\left(\xi + \tfrac{1}{2}\right)^{1/3} / \sqrt{4\eta} \quad . \quad (27)$$

The concentration at the center of the patch is,

$$S(0,\eta) = \operatorname{erf}(z) - \frac{2}{\sqrt{\pi}} z e^{-z^2} , \quad (28)$$

with

$$z = 3 / \left(2^{4/3} \eta^{1/2}\right) . \quad (29)$$

If the scale, or size, of the patch is determined from

$$S\left(\tfrac{1}{2}\ell, \eta\right) = a^* S(0, \eta), \qquad (30)$$

then equations (25) through (30) can be solved numerically for ℓ/ℓ_0 as a function of $\eta\,(t/t_0)$ for any particular choice of a^*. If, however, the scale is defined, as before, in terms of the variance, it can be shown (Appendix B) that

$$\sigma^2(\eta) = \frac{1}{12} + \left(\frac{280}{243}\right)\eta^3, \qquad (31)$$

and thus that

$$\ell/\ell_0 = \sqrt{1 + \left(\frac{3360}{243}\right)\eta^3} \; . \qquad (32)$$

The scale size, ℓ/ℓ_0, as determined from equation (30), is plotted in Figure (2) as a function of t/t_0 for $a^* = 0.1$ and 0.25 as well as the scale size defined in terms of the variance, equation (32). From Figure (2) it is seen that the scale is approximately equal to ℓ_0 until $t/t_0 \approx 0.02$ and then begins to grow rapidly.

In contrast to the constant eddy diffusivity model, the variance scale is larger than 0.10 and 0.25 concentration scales. This can be understood in terms of the eddy diffusivity model, equation (23). The rate of diffusion increases with increasing distance from the center of the patch. This tends to give slow diffusion near the center of the patch and fast diffusion near the edges.

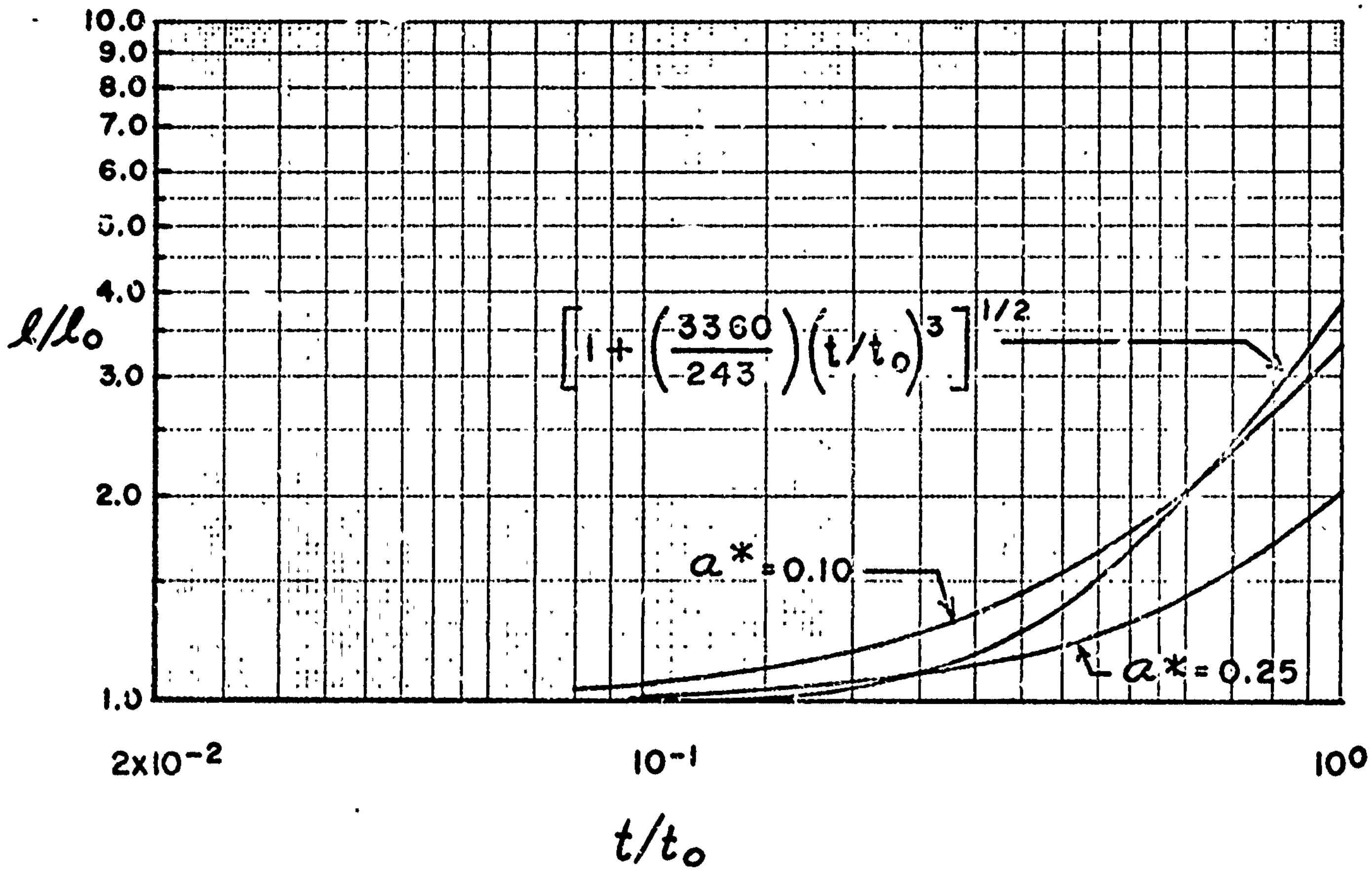


Figure 2. Variation of the dimensionless patch size (ℓ/ℓ_o) as a function of the dimensionless time (t/t_o). The "local" 4/3 law ($\mu = \xi^{4/3}$) has been used. The different curves correspond to the different definitions of the patch size discussed in the text.

This implies that there is a much sharper "edge" to the patch than in the case of a constant eddy diffusivity and that there will be a long "tongue" of material at low concentration extending out from the edge. Since the material in the "tongue" is diffusing rapidly compared to the material at the center, the "tongue" grows rapidly. Now in calculating the variance the material at large ξ is heavily weighted (by ξ^2) and thus contributes more than material near the center. Because of the sharp "edge" and long "tongue", a large difference between variance scale and constant concentration scale is to be expected.

A reasonable approximation in this case would appear to be to take the scale as the variance scale. The mean concentration is then

$$\langle S \rangle = \left[1 + \left(\frac{3360}{243}\right)\left(t/t_0\right)^3 \right]^{-1/2} . \quad (33)$$

3.3 Integral Scale Eddy Diffusivity Model

Finally, it can be argued that the appropriate scale for the diffusion is constant at any instant of time and is, in fact, the "size" of the patch. Taking

$$\mathcal{K} = \tilde{\epsilon}\, \ell^{4/3}, \tag{34}$$

and

$$\mathcal{K}_0 = \tilde{\epsilon}\, \ell_0^{\,4/3}, \tag{35}$$

$$\mu = \left(\ell / \ell_0 \right)^{4/3}, \tag{36}$$

with $\ell(\eta)$ defined by, say,

$$S(\tfrac{1}{2}\ell, \eta) = a^* S(0, \eta), \tag{37}$$

or perhaps in terms of the variance scale,

$$\ell / \ell_0 = \sqrt{\sigma^2 / \sigma_0^2}\ . \tag{38}$$

The mathematical problem is now to solve

$$\frac{\partial S}{\partial \eta} = \mu(\eta)\, \frac{\partial^2 S}{\partial \xi^2}, \tag{39}$$

$$S(\xi, 0) = 1\ , \quad -1/2 \le \xi \le 1/2. \tag{40}$$

Since μ is only a function of η, it is possible to eliminate $\mu(\eta)$ from equation (39) by a suitable coordinate transformation. Define the weighted time coordinate τ by

$$\tau = \int \mu(\eta)\, d\eta\,, \tag{41}$$

then

$$\frac{\partial}{\partial \eta} = \frac{\partial \tau}{\partial \eta}\frac{\partial}{\partial \tau} = \mu(\eta)\frac{\partial}{\partial \tau}, \tag{42}$$

and equation (39) becomes

$$\frac{\partial S}{\partial \tau} = \frac{\partial^2 S}{\partial \xi^2}\,, \tag{43}$$

with the same initial condition.

This is, of course, the diffusion equation for a constant eddy diffusivity. The solution is equation (12) with η replaced by τ. In order to interpret the results it is necessary to transform back from the τ to the η variable. (Recall that η is directly proportional to the time while τ is a weighted integral of the time.) The diffusion scale (ℓ/ℓ_0) can be calculated, whatever its definition, as a function of τ. Then the eddy diffusivity can be calculated from (ℓ/ℓ_0). Now, the eddy diffusivity must be the same in the η and τ coordinates when it is calculated at corresponding (not equal) values of η and τ. The correspondence is that determined by the transformation, equation (41). From the differential form of this transformation

$$d\tau = \mu(\eta)\, d\eta , \qquad (44)$$

η can be obtained by using the above principle. Thus

$$\mu(\tau) \equiv \mu(\eta) , \qquad (45)$$

and corresponding η and τ, so that

$$\frac{d\tau}{\mu(\tau)} = d\eta , \qquad (46)$$

or

$$\eta = \int_0^{\tau} \frac{d\tau'}{\mu(\tau')} . \qquad (47)$$

Using a constant concentration to define the scale, values of ℓ/ℓ_c as a function of τ are known from the numerical solution of equations (12), (14) and (15). The: ·ing equations (36) and (47) a numerical integration yields $\eta(\tau)$. This calculation has been carried out with a^* (the value of the concentration defining the edge of the patch) equal to 0.1 and 0.25. Since (ℓ/ℓ_0) and $S(0,\tau)$ are known as functions of τ, it is easy to tabulate (ℓ/ℓ_0) as a function of η and also $S(0,\eta)$.

On the other hand, if the scale is defined in terms of the variance, the transformation can be inverted analytically. In terms of the τ variable, the variance scale is from equation (18)

$$\ell/\ell_0 = (1+24\tau)^{1/2} . \tag{48}$$

Then, equation (36),

$$\mu(\tau) = (\ell/\ell_0)^{4/3} = (1+24\tau)^{2/3} , \tag{49}$$

so that

$$\eta = \int_0^\tau \frac{d\tau'}{\mu(\tau')} = \int_0^\tau (1+24\tau')^{-2/3} d\tau'$$

$$= \left(\frac{1}{8}\right)\left[(1+24\tau)^{1/3} - 1\right] . \tag{50}$$

Therefore, solving for τ as a function of η

$$\tau = \left(\frac{1}{24}\right)\left[(1+8\eta)^3 - 1\right] , \tag{51}$$

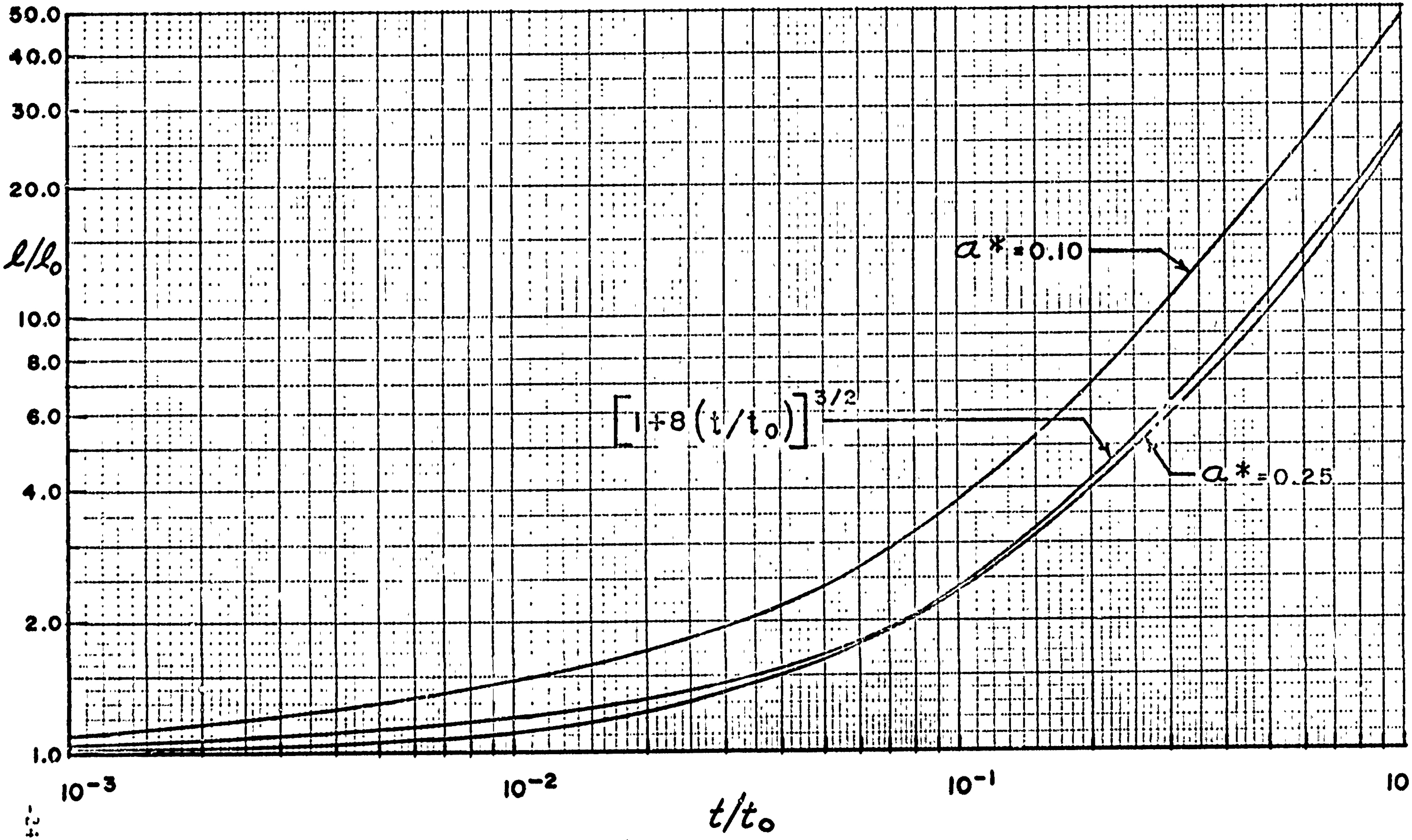
and substituting back into equation (48)

$$\ell/\ell_0 = (1+8\eta)^{3/2} . \tag{52}$$

Figure (3) is a plot of (ℓ/ℓ_0) as a function of (t/t_0) for the constant concentration scales, $a^* = 0.1$ and 0.25, and the variance scale. Again the largest scale is that obtained from the 1/10 concentration point. The variance scale and the 1/4 concentration point scale are very close, within 10%. It appears to be reasonable to use the variance definition of the scale. The mean concentration is the reciprocal of ℓ/ℓ_0 and is

$$\langle S \rangle = \left[1 + 8 (t/t_0) \right]^{-3/2} . \qquad (53)$$

Figure 3. Variation of the dimensionless patch size (ℓ/ℓ_o) as a function of the dimensionless time (t/t_o). The "global" 4/3 law ($\epsilon = [\ell/\ell_o]^{4/3}$) has been used. The different curves correspond to the different definitions of the patch size discussed in the text.



4.0 Interpretation of the Numerical Results

4.1 Theoretical Considerations

Although the constant eddy diffusivity model has often been used, it is clear, on both experimental and theoretical grounds, that the "4/3 law" provides a better model of the diffusion process. While choosing a scale proportional to the distance from the center of the patch (a local scale) may be a reasonable approximation for small times, the integral scale is probably the most reasonable scale to use in the 4/3 law. Figure (4) shows the variation of concentration at the center of the patch $S(0,\tau)$ and the mean concentration $\langle S \rangle$ with time (t/t_0) for all three models. From now on all the discussion will center on the integral scale 4/3 law, $\mu = (l/l_0)^{4/3}$.

In order to interpret these results, the initial length scale, l_0, and initial time scale t_0 must be determined. There is no problem with the length scale; l_0 should be the initial size of the diffusing patch. As for t_0,

$$t_0 \equiv l_0^2 / K_0 \; ; \qquad (54)$$

the initial eddy diffusivity

$$K_0 = \tilde{\epsilon}\, l_0^{4/3} \; , \qquad (55)$$

is required for definition. In order to pick a value of $\tilde{\epsilon}$, experimental data must be used.

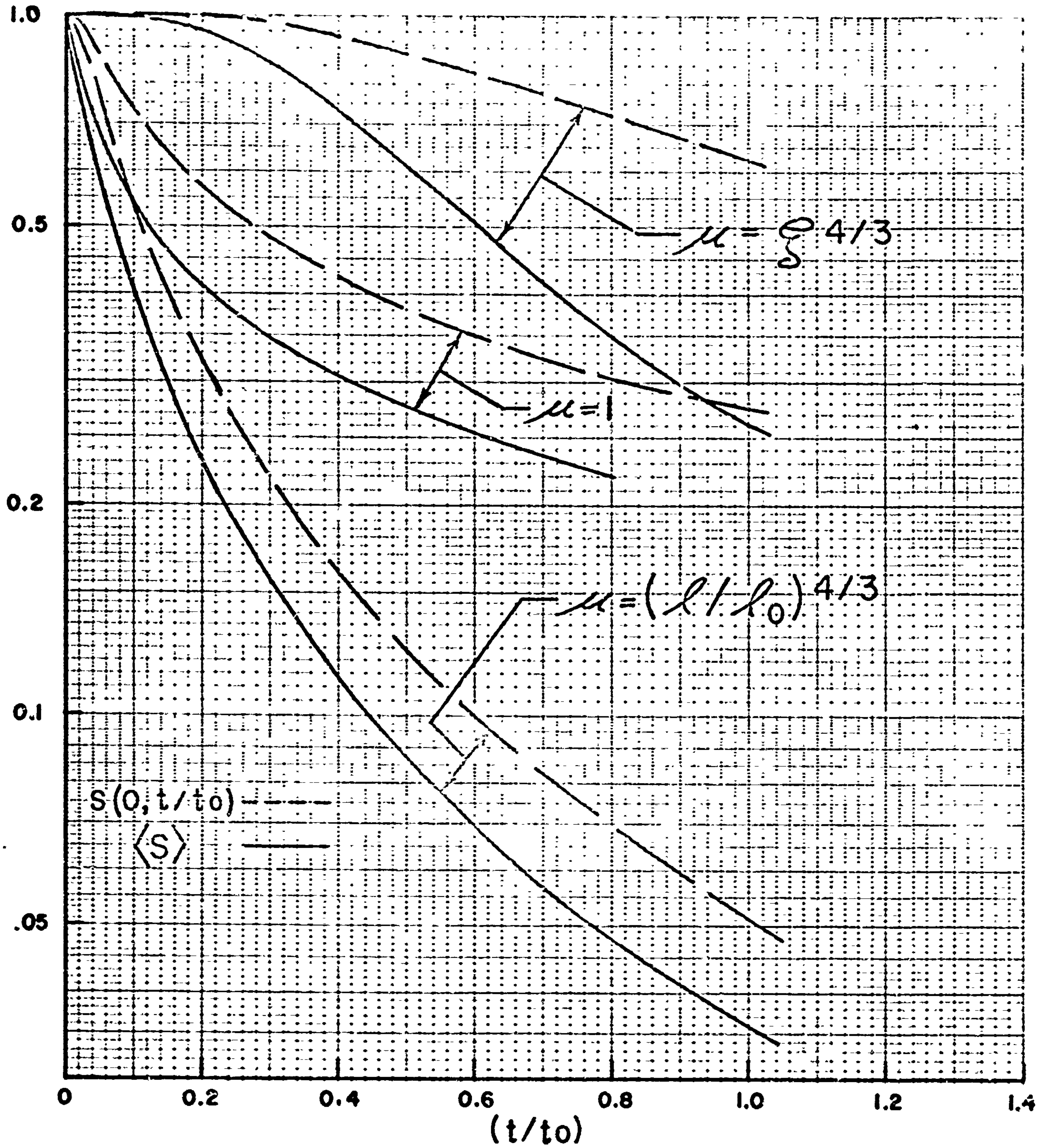


Figure 4. Variation of the concentration at the center of the patch (dashed line) and the mean concentration (solid line) as a function of the dimensionless time (t/t_0). The curves as labeled correspond to the three different diffusion laws discussed in the text.

Most experimental studies of horizontal diffusion in the ocean have been conducted in near surface waters, if not on the surface, see Ozmidov (1959) and Okubo & Pritchard (1969). Although there is considerable scatter in the data, a reasonable fit appears to be given by equation (55) with $\tilde{\varepsilon} \approx 0.01 \ cm^{2/3}/sec$ (ℓ in cm).

For the present purposes these measurements near the surface are not satisfactory. The near surface waters are disturbed by wind driven currents and surface waves, for example, and generally are not stratified. It would be desirable to have an extensive set of measurements of the diffusion coeffcients as a function of scale for various depths and Väisälä periods. Such experimental results are not available. There appears to be only a few diffusion measurements at depth. Fortunately, some additional measurements are available, which, properly interpreted, will give further information on the variation of $\mathcal{K}$ with depth (D) and Väisälä frequency (N).

Batchelor has shown (see Hinze, 1959) that in homogeneous turbulence, for intermediate times,

$$\mathcal{K} = A \, \epsilon^{1/3} \ell^{4/3} \qquad , \qquad (56)$$

i.e., that

$$\tilde{\varepsilon} = A \, \epsilon^{1/3} \qquad , \qquad (57)$$

where A is a constant and ϵ is the dissipation rate (cm^2/sec^3).

Ozmidov (1960) suggests that $\mathcal{A}$ is approximately equal to 0.1. As will be seen below, $\epsilon \approx 10^{-2}/cm^2/sec^3$ in near surface waters of the ocean with moderate wave heights. Assuming that $\tilde{\epsilon} = 10^{-2}$ and $\epsilon = 10^{-2}$, it is found that $\mathcal{A} \approx 0.05$. Throughout the remainder of this report it will be assumed that

$$\mathcal{A} = 0.05 , \qquad (58)$$

therefore

$$\mathcal{K} = 0.05\, \epsilon^{1/3} \ell^{4/3} . \qquad (59)$$

Measurements of the energy dissipation rate, ϵ, as a function of D and N thus yields information about the corresponding variation of $\mathcal{K}$.

It is desired to use equation (59) to calculate the diffusion coefficient as a function of the energy dissipation rate, ϵ, which, in turn, is a function of depth and Väisälä period. The validity of the 4/3 law has been demonstrated experimentally for diffusion in near surface waters, and the constant in equation (59) has been obtained from this data. In order to apply this relation to intermediate depths, it is necessary, at least, to show that the flows have similar character and that the application of (59) is consistent with the limited data available.

Ozmidov (1965) has suggested that a stable stratification can have the effect of suppressing turbulence (at least the vertical motions of the turbulence). It is, of course, well known that in a stratified fluid with shear, there exists a critical Richardson number, R_i^*, such that if

$$R_i \equiv -\left(\frac{g}{\rho}\right)\left(\frac{\partial \rho}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial z}\right)^2 < R_i^*, \qquad (60)$$

the flow is unstable to small disturbances and presumably an unstable disturbance will lead to transition to turbulence, (see Miles & Howard, 1964 or, for a comprehensive review of the stability of stratified flows, Drazin and Howard, 1966). Miles & Howard suggest that $R_i^* = 1/4$. If R_i is the flux Richardson number, see Lumley & Panofsky (1964), then $R_i^* =$ (eddy conductivity/ eddy diffusivity) $\approx$ 1. Experimental results can be interpreted as giving critical values, $1/4 \lesssim R_i^* \lesssim 1$.

Ozmidov argues that if

$$\epsilon \sim u^3/\ell \qquad (61)$$

where u is the turbulent intensity and ℓ is a turbulent scale, then

$$\Delta u/\Delta \ell \sim \epsilon^{1/3} \ell^{-2/3}. \qquad (62)$$

So that, if ℓ is the vertical scale of the turbulence,

$$\frac{\partial u}{\partial z} \doteq C \epsilon^{1/3} \ell^{-2/3} , \qquad (63)$$

with C a constant. The Richardson number of the turbulence is

$$R_i = N^2 \ell^{4/3} / C^2 \epsilon^{2/3} , \qquad (64)$$

after setting

$$N^2 = -\left(\frac{g}{\rho}\right)\left(\frac{\partial \rho}{\partial z}\right) . \qquad (65)$$

For the turbulence to exist, $R_i \leq R_i^*$, i.e.,

$$\left(N^2 \ell^{4/3} / C^2 \epsilon^{2/3}\right) \leq R_i^* . \qquad (66)$$

Therefore there exists a critical length scale, ℓ_c ,

$$\ell_c = \left[R_i^* C^2 \epsilon^{2/3} / N^2\right]^{3/4} , \qquad (67)$$

such that if $\ell \leq \ell_c$, the turbulence is three-dimensional and perhaps isotropic. If $\ell > \ell_c$ the stable stratification will inhibit the vertical component of the turbulent motion and motion on this and larger scales will be two-dimensional turbulence (a random field of internal waves?)

Lumley (1964) has also suggested that, in a stratified flow, there exists a buoyancy subrange as well as an equilibrium subrange and that the wave number separating these ranges

$$k_b = C_b^{3/4} N^{3/4} \epsilon^{-1/2} , \qquad (68)$$

where C_b is a constant of order unity. If then

$$C_b^{-1} = R_i^* c^2 , \qquad (69)$$

$$k_b^{-1} = \ell_b = \ell_c . \qquad (70)$$

The equilibrium subrange of the turbulence exists for wavenumbers,

$$k_b < k < k_* , \qquad (71)$$

where k_* is the Kolmogorov microscale

$$k_* = \nu^{-3/4} \epsilon^{1/4} , \qquad (72)$$

with ν the kinematic viscosity.

If $k_b = k_*$ the equilibrium subrange disappears and possible, there can exist only a random field of internal waves. For even small scale turbulence to exist, there would have to be an increase in the shear corresponding to an increase in ϵ , so that ℓ_b would become larger than ℓ_* . This general picture is in agreement with the observations of Woods & Wiley (19 2) which show that the appearance of turbulent patches is due to short internal waves in the microstructure, giving an increase in the local shear.

The critical dissipation rate for the existence of an equilibrium subrange can be estimated by equating k_b and k_* ,

$$N^{3/2}\epsilon_c^{-1/2} = \nu^{3/4}\epsilon_c^{-1/4}, \qquad (73)$$

or

$$\epsilon_c = \nu N^2, \qquad (74)$$

and for water ($\nu \approx 10^{-2}$ cm^2/sec) and $10^{-2} \lesssim N^2 \lesssim 10^{-3}$ sec^{-2}

$$\epsilon_c \approx 10^{-7} \ \mathrm{cm}^2/\mathrm{sec}^3. \qquad (75)$$

These considerations suggest an interesting experiment. Produce a turbulent field in a stratified fluid, perhaps by allowing flow past a grid. Initially ϵ will be large and an equilibrium range will exist. If ϵ is large enough and the experimental facility (tunnel) is not very large, ℓ_b may be larger than the diameter of the tunnel and a buoyancy subrange may not exist. Downstream from the grid the turbulent energy and also ϵ will have decreased; a buoyancy subrange should then exist. Further downstream, ϵ will become less than ϵ_c and, if this model is correct, the three-dimensional character of the turbulence will entirely disappear leaving a random field of 2-dimensional turbulence, possibly a random field of internal waves.

It would also be interesting to calculate, say from Ko's model, the point at which the energy dissipation rate in the wake drops below ϵ_c .

Typical values of ℓ_c ($R_i^* = C = 1$) are shown below. Near surface values of ϵ are of the order of 10^{-2} dropping to 10^{-4} at depths.

ℓ_c (cm)

ϵ (cm^2/sec^3) / N (sec^{-1})	10^{-2}	10^{-3}	10^{-4}
10^{-2}	100	32	10
10^{-3}	3200	1000	320

It is clear that a strong thermocline ($N \sim 10^{-2}$) and/or small energy dissipation rate will strongly suppress three-dimensional turbulence. Therefore, the motion at intermediate depths is likely to be two-dimensional, at least for scales larger than a few meters. However, this is also the case near the surface where the vertical extent of the eddies will be of the order of the distance to the surface. Since most ocean diffusion experiments were carried out at depths of only a few meters, the large scale motions are likely to have been two-dimensional in these experiments also. Thus there is at least no inconsistency in the character of the motions at intermediate depths and near the surface.

4.2 Experimental Results

Woods & Fosberry (1967) have made visual observations of dye motions in the thermocline fine structure. They state that λ, the minimum eddy size observed varied from about 1 mm to 10 cm, depending on the depth and stability. If λ is taken to be the Kolmogorov micro-scale then

$$\epsilon = \nu^3 / \lambda^4 , \qquad (76)$$

and corresponding to 1 mm $\lesssim \lambda \lesssim$ 10 cm,

$$10^{-10} \ cm^2/sec^3 \lesssim \epsilon \lesssim 10^{-2} \ cm^2/sec^3 .$$

Hale (1971) has observed visually the spread of the dye above, in, and below a thermocline in fresh water. He states that the turbulent motion appears to have maximum energy near the surface where it is 3-dimensional. The turbulent intensity decreases with depth; in the thermocline the motion is largely 2-dimensional and is perhaps not turbulent at all. Below the thermocline the intensity increases again and the motion becomes 3-dimensional once more. Diffusion coefficients measured near the surface were stated to be unrelated to those at depth.

Webster (1969) has made horizontal current measurements at a site on the continental slope (39° 20' N, 70° W) about 175 km north of the mean axis of the Gulf Stream. The water depth is 2600 m and measurements were made from near surface (6 m) to a depth of about 2000 m. He found that some of the spectra of kinetic energy density decreased as $f^{-5/3}$ for $f \gtrsim 0.1$ cycle/hour. Assuming that this was an inertial subrange, Webster calculated the energy dissipation rate as a function of depth and found that the ϵ thus calculated was proportional to N. Webster calculated values of ϵ 5×10^{-4} cm^2/sec^3 at D = 10 m, 10^{-4} cm^2/sec^3 at D = 100 m and 10^{-5} cm^2/sec^3 at D = 1000 m. A complete listing of Webster's calculated values of ϵ is given in Table 1.

Pochapsky (1972) measured the energy densities of vertical and relative horizontal fluid velocities with neutrally buoyant floats. The measurements, made at a number of locations, yielded spectra of the horizontal velocity having a frequency variation between f^{-2} and f^{-3}. Pochapsky argued that most of the energy in this frequency range (0.1 to 10 cph) was associated with internal waves, not turbulence.

Grant, Stewart & Moillet (1962) have reported measurements of turbulence spectra in near surface waters of a tidal channel. From each of their individual spectra they calculated energy dissipation rates. These ranged from 0.02 to 1.0 cm^2/sec^3 with an average of about 0.34 cm^2/sec^3. These results are listed in Table 1.

Stewart and Grant (1962) have reported measurements of ϵ at various depths from 1 to 15 m and with various wave heights in the 0.1 to 0.9 m range. The energy dissipation rate showed a tendency to decrease

TABLE I

Summary of reported measurements in the ocean of the energy dissipation rate

Data of Webster (1969)

D(m)	$\bar{u}$(cm/sec)	$\epsilon \times 10^5$(cm^2/sec)
6	37.2	40.50
7	48.8	48.30
50	16.1	6.84
64	16.7	7.41
88	29.3	5.32
98	28.9	8.86
104	10.9	4.68
106	10.7	10.13
120	13.4	7.18
450	9.6	2.05
492	7.3	2.60
502	5.9	1.52
511	11.9	3.49
522	5.4	2.58
930	6.7	2.28
940	5.4	1.07
950	5.7	1.23
1001	5.2	0.82
1013	7.7	1.38
1950	5.4	0.78
2002	4.2	1.06
2020	6.9	2.66
2026	3.7	2.11

TABLE I (cont)

Data of Grant, Stewart & Moillet (1962)

Measured in a tidal channel -9 independent measurements

ϵ (cm^2/sec^3)

0.610, 1.02, 0.395, 0.121, 0.235, 0.147, 0.044, 0.0187, 0.441

Data of Stewart and Grant (1962)

Depth (m)	Wave Height (m)	ϵ (cm^2/sec^3)
1.0	0.5	4.2×10^{-2}
1.5	0.2	1.5×10^{-2}
1.5	0.4	2.3
2.0	0.1	5.2×10^{-3}
2.0	0.3	2.9×10^{-2}
2.0	0.5	2.2
2.0	0.9	4.5
12.0	0.5	2.5×10^{-4}
15.0	0.1	1.1×10^{-3}

TABLE I (cont)

Data of Grant, Moillet & Vogel (1968)

Depth (m)	N^2 (sec^{-2})	ϵ (cm^2/sec^3)	γ	$\bar{\epsilon} \equiv \gamma \epsilon$ (cm^2/sec^3)
15	1.56×10^{-5}	2.5×10^{-2}	1.00	2.5×10^{-2}
27	1.56	5.2×10^{-3}	1.00	5.2×10^{-3}
43	1.56	3.0	0.77	2.3
58	5.80×10^{-4}	4.8	0.77	3.7
73	2.05	1.9	0.53	1.0
89	1.24	1.1	0.31	3.4×10^{-4}
90	1.24	1.0	0.31	3.1
90	1.24	4.8	0.31	1.5

with increasing depth and decreasing wave height. Values of ϵ ranged from 4.5×10^{-2} to $2.5 \times 10^{-4}\ cm^2/sec^3$. The values are given in Table 1.

Grant, Moillet and Vogel (1968) have measured ϵ at a number of depths in and above the thermocline. They found that ϵ decreased with increasing depth and that, while the water above the thermocline was essentially always turbulent, the turbulence was intermittent in the thermocline and tended to die out with increasing depth. Figure (5) shows the variation with depth of the density profile, the energy dissipation rate ϵ, the intermittency factor γ and the "average" energy dissipation rate $\bar{\epsilon}$ $(\bar{\epsilon} \equiv \gamma \epsilon)$ taken from the paper of Grant et al and N^2 calculated from the density profile. It can be seen that ϵ differs by at least a factor of 5 across the thermocline. Numerical values are given in Table 1.

Finally there is the measurement of the diffusion coefficient at 300 m depth by Schuert (1970). Schuert fitted his experimental results with a theoretical model of point source diffusing according to the 4/3 law and found that the measured value of $\tilde{\epsilon}$ was approximately 1/10 of the values obtained in surface experiments. Schuert found $\tilde{\epsilon} = 6.6 \times 10^{-3}\ cm^{2/3}/sec$, corresponding to $\epsilon = 2.3 \times 10^{-4}\ cm^2/sec^3$ $(A = 0.05)$ at a depth of 300 m where $N^2 \approx 4 \times 10^{-5}\ sec^{-2}$, (calculated from Figure 1 of Schuert's paper).

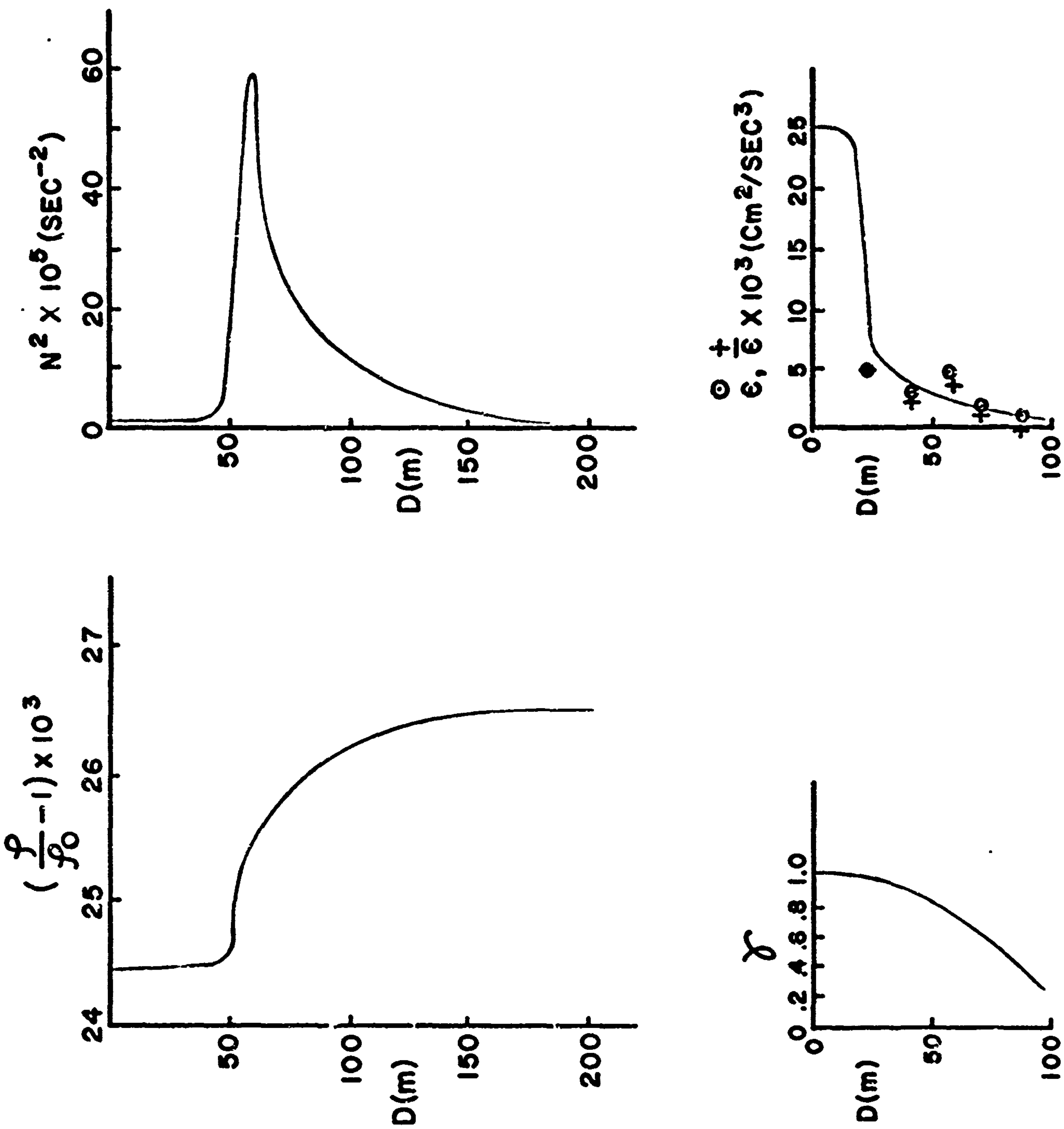


Figure 5. Example of the variation of density, ρ, square of Vaisälä frequency, N^2, intermittency, γ, energy dissipation rate, ϵ, and average energy dissipation rate $\bar{\epsilon}$ $(\bar{\epsilon} \equiv \gamma\epsilon)$. with depth. Data of Grant, Moillet and Vogel (1968).

It is to be expected that the energy dissipation rates in the tidal channel, as measured by Grant, Stewart & Moillet (1962) would be considerably larger than those typical of the open ocean. The current in the channel was several knots where the velocity in the open ocean might be expected to be about 1/4 knot. For this reason the tidal current measurements may be considered the probable upper limit to ϵ .

Near surface diffusion experiments, as well as turbulence measurements, are likely to be made under conditions of light winds and small to moderate seas. The measurements of Stewart & Grant (1962) are probably representative of "normal" conditions. A representative value of ϵ near the surface would appear to be $10^{-2}\ cm^2/sec^3$.

To summarize, values of ϵ measured at depth tend to lie in the range 10^{-3} - $10^{-4}\ cm^2/sec^3$, corresponding, using equation (59), to values of $\tilde{\epsilon}$ in the range 5×10^{-3} - $2 \times 10^{-3}\ cm^{2/3}/sec$. This is in reasonable agreement with Schuert's measured value of $\tilde{\epsilon} = 6.6 \times 10^{-3}\ cm^{2/3}/sec$ at a depth of 300 meters. These values of $\tilde{\epsilon}$ are about 10 times smaller than that observed for near surface experiments.

5.0 Summary and Discussion

The theoretical considerations and limited experimental data discussed in the previous sections suggest that:

1. For a patch scale $\ell > \ell_c$ (equation 67), the diffusion is two-dimensional.

2. The vertical diffusion coefficient

$$K_z = 0.05\, \epsilon^{1/3} \ell^{4/3} \text{ for } \ell \le \ell_c, \quad (77)$$

$$K_z = 0.05\, \epsilon^{1/3} \ell_c^{4/3} \text{ for } \ell > \ell_c. \quad (78)$$

3. The horizontal diffusion coefficient

$$K = 0.05\, \epsilon^{1/3} \ell^{4/3} \quad (79)$$

for ℓ larger than the microscale and smaller than the energy-containing eddies.

4. The energy dissipation rate ϵ, varies with depth and Väisälä frequency. Approximate values ϵ and the corresponding values of $\tilde{\epsilon}$ for different depths are

	ϵ (cm^2/sec^3)	$\tilde{\epsilon}$ ($cm^{2/3}/sec$)
Near surface	10^{-2}	10^{-2}
At moderate depths	10^{-3} - 10^{-4}	5×10^{-3} - 2×10^{-3}

Consider now a patch with an initial size, ℓ_0 = 100 m. The diffusion of this patch will be essentially 2-dimensional. The length scale is of course $\ell_0 = 10^4$ cm. The time scale is

$$t_0 = \ell_0^2 / K_0 \, , \qquad (80)$$

Assuming that the integral 4/3 law is the appropriate model, $K_0 = \tilde{\epsilon}\, \ell_0^{4/3}$ and the values of t_0 near the surface and at depth are:

	$\tilde{\epsilon}$ ($cm^{2/3}$/sec)	t_0 (sec)
near surface	10^{-2}	2×10^4
at depth	3×10^{-3}	6×10^4

Therefore the time scale is about 5.5 hours near the surface and 16.5 hours at moderate depths. The average concentration drops to about 1/10 the initial value at $t/t_0 \approx$ 0.45, i.e., after about 2 hours near the surface and 7-1/2 hours at moderate depths.

The calculations and results presented and discussed in the previous sections are based on a simple model of the diffusion process in the ocean. There are a number of important phenomena which have not been included in this model.

The calculations are of the mean (ensemble average) concentration as a function of space and time. In any particular realization of this field there will be statistical variations from the mean. The larger eddies will be nearly frozen over periods of time of the order of $t_{\bullet}$. For example, turbulent velocities will be of the order of 10 cm/sec and for eddies of the scale of 1/2 km, the time scale will be of the order of 1-1/2 hours. The frozen or nearly frozen eddies will cause meanders of the wake which cannot be predicted from this model. This model can predict the variation of the concentration off the axis of the wake but not the meanders of the wake axis. See Schuert (1970) for an example of the meandering drift of a single patch.

The model does not include the effect of geostrophic currents or current shear on the diffusion, although the current velocities (a few cm/sec) are comparable to the turbulent velocity. Nor have the effects of inertial waves which produce fluid velocities comparable to that of the turbulent flow field been included in the model" [Pochapsky (1972) has measured fluid velocities due to inertial waves in the range 0.5 to 10.0 cm/sec].

Finally and perhaps most importantly, this model does not include the intermittency of the turbulence. It is observed that in a thermocline turbulence appears intermittently, i.e., in patches. Woods & Fosberry (1967) and Hale (1971) both comment on this patchiness. The measurements of Grant, Moillet & Vogel (1968) clearly show the intermittent character of the turbulence with the intermittency factor, $\gamma \approx 0.3$ at $D = 90$ meters.

Very recently, Woods & Wiley (1972) have suggested a mechanism for mixing in the interior of all statistically stable flows. This mechanism is based on dye studies of the microstructure in the thermocline. Woods & Wiley call this mechanism billow turbulence and describe it as ".... free shear turbulence modified by a density gradient and initiated by Kelvin-Helmholtz instability".

The Woods & Fosberry mechanism can be described as follows:

> The thermocline consists of thin sheets, which have relatively large density changes across them and more uniform layers 1 - 2 meters thick. There is a velocity shear across the layers due to the geostrophic current. Internal gravity waves propagate along the sheets. It is observed that when the sheets are about 10 cm thick, short ($\lambda \sim 75$ cm), steep internal waves propagating on the sheet produce enough additional shear to make the layer dynamically unstable ($R_i < R_i^*$). The crests "roll up" and produce a sequence of turbulent billows. Neighboring billows coalesce, driven by the shear, producing a turbulent patch of the order of 3 to 5 meters in horizontal extent and approximately 1 - 2 sheet thickness (10 - 20 cm) in vertical extent. The patch grows somewhat by entrainment and the turbulence begins to decay. The interior of the patch is relatively homogeneous but there are large density gradients at the top and bottom edges of the patch. That is, there are now two very thin sheets at the top and bottom of the patch. These sheets spread by molecular diffusion until they are about 10 cm thick and then the process repeats. This mechanism produces patches of fine scaled, decaying homogeneous, isotropic turbulence. This mechanism is assumed to be operative and controlling the formation of turbulence throughout most of the ocean, i.e., outside of the near-shore areas and away from the major ocean currents.

If this mechanism does describe the generation and distribution of turbulence in the ocean, and the evidence that it is an accurate picture of, at least part, of reality is impressive, then our model of turbulent diffusion in the ocean must be drastically revised. The diffusion model would be only appropriate as an <u>average</u> over a relatively long time scale (long compared to the period of patch gestation).

A more accurate model would include large scale advection of the passive scale by internal and inertial waves and "two-dimensional" turbulence (if it exists). At a finer scale, a sparse, random distribution of homogeneous, isotropic turbulent "pancakes" would stir up the scalar locally. There must be a gap of about one decade in the energy spectrum of the fluid velocities since the shortest internal waves with appreciable energy (in this model) have a wavenumber

$$k_{max} \approx \frac{1}{75} \approx 0.01 \ \text{cm}^{-1} , \qquad (81)$$

and the largest turbulence scales have a wavenumber

$$k_{min} \approx \frac{1}{10} = 0.1 \ \text{cm}^{-1} . \qquad (82)$$

A number of approaches to this model incorporating large scale advection and localized random mixing are possible. A description of these approaches will be the subject of later reports.

APPENDIX A

The Diffusion Equation for a Passive Scalar

The discussion of diffusion given here is based upon the general discussion of this phenomena given by Landau & Lifshitz (1959) but is adapted to the case of particular interest, namely diffusion in the ocean.

Consider a mixture of two fluids. The equations of mass conservation (continuity) and momentum balance (Navier-Stokes) remain unchanged.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 . \qquad \text{(a-1)}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} =$$

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \left(\frac{1}{3}\mu + \eta\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right), \qquad \text{(a-2)}$$

where ρ is the total density,

u_i is the i^{th} component of the velocity,

p is the pressure,

μ is the 1st or shear coefficient of viscosity,

η is the 2nd or bulk coefficient of viscosity,

x_j is the j^{th} space coordinate, and

t is the time.

If the relative concentration of the two fluids in a given volume were constant, the motion would simply distort the volume and would be absolutely <u>reversible.</u> The <u>total</u> change in concentration

$$\frac{DS}{Dt} = 0, \qquad (a-3)$$

with S the concentration i.e. the ratio of the mass of one component to the total mass. Using (a-1) this is equivalent to

$$\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x_i}(\rho S u_i) = 0. \qquad (a-4)$$

When there occurs an interchange of material on the molecular scale, as is almost always the case, there is an <u>irreversible</u> change in concentration. The rate of change of concentration then depends on the flux of material at the molecular level.

Assuming that such transfer occurs, the transport equation for S must be modified.

After some algebra, one finds

$$\rho\left(\frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left\{\rho d\left[\frac{\partial S}{\partial x_j} + \left(\frac{K_T}{T}\right)\frac{\partial T}{\partial x_j} + \left(\frac{K_p}{p}\right)\frac{\partial p}{\partial x_j}\right]\right\}, \qquad (a-5)$$

with

d the diffusion coefficient,

K_T the thermal diffusion ratio,

K_p the pressure diffusion ratio, and

T the temperature.

The second and third terms on the right hand side of (a-5) give the change in concentration due to thermal and pressure diffusion. In the cases of interest there is no measurable difference in diffusivities due to differences in physical or chemical composition. Therefore

$$K_T = K_p = 0 \qquad (a\text{-}6)$$

to the degree of approximation considered here. In this case,

$$\rho\left(\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\rho d \frac{\partial s}{\partial x_j}\right). \qquad (a\text{-}7)$$

This reduces, in the case of a stratified fluid (density gradient in the 3 direction) to

$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = d \frac{\partial^2 s}{\partial x_j \partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_j}(\rho d)\frac{\partial s}{\partial x_j}. \qquad (a\text{-}8)$$

If the relative change in desnity in the 3 direction is small, and the fluid is homogeneous in a horizontal plane (1-2 plane) then (a-8) reduces to

$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = d \frac{\partial^2 s}{\partial x_j \partial x_j}. \qquad (a\text{-}9)$$

Now assume that the fluid velocity, u_j, has a mean (ensemble) and fluctuating part, i.e.

$$u_j = U_j + u_j' \,, \qquad (a-10)$$

where, $\langle \ \rangle$ denoting an ensemble average,

$$\langle u_j \rangle = U_j \,. \qquad (a-11)$$

Assuming that

$$S = \overline{S} + s \,, \qquad (a-12)$$

with

$$\langle s \rangle = \overline{S} \,, \qquad (a-13)$$

equation (a-9) becomes, after taking averages and assuming the statistical independence of u_j' and s

$$\frac{\partial \overline{S}}{\partial t} + U_j \frac{\partial \overline{S}}{\partial x_j} + \left\langle u_j' \frac{\partial s}{\partial x_j} \right\rangle = d \frac{\partial^2 \overline{S}}{\partial x_j \partial x_j} \,. \qquad (a-14)$$

Invoking an "eddy diffusivity" or mean gradient hypothesis

$$\left\langle u_j' \frac{\partial s}{\partial x_j} \right\rangle = -K_j' \frac{\partial \overline{S}}{\partial x_j} \,. \qquad (a-15)$$

equation (a-14) becomes

$$\frac{\partial \overline{S}}{\partial t} + U_j \frac{\partial \overline{S}}{\partial x_j} = d \frac{\partial^2 \overline{S}}{\partial x_j \partial x_j} + K_j' \frac{\partial \overline{S}}{\partial x_j} . \qquad (a-16)$$

If the fluid is stratified, there is no motion in the 3 direction, equation (a-16) can be written in the form

$$\frac{\partial \overline{S}}{\partial t} + U_j \frac{\partial \overline{S}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K_j \frac{\partial \overline{S}}{\partial x_j}\right), \qquad (a-17)$$

with

$$K_j = d + K_j' . \qquad (a-18)$$

Assuming that the mean flow is zero,

$$U_j \equiv 0 , \qquad (a-19)$$

this reduces to equation (i).

APPENDIX B

Consider the equation

$$\frac{\partial S}{\partial \eta} = \frac{\partial}{\partial \xi}\left(\xi^{4/3}\frac{\partial S}{\partial \xi}\right), \qquad (b-1)$$

with the initial condition

$$S(\xi, 0) = 1 \quad , \quad -1/2 \leq \xi \leq 1/2 . \qquad (b-2)$$

First, the fundamental solution will be found. Let

$$\xi = y^3 . \qquad (b-3)$$

Under this transformation equation (b-1) becomes

$$\frac{\partial S}{\partial \eta} = \left(\frac{2}{9y}\right)\frac{\partial S}{\partial y} + \left(\frac{1}{9}\right)\frac{\partial^2 S}{\partial y^2} . \qquad (b-4)$$

Because (b-4) differs only in the $(\partial S/\partial y)$ term from the standard form of the diffusion equation, it is expected that its fundamental solution will be similar to that of the usual diffusion equation.

Therefore, let

$$S^*(y, \eta) = \left[\frac{c}{f(\eta)}\right] e^{-y^2/b\eta} , \qquad (b-5)$$

with b and c constants and $f(\eta)$ an, as yet, undetermined function. Substituting this ansatz into equation (b-1), it is found that

$$c\left\{\left(\frac{1}{f}\right)\left[\frac{-f'}{f}+\frac{2}{3b\eta}\right]-\left(\frac{y^2}{bf\eta^2}\right)\left[\frac{4}{9b}-1\right]\right\}e^{-y^2/b\eta}=0. \qquad (b-6)$$

In order for this to hold for all y and η,

$$b = 4/9, \qquad (b-7)$$

$$f = \eta^{3/2}. \qquad (b-8)$$

Thus

$$S^*(\xi,\eta) = \left(\frac{c}{\eta^{3/2}}\right)e^{-(9\xi^{2/3}/4\eta)}. \qquad (b-9)$$

It is clear that if $\xi \neq 0$,

$$\lim_{\eta\to 0} S^*(\xi,\eta) = 0, \qquad (b-10)$$

and

$$\lim_{\eta\to 0} S^*(0,\eta) \to \infty, \qquad (b-11)$$

showing that S^* has the character of a δ function.

It is also necessary that

$$\int_{-\infty}^{\infty} S^*(\xi,\eta)\,d\xi = 1 . \qquad (b\text{-}12)$$

Equation (b-12) is

$$\left(\frac{C}{\eta^{3/2}}\right)\int_{-\infty}^{\infty} e^{-(9\xi^{3/2}/4\eta)}\,d\xi = 1 . \quad (b\text{-}13)$$

Let

$$z = \left(\frac{9}{4\eta}\right)^{3/2}\xi , \qquad (b\text{-}14)$$

and equation (b-13) becomes

$$\left(\frac{16C}{27}\right)\int_{0}^{\infty} e^{-z^{2/3}}\,dz = 1 , \qquad (b\text{-}15)$$

so that, with (Gradshteyn & Ryzhik, 1965, 3.326)

$$\int_{0}^{\infty} e^{-z^{2/3}}\,dz = \left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right), \qquad (b\text{-}16)$$

$$C = 9/4\sqrt{\pi} . \qquad (b\text{-}17)$$

With the initial condition (b-2), $S(\xi,\eta)$ can be easily found by integrating S^* over the interval, thus

$$S(\xi,\eta) = \int_{-1/2}^{1/2} S^*(\xi-\alpha,\eta)\,d\alpha$$

$$= \frac{9}{4\pi^{1/2}\eta^{3/2}} \int_{-1/2}^{1/2} e^{-9(\xi-\alpha)^{2/3}/4\eta}\,d\alpha$$

$$= \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} z^2 e^{-z^2}\,dz, \qquad \text{(b-18)}$$

under the transformation

$$z = 3(\xi-\alpha)^{1/3}/2\eta^{1/2}. \qquad \text{(b-19)}$$

Then, with

$$z_1 = 3(\xi-\tfrac{1}{2})^{1/3}/2\eta^{1/2}, \qquad \text{(b-20)}$$

$$z_2 = 3(\xi+\tfrac{1}{2})^{1/3}/2\eta^{1/2}. \qquad \text{(b-21)}$$

$$S(\xi,\eta) = \frac{1}{2}\Big\{ \operatorname{erf}(z_2) - \operatorname{erf}(z_1)$$

$$+ \frac{2}{\sqrt{\pi}}\Big[z_1 e^{-z_1^2} - z_2 e^{-z_2^2} \Big]\Big\}. \qquad (b\text{-}22)$$

The concentration at the center of the patch is given by

$$S(0,\eta) = \operatorname{erf}\left[\frac{3}{2^{4/3}\eta^{1/2}}\right] - \left[\frac{3}{2^{1/3}\pi^{1/2}\eta^{1/2}}\right] e^{-(9/2^{8/3}\eta)}. \qquad (b\text{-}23)$$

As for the scale or size of the patch, it can be defined as in the case of the constant eddy diffusivity,

$$S(\tfrac{1}{2}\ell, \eta) = a^* S(0,\eta), \qquad (b\text{-}24)$$

with $0 < a^* < 1$. Equation (b-24) can be solved numerically for ℓ as a function of η , using (b-22) and (b-23).

Defining the variance as usual by

$$\sigma^2 = \int_{-\infty}^{\infty} \xi^2 S(\xi,\eta)\, d\xi, \qquad (b\text{-}25)$$

Defining the variance as usual by

$$\sigma^2 \equiv \int_{-\infty}^{\infty} \xi^2 S(\xi, \eta)\, d\xi , \qquad (b\text{-}26)$$

it can be seen, substituting from equation (b-18) that,

$$\sigma^2 = \frac{9}{4\pi^{1/2}\eta^{3/2}} \int_{-\infty}^{\infty} \xi^2 \int_{-1/2}^{1/2} e^{-9(\xi-\alpha)^{2/3}/4\eta}\, d\alpha\, d\xi . \quad (b\text{-}27)$$

With the substitution

$$y^2 = \left(\frac{9}{4\eta}\right)(\xi - \alpha)^{2/3} , \qquad (b\text{-}28)$$

and an interchange in the order of integrations

$$\sigma^2 = \frac{9}{4\pi^{1/2}\eta^{3/2}} \int_{-1/2}^{1/2} \left\{ \left(\frac{8}{9}\right) \eta^{3/2} \alpha^2 I_2 + \left(\frac{128}{243}\right) \eta^3 \alpha I_5 + \left(\frac{512}{6561}\right) \eta^{9/2} I_8 \right\} d\alpha , \quad (b\text{-}29)$$

where,

$$I_n \equiv \int_{-\infty}^{\infty} y^n e^{-y^2}\, dy . \qquad (b\text{-}30)$$

Carrying out the integrations

$$\sigma^2 = \frac{1}{12} + \left(\frac{280}{243}\right) \eta^3 . \qquad (b\text{-}31)$$

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