AD-759 989

TOLERANCE LIMITS FOR THE RAYLEIGH (RADIAL NORMAL) DISTRIBUTION WITH EMPHASIS ON THE CEP

Marlin A. Thomas, et al

Naval Weapons Laboratory
Dahlgren, Virginia

May 1973

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Weapons Laboratory Dahlgren, Virginia 22448	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE TOLERANCE LIMITS FOR THE RAYLEIGH (RADIAL NORMAL) DISTRIBUTION WITH EMPHASIS ON THE CEP	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)	
5. AUTHOR(S) (First name, middle initial, last name) Marlin A. Thomas, John R. Crigler, Gary W. Gemmill & Audrey E. Taub	
6. REPORT DATE May 1973	7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS
8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NWL TR-2946 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited.	
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

The weapon systems analyst is often confronted with the problem of estimating the radius of a mean centered circle which will include 50% of the future rounds from a particular weapon under specified conditions. If the fall of shot tends to follow a circular normal distribution with standard deviation σ, this is usually accomplished by estimating σ with $\hat{\sigma}$ from the results of test firings and then forming CEP = $1.1774\hat{\sigma}$. CEP, the parameter estimated, is the radius of a mean centered circle which includes 50% of the bivariate probability which, of course, is taken to mean 50% of the future rounds from this weapon under similar conditions. While CEP is a valid point estimate of the radius of the 50% circle (provided $\hat{\sigma}$ is a valid estimate of σ), it does not provide the analyst with any measure of confidence concerning his statement.

Statements of confidence concerning the percent of a population which lies within a circle of given radius are formulated in this report through the concept of statistical tolerance limits. The results will enable the analyst to ascertain (with the aid of tables) the confidence with which he can state that a circle of radius CEP contains at least 50% of the population. This confidence is shown to be quite low (at most .50 unless one has complete knowledge about the population parameter σ) and can be increased only by increasing the multiplying constant for $\hat{\sigma}$ above the customary 1.1774. Tables of such constants (tolerance limit factors) are provided which will enable the analyst to obtain more reasonable levels of confidence not only for 50% of the population but also for 75%, 90%, 95%, and 99%.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

UNCLASSIFIED

Security Classification

NWL TECHNICAL REPORT TR-2946

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Marlin A. Thomas
John R. Crigler
Gary W. Gemmill
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NAVAL WEAPONS LABORATORY

DAHLGREN, VIRGINIA 22448

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FOREWORD

The work covered in this Technical Report was performed in the Mathematical Statistics and Systems Simulation Branch (KCM), Operations Research Division, Warfare Analysis Department. The date of completion was 26 March 1973.

This report was reviewed by Mr. Carl M. Hynden, Jr.

Released by:

Ralph A. Niemann

RALPH A. NIEMANN, Head
Warfare Analysis Department

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Statements of confidence concerning the percent of a population which lies within a circle of given radius are formulated in this report through the concept of statistical tolerance limits. The results will enable the analyst to ascertain (with the aid of tables) the confidence with which he can state that a circle of radius $\widehat{CEP}$ contains at least 50% of the population. This confidence is shown to be quite low (at most .50 unless one has complete knowledge about the population parameter σ) and can be increased only by increasing the multiplying constant for $\hat{\sigma}$ above the customary 1.1774. Tables of such constants (tolerance limit factors) are provided which will enable the analyst to obtain more reasonable levels of confidence not only for 50% of the population but also for 75%, 90%, 95%, and 99%.

CONTENTS

INTRODUCTION

The CEP concept (Circular Probable Error, Circular Error Probable, Circle of Equal Probability) is well known to the weapon systems analyst. Quite briefly, this parameter is the radius of a mean-centered circle which includes 50% of the bivariate probability, or in terms of a particular weapon, it is the radius of a circle within which 50% of the rounds will fall. To estimate the CEP for a particular weapon (at a specified weapon to target range), n rounds are fired at a target arbitrarily placed at the center of the Cartesian coordinate system. The results of these firings are simply the miss distances of the rounds from the target center in the x and y directions, usually denoted by $\{x_i, y_i\}_{i=1}^{n}$. These n pairs of miss distances are then used to compute an estimate of the CEP, say $\hat{CEP}$, which is taken as the radius of a circle within which 50% of the future rounds from this weapon will fall. $\hat{CEP}$ is, of course, only a point estimate of CEP, and it will vary from sample to sample. Also, since $\hat{CEP}$ is a continuous random variable, the probability that a circle of radius $\hat{CEP}$ will encompass exactly 50% of the future rounds is zero. Hence, if one wishes to measure the precision of this estimator, it appears that he should consider the probability that a circle of radius $\hat{CEP}$ will encompass <u>at least</u> 50% of the future rounds. The purpose of this report is to consider this probability and show that for any finite n it is quite low, about .40 to .50. A procedure is then suggested which will enable one to increase this probability (or confidence) to more reasonable levels, say .75, .90, .95, or .99. The suggested procedure involves the formulation of tolerance limits for the Rayleigh (or radial normal) distribution and is sufficiently general to be useful to those other than the weapon systems analyst.

RELATIONSHIP BETWEEN CEP AND σ

The use of CEP as a measure of weapon accuracy requires the assumption that the miss distances (X,Y) are distributed according to the uncorrelated bivariate normal distribution with mean at the origin (target center) and common variance σ^2 in both directions. Hence, this assumption will be used throughout this report; that is, it will be assumed that the density of miss distances about the target is given by

$$f(x,y) = (2\pi\sigma^2)^{-1} e^{-(x^2+y^2)/2\sigma^2}, \quad -\infty < x,\ y < \infty . \qquad (1)$$

Using (1), it is easy to derive the density of the radial error (or radial miss distance) $R = (X^2 + Y^2)^{\frac{1}{2}}$ which is

$$g(r) = (r/\sigma^2)\, e^{-r^2/2\sigma^2},\ r > 0\ . \tag{2}$$

This distribution of the radial errors is referred to as either the radial normal distribution or the Rayleigh distribution and is the basis for the relationship between CEP and σ. To explore this further, consider the probability that the radial error R is less than t which is expressed as

$$P\{R < t\} = \int_0^t g(r)\, dr = 1-e^{-t^2/2\sigma^2}. \tag{3}$$

To find the relationship between CEP and σ, one merely solves

$$P\{R < CEP\} = .50 \tag{4}$$

for CEP in terms of σ. It turns out that

$$CEP = (-2 \ln .50)^{\frac{1}{2}}\, \sigma = 1.1774\sigma \tag{5}$$

which is a well known relation. Hence, if the population variance σ^2 were known for a weapon, a circle of radius 1.1774σ = CEP contains 50% of the bivariate probability, i.e., 50% of the future rounds will fall in a circle of radius CEP. Unfortunately σ^2 and hence CEP are never known and must be estimated from test firing. As aforementioned, this means that the estimated CEP, $\widehat{CEP}$, is a random variable which varies from sample to sample and that for any particular sample, the probability that a circle of radius $\widehat{CEP}$ encompasses exactly 50% of the bivariate probability is zero. However, consider the question of how much probability or confidence is afforded with a sample of size n when making the statement that at least 50% of the future (or population) rounds will fall within $\widehat{CEP}$. The answer to this question will be presented in the next two sections.

UPPER TOLERANCE BOUND FOR THE RAYLEIGH DISTRIBUTION

Making confidence statements concerning the percent of the population which lies below an estimate of the CEP involves the concept of an upper tolerance bound. In the more general sense, an upper tolerance bound, $U(P,\gamma)$ is a point defined such that at least 100P% of the population lies below it with 100γ% confidence. (See, for example, Bowker and Lieberman (1972) and Proschan (1953).) It is constructed in the following manner: A random sample of size n is extracted from the population, and these sample values are used to compute an estimate(s) of the unknown population parameter(s); $U(P,\gamma)$, is then formulated as a function of the estimate(s). For the case at hand where one is sampling from the bivariate normal distribution with common variance, there is only one unknown parameter, namely σ, and the upper tolerance bound will be formulated as a function of the estimate of this parameter. Hence, to explore an upper tolerance bound in this case will first require an appropriate estimator for σ.

Various estimators for σ are available, and these are discussed in detail by Moranda (1959). One of the estimators discussed by Moranda is the maximum likelihood estimator which is the form

$$\hat{\sigma} = \{\sum_{i=1}^{n}(X_i^2 + Y_i^2)/2n\}^{\frac{1}{2}} \tag{6}$$

where X_i and Y_i are random variables designating the i^{th} miss distance in the x and y directions, respectively. It is easily shown that this estimator is sufficient for σ so the upper tolerance bound should be of the form $U(P,\gamma) = k(P,\gamma,n)\hat{\sigma}$. The constant $k(P,\gamma,n)$ (tolerance limit factor) is to be determined such that one is 100γ% confident that at least 100P% of the population lies below $U(P,\gamma)$. Deleting the arguments for notational simplicity, $k(P,\gamma,n)$ is sought such that

$$P\{\int_0^{k\hat{\sigma}} (r/\sigma^2)\, e^{-r^2/2\sigma^2}\, dr \geq P\} = \gamma. \tag{7}$$

Straightforward manipulations of (7) leads to

$$P\{\hat{\sigma}^2 \geq [-2\sigma^2 \ln (1-P)]/k^2\} = \gamma. \tag{8}$$

Recalling the form of $\hat{\sigma}$ in (6), it is easily shown that the density of $W = \hat{\sigma}^2$ is given by

$$h(w) = [(\sigma^2/n)^n \Gamma(n)]^{-1} w^{n-1} e^{-nw/\sigma^2}, \; w > 0 \tag{9}$$

where

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx . \tag{10}$$

Hence, equation (8) can be written as

$$\int_0^v h(w)\, dw = 1 - \gamma \tag{11}$$

where $v = [-2\sigma^2 \ln (1-P)]/k^2$ and $h(w)$ is given in (9). It appears from (11) above that k is a function of the unknown parameter σ. However, a simple transformation reveals that it is not. Letting

$z = w/\sigma^2$ in (11), one obtains

$$\int_0^{v'} n^n/\Gamma(n)\, z^{n-1} e^{-nz} dz = 1 - \gamma \tag{12}$$

where $v' = [-2 \ln (1-P)]/k^2$. Hence, the tolerance limit factor k is a function only of P, γ and n. Thus tabular values of k as a function of these quantities will enable one to make exact tolerance statements concerning the Rayleigh distribution. Before discussing the computation, tabulation and use of such values of k, it will be instructive to answer the question posed at the end of the last section, namely, how much confidence is afforded with a sample of size n when making the statement that at least 50% of the population lies within a circle of radius $\widehat{CEP}$.

CONFIDENCE AFFORDED USING $k = 1.1774$ FOR $P = .50$

Equation (12) in the last section was developed to evaluate k for specified values of P, γ and n. It can also be used to evaluate γ for any P, k, and n. Consider now using it in the latter sense to answer the question posed at the end of the last two sections. Assuming that σ is estimated by the method of maximum likelihood, that is, that $\hat{\sigma}$ has the form in (6), then the maximum likelihood estimator for CEP would be simply $\widehat{CEP} = 1.1774\hat{\sigma}$. The confidence afforded using a CEP estimate of this form can be ascertained by using $k = (-2 \ln .50)^{\frac{1}{2}}$ ($= 1.1774$ to five significant digits) and $P = .50$ in equation (12) and solving for γ for various values of n. One notes first that using this value of k, the upper limit of the integral in (12) is equal to one. Hence, the confidence, γ, can be ascertained by solving

$$\int_0^1 n^n/\Gamma(n)\, z^{n-1}\, e^{-nz}\, dz = 1 - \gamma \tag{13}$$

for γ.

Equation (13) was solved numerically for γ for $n = 2(1)25(5)100(10)200(50)300(100)1000, \infty$. The results are set out in Table 1 and reveal some interesting results. First, the confidence with which a circle of radius $\widehat{CEP}$ contains at least 50% of the population lies between .4060 and .4958 for all n between 2 and 1000. With $n = \infty$, that is, when the parameter σ is known exactly, the confidence is one. The reason for this is quite clear if one examines the integrand in (13). It is recognizable as a gamma density with a mean of one and variance $1/n$. Hence, one is always integrating up to the mean. For small n, this distribution is skewed to the right, but as n increases, it becomes symmetrical about the mean so that the integral from zero to one approaches .50. For $n = \infty$, the variance of this gamma density is zero, that is, the entire density is concentrated at the mean which is one. Hence, the integral from zero to one is zero which results in a gamma equal to 1.0000. This all means that in estimating the CEP with $\widehat{CEP} = 1.1774\hat{\sigma}$, the statement that at least 50% of the population lies within a circle of radius $\widehat{CEP}$ can be made with confidence between .4060 and .5000 unless one has complete knowledge about the population parameter σ. To increase this confidence to a more reasonable level, say .75 and above, one must increase the factor k above the customary 1.1774.

Table 1

CONFIDENCE AFFORDED USING k = 1.1774 FOR P = .50

n	γ	n	γ
2	.4060	60	.4828
3	.4232	65	.4835
4	.4335	70	.4841
5	.4405	75	.4846
6	.4457	80	.4851
7	.4497	85	.4856
8	.4530	90	.4860
9	.4557	95	.4864
10	.4579	100	.4867
11	.4599	110	.4873
12	.4616	120	.4879
13	.4631	130	.4883
14	.4644	140	.4888
15	.4657	150	.4891
16	.4667	160	.4895
17	.4677	170	.4898
18	.4686	180	.4901
19	.4695	190	.4904
20	.4703	200	.4906
21	.4710	250	.4916
22	.4716	300	.4923
23	.4723	400	.4934
24	.4728	500	.4941
25	.4734	600	.4946
30	.4757	700	.4950
35	.4775	800	.4953
40	.4790	900	.4956
45	.4802	1000	.4958
50	.4812	∞	1.0000
55	.4821		

This can be done by fixing γ in (12) to the desired confidence, setting $P = .50$ and solving for k for selected values of n. Of course, equation (12) is valid for general P (vice $P = .50$) which upon solving for k would enable one to find the radius of a circle which would include at least 100P% of the population with 100γ% confidence. Such tabular values of k for selected values of P, γ, and n are provided in the next section.

TABULATION OF TOLERANCE LIMIT FACTORS

In order to obtain a set of tolerance limit factors for $P = .50$ and other reasonable values of P, equation (12) was solved for k for $P = .50, .75, .90, .95, .99$; $\gamma = .75, .90, .95, .99$; and $n = 2(1)25(5)100(10)200(50)300(100)1000,\infty$. The solutions were obtained using Simpson's integration rule and successive binary cuts beginning with an appropriate starting value for the upper limit v'. The computations were performed on the CDC 6700 at the Naval Weapons Laboratory. Tolerances were set to provide an accuracy in k of four decimal digits; the values of k are set out in Table 2 for the above listed values of P, γ, and n.

As an example of using this table, suppose ten rounds are fired at a target to obtain the radius of a circle about the target center which will include at least 50% of the future rounds (under similar conditions) from this weapon with 95% confidence. The miss distances from the target in the x (cross range) and y (range) directions are shown below. All measurements are in feet.

x	y
54.7	87.8
-20.1	-178.1
-37.3	-5.6
-136.3	-214.1
-8.1	-23.4
95.8	97.5
91.8	-79.1
116.3	-52.8
-144.5	94.6
75.9	167.3

It can be verified that $\sum_{i=1}^{10} (x_i^2 + y_i^2) = 222537.8$ and, hence, that

Table 2

TOLERANCE LIMIT FACTORS FOR THE RAYLEIGH DENSITY

n \ P	γ=.75 .50	.75	.90	.95	.99	γ=.90 .50	.75	.90	.95	.99
2	1.6983	2.4018	3.0953	3.5306	4.3775	2.2833	3.2291	4.1616	4.7469	5.8854
3	1.5517	2.1944	2.8281	3.2258	3.9996	1.9426	2.7472	3.5406	4.0384	5.0071
4	1.4789	2.0915	2.6955	3.0746	3.8120	1.7828	2.5212	3.2493	3.7062	4.5952
5	1.4345	2.0826	2.6144	2.9821	3.6974	1.6880	2.3872	3.0766	3.5093	4.3510
6	1.4041	1.9856	2.5591	2.9190	3.6191	1.6245	2.2974	2.9608	3.3772	4.1872
7	1.3818	1.9541	2.5184	2.8726	3.5616	1.5785	2.2323	2.8769	3.2315	4.0686
8	1.3646	1.9298	2.4871	2.8368	3.5172	1.5433	2.1826	2.8129	3.2085	3.9780
9	1.3508	1.9103	2.4620	2.8082	3.4818	1.5155	2.1432	2.7621	3.1506	3.9063
10	1.3395	1.8944	2.4415	2.7848	3.4528	1.4928	2.1111	2.7207	3.1033	3.8477
11	1.3301	1.8810	2.4242	2.7651	3.4284	1.4738	2.0842	2.6861	3.0639	3.7988
12	1.3220	1.8696	2.4095	2.7484	3.4076	1.4577	2.0615	2.6568	3.0304	3.7572
13	1.3150	1.8597	2.3968	2.7338	3.3895	1.4438	2.0418	2.6314	3.0015	3.7214
14	1.3089	1.8510	2.3856	2.7211	3.3738	1.4316	2.0246	2.6093	2.9762	3.6901
15	1.3035	1.8434	2.3757	2.7098	3.3598	1.4209	2.0094	2.5897	2.9539	3.6624
16	1.2986	1.8366	2.3669	2.6998	3.3473	1.4114	1.9960	2.5724	2.9341	3.6379
17	1.2943	1.8304	2.3590	2.6908	3.3362	1.4028	1.9839	2.5568	2.9163	3.6158
18	1.2904	1.8249	2.3518	2.6826	3.3260	1.3951	1.9729	2.5427	2.9002	3.5959
19	1.2868	1.8198	2.3453	2.6751	3.3168	1.3880	1.9630	2.5298	2.8856	3.5777
20	1.2835	1.8152	2.3393	2.6683	3.3083	1.3816	1.9539	2.5181	2.8722	3.5612
21	1.2805	1.8109	2.3338	2.6620	3.3006	1.3757	1.9455	2.5074	2.8600	3.5459
22	1.2777	1.8070	2.3288	2.6563	3.2934	1.3702	1.9378	2.4974	2.8486	3.5319
23	1.2751	1.8033	2.3241	2.6509	3.2867	1.3652	1.9307	2.4882	2.8382	3.5189
24	1.2727	1.7999	2.3197	2.6459	3.2806	1.3605	1.9241	2.4797	2.8284	3.5068
25	1.2705	1.7967	2.3156	2.6412	3.2748	1.3562	1.9179	2.4718	2.8193	3.4956
30	1.2612	1.7836	2.2986	2.6219	3.2508	1.3380	1.8923	2.4387	2.7817	3.4489
35	1.2541	1.7736	2.2858	2.6072	3.2326	1.3244	1.8729	2.4138	2.7532	3.4136
40	1.2485	1.7657	2.2756	2.5956	3.2182	1.3135	1.8576	2.3941	2.7308	3.3857
45	1.2440	1.7593	2.2673	2.5862	3.2065	1.3047	1.8452	2.3780	2.7124	3.3630
50	1.2402	1.7539	2.2604	2.5782	3.1967	1.2974	1.8348	2.3647	2.6972	3.3442

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE RAYLEIGH DENSITY

	γ=.75					γ=.90				
n \ P	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
55	1.2370	1.7493	2.2545	2.5715	3.1883	1.2912	1.8260	2.3533	2.6842	3.3281
60	1.2342	1.7454	2.2494	2.5657	3.1811	1.2858	1.8184	2.3435	2.6730	3.3142
65	1.2317	1.7419	2.2449	2.5606	3.1748	1.2811	1.8117	2.3349	2.6633	3.3021
70	1.2295	1.7388	2.2410	2.5561	3.1692	1.2769	1.8058	2.3274	2.6546	3.2914
75	1.2276	1.7361	2.2375	2.5521	3.1642	1.2732	1.8006	2.3206	2.6469	3.2818
80	1.2259	1.7336	2.2343	2.5485	3.1598	1.2699	1.7959	2.3145	2.6340	3.2732
85	1.2243	1.7314	2.2314	2.5452	3.1557	1.2669	1.7916	2.3090	2.6337	3.2654
90	1.2229	1.7294	2.2288	2.5422	3.1520	1.2641	1.7877	2.3040	2.6280	3.2584
95	1.2216	1.7276	2.2264	2.5395	3.1487	1.2616	1.7842	2.2994	2.6228	3.2519
100	1.2204	1.7258	2.2242	2.5370	3.1456	1.2593	1.7809	2.2952	2.6180	3.2459
110	1.2182	1.7228	2.2203	2.5326	3.1400	1.2552	1.7751	2.2877	2.6094	3.2353
120	1.2163	1.7202	2.2169	2.5287	3.1352	1.2516	1.7700	2.2812	2.6020	3.2261
130	1.2147	1.7179	2.2140	2.5253	3.1310	1.2485	1.7656	2.2755	2.5955	3.2181
140	1.2133	1.7158	2.2113	2.5223	3.1273	1.2457	1.7617	2.2705	2.5898	3.2109
150	1.2120	1.7140	2.2090	2.5196	3.1239	1.2432	1.7582	2.2660	2.5846	3.2046
160	1.2100	1.7123	2.2068	2.5172	3.1209	1.2410	1.7551	2.2619	2.5800	3.1988
170	1.2097	1.7108	2.2049	2.5150	3.1182	1.2390	1.7522	2.2582	2.5758	3.1936
180	1.2088	1.7095	2.2031	2.5130	3.1157	1.2372	1.7496	2.2549	2.5720	3.1888
190	1.2079	1.7082	2.2015	2.5111	3.1134	1.2355	1.7472	2.2518	2.5684	3.1845
200	1.2071	1.7071	2.2000	2.5094	3.1113	1.2339	1.7450	2.2490	2.5652	3.1805
250	1.2038	1.7024	2.1940	2.5026	3.1029	1.2276	1.7361	2.2370	2.5522	3.1643
300	1.2014	1.6990	2.1897	2.4976	3.0967	1.2230	1.7297	2.2292	2.5426	3.1525
400	1.1980	1.6943	2.1836	2.4907	3.0881	1.2167	1.7206	2.2176	2.5294	3.1361
500	1.1958	1.6911	2.1795	2.4860	3.0822	1.2124	1.7146	2.2097	2.5205	3.1250
600	1.1941	1.6888	2.1765	2.4825	3.0780	1.2092	1.7101	2.2040	2.5139	3.1169
700	1.1929	1.6870	2.1741	2.4799	3.0747	1.2068	1.7067	2.1955	2.5089	3.1106
800	1.1918	1.6655	2.1723	2.4778	3.0720	1.2048	1.7039	2.1960	2.5048	3.1056
900	1.1910	1.6843	2.1707	2.4760	3.0699	1.2032	1.7016	2.1930	2.5014	3.1014
1000	1.1903	1.6833	2.1694	2.4745	3.0680	1.2019	1.6997	2.1906	2.4986	3.0979
∞	1.1774	1.6651	2.1460	2.4477	3.0348	1.1774	1.6651	2.1460	2.4477	3.3048

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE RAYLEIGH DENSITY

	γ=.95					γ=.99				
n \ P	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
2	2.7935	3.9503	5.0911	5.8070	7.1999	4.3201	6.1095	7.8739	8.9811	11.1353
3	2.2552	3.1893	4.1103	4.6883	5.8128	3.0879	4.3670	5.6281	6.4196	7.9593
4	2.0146	2.8490	3.6718	4.1881	5.1927	2.5953	3.6704	4.7303	5.3955	6.6897
5	1.8757	2.6526	3.4186	3.8994	4.8347	2.3277	3.2919	4.2425	4.8391	5.9998
6	1.7842	2.5232	3.2519	3.7092	4.5989	2.1583	3.0523	3.9337	4.4869	5.5631
7	1.7186	2.4305	3.1324	3.5730	4.4300	2.0407	2.8860	3.7194	4.2425	5.2601
8	1.6691	2.3604	3.0421	3.4699	4.3022	1.9535	2.7626	3.5604	4.0611	5.0352
9	1.6301	2.3053	2.9711	3.3889	4.2017	1.8859	2.6671	3.4373	3.9207	4.8611
10	1.5985	2.2606	2.9134	3.3232	4.1202	1.8322	2.5911	3.3394	3.8090	4.7226
11	1.5722	2.2235	2.8656	3.2685	4.0525	1.7878	2.5284	3.2585	3.7168	4.6082
12	1.5500	2.1920	2.8250	3.2223	3.9952	1.7505	2.4756	3.1906	3.6392	4.5121
13	1.5309	2.1650	2.7903	3.1827	3.9461	1.7190	2.4311	3.1332	3.5738	4.4310
14	1.5143	2.1415	2.7600	3.1481	3.9032	1.6915	2.3922	3.0830	3.5166	4.3601
15	1.4996	2.1208	2.7333	3.1176	3.8654	1.6678	2.3586	3.0397	3.4671	4.2988
16	1.4866	2.1024	2.7096	3.0906	3.8319	1.6465	2.3285	3.0010	3.4230	4.2440
17	1.4750	2.0860	2.6884	3.0665	3.8020	1.6278	2.3020	2.9668	3.3840	4.1957
18	1.4645	2.0711	2.6692	3.0446	3.7748	1.6109	2.2781	2.9360	3.3489	4.1521
19	1.4550	2.0577	2.6519	3.0248	3.7503	1.5955	2.2564	2.9080	3.3170	4.1126
20	1.4463	2.0454	2.6361	3.0068	3.7280	1.5818	2.2370	2.8830	3.2884	4.0771
21	1.4383	2.0341	2.6215	2.9902	3.7074	1.5690	2.2189	2.8597	3.2619	4.0443
22	1.4310	2.0237	2.6081	2.9749	3.6884	1.5574	2.2025	2.8386	3.2378	4.0144
23	1.4242	2.0141	2.5958	2.9608	3.6709	1.5467	2.1874	2.8191	3.2155	3.9868
24	1.4179	2.0052	2.5843	2.9477	3.6547	1.5367	2.1732	2.8008	3.1947	3.9610
25	1.4120	1.9969	2.5736	2.9355	3.6396	1.5275	2.1603	2.7841	3.1756	3.9373
30	1.3878	1.9626	2.5294	2.8851	3.5771	1.4896	2.1066	2.7150	3.0968	3.8396
35	1.3695	1.9368	2.4961	2.8471	3.5300	1.4613	2.0666	2.6634	3.0380	3.7567
40	1.3552	1.9165	2.4699	2.8173	3.4930	1.4392	2.0354	2.6232	2.9921	3.7097
45	1.3435	1.9000	2.4486	2.7930	3.4629	1.4214	2.0102	2.5907	2.9550	3.6638
50	1.3338	1.8862	2.4309	2.7728	3.4378	1.4066	1.9892	2.5637	2.9242	3.6256

Table 2 (continued)

TOLERANCE LIMIT FACTORS FOR THE RAYLEIGH DENSITY

	γ=.95					γ=.99				
n \ P	.50	.75	.90	.95	.99	.50	.75	.90	.95	.99
55	1.3255	1.8746	2.4159	2.7556	3.4166	1.3941	1.9716	2.5410	2.8983	3.5935
60	1.3184	1.8645	2.4030	2.7409	3.3993	1.3834	1.9564	2.5214	2.8760	3.5658
65	1.3122	1.8558	2.3917	2.7280	3.3823	1.3741	1.9432	2.5044	2.8566	3.5417
70	1.3067	1.8480	2.3817	2.7166	3.3682	1.3658	1.9316	2.4894	2.8395	3.5206
75	1.3019	1.8411	2.3728	2.7065	3.3556	1.3586	1.9213	2.4761	2.8244	3.5018
80	1.2975	1.8349	2.3648	2.6974	3.3444	1.3520	1.9120	2.4642	2.8107	3.4848
85	1.2935	1.8293	2.3576	2.6892	3.3342	1.3461	1.9037	2.4534	2.7984	3.4696
90	1.2899	1.8242	2.3510	2.6817	3.3249	1.3407	1.8960	2.4436	2.7872	3.4558
95	1.2866	1.8196	2.3450	2.6748	3.3164	1.3358	1.8891	2.4346	2.7770	3.4431
100	1.2836	1.8153	2.3395	2.6685	3.3085	1.3313	1.8828	2.4265	2.7677	3.4315
110	1.2732	1.8076	2.3297	2.6573	3.2546	1.3233	1.8715	2.4119	2.7511	3.4110
120	1.2735	1.8010	2.3212	2.6476	3.2826	1.3164	1.8617	2.3993	2.7367	3.3932
130	1.2694	1.7953	2.3137	2.6391	3.2721	1.3104	1.8532	2.3884	2.7242	3.3777
140	1.2658	1.7902	3.3072	2.6316	3.2628	1.3050	1.8456	2.3786	2.7131	3.3638
150	1.2626	1.7856	2.3012	2.6249	3.2545	1.3003	1.8389	2.3700	2.7033	3.3517
160	1.2597	1.7815	2.2960	2.6188	3.2470	1.2960	1.8329	2.3622	2.6944	3.3406
170	1.2571	1.7778	2.2912	2.6134	3.2402	1.2922	1.8274	2.3551	2.6863	3.3307
180	1.2547	1.7744	2.2868	2.6084	3.2340	1.2886	1.8224	2.3487	2.6790	3.3216
190	1.2525	1.7713	2.2828	2.6038	3.2283	1.2854	1.8179	2.3428	2.6723	3.3133
200	1.2504	1.7684	2.2791	2.5996	3.2231	1.2824	1.8137	2.3374	2.6661	3.3056
250	1.2423	1.7568	2.2642	2.5826	3.2021	1.2706	1.7968	2.3157	2.6414	3.2749
300	1.2363	1.7484	2.2533	2.5702	3.1867	1.2619	1.7846	2.2999	2.6234	3.2526
400	1.2281	1.7368	2.2383	2.5531	3.1654	1.2499	1.7676	2.2781	2.5985	3.2217
500	1.2225	1.7289	2.2281	2.5415	3.1511	1.2418	1.7562	2.2634	2.5817	3.2009
600	1.2184	1.7231	2.2207	2.5330	3.1406	1.2360	1.7479	2.2527	2.5695	3.1858
700	1.2153	1.7187	2.2149	2.5265	3.1325	1.2314	1.7415	2.2444	2.5601	3.1741
800	1.2128	1.7151	2.2104	2.5212	3.1260	1.2278	1.7364	2.2378	2.5525	3.1648
900	1.2107	1.7121	2.2066	2.5169	3.1206	1.2248	1.7321	2.2323	2.5463	3.1570
1000	1.2089	1.7097	2.2034	2.5132	3.1160	1.2223	1.7286	2.2278	2.5410	3.1505
∞	1.1774	1.6651	2.1460	2.4477	3.0348	1.1774	1.6651	2.1460	2.4477	3.0348

$$\hat{\sigma} = 105.4841$$

and

$$\hat{CEP} = 1.1774\hat{\sigma} = 124.1970 .$$

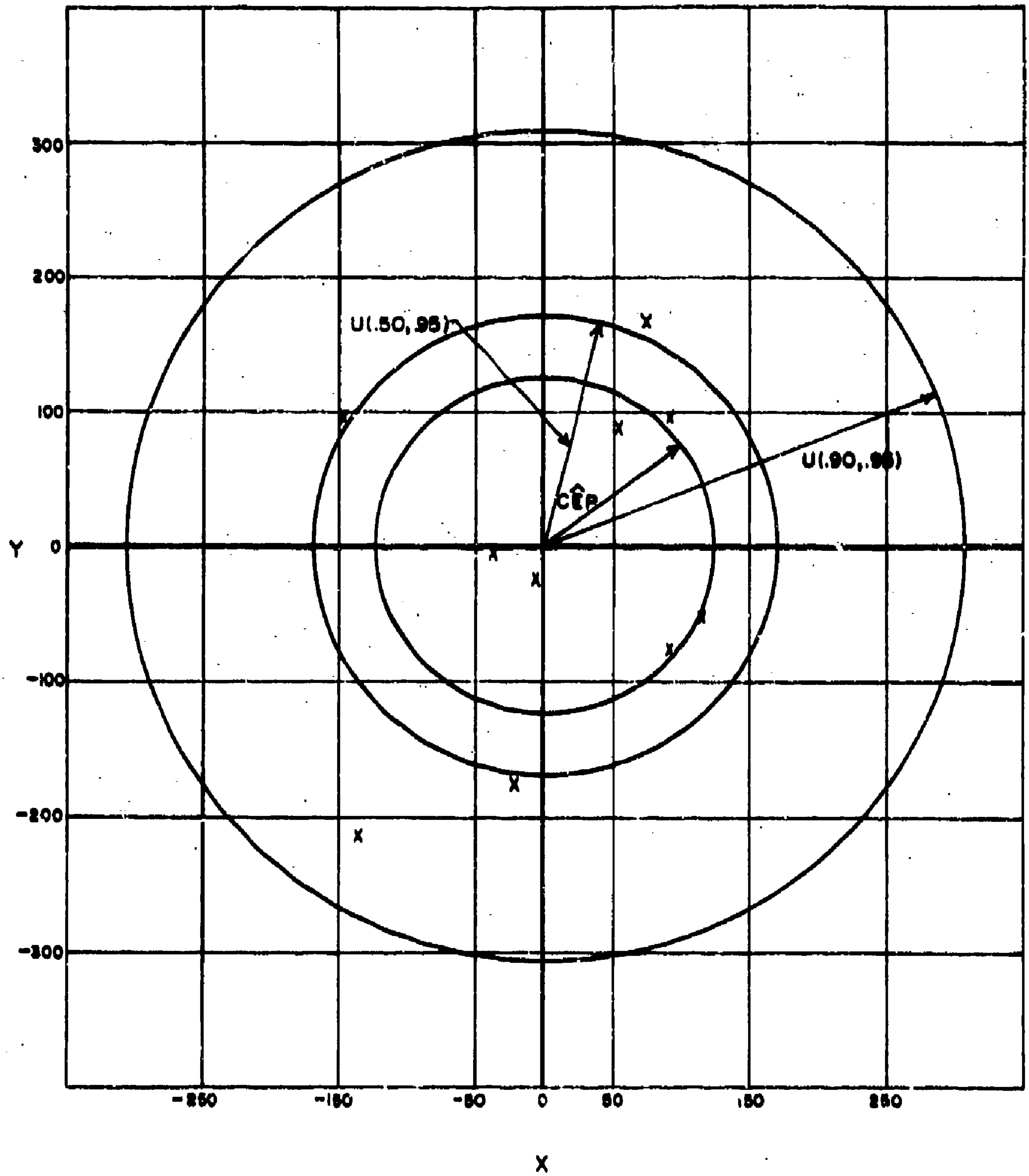
From Table 1, it is seen that one is only 46% confident that a circle

of radius $\hat{CEP}$ = 124.1970 will include at least 50% of the rounds from this weapon under similar conditions. To increase the confidence to 95% as specified, one refers to Table 2 under $P = .50$, $\gamma = .95$ and $n = 10$ to find the tolerance limit factor $k = 1.5985$ (vice 1.1774). This is then multiplied times $\hat{\sigma}$ to obtain $U(.50, .95) = k(.50, .95, 10)\,\hat{\sigma} = (1.5985)(105.4841) = 168.6163$. Hence, a circle of radius 168.6 feet will include at least 50% of the future rounds from this weapon under similar conditions with 95% confidence. Should one want the radius of a circle which will include at least 90% of the rounds from this weapon (vice 50%) under similar conditions with 95% confidence he simply refers to Table 2 under $P = .90$, $\gamma = .95$, and $n = 10$ to find the tolerance limit factor $k(.90, .95, 10) = 2.9134$. This is then multiplied times $\hat{\sigma}$ to obtain $U(.90, .95) = 307.3174$. The ten hit points and the above three circles are illustrated in Figure 1.

CONCLUSIONS

The tables contained in this report provide the weapon systems analyst (and others who work with Rayleigh data) with valuable tools for (1) ascertaining the confidence with which the customary CEP statements are made and for (2) increasing the confidence through tolerance limit factors for at least 75, 90, 95, and 99 percent of the population as well as 50%.

FIGURE 1



U(.50,.95)
CÊP
U(.90,.95)
300
200
100
0
-100
-200
-300
Y
-250
-150
-50
0
50
150
250
X

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