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SOME APPLICATIONS OF COMPETITIVE PRICES TO DYNAMIC PROGRAMMING PROBLEMS UNDER UNCERTAINTY

Jack Schechtman

California University

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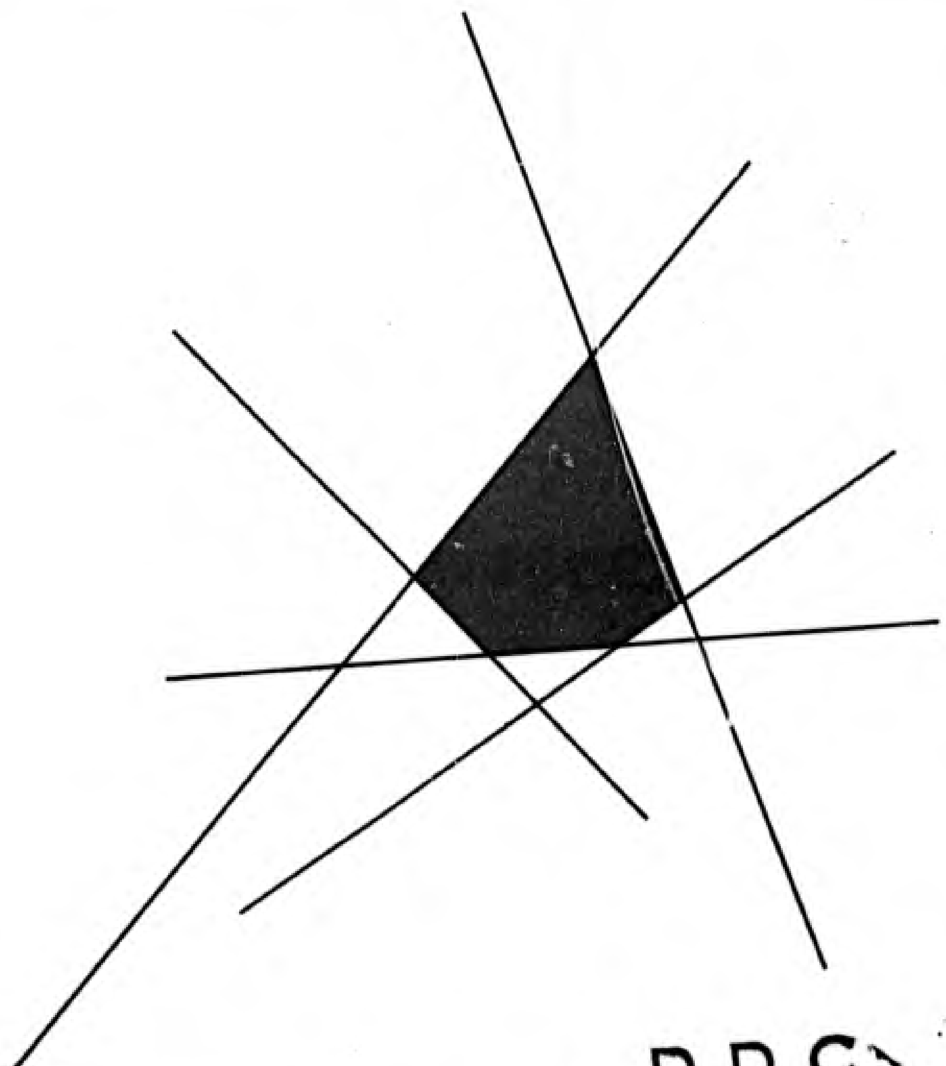

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DEDICATION

To Sandra, Deborah and Lilian.

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ABSTRACT

We will be concerned with one-good economy. The good can be used at any period of time for production or consumption. If $\underline{x}$ units are put into production in period $\underline{t}$ then $f^t(x;\omega^t)$ units become available as outputs in period $\underline{t+1}$; where ω^t is a random variable with known distribution. If $\underline{c}$ units are consumed in period $\underline{t}$ this produces $u^t(c)$ units of satisfaction or utility to the society in that period. Our main interest is the study of qualitative properties of optimal solutions for a problem in which we maximize the total expected utility accumulated in $\underline{t}$ periods.

In deterministic cases, i.e., the output is known with certainty, the basic tools are *prices* and *competitive policies*. D. Gale has suggested the introduction of similar concepts for the stochastic cases, and it turns out that after this is done, we get a better understanding of the problem and a powerful tool in the proof of theorems. Our main purpose here is to introduce the appropriate *price* concept and then exploit it in several directions to obtain new information on various stochastic problems.

The concepts of *price* and *competitive policy* are introduced, and it is shown that a *competitive policy* is optimal. For $f(x;\omega)$ and $u(c)$ increasing, differentiable and strictly concave, optimality conditions are obtained and it is shown that every optimal policy is *competitive*. For the case in which $f(x;\omega) = g(x) + \omega$, $f'(x;\omega)$ and $u'(c)$ are convex functions we obtain a result that permits us to compare optimal consumption policies with the corresponding policies of a deterministic

case in which $f(x;\omega) = g(x) + \bar{\omega}$, where $\bar{\omega} = E\omega$. Finally the case $f(x;\omega) = x + \omega$ is studied in more details and it is shown that the limiting policy satisfies the following inequalities $0 < c(y) < \bar{\omega}$ for $0 < y < +\infty$, and that $\lim_{y \to +\infty} c(y) = \bar{\omega}$.

0. INTRODUCTION

We will be concerned with one-good economy. The good can be used at any period of time for production or consumption. If $\underline{x}$ units are put into production in period $\underline{t}$ then $f^t(x;\omega^t)$ units become available as outputs in period $\underline{t+1}$; where ω^t is a random variable with known distribution. If $\underline{c}$ units are consumed in period $\underline{t}$ this produces $u^t(c)$ units of satisfaction or utility to the society in that period. Our main interest is the study of qualitative properties of optimal solutions for a problem in which we maximize the total expected utility accumulated in $\underline{t}$ periods.

In deterministic cases, i.e., the output is known with certainty, the basic tools are *prices* and *competitive policies*. D. Gale has suggested the introduction of similar concepts for the stochastic cases, and it turns out that after this is done, we get a better understanding of the problem and a powerful tool in the proof of theorems. Our main purpose here is to introduce the appropriate *price* concept and then exploit it in several directions to obtain new information on various stochastic problems.

This thesis is divided into 5 sections and 1 appendix. In Section 1 we introduce the concepts of *price* and *competitive policy*, and it is shown using the fundamental equation of Dynamic Programming that *competitive* policy is optimal. In Section 2 for $f(x;\omega)$ and $u(c)$ increasing, differentiable and strictly concave, optimality conditions are obtained and it is shown that any optimal policy is *competitive*. Section 3 is devoted to the study of the basic properties of prices and optimal policies. In Section 4 we study the case $f(x;\omega) = g(x) + \omega$ which has applications to the income fluctuation problem considered by M. E. Yaari [16] and B. Miller [9]. In this section we generalize one result obtained by L. J. Mirman [10] which permits us to compare optimal consumption policies with the corresponding policies of a deterministic case in which $f(x;\omega) = g(x) + \bar{\omega}$, where $\bar{\omega} = E\omega$. Bounds for limiting policies and sufficient conditions for capital growing as it is defined in E. S. Phelps [11] are also obtained. Finally in Section 5 we consider the "income fluctuation problem" in which $f(x;\omega) = x + \omega$. This problem was introduced by M. E. Yaari [16], and we obtain the surprising fact that the limiting policy satisfies the following inequality $0 < c(y) < \bar{\omega}$, for $0 < y < +\infty$ and it is also shown that $\lim_{y \to \infty} c(y) = \bar{\omega}$.

Lemmas, theorems and corollaries are numbered by section in the order that they appear and the end of a proof is represented by ■ .

1. COMPETITIVE POLICIES

In this section we consider a very general one-good economy. The good can be used at any period of time either for production or consumption. If x units are put into production in period t then $f^t(x;\omega^t)$ units of the same good are available as outputs in period $t+1$, where ω^t is a random variable with known distribution.* If c units are consumed in period t this produces $u^t(c)$ units of satisfaction, or utility, to the society in that period. The functions $f^t(x;\omega^t)$ and $u^t(c)$ are called the production and utility function. In general it will be assumed that the random variables ω^t are independent and $f^t(x;\omega^t) \geqq 0$, with strict inequality if $x > 0$. Furthermore we will also assume that $f^t(x;\omega^t)$ and $u^t(c)$ are continuous function.

Our problem is: given that we have y_0 units of the good at $t = 1$, how should we consume, invest, in each period in order that we maximize the total expected utility accumulated in T periods. We should mention that the definitions and theorems are very natural extensions to the stochastic case, of the definitions and theorems given in D. Gale [5] for the corresponding deterministic case.

In order to simplify the exposition, we will assume that ω^t are i.i.d., $u^t(c) \equiv \delta^t u(c)$ and $f^t(x;\omega^t) \equiv f(x;\omega)$ for all t where δ is a discount factor which can be any nonnegative number.

Let $V_t(y)$ be the maximum expected utility that we get when y is the stock available and we have t periods to go. Then, the usual Dynamic Programming formulation leads us to the following functional equations:

$$V_t(y) = \max_{\substack{c+x=y \\ c\geqq 0, x\geqq 0}} \{u(c) + \delta \underset{\omega}{E} V_{t-1}(f(x;\omega))\} \qquad \text{for } t > 1$$

---

*For a given x, $f^t(x;\omega^t)$ is a random variable.

and

$$V_1(y) = \max_{\substack{c+x=y \\ c \geq 0, x \geq 0}} \{u(c)\} .$$

Where E_ω stands for the expectation operator, and will be simply denoted by E when the random variable is easily understood from the text.

The amounts that we decide to save and consume when y is the stock available when t periods are left, will be denoted by the pair

$$(x_t(y), c_t(y)) .$$

Definition:

A T *period policy* is a sequence of pair of functions $\{(x_t(y), c_t(y))\}_{t=1,\ldots,T}$ such that

$$x_t(y) + c_t(y) = y$$

$$x_t(y) \geq 0 \ , \ c_t(y) \geq 0 \qquad \text{for} \quad t = 1, \ldots, T .^*$$

Definition:

A T *period policy* $\{(x_t(y), c_t(y))\}_{t=1,\ldots,T}$ is *optimal* if $(x_t(y), c_t(y))$ is a solution of

$$V_t(y) = \max_{\substack{c+x=y \\ c \geq 0, x \geq 0}} \{u(c) + \delta E V_{t-1}(f(x;\omega))\}$$

for all t and y.

Definition:

A T *period policy* $\{(x_t(y), c_t(y))\}_{t=1,\ldots,T}$ *is competitive* if there exists a sequence of nonnegative functions $\{p_t(y)\}_{t=1,\ldots,T}$ such that:

---

*It is also assumed that $x_t(y)$ and $c_t(y)$ are Borel measurable functions.

(1.1)
$$c_t(y) \text{ maximizes } u(c) - p_t(y)c \quad \text{subject to } c \geqq 0$$

(1.2)
$$x_t(y) \text{ maximizes } \delta E(p_{t-1}(f(x_t(y);\omega))f(x;\omega)) - p_t(y)x \quad \text{subject to } x \geqq 0$$

(1.3)
$$p_0(y) \equiv 0 .^*$$

Conditions (1.1) and (1.2) have natural economic interpretations. If the numbers $p_t(y)$ are taken as the prices of one unit of goods when y is the stock available and t periods are left then (1.2) obviously states that $x_t(y)$ is chosen to maximize the expected profit, while condition (1.1) is the classical condition of consumption theory which, at least when $u(c)$ is differentiable, says that price is equal to marginal utility and (1.3) is an ending condition.

<u>Theorem 1.1</u>

If a policy $\{(x_t(y),c_t(y))\}_{t=1,\ldots,T}$ is competitive then it is optimal, and furthermore y maximizes

$$V_t(z) - p_t(y)z \quad \text{subject to } z \geqq 0 .$$

<u>Proof</u>:

The proof will be done by induction.

I. For $T = 1$ we have,

$$V_1(y) = \max_{\substack{c+x=y \\ c\geqq 0,x\geqq 0}} \{u(c)\} .$$

Let (x,c) be any pair such that $x + c = y$, $x \geqq 0$, $c \geqq 0$. Now, from (1.1) we get

---

*It is also assumed that $p_t(y)$ is a Borel measurable function.

$$u(c) - p_1(y)c \leqq u(c_1(y)) - p_1(y)c_1(y) \quad \text{for all} \quad c \geqq 0 .$$

Thus,

$$u(c_1(y)) - u(c) \geqq p_1(y)(c_1(y) - c) \tag{1.4}$$

$$= p_1(y)(x - x_1(y)) .$$

From (1.2), we have

$$\delta E p_0(f(x_1(y);\omega))f(x_1(y);\omega) - p_1(y)x_1(y) \geqq$$

$$\delta E p_0(f(x_1(y);\omega))f(x;\omega) - p_1(y)x \quad \text{for all} \quad x \geqq 0 .$$

Hence, using (1.3)

$$p_1(y)(x - x_1(y)) \geqq 0 . \tag{1.5}$$

Finally, from (1.4) and (1.5), it follows that

$u(c_1(y)) \geqq u(c)$ for all $0 \leqq c \leqq y$. So, $(x_1(y),c_1(y))$ is optimal.

To show that y maximizes

$$V_1(z) - p_1(y)z$$

subject to $z \geqq 0$,

let $(x_1(z),c_1(z))$ be an optimal solution to

$$V_1(z) = \max_{\substack{c+x=z \\ c\geqq 0,x\geqq 0}} \{u(c)\} .$$

Now, from (1.4) and (1.5) replacing c by $c_1(z)$ and x by $x_1(z)$ we get

$$u(c_1(y)) - u(c_1(z)) \geqq p_1(y)(c_1(y) - c_1(z)) \tag{1.6}$$

and

$$p_1(y)(x_1(z) - x_1(y)) \geq 0 . \tag{1.7}$$

Adding (1.6) and (1.7) we have

$$u(c_1(y)) - p_1(y)y \geq u(c_1(z)) - p_1(y)z \quad \text{for all} \quad z \geq 0 .$$

So,

$$V_1(y) - p_1(y)y \geq V_1(z) - p_1(y)z \quad \text{for all} \quad z \geq 0 .$$

II. Assume it is true for $T = t - 1$, i.e., any competitive policy for a $t - 1$ period is optimal, and y maximizes

$$V_{t-1}(z) - p_{t-1}(y)z$$

$$\text{subject to } z \geq 0 .$$

Now,

$$V_t(y) = \max_{\substack{c+x=y \\ c\geq 0, x\geq 0}} \{u(c) + \delta EV_{t-1}(f(x;\omega))\} .$$

Let (x,c) be any pair such that $c + x = y$, $x \geq 0$, $c \geq 0$. Thus, from (1.1) we get

$$u(c) - p_t(y)c \leq u(c_t(y)) - p_t(y)c_t(y) \quad \text{for all} \quad c \geq 0 .$$

Or,

$$\begin{aligned} u(c_t(y)) - u(c) &\geq p_t(y)(c_t(y) - c) \\ &= p_t(y)(x - x_t(y)) . \end{aligned} \tag{1.8}$$

From (1.2) we get

$$\delta E p_{t-1}(f(x_t(y);\omega)f(x_t(y);\omega) - p_t(y)x_t(y) \geqq$$

$$\delta E p_{t-1}(f(x_t(y);\omega)f(x;\omega) - p_t(y)x \quad \text{for all } x \geqq 0 .$$

So,

$$p_t(y)(x - x_t(y)) \geqq \delta E p_{t-1}(f(x_t(y);\omega)f(x;\omega) - \delta E p_{t-1}(f(x_t(y);\omega)f(x_t(y);\omega) . \tag{1.9}$$

Now, from the induction hypothesis, y maximizes

$$V_{t-1}(z) - p_{t-1}(y)z$$

$$\text{subject to } z \geqq 0 .$$

Thus,

$$V_{t-1}(f(x_t(y);\omega)) - p_{t-1}(f(x_t(y);\omega))f(x_t(y);\omega) \geqq$$

$$V_{t-1}(f(x;\omega)) - p_{t-1}(f(x_t(y);\omega))f(x;\omega) .$$

Or, taking the expected value, rearranging and multiplying by δ we get

$$\delta E p_{t-1}(f(x_t(y)\omega))f(x;\omega) - \delta E p_{t-1}(f(x_t(y);\omega)f(x_t(y);\omega) \geqq \delta E V_{t-1}(f(x;\omega)) - \delta E V_{t-1}(f(x_t(y);\omega)) . \tag{1.10}$$

From (1.8), (1.9) and (1.10) we get

$$u(c_t(y)) - u(c) \geqq \delta E V_{t-1}(f(x;\omega)) - \delta E V_{t-1}(f(x_t(y);\omega) .$$

Or,

$$u(c_t(y)) + \delta E V_{t-1}(f(x_t(y);\omega)) \geqq u(c) + \delta E V_{t-1}(f(x;\omega)) .$$

So, $(x_t(y),c_t(y))$ is optimal.

To show that y maximizes

$$V_t(z) - p_t(y)z$$

$$\text{subject to } z \geq 0 ,$$

let $(x_t(z), c_t(z))$ be an optimal solution to

$$V_t(z) = \max_{\substack{c+x=z \\ c\geq 0, x\geq 0}} \{u(c) + \delta EV_{t-1}(f(x;\omega))\} .$$

From (1.1) and (1.2) we have

(1.11) $$u(c_t(y)) - u(c_t(z)) \geq p_t(y)(c_t(y) - c_t(z)) .$$

and

(1.12) $$\delta Ep_{t-1}(f(x_t(y);\omega))f(x_t(y);\omega) - \delta Ep_{t-1}(f(x_t(y);\omega)f(x_t(z);\omega)$$
$$\geq p_t(y)(x_t(y) - x_t(z)) .$$

But from (1.10) replacing x by $x_t(z)$ and rearranging we get

(1.13) $$\delta EV_{t-1}(f(x_t(y);\omega)) - \delta EV_{t-1}(f(x_t(z);\omega)) \geq$$
$$\delta Ep_{t-1}(f(x_t(y);\omega))f(x_t(y);\omega) - \delta Ep_{t-1}(f(x_t(y);\omega))f(x_t(z);\omega) .$$

Thus, from (1.13) and (1.12) it follows that

(1.14) $$\delta EV_{t-1}(f(x_t(y);\omega)) - \delta EV_{t-1}(f(x_t(z);\omega) \geq p_t(y)(x_t(y) - x_t(z)) .$$

Now, adding (1.11) and (1.14) we get

$$u(c_t(y)) + \delta EV_{t-1}(f(x_t(y);\omega)) - u(c_t(z)) - \delta EV_{t-1}(f(x_t(z);\omega))$$
$$\geq p_t(y)(y - z)$$

or,

$$V_t(y) - p_t(y)y \geq V_t(z) - p_t(y)z \quad \text{for all } z \geq 0 \blacksquare$$

2. CASE OF CONCAVE UTILITY AND PRODUCTION FUNCTION

From now on we will assume that the functions $f(x;\omega)$ and $u(c)$ are strictly concave, increasing and differentiable. The reason for making this assumption, is to obtain the converse of Theorem 1.1.

First we will show that $V_t(y)$ inherit the basic properties of the function $u(c)$.

Theorem 2.1

The function $V_t(y)$ is increasing and strictly concave for all t .

Proof:

The first part is immediate, the second, will be done by induction.

I. For $t = 1$

$$V_1(y) = u(y)$$

since $u(c)$ is increasing. So, $V_1(y)$ is strictly concave, since by assumption $u(c)$ is strictly concave.

II. Assume that $V_{t-1}(y)$ is strictly concave. Now,

$$V_t(y) = \max_{\substack{c+x=y \\ c\geq 0, x\geq 0}} \{u(c) + \delta E V_{t-1}(f(x;\omega))\} \tag{2.1}$$

Pick y_1 , $y_2 \in [0,+\infty)$, and let $(x_t(y_1),c_t(y_1))$ and $(x_t(y_2),c_t(y_2))$ be the corresponding solutions to (2.1). Let, $\bar{y}$ be a convex combination of y_1 and y_2 , i.e., $\bar{y} = \lambda_1 y_1 + \lambda_2 y_2$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.
Then,

$$V_t(\bar{y}) = \max_{\substack{c+x=\bar{y} \\ c\geq 0, x\geq 0}} \{u(c) + \delta \phi_{t-1}(x)\} .$$

Where,

$$\phi_{t-1}(x) \equiv EV_{t-1}(f(x;\omega)) ,$$

which is strictly concave from the induction hypothesis. Since,

$$(\lambda_1 x_t(y_1) + \lambda_2 x_t(y_2) , \lambda_1 c_t(y_1) + \lambda_2 c_t(y_2))$$

is a feasible solution to (2.1), it follows that,

$$V_t(\bar{y}) \geq u(\lambda_1 c_t(y_1) + \lambda_2 c_t(y_2)) + \delta\phi_{t-1}(\lambda_1 x_t(y_1) + \lambda_2 x_t(y_2))$$

$$> \lambda_1 u(c_t(y_1)) + \lambda_2 u(c_t(y_2)) + \delta\lambda_1\phi_{t-1}(x_t(y_1)) + \delta\lambda_2\phi_{t-1}(x_t(y_2))$$

$$= \lambda_1 V_t(y_1) + \lambda_2 V_t(y_2) \blacksquare$$

It is an immediate consequence of the previous theorem that the solution of (2.1) is unique. Hence, $x_t(y)$ and $c_t(y)$ are well defined functions.

<u>Theorem 2.2</u>

The function $V_t(y)$ is differentiable for $y > 0$, and for all t .

<u>Proof</u>:

I. for $t = 1$ we have,

$$V_1(y) = u(y) .$$

So, $V_t(y)$ is differentiable for $y > 0$, since by assumption $u(c)$ is differentiable for $c > 0$.

II. Assume that $V_{t-1}(y)$ is differentiable for $y > 0$. To prove that $V_t(y)$ is differentiable for $y > 0$, we must show that $V_t(y)$ has a unique support for all $y > 0$.

Let p be a support of $V_t(y)$ at y . Then,

$$V_t(y') - V_t(y) \leq p(y' - y), \quad \text{for all } y' \geq 0 .$$

Or,

(2.2) $$V_t(y') - py' \leqq V_t(y) - py .$$

Now, choose any c' , $x' \geqq 0$ and let $y' = c' + x'$. Then,

(2.3) $$u(c') + \delta\phi_{t-1}(x') \leqq V_t(y')$$

from the definition of $V_t(y)$. So, from (2.2) and (2.3) we get

(2.4) $$\begin{aligned} u(c') + \delta\phi_{t-1}(x') - p(c' + x') &\leqq V_t(y') - py' \\ &\leqq V_t(y) - py \\ &= u(c_t(y)) + \delta\phi_{t-1}(x_t(y)) - p(x_t(y) + c_t(y)) . \end{aligned}$$

Thus,

(2.5) $$u(c') - pc' + \delta\phi_{t-1}(x') - px' \leqq u(c_t(y)) - pc_t(y) + \delta\phi_{t-1}(x_t(y)) - px_t(y) .$$

Now, either $x_t(y) > 0$ or $c_t(y) > 0$ since $y = c_t(y) + x_t(y) > 0$. Suppose $c_t(y) > 0$, then for $x' = x_t(y)$ in (2.5), we get

(2.6) $$u(c') - pc' \leqq u(c_t(y)) - pc_t(y) \quad \text{for all } c' \geqq 0 .$$

Or, $u(c) - pc$ has a maximum at $c_t(y) > 0$. Then,

(2.7) $$u'(c_t(y)) = p .$$

In an analagous way for $x_t(y) > 0$ we get

(2.8) $$\delta\phi_{t-1}(x') - px' \leqq \delta\phi_{t-1}(x_t(y)) - px_t(y) , \quad \text{for all } x' \geqq 0 .$$

Or, $\phi_{t-1}(x) - px$ has a maximum at $x_t(y) > 0$. So,

(2.9) $$p = \delta\phi'_{t-1}(x_t(y)) \blacksquare$$

In spite of the fact that $V_t(y)$ is differentiable for $y > 0$, the function $c_t(y)$ need not be. Consider the following case:

$$f(x;\omega) = x + \omega ,$$

$$u(c) = \log c$$

and, ω assumes the values $a_1 = 1$, $a_2 = 2$ with corresponding probabilities $\pi_1 = 1/2$ and $\pi_2 = 1/2$. Now,

$$V_2(y) = \max_{\substack{c+x=y \\ c\geq 0, x\geq 0}} \{\log c + E \log (x + \omega)\} .$$

Which is equivalent to maximize

$$\log c + \sum_{i=1}^{2} \pi_i \log (y - c + a_i) \quad \text{for } c \geq 0 . \tag{2.10}$$

Taking the derivatives in (2.10), we get

$$\frac{1}{c} = \sum_{i=1}^{2} \frac{\pi_i}{y - c + a_i} . \tag{2.11}$$

Making $c = y$ in (2.11) we get a critical value

$$\bar{y} = \frac{1}{\sum_{i=1,2} \frac{\pi_i}{a_i}} = \frac{4}{3} ,$$

below which, we consume all the stock of goods available, i.e.,

$$c_2(y) = y \quad \text{for } 0 \leq y \leq \bar{y} .$$

For $y \geq \bar{y}$ the solution of (2.10) is given by

$$c_2(y) = \frac{6y + 9 - \sqrt{4y^2 + 12y + 17}}{8}$$

Now, taking the right and left hand side derivatives of $c_2(y)$, at the point $\bar{y}$ we get:

$$\left.\frac{dc_2(y)}{dy}\right|_{\bar{y}}^{+} = \frac{1280}{2432}$$

and

$$\left.\frac{dc_2(y)}{dy}\right|_{\bar{y}}^{+} = 1 .$$

Which shows that $c_2(y)$ is nondifferentiable at the point $\bar{y}$. This example could be generalized for the case in which ω is any discrete random variable with a finite range.

To prove the converse of Theorem 1.1, we need to define the prices $p_t(y)$. The natural candidates are of course, $V_t'(y)$.

For some problems, when we follow the optimal policy, the level of stock is always positive. Hence, for those cases we do not need to define the prices $p_t(0)$. On the other hand, when this is not the case, in order to define the prices $p_t(0)$, we must show that the functions $V_t(y)$ have a derivative from the right at the point $y = 0$.

If $f'(o;\omega) = +\infty$, or $u'(o) = \infty$ and $f(o;\omega) = 0$ with positive probability, it can be shown from (2.7) and (2.9), that for the optimal policy $x_t(y) > 0$ whenever $y > 0$. In the Appendix it is shown that the functions $V_t(y)$ are differentiable from the right for $y = 0$, whenever $u'(0) < +\infty$ and $f'(0;\omega) < +\infty$.

Definition:

A problem is admissible whenever either the level of stock is always positive for the optimal policy, or the functions $V_t(y)$ are differentiable for $y = 0$.

Theorem 2.3

If a problem is admissible then the optimal policy is competitive. Furthermore $(x_t(y), c_t(y))$ satisfies the following relations

(2.12) $$p_t(y) \geqq u'(c_t(y)) \quad \text{with equality if } c_t(y) > 0 .$$

and

(2.13) $$p_t(y) \geqq \delta E p_{t-1}(f(x_t(y);\omega)) f'(x_t(y);\omega))$$

with equality if $x_t(y) > 0$. Where $p_t(y) = V_t'(y)$.

Proof:

In (2.5) replacing p by $p_t(y)$, $\phi_{t-1}(x')$ by $EV_{t-1}(f(x';\omega))$ we get

(2.14) $$u(c') - p_t(y)c' + \delta EV_{t-1}(f(x';\omega)) - p_t(y)x'$$

$$\leqq u(c_t(y)) - p_t(y)c_t(y) + \delta EV_{t-1}(f(x_t(y);\omega) - p_t(y)x_t(y) .$$

Now, setting $x' = x_t(y)$ we get

(2.15) $$u(c') - p_t(y)c' \leqq u(c_t(y)) - p_t(y)c_t(y) \quad \text{for all } c' \geqq 0 .$$

So, the competitive condition (1.1) holds. To obtain (2.12) we observe that $c_t(y)$ maximizes

$$u(c) - p_t(y)c$$

$$\text{subject to } c \geqq 0 .$$

Hence,

$$p_t(y) \geqq u'(c_t(y)) ,$$

with equality if $c_t(y) > 0$. In a similar way, setting in (2.14) $c' = c_t(y)$ we get

$$\delta EV_{t-1}(f(x';\omega)) - p_t(y)x'$$

$$\leqq \delta EV_{t-1}(f(x_t(y);\omega) - p_t(y)x_t(y) \ .$$

That is, $x_t(y)$ maximizes

$$\delta EV_{t-1}(f(x;\omega)) - p_t(y)x$$

subject to $x \geqq 0$.

So,

$$p_t(y) \geqq \delta Ep_{t-1}(f(x_t(y);\omega))f'(x_t(y);\omega) \ , \tag{2.16}$$

with equality if $x_t(y) > 0$, which is (2.13).

Now, multiplying (2.16) by $x - x_t(y)$* , and using the concavity of $f(x;\omega)$ we have

$$(x - x_t(y))p_t(y) \geqq \delta Ep_{t-1}(f(x_t(y);\omega))f'(x_t(y);\omega)(x - x_t(y))$$

$$\geqq \delta Ep_{t-1}(f(x_t(y);\omega))(f(x;\omega) - f(x_t(y);\omega)) \ .$$

Or,

$$Ep_{t-1}(f(x;\omega))f(x;\omega)) - p_t(y)x \leqq Ep_{t-1}(f(x_t(y);\omega)) - p_t(y)x_t(y)$$

for all $x \geqq 0$, that is, competitive condition (1.2) holds ∎

From now on the derivatives of $V_t(y)$ will be denoted by $p_t(y)$, and will be called prices.

The prices $p_t(y)$ are continuous decreasing functions since they are the derivatives of concave functions. The continuity is defined on $(0,+\infty)$, and on $[0,+\infty)$ if $V_t(y)$ are differentiable at $y = 0$.**

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*If $x_t(y) > 0$, (2.16) is an equality; and for $x_t(y) = 0$, $x - x_t(y) \geqq 0$ and hence the direction of the inequality does not change.

**For a proof see W. Fenchel (4).

3. PROPERTIES OF $x_t(y)$, $c_t(y)$ AND $p_t(y)$.

From now on we will assume that all problems are admissible. In all proofs we will not make any distinction between the cases in which the level of stock is always positive and those for which $p_t(0)$ is well defined.

Theorem 3.1

The functions $x_t(y)$, $c_t(y)$ are nondecreasing functions. Furthermore, $x_t(y)(c_t(y))$ is increasing whenever $x_t(y) > 0(c_t(y) > 0)$.

Proof:

Let $y' > y \geq 0$. First we will show that $c_t(y') \geq c_t(y)$. Consider the following cases:

Case i.

$c_t(y') > 0$ and $c_t(y) > 0$. Then from the optimality conditions (2.12), it follows that

$$p_t(y') = u'(c_t(y'))$$

and

$$p_t(y) = u'(c_t(y)) .$$

Thus,

$$u'(c_t(y')) < u'(c_t(y))$$

by monotonicity of $p_t(y)$.

So,

$$c_t(y') > c_t(y)$$

by monotonicity of $u'(c)$.

Case ii.

$c_t(y') \geqq 0$ and $c_t(y) = 0$. It immediately follows that

$$c_t(y') \geqq c_t(y) .$$

Now, we will show that $c_t(y') = 0$ and $c_t(y) > 0$ is not possible. From (2.12) we have

$$p_t(y') \geqq u'(0)$$

and

$$p_t(y) = u'(c_t(y)) .$$

But

$$p_t(y') < p_t(y) .$$

Hence,

$$u'(0) < u'(c_t(y)) .$$

Which is a contradiction. An analagous proof holds for $x_t(y)$, except that we use the optimality condition (2.13).■

Theorem 3.2*

The functions $x_t(y)$ and $c_t(y)$ are continuous.

Proof:

From Theorem 3.1, $x_t(y)$ and $c_t(y)$ are nondecreasing functions. Thus for $\bar{y} \geqq 0$

$$\lim_{y \uparrow \bar{y}} c_t(y) = c_t(\bar{y}^-) \leqq c_t(\bar{y}^+) = \lim_{y \downarrow \bar{y}} c_t(y)$$

---

*The same proof given by W. A. Brock and L. J. Mirman [2].

and

$$\lim_{y \uparrow \bar{y}} x_t(y) = x_y(\bar{y}^-) \leqq x_t(\bar{y}^+) = \lim_{y \downarrow \bar{y}} x_t(y) .$$

However

$$x_t(y) + c_t(y) = y .$$

Hence,

$$x_t(\bar{y}^-) + c_t(\bar{y}^-) = y$$

and

$$x_t(\bar{y}^+) + c_t(\bar{y}^+) = y .$$

From which we get

$$[x_y(\bar{y}^-) - x_t(\bar{y}^+)] + [c_t(\bar{y}^-) - c_t(\bar{y}^+)] = 0 .$$

Since, both $x_t(y)$ and $c_t(y)$ are nondecreasing in y, both terms of the sum must be equal to zero. Hence

$$c_t(\bar{y}^-) = c_t(\bar{y}^+)$$

and

$$x_t(\bar{y}^-) = x_t(\bar{y}^+) .$$

Which implies the continuity of $c_t(y)$ and $x_t(y)$.

<u>Theorem 3.3</u>.

$p_t(y) \geqq p_{t-1}(y)$ for all $t \geqq 2$.*

---

*In general we need to prove this only for $y > 0$. By continuity of $p_t(y)$ it will follow for $y = 0$.

Proof:

The proof will be done by induction.

I. For $t = 2$, we have from the optimality conditions (2.12) that

$$p_2(y) \geqq u'(c_2(y))$$

and

$$p_1(y) = u'(y) .$$

Hence,

$$p_2(y) \geqq p_1(y)$$

since, $c_2(y) \leqq y$ and $u'(c)$ is a decreasing function.

II. Assume that it is true for $t - 1$, i.e., $p_{t-1}(y) \geqq p_{t-2}(y)$. We need to consider two cases depending on the values that $x_t(y)$ and $x_{t-1}(y)$ assume.

Case i.

$0 \leqq x_t(y) \leqq y$, $0 < x_{t-1}(y) \leqq y$. Then, from the optimality condition (2.13) we get

(3.1) $$p_t(y) \geqq \delta E p_{t-1}(f(x_t(y);\omega))f'(x_t(y);\omega)$$

and

(3.2) $$p_{t-1}(y) = \delta E p_{t-2}(f(x_{t-1}(y);\omega))f'(x_{t-1}(y);\omega)) .$$

Now, suppose that $p_t(y) < p_{t-1}(y)$. Hence,

(3.3) $$c_t(y) \geqq c_{t-1}(y)$$

and consequently

(3.4) $$x_t(y) \leqq x_{t-1}(y) .$$

From (3.1) and the induction hypothesis we get

(3.5)
$$\begin{aligned} p_t(y) &\geqq \delta E p_{t-1}(f(x_t(y);\omega))f'(x_t(y);\omega)) \\ &\geqq \delta E p_{t-2}(f(x_t(y);\omega))f'(x_t(y);\omega)) \end{aligned}$$

So, from (3.5), (3.4) and the monotonicity of $p_t(y)$ and $f'(x;\omega)$, we have

$$\begin{aligned} p_t(y) &\geqq \delta E p_{t-2}(f(x_{t-1}(y);\omega))f'(x_{t-1}(y);\omega) \\ &= p_{t-1}(y) \quad \text{by} \quad (3.2). \end{aligned}$$

Which is a contradiction.

Case ii.

$0 \leqq x_t(y) \leqq y$, $x_{t-1}(y) = 0$. From (2.12) we have,

$$p_t(y) \geqq u'(c_t(y))$$

and

$$p_{t-1}(y) = u'(y)$$

Hence,

$$p_t(y) \geqq p_{t-1}(y)$$

since, $c_t(y) \leqq y$ and $u'(c)$ is a decreasing function ∎

It is an immediate consequence the following corollary

Corollary 3.4.

$c_t(y) \leqq c_{t-1}(y)$ and $x_t(y) \geqq x_{t-1}(y)$ for all $t \geqq 2$.

We can now prove a result which will be needed later when we consider infinite horizon problems.

Theorem 3.5.

The limiting policy $x(y) = \lim_{t\to\infty} x_t(y)$ and $c(y) = \lim_{t\to\infty} c_t(y)$ exist and are nondecreasing continuous functions. Furthermore the sequence of functions $x_t(y)$ and $c_t(y)$ converge uniformly in any finite closed interval.

Proof:

From Corollary 3.4 we have

$$x_{t+1}(y) \geqq x_t(y) \quad \text{for all} \quad t\ ,\ y \geqq 0$$

and

$$c_{t+1}(y) \leqq c_t(y) \quad \text{for all} \quad t\ ,\ y \geqq 0\ .$$

But $c_t(y) \geqq 0$ and $x_t(y) \leqq y$ for all t. Hence,

$$x(y) = \lim_{t\to\infty} x_t(y)$$

and

$$c(y) = \lim_{t\to\infty} c_t(y)$$

are well defined functions. Now, since $x_t(y)$, $c_t(y)$ are nondecreasing continuous functions, it follows that $x(y)$ and $c(y)$ are also nondecreasing functions. The continuity of $x(y)$ and $c(y)$ will follow by the same arguments used in Theorem 3.2. From the fact that a sequence of monotonic functions converging pointwise to a continuous function on a closed interval converges uniformly the theorem is finally proved ∎

The derivatives of the utility functions $u(c) = \log c$, $u(c) = c^{\gamma} 0 < \gamma < 1$, $u(c) = ke^{-c}$ and the production function $f(x;\omega) = \omega x^{\gamma} 0 < \gamma < 1$ usually found in the literature are convex functions. For those cases we can state the following theorem.

<u>Theorem 3.6</u>.

If $u'(c)$ and $f'(x;\omega)$ are convex functions then $p_t(y)$ is also convex for all t .

<u>Proof</u>:

For $t = 1$ it is immediate, since

$$p_1(y) = u'(c) .$$

Now, assume that $p_{t-1}(y)$ is convex; and suppose that $p_t(y)$ is not convex, i.e., there exist y_1 and $y_2 \varepsilon (0,+\infty)$, and λ_1 , $\lambda_2 \geqq 0$, $\lambda_1 + \lambda_2 = 1$ such that for $\bar{y} = \lambda_1 y_1 + \lambda_2 y_2$ we have

(3.6) $$p_t(\bar{y}) > \lambda_1 p_t(y_1) + \lambda_2 p_t(y_2) .$$

We need to consider three cases depending on the values that $x_t(y)$ assumes at y_1 , y_2 and $\bar{y}$.

<u>Case i</u>.

$0 \leqq x_t(y_1) \leqq y_1$, $0 \leqq x_t(y_2) \leqq y_2$, $0 < x_t(\bar{y}) < \bar{y}$, and whenever the three inequalities holds. Then, from (2.12) and (2.13) it follows that

(3.7) $$\begin{aligned} p_t(y_1) &\geqq u'(c_t(y_1)) \\ p_t(y_2) &\geqq u'(c_t(y_2)) \\ p_t(\bar{y}) &= u'(c_t(\bar{y})) \end{aligned}$$

and

$$\lambda_1 p_t(y_1) \geqq \delta\lambda_1 Ep_{t-1}(f(x_t(y_1);\omega))f'(x_t(y_1);\omega))$$

(3.8)
$$\lambda_2 p_t(y_2) \geqq \delta\lambda_2 Ep_{t-1}(f(x_t(y_2);\omega))f'(x_t(y_2);\omega))$$

$$p_t(\bar{y}) = \delta Ep_{t-1}(f(x_t(\bar{y});\omega))f'(x_t(\bar{y});\omega))$$

Now, from (3.6) and (3.7) we have

$$u'(c_t(\bar{y})) > \lambda_1 u'(c_t(y_1)) + \lambda_2 u'(c_t(y_2))$$

$$\geqq u'(\lambda_1 c_t(y_1) + \lambda_2 c_t(y_2))$$

by convexity of $u'(c)$. Thus,

$$c_t(\bar{y}) \leq \lambda_1 c_t(y_1) + \lambda_2 c_t(y_2)$$

by monotonicity of $u'(c)$. And consequently

(3.9)
$$x_t(\bar{y}) \geq \lambda_1 x_t(y_1) + \lambda_2 x_t(y_2) .$$

Now, from (3.6) and (3.8) we get

$$p_t(\bar{y}) > \delta\lambda_1 Ep_{t-1}(f(x_t(y_1);\omega))f'(x_t(y_1);\omega))$$

(3.10)
$$+ \delta\lambda_2 Ep_{t-1}(f(x_t(y_2);\omega))f'(x_t(y_2);\omega))$$

$$\geqq \delta Ep_{t-1}(f(\lambda_1 x_t(y_1) + \lambda_2 x_t(y_2));\omega))f'(\lambda_1 x_t(y_1) + \lambda_2 x_t(y_2);\omega)) .$$

Where in the last inequality we used the fact that

$$\phi'_{t-1}(x) = Ep_{t-1}(f(x;\omega))f'(x;\omega)$$

is convex.* Finally from (3.10) and (3.9) we have

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*$\psi(x) \equiv p_{t-1}(f(x;\omega))f'(x;\omega)$ as a product of nonnegative, decreasing and convex functions is also convex. So, $Ep_{t-1}(f(x;\omega))f'(x;\omega)$ is convex.

$$p_t(\bar{y}) > \delta E p_{t-1}(f(x_t(\bar{y});\omega))f'(x_t(\bar{y});\omega)$$

$$= p_t(\bar{y})$$

by (3.7). Which is a contradiction.

<u>Case ii</u>.

$x_t(y_1) = 0$, $0 \leq x_t(y_2) < y_2$, $x_t(\bar{y}) = 0$. Then from (2.12) we have

$$p_t(y_1) = u'(y_1)$$

$$p_t(y_2) = u'(c_t(y_2)) \tag{3.11}$$

$$p_t(\bar{y}) = u'(\bar{y}) .$$

So, from (3.6) and (3.11) we get

$$u'(y) > \lambda_1 u'(y_1) + \lambda_2 u'(c_t(y_2))$$

$$\geq \lambda_1 u'(y_1) + \lambda_2 u'(y_2)$$

$$\geq u'(\bar{y}) .$$

<u>Case iii</u>.

$x_t(y_1) = y_1$, $0 < x_t(y_2) \leq y_2$ and $x_t(\bar{y}) = \bar{y}$. The proof is the same as in Case i except that (3.9) is obtained in the following way:

$$\bar{y} = \lambda_1 y_1 + \lambda_2 y_2 .$$

But, $x_t(y_1) = y_1$, $x_t(\bar{y}) = \bar{y}$ and $x_t(y_2) \leq y_2$. So,

$$x_t(\bar{y}) \geq \lambda_1 x_t(y_1) + \lambda_2 x_t(y_2) \blacksquare$$

4. THE CASE OF $f(x;\omega) = g(x) + \omega$

One reason for studying this case is its application to the "income fluctuation problem" in which a consumer is faced with the following situation: He has a t period planning horizon and at each period he must decide how much to consume and save, in order to maximize his total expected utility. If x is the amount saved at period t, then the available income in the next period is $rx + \omega^t$, i.e., $f(x;\omega^t) = rx + \omega^t$, where ω^t is a nonnegative bounded random variable and $r \geq 1$ is the interest factor of a riskless investment. B. Miller [9] considers the case in which $u(c) = \log c$, borrowing is allowed with some constraints, ω^t are independent and special constraints are imposed on the consumption. M. E. Yaari [16] considers the case $r = 1$, ω^t are i.i.d., borrowing is allowed at zero interest rate and no constraints are imposed on the consumption, (this case will be considered in more detail in Section 5). The "income fluctuation problem" differs from those considered by E. S. Phelps [10] and N. H. Hakansson [8] in that they consider r as a random variable and ω^t is known with certainty. From now on when we talk about an "income fluctuation problem," it is understood that no borrowing is allowed, $0 \leq c_t(y) \leq y$ and ω^t are i.i.d. random variables with range $[a,A]$, $a \geq 0$ and $A < +\infty$.*

Our main interest in this section is to obtain bounds for the limiting policy $c(y)$ (by comparing it with various cases of perfect certainty), which we know from Theorem 3.5, is a continuous nondecreasing function. First we introduce some notation:

Notation

i) If $f(x;\omega) = g(x) + a$, then the corresponding prices and policies will be denoted by $p_t^a(y)$ and $\left(x_t^a(y), c_t^a(y)\right)$.

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*The method used here could be applied to the stationary case of B. Miller [9] with borrowing.

ii) If $f(x;\omega) = g(x) + \bar{\omega}$, then the corresponding prices and policies will be denoted by $p_t^{\bar{\omega}}(y)$ and $\left(x_t^{\bar{\omega}}(y), c_t^{\bar{\omega}}(y)\right)$.

iii) If $f(x;\omega) = g(x) + A$, then the corresponding prices and policies will be denoted by $p_t^A(y)$ and $\left(x_t^A(y), c_t^A(y)\right)$,

iv) and as the usual $p_t(y)$ and $(x_t(y), c_t(y))$ stands for the case $f(x;\omega) = g(x) + \omega$.

The optimality condition (2.12) and (2.13) are for i, ii, iii

$$\text{(4.1)} \qquad \begin{aligned} p_t^v(y) &\geqq \delta g'\left(x_t^v(y)\right) p_{t-1}^v\left(g\left(x_t^v(y)\right) + v\right) \\ p_t^v(y) &\geqq u'\left(c_t^v(y)\right) \end{aligned}$$

with $v = a$, $\bar{\omega}$ and A respectively, and for iv,

$$\text{(4.2)} \qquad \begin{aligned} p_t(y) &\geqq \delta g'(x_t(y)) E p_{t-1}(g(x_t(y)) + \omega) \\ p_t(y) &\geqq u'(c_t(y)) . \end{aligned}$$

Now we will prove a theorem that will give a lower bound for $c_t(y)$, i.e., we will compare $c_t(y)$ with the optimal consumption policy $c_t^a(y)$ of a "pessimistic consumer" who assumes $\omega \equiv a$.

<u>Theorem 4.1</u>

$p_t(y) \leqq p_t^a(y)$ for all t , and consequently $c_t(y) \geqq c_t^a(y)$ for all t .

<u>Proof</u>:

The proof will be done by induction.

I. For $t = 1$ it is true since $p_1(y) = p_1^a(y) = u'(y)$.

II. Assume it is true for $t - 1$, i.e., $p_{t-1}(y) \leqq p_{t-1}^a(y)$. We need to consider two cases depending on the values that $x_t(y)$ assumes.

Case i.

$x_t(y) = 0$. Then $c_t(y) = y$ and $p_t(y) = u'(y)$. So, $p_t(y) \leq p_t^a(y)$ since $c_t^a(y) \leq y$ and $u'(c)$ is decreasing function.

Case ii.

$x_t(y) > 0$. By contradiction suppose that $p_t(y) > p_t^a(y)$. Then,

(4.3) $$c_t(y) \leq c_t^a(y)$$

and

(4.4) $$x_t(y) \geq x_t^a(y) .$$

Now, from the optimality condition (4.2) we get

(4.5) $$p_t(y) = \delta g'(x_t(y))Ep_{t-1}(g(x_t(y)) + \omega) .$$

Thus,

(4.6) $$p_t(y) \leq \delta g'\left(x_t^a(y)\right)Ep_{t-1}\left(g\left(x_t^a(y)\right) + \omega\right) ,$$

since from (4.4) $x_t(y) \geq x_t^a(y)$ and $g'(x)$ and $p_{t-1}(y)$ are decreasing functions. Now, from (4.6) using the fact that $p_{t-1}(y)$ is decreasing, and the induction hypothesis we get

$$\begin{aligned} p_t(y) &\leq \delta g'\left(x_t^a(y)\right)p_{t-1}\left(g\left(x_t^a(y)\right) + a\right) \\ &\leq \delta g'\left(x_t^a(y)\right)p_{t-1}^a\left(g\left(x_t^a(y)\right) + a\right) \\ &\leq p_t^a(y) \quad \text{by (4.1)} , \end{aligned}$$

which is a contradiction.■

An immediate consequence is the following corollary.

Corollary 4.2.

If $c(y) = \lim_{t\to\infty} c_t(y)$ and $c^a(y) = \lim_{t\to\infty} c_t^a(y)$ then $c^a(y) \leqq c(y)$.

In a similar way we can prove the following theorem and corollary that compare $c_t(y)$ with the consumption $c_t^A(y)$ of an "optimistic consumer" who assumes $\omega \equiv A$.

Theorem 4.3.

$p_t(y) \geqq p_t^A(y)$ for all t and consequently $c_t(y) \leqq c_t^A(y)$ for all t .

Corollary 4.4.

If $c(y) = \lim_{t\to\infty} c_t(y)$ and $c^A(y) = \lim_{t\to\infty} c_t^A(y)$ then $c(y) \leqq c^A(y)$.

If we consider the cases in which $g'(x)$ and $u'(c)$ are convex functions, we will be able to compare, the optimal policy $c_t(y)$ with the corresponding solution $c_t^{\bar{\omega}}(y)$ of the deterministic problem in which ω is replaced by $\bar{\omega} = E\omega$, i.e., we will show that $c_t(y) \leqq c_t^{\bar{\omega}}(y)$. This result generalizes the one obtained by L. J. Mirman [10] page 184.

Theorem 4.5.

If $g'(x)$ and $u'(c)$ are convex functions then $p_t(y) \geqq p_t^{\bar{\omega}}(y)$, and consequently $c_t(y) \leqq c_t^{\bar{\omega}}(y)$.

I. It is true for $t = 1$, since $p_1(y) = p_1^{\bar{\omega}}(y) = u'(y)$.

II. Assume that it is true for $t - 1$. We need to consider two cases depending on the values that $x_t^{\bar{\omega}}(y)$ assumes.

Case i.

$x_t^{\bar{\omega}}(y) > 0$. By contradiction suppose that $p_t(y) < p_t^{\bar{\omega}}(y)$. Thus,

$$c_t(y) \geqq c_t^{\bar{\omega}}(y) , \tag{4.7}$$

and consequently

$$x_t(y) \leqq x_t^{\bar{\omega}}(y) . \tag{4.8}$$

Now, from (4.2), (4.8) and the fact that $g(x)$ is increasing, $g'(x)$ and $p_{t-1}(y)$ are decreasing functions we get

$$\begin{aligned} p_t(y) &\geqq \delta g'(x_t(y))Ep_{t-1}(g(x_t(y)) + \omega) \\ &\geqq \delta g'(x_t^{\bar{\omega}}(y))Ep_{t-1}(g(x_t^{\bar{\omega}}(y)) + \omega) . \end{aligned} \tag{4.9}$$

But, from Theorem 3.6 we have that $p_t(y)$ is a convex function. Hence from (4.9) using the convexity and the induction hypothesis we get

$$\begin{aligned} p_t(y) &\geqq \delta g'(x_t^{\bar{\omega}}(y))Ep_{t-1}(g(x_t^{\bar{\omega}}(y)) + \omega) \\ &\geqq \delta g'(x_t^{\bar{\omega}}(y))p_{t-1}(g(x_t^{\bar{\omega}}(y)) + \bar{\omega}) \\ &\geqq \delta g'(x_t^{\bar{\omega}}(y))p_{t-1}^{\bar{\omega}}(g(x_t^{\bar{\omega}}(y)) + \bar{\omega}) \\ &= p_t^{\bar{\omega}}(y) \end{aligned}$$

by (4.2), which is a contradiction.

<u>Case ii</u>.

$x_t^{\bar{\omega}}(y) = 0$. Then,

$$c_t^{\bar{\omega}}(y) = y$$

and

$$p_t^{\bar{\omega}}(y) = u'(y) .$$

Now,

$$p_t(y) \geqq u'(c_t(y)) .$$

So,

$$p_t(y) \geq p_t^{\bar{\omega}}(y)$$

since $c_t(y) \leq y$ and $u'(c)$ is a decreasing function ∎

And it is an immediate consequence the following corollary.

<u>Corollary 4.6</u>.

If $c^{\bar{\omega}}(y) = \lim_{t\to\infty} c_t(y)$ then $c(y) \leq c^{\bar{\omega}}(y)$.

<u>Theorem 4.7</u>.

The limiting policy satisfies the following inequalities

(4.10) $$c^a(y) \leq c(y) \leq c^A(y) .$$

Moreover if $u'(c)$ and $g'(x)$ are convex functions then

(4.11) $$c^a(y) \leq c(y) \leq c^{\bar{\omega}}(y) .$$

<u>Proof</u>:

It is an immediate consequence of Corollary 4.2, Corollary 4.4 and Corollary 4.6 ∎

<u>Example</u>:

As an example of application of the last theorem consider the following "income fluctuation problem"

$$f(x;\omega) = rx + \omega , \; r \geq 1$$

$$u(c) = \lambda c^\gamma , \; \lambda > 0 \quad 0 < \gamma < 1$$

and we assume that $\delta r^\gamma < 1$, $\delta r > 1$ and $\omega \in [0,A]$.

The limiting policies $c^{a}(y)$ and $c^{\bar{\omega}}(y)$ can be computed* and they are

$$c^{a}(y) = \left(1 - (\delta r^{\gamma})^{\frac{1}{1-\gamma}}\right)y$$

and

$$\begin{cases} c^{\bar{\omega}}(y) \leqq y & \text{for} \quad 0 \leqq y \leqq \bar{\omega} \\ c\dot{\omega}(y) = \left(1 - (\delta r^{\gamma})^{\frac{1}{1-\gamma}}\right)\left(y + \dfrac{\bar{\omega}}{\gamma - 1}\right) & \text{for} \quad y \geqq \bar{\omega} . \end{cases}$$

so,

$$\begin{cases} \left(1 - (\delta r^{\gamma})^{\frac{1}{1-\gamma}}\right)y \leqq c(y) \leqq y & \text{for} \quad 0 \leqq y \leqq \bar{\omega} . \\ \left(1 - (\delta r^{\gamma})^{\frac{1}{1-\gamma}}\right)y \leqq c(y) \leqq \left(1 - (\delta r^{\gamma})^{\frac{1}{1-\gamma}}\right)\left(y + \dfrac{\bar{\omega}}{r - 1}\right) \end{cases}$$

for $y \geqq \bar{\omega}$.

<u>Lemma 4.8</u>.

If the sequence of numbers $\{p_t(y)\}$ is bounded above then $x(y) = \lim_{t\to\infty} x_t(y)$ satisfies the following relation

(4.12) $$p_t(y) \geqq \delta g'(x(y))Ep(g(x(y)) + \omega) ,$$

where $p(y) = \lim_{t\to\infty} p_t(y)$.

<u>Proof</u>:

From (4.2) we have,

$$p_t(y) \geqq \delta g'(x_t(y))Ep_{t-1}(g(x_t(y)) + \omega) .$$

But, from Corollary 3.4 $x_t(y) \leqq x(y)$, so

(4.13) $$p_t(y) \geqq \delta g'(x(y))Ep_{t-1}(g(x(y)) + \omega) .$$

---

*E. S. Phelps [11] page 741.

Now, the left hand side of (4.13) converges, since $\{p_t(y)\}$ is a nondecreasing sequence (Theorem 3.3) bounded above. Hence, taking the limit as $t \to \infty$ and using the Theorem of monotonic convergence* we get (4.12) ■

Another way to study the limiting policy $c(y)$ is using the relation (4.12). But in order to apply it, we first need to find a bound for $p_t(y)$. In some cases such a bound can be obtained from those of $p_t^a(y)$ since $p_t(y) \leqq p_t^a(y)$ for all t. If $g'(x) > 1$, $u(c)$ is bounded above and below then the following theorem will give us a bound for $p_t(y)$ that does not depend on the computation of $p_t^a(y)$.

Theorem 4.9**

If $u(c)$ is bounded above and below, $g'(x) \geqq r > 1$, $\delta = 1$ then

$$p_t^a(y) \leqq \frac{M}{y}$$

where $M = \frac{r}{r-1}(\eta - u(0))$, and η is such that $u(c) \leqq \eta$ for all c.

For the remainder of this section we will assume that $p_t(y)$ is bounded in a convenient interval.

Now we are able to obtain an upper bound for $c(y)$ of an "income fluctuation problem." This bound does not depend on the computation of $c^A(y)$ and it is also independent of the utility function.

Theorem 4.10.

If $g(x) = rx$, $\delta r \geqq 1$, then the limiting policy satisfies the following inequality

$$c(y) \leqq \frac{r-1}{r} y + \frac{A}{r}.$$

*E. C. Titchmarsh [15].

**For - proof see D. Gale and W. R. Sutherland [6].

Proof:

From (4.12) we have

$$p(y) \geqq \delta r E p(rx(y) + \omega)$$

$$\geqq Ep(rx(y) + \omega)$$

since $\delta r \geqq 1$. Hence,

$$p(y) \geqq p(rx(y) + A)$$

and

$$y \leqq r(y - c(y)) + A$$

by monotonicity of $p(y)$.* So,

$$c(y) \leqq \frac{r-1}{r} y + \frac{A}{r} \blacksquare$$

A sharper bound can be obtained for the cases in which $u'(c)$ is convex.

Theorem 4.11.

If $g(x) = rx$, $\delta r \geqq 1$, and $u'(c)$ is convex then

$$c(y) \leqq \frac{r-1}{r} y + \frac{\bar{\omega}}{r} .$$

Proof:

From (4.2) we get

$$p_t(y) \geqq \delta r E p_{t-1}(rx_t(y) + \omega) \tag{4.14}$$

$$\geqq Ep_{t-1}(rx_t(y) + \omega)$$

$$\geqq p_{t-1}(rx_t(y) + \bar{\omega})$$

*p(y) is the limit of a sequence $\{p_t(y)\}$ of decreasing continuous function and hence nonincreasing.

by convexity of $p_{t-1}(y)$.* Now taking the limit as $t \to \infty$ we get

$$p(y) \geqq p(rx(y) + \bar{\omega}) .$$

Hence,

$$y \leqq r(y - c(y)) + \bar{\omega}$$

by monotonicity of $p(y)$. So,

$$c(y) \leqq \frac{r - 1}{r} y + \frac{\bar{\omega}}{r} \blacksquare$$

Now, we will introduce the concept of capital growing as it is defined by S. E. Phelps [11] .

Definition.

If y^t (observe the superscript) is the stock of capital at the beginning of period t , when we follow a policy $(x(y),c(y))$, we will say that *capital grows* for this policy if $E[y^{t+1}] \geqq y^t$.

Theorem 4.12.

If $\delta g'(x) \geqq 1$, $g'(x)$ and $u'(c)$ are convex functions then for the limiting policy $(x(y),c(y))$ the capital grows.

Proof:

From (4.2) we have

$$p_t(y) \geqq \delta g'(x_t(y))Ep_{t-1}(g(x_t(y)) + \omega) \tag{4.15}$$

$$\geqq Ep_{t-1}(g(x_t(y)) + \omega)$$

since $\delta g'(x) \geqq 1$. From (4.15) using the fact that $x_t(y) \leqq x(y)$, $g(x)$ is increasing and $p_{t-1}(y)$ is decreasing we get

*Theorem 3.6.

$$p_t(y) \geq Ep_{t-1}(g(x_t(y)) + \omega) \tag{4.16}$$

$$\geq Ep_{t-1}(g(x(y)) + \omega)$$

$$\geq p_{t-1}(E(g(x(y)) + \omega)$$

by convexity of $p_{t-1}(y)$. Now, taking the limit as $t \to \infty$ we get

$$p(y) \geq p(E(g(x(y)) + \omega)) ,$$

and finally from the monotonicity of $p(y)$ it follows that

$$y \leq E(g(x(y)) + \omega) \blacksquare$$

We finish this section with the following example,

<u>Example</u>:

Consider the following "income fluctuation problem"

$$f(x;\omega) = rx + \omega ,$$

$$u(c) = \log c ,$$

and $0 < \delta < 1$. For $u(c) = \log c$, $p_t^a(y)$ can be computed and it is equal to

$$p_t^a(y) = \frac{1 + \delta + \dots + \delta^{n-1}}{y} .$$

Then,

$$p_t(y) \leq p_t^a(y)$$

$$\leq \frac{1}{(1 - \delta)y} .$$

So, $p_t(y)$ converges. Now using the convexity of $u'(c)$ and Theorem 4.12 we will get that for the limiting policy $(x(y),c(y))$ the capital grows whenever $\delta r \geq 1$.* Now, from Corollary 4.2 and Theorem 4.11, it follows that for $\delta r \geq 1$

$$(1 - \delta)y \leq c(y) \leq \frac{r - 1}{r} y + \frac{\bar{\omega}}{r} .$$

We should also mention that we could prove using the concept of "competitive policy" that the limiting policy $(x(y),c(y))$ is also optimal. But this would take us too far from the purpose of this paper.

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*This is the result obtained by B. Miller [9] page 19.

5. CASE OF $f(x;\omega) = x + \omega$.

In this section we will consider the "income fluctuation problem" with $r = 1$ and $\delta = 1$. Now, when the consumer saves an amount x at period t , the available income in the next period is $x + \omega^t$, i.e., $f(x;\omega^t) = x + \omega^t$.

When y is the initial income, the earning $\omega^1, \ldots, \omega^t$ are known with certainty and $u(c)$ is strictly concave, it is known that it is optimal to set the consumption level $c_t(y)$ at

$$c_t(y) = \min\left(y, \frac{\omega^1 + \ldots + \omega^t}{t}\right).$$

If $\dfrac{\omega^1 + \ldots + \omega^t}{t}$ converges to an average $\bar{\omega}^t$ as $t \to \infty$, then the optimal consumption level will converge to $\min(y,\bar{\omega})$. In the stochastic case, in which ω^t are i.i.d. random variables, it is natural to expect that $c_t(y)$ will converge to $c(y) = \min(y,\bar{\omega})$, since the expected average income $\dfrac{\omega^1 + \ldots + \omega^t}{t}$ will converge to $\bar{\omega} = E\omega^t$ by the law of large numbers. In this section we will show that this is not true whenever $u'(0) < \infty$ and $\omega^t \equiv \omega$ are nondegenerate random variable. In fact, we will obtain the surprising result that $0 < c(y) < \bar{\omega}$ for $0 < y < \infty$. M. E. Yaari [16] considers the case $r = 1$, $\delta = 1$, borrowing is allowed at zero interest rate, no constraints are imposed on $c_t(y)$, i.e., $c_t(y) \in (-\infty,+\infty)$ and obtains the result that $c(y) = \lim_{t\to\infty} c_t(y) = \bar{\omega}$ for all y .

The range of the random variable $\omega \equiv \omega^t$ is an interval $[0,A]$, $0 < A < +\infty$, and if $\nu(0,\varepsilon) \equiv \text{Prob}[\omega \leq \varepsilon], \varepsilon \geq 0$, it will be assumed that $1 > \nu(0,\varepsilon) > 0$ for $A > \varepsilon > 0$. We also assume that $u'(0) < +\infty$ and without loss of generality that $u(0) = 0$.

The functional equations of the Dynamic Programming formulation are now,

$$(5.1)\qquad \begin{cases} V_t(y) = \max\limits_{\substack{c+x=y \\ c\geq 0,\, x\geq 0}} \{u(c) + EV_{t-1}(x+\omega)\} & \text{for } t > 1 \\[2em] V_1(y) = \max\limits_{\substack{c+x=y \\ c\geq 0,\, x\geq 0}} \{u(c)\}, & \end{cases}$$

and the optimality conditions are

$$(5.2)\qquad p_t(y) \geqq Ep_{t-1}(x_t(y) + \omega)$$

with equality if $x_t(y) > 0$, and

$$(5.3)\qquad p_t(y) \geqq u'(c_t(y))$$

with equality if $c_t(y) > 0$.

<u>Theorem 5.1</u>

The function $V_t(y)$ is differentiable for $y \geqq 0$, and for all t . Furthermore $V_t'(0) = u'(0)$.

<u>Proof</u>:

For $y > 0$ it has been proved in Theorem 2.2. For $y = 0$ Theorem A2 holds, since $u'(0) < +\infty$, $u(0) = 0$ and $f'(x;\omega) = 1$ ■ *

<u>Theorem 5.2.</u>

The policy $c_t(y)$ satisfies the following inequalities

$$(5.4)\qquad \frac{y}{t} \leqq c_t(y) \leqq \frac{y}{t} + \frac{t-1}{t} A \qquad \text{for all } t .$$

Moreover if $u'(c)$ is convex then

$$(5.5)\qquad \frac{y}{t} \leqq c_t(y) \leqq \frac{y}{t} + \frac{t-1}{t} \bar{\omega} \qquad \text{for all } t .$$

---

*Hence the problem is admissible.

Proof:

From Theorem 4.1 and Theorem 4.3 we have without any assumption about $u'(c)$ that

$$c_t^a(y) \leq c_t(y) \leq c_t^A(y) \quad \text{for all } t .$$

But, it is known that $c_t^a(y) = \frac{y}{t}$ and $c_t^A(y) = \min\left(y, \frac{y}{t} + \frac{t-1}{t} A\right)$. So, (5.4) holds. Now, for $u'(c)$ convex, using Theorem 4.5 we get

$$c_t(y) \leq c_t^{\bar{\omega}}(y) \quad \text{for all } t .$$

Hence,

$$c_t(y) \leq \frac{y}{t} + \frac{t-1}{t} \bar{\omega}$$

since it is known that $c_t(y) = \min\left(y, \frac{y}{t} + \frac{t-1}{t} \bar{\omega}\right)$. So, (5.5) holds ∎

Now we will obtain the surprising fact that $c(y)$ is uniformly bounded.

Corollary 5.3.

The limiting policy $c(y)$ satisfies the following inequalities

$$0 \leq c(y) \leq A .$$

Moreover if $u'(c)$ is convex then

$$0 \leq c(y) \leq \bar{\omega} .$$

Proof:

It follows immediately from Theorem 5.2 by taking limits as $t \to \infty$ in (5.4) and (5.5) ∎

Lemma 5.4.

The sequence of functions $\{p_t(y)\}$ converges uniformly in any finite and closed interval to a continuous nonincreasing function $p(y)$.

Proof:

It converges pointwise, since

$$p_t(y) \geqq p_{t-1}(y)$$

and

$$\begin{aligned} p_t(y) &= V_t'(y) \\ &\leqq V_t'(0) \\ &= u'(0) \end{aligned}$$

by concavity and Theorem 5.1. Now, from the fact that $c_t(y) \geqq \frac{y}{t} > 0$ for $y > 0$ (Theorem 5.2), and the optimality condition (5.3) we have that

$$p_t(y) = u'(c_t(y)) \quad \text{for all } y .$$

Taking the limit and using the fact that $u'(c)$ is a continuous function and $c_t(y)$ converges by Theorem 3.5, we will have that

$$p(y) = u'(c(y)) .$$

So, $p(y)$ is continuous, since $c(y)$ is continuous from Theorem 3.5. Now, the theorem follows from the fact that $\{p_t(y)\}$ is a sequence of monotonic functions converging pointwise to a continuous function, consequently it converges uniformly in any finite closed interval ∎

Lemma 5.5.

The limiting policy $(x(y),c(y))$ satisfies the following relations

(5.6) $$p(y) = u'(c(y))$$

and

(5.7) $$p(y) \geq Ep(x(y) + \omega)$$

with equality if $x(y) > 0$.

Proof:

From the Optimality conditions (5.3) and (5.2) we get

$$p_t(y) = u'(c_t(y))$$

since $c_t(y) > 0$ for $y > 0$ and $p_t(0) = u'(0)$, and

$$p_t(y) \geq Ep_{t-1}(x_t(y) + \omega)$$

with equality if $x_t(y) > 0$. Now, taking the limit and using the fact that $\{p_t(y)\}$ converges uniformly in any finite closed interval we get (5.6) and (5.7)■

Lemma 5.6.

If $c(y) = 0$ for some $y > 0$ then $c(y) \equiv 0$.

Proof:

Let $\bar{y} = \sup \{y \mid c(y) = 0\}^*$. If $\bar{y} = +\infty$ we have done. Now suppose that $\bar{y} < +\infty$. Then,

$$p(\bar{y}) = Ep(\bar{y} + \omega)$$

and

$$p(\bar{y}) = u'(c(\bar{y}))$$

$$= u'(0) .$$

---

*The supremum is actually achieved from the continuity of $c(y)$.

Now, for $\omega > 0$, $c(\bar{y} + \omega) > 0$ and $p(\bar{y} + \omega) < u'(0)$. Let ε be any number such that $0 < \varepsilon < A$. Then

$$
\begin{aligned}
p(\bar{y}) &= Ep(\bar{y} + \omega) \\
&= \int_0^{\varepsilon} p(\bar{y} + \omega)\nu(d\omega) + \int_{\varepsilon}^{A} p(\bar{y} + \omega)\nu(d\omega) \\
&\leq u'(0)\nu(0,\varepsilon) + p(\bar{y} + \varepsilon)(1 - \nu(0,\varepsilon)) \\
&< u'(0)\nu(0,\varepsilon) + u'(0)(1 - \nu(0,\varepsilon)) \\
&< u'(0)
\end{aligned}
$$

which is a contradiction ■

<u>Theorem 5.7</u>.

If $c(y) > 0$ for $y > 0$ then $p(y)$ is strictly decreasing.

<u>Proof</u>:

By contradiction suppose not, i.e., there exits $y' > y \geq 0$ such that

(5.8) $$p(y') = p(y) = \bar{p} .$$

Let,

$$\bar{y} = \min \{y \mid p(y) = \bar{p} , 0 \leq y \leq y'\}.$$

Now, $\bar{y} > 0$ since $c(y') > 0$ and

$$
\begin{aligned}
p(\bar{y}) &= p(y') \\
&= u'(c(y')) \\
&< u'(0) .
\end{aligned}
$$

Hence

(5.9) $$p(y) > p(\bar{y}) \quad \text{for} \quad y < \bar{y} .$$

From (5.2) we have that

(5.10) $$p(\bar{y}) \geqq Ep(x(\bar{y}) + \omega)$$

and

(5.11) $$p(y') = Ep(x(y') + \omega) .$$

Hence, from (5.8), (5.10) and (5.11) we have that

$$Ep(x(y') + \omega) \geqq Ep(x(\bar{y}) + \omega) .$$

So,

(5.12) $$E[p(x(y') + \omega) - p(x(\bar{y}) + \omega)] \geqq 0 .$$

But

(5.13) $$E[p(x(y') + \omega) - p(x(\bar{y}) + \omega)] \leqq 0$$

since $x(y') > x(\bar{y})$ and $p(y)$ is nondecreasing. From (5.12) and (5.13) we have

$$E[p(x(y') + \omega) - p(x(\bar{y}) + \omega)] = 0 .$$

Hence,

(5.14) $$p(x(y') + \omega) = p(x(\bar{y}) + \omega) \quad \text{for} \quad \omega = 0 ,$$

since $\psi(\omega) \equiv p(x(y') + \omega) - p(x(\bar{y}) + \omega)$ is nonpositive, continuous and $\nu(0,\omega) > 0$ for $\omega > 0$. From the fact that $p(\bar{y}) = p(y')$ it follows that

(5.15) $$x(y') = x(\bar{y}) + y' - \bar{y} .$$

Thus from (5.15) and (5.14) we have

$$p(x(\bar{y})) = p(x(\bar{y}) + y' - \bar{y}) ,$$

and consequently for $y \in [x(\bar{y}) , x(\bar{y}) + y' - \bar{y}]$ we have

$$p(y) = p(x(\bar{y})) .$$

If $x(\bar{y}) = 0$, then $c(y) = 0$ for $y \in [x(\bar{y}) , x(\bar{y}) + y' - \bar{y}]$ which contradicts the fact that $c(y) > 0$ for $y > 0$. For $x(\bar{y}) > 0$ consider the following cases:

Case i.

$x(\bar{y}) + y' - \bar{y} \geq \bar{y}$. Then it follows that $p(x(\bar{y})) = p(\bar{y})^*$ which contradicts (5.9) since $x(\bar{y}) < \bar{y}$.

Case ii.

$x(y) + y' - \bar{y} < \bar{y}$. Now repeat the proof starting with the new interval $[x(\bar{y}),x(\bar{y}) + y' - \bar{y}]$ and computing the new

$$\bar{y} = \min \{y \mid p(y) = p(x(\bar{y})) , y \geq 0 , y \leq x(\bar{y})\} .$$

The repetition of Case ii will generate a sequence of intervals which are nonoverlapping, have nondecreasing lengths and are all contained in the interval $[0,y']$. So, Case ii can be repeated only a finite number of times ∎

Now we will show that $\lim\limits_{T\to\infty} \frac{V_T(y)}{T} = u(\bar{\omega})$.

Lemma 5.8.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} \geq u(\bar{\omega}) , \text{ where } \bar{\omega} = E\omega .^{**}$$

---

* $\bar{y} \in [x(\bar{y}) , x(\bar{y}) + y' - \bar{y}]$ and hence $p(x(\bar{y})) = p(\bar{y})$.

**This proof is similar to the one given in M. E. Yaari [16].

Proof:

It suffices to show that $\lim_{T\to\infty} \frac{V_T(0)}{T} \geqq u(\bar{\omega})$, since $V_T(y) \geqq V_T(0)$. For $V_T(0)$, consider the policy $c_t(y) = \min (y,\bar{\omega})$ for all $1 \leqq t \leqq T$. If y^t (observe the superscript) is the stock level at the beginning of period t when we follow the policy $c_t(y) = \min (y,\bar{\omega})$, then

$$y^t = \max (y^{t-1} + z^t,0) \qquad t \geqq 2$$

$$y^1 = 0 .$$

Where $z^t = \omega - \bar{\omega}$, and hence $Ez^t = 0$.

When $y^k = 0$, we have at the k^{th} period (trial)* a renewal point, since at this point the process will start all over again. If $N(T)$ is the number of renewals in T periods, then the expected return from the policy $c_t(y) = \min (y,\bar{\omega})$ for a T period problem is greater or equal to $u(\bar{\omega})(T - E(N(T)) - 1)$. Hence

$$V_T(0) \geqq u(\bar{\omega})(T - E(N(U)) - 1) .$$

Thus to show that

$$\lim_{T\to\infty} \frac{V_T(0)}{T} \geqq u(\bar{\omega})$$

it suffices to show that

$$\lim_{T\to\infty} \frac{EN(U)}{T} = 0 .$$

Let τ_i be the number of Trials between the i^{th} and $(i - 1)^{th}$ renewals. From the Renewal Theory we have

$$\lim_{T\to\infty} \frac{EN(T)}{T} = \frac{1}{E\tau_1} \qquad \text{if } E\tau_1 < \infty$$

---

*Trial is the usual terminology in Renewal Theory.

and

$$\lim_{T\to\infty} \frac{EN(T)}{T} = 0 \quad \text{if} \quad E\tau_1 = \infty \,.^{*}$$

But, from the Random Walk Theory it can be shown that $E\tau_1 = \infty$ if $Ez^t = 0$.**
So, the lemma holds∎

Lemma 5.9.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} \leqq u(\bar{\omega}) \,.$$

Proof:

It suffices to show that

$$V_T(y) \leqq Tu\left(\frac{y}{T} + \frac{(T-1)}{T}\bar{\omega}\right) .$$

It is true for $T = 1$, since $V_1(y) \leqq u(y)$. Assume it is true for t , i.e.,

(5.16) $$V_T(y) \leqq tu\left(\frac{y}{t} + \frac{t-1}{t}\bar{\omega}\right) \quad \text{for all } y \,.$$

Now,

(5.17) $$V_{t+1}(y) = u(c_t(y)) + EV_t(x_t(y) + \omega)$$

$$\leqq u(c_t(y)) + V_t(x_t(y) + \bar{\omega})$$

by concavity of $V_t(y)$. Thus, from (5.16) and (5.17) we get

(5.18) $$V_{t+1}(y) \leqq u(c_t(y)) + tu\left(\frac{1}{t}(x_t(y) + \bar{\omega}) + \frac{t-1}{t}\bar{\omega}\right)$$

$$= (t+1)\left(\frac{1}{t+1}u(c_t(y)\right) + \frac{t}{t+1}u\left(\frac{1}{t}(x_t(y) + t\bar{\omega}\right)$$

$$(t+1)u\left(\frac{1}{t+1}(y + t\bar{\omega})\right)$$

*For a proof see S. Ross [13].
**See W. Feller [3] page 397.

by concavity of $u(c)$ ■

From the previous two lemmas we can state the following Theorem.

Theorem 5.10.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} = u(\bar{\omega}) \qquad \text{for all } y .$$

The sequence $V_t(y)$ diverges as $t \to \infty$. In order to study the behavior of the limiting policy $c(y)$ we will define a special function, $h_t(y)$, that has the same properties of $V_t(y)$, but which sequence converges.

Definition.

Let $h_t(y) \equiv V_t(y) - V_t(0)$.

Using the fact that

$$\begin{aligned} V_t(0) &= u(0) + EV_{t-1}(\omega) \\ &= EV_{t-1}(\omega) , \end{aligned}$$

since by assumption $u(0) = 0$, it can be shown that the function $h_t(y)$ satisfies the following relations

$$h_t(y) + V_t(0) - V_{t-1}(0) = \max_{\substack{c+x=y \\ c\geq 0, x\geq 0}} \{u(c) + Eh_{t-1}(x + \omega)\}$$

(5.19)

$$h_1(y) = V_1(y) = \max_{\substack{c+x=y \\ c\geq 0, x\geq 0}} \{u(c)\}$$

and

(5.20) $$V_t(y) = h_t(y) + E \sum_{k=1}^{t-1} h_k(\omega) .$$

Theorem 5.11.

The function $h_t(y)$ is strictly concave, increasing and differentiable for all t . Furthermore

$$h_t'(y) = p_t(y)$$

and if $(x_t(y), c_t(y))$ is the solution of (5.1), it is also the solution of (5.19).

<u>Proof</u>:

It is an immediate consequence of the fact that $h_t(y)$ and $V_t(y)$ differ by a constant∎

Now, we will state without proof the following lemma (for a proof see W. Rudin [14]).

<u>Lemma 5.12</u>.

Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_o)\}$ converges for some x_0 on $[a,b]$. If the sequence of derivatives $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad a \leq x \leq b .$$

<u>Theorem 5.13</u>.

The sequence of functions $h_t(y)$ converges uniformly in any finite interval $[0,b]$. Furthermore if $h(y) = \lim_{t\to\infty} h_t(y)$, then

$$h'(y) = \lim_{t\to\infty} h_t(y) = p(y)$$

<u>Proof</u>:

I. $h_t(0) = 0$ for all t . Hence

$$\lim_{t\to\infty} h_t(0) = 0$$

II. $h_t(y)$ is differentiable for all t , $y \geq 0$.

III. $h_t'(y) = p_t(y)$ converges uniformly on any finite interval $[0,b]$ by Lemma 5.4.

Hence, the theorem follows from the previous lemma ∎

It is an immediate consequence the following corollary.

<u>Corollary 5.14</u>.

$$\lim_{t\to\infty} Eh_t(\omega) = Eh(\omega) .$$

<u>Lemma 5.15</u>.

$$Eh(\omega) = u(\bar{\omega}) .$$

<u>Proof</u>:

From (5.20) we have

$$V_t(y) = h_t(y) + \sum_{k=1}^{t-1} Eh_k(\omega) .$$

Now, dividing by t , and taking the limit we get

$$u(\bar{\omega}) = Eh(\omega) .$$

Where we have used the following facts:

I. $\lim\limits_{t\to\infty} \dfrac{V_t(y)}{t} = u(\bar{\omega})$ from Theorem 5.10.

II. $\lim\limits_{t\to\infty} \dfrac{\sum_{k=1}^{t-1} Eh_k(\omega)}{t} = Eh(\omega)$ from corollary 5.14.

III. $\lim_{t\to\infty} \frac{h_t(y)}{t} = 0$, since $h_t(y)$ is uniformly bounded in any finite interval■

Lemma 5.16.

$$\lim_{t\to\infty} V_t(0) - V_{t-1}(0) = u(\bar{\omega})$$

Proof:

From the definitions of $V_t(0)$, $V_{t-1}(0)$ and $h_{t-1}(y)$ we have,

$$V_t(0) - V_{t-1}(0) = u(0) + EV_{t-1}(\omega) - V_{t-1}(0)$$

$$= Eh_{t-1}(\omega) .$$

Now, taking the limit and using Corollary 5.14, Lemma 5.16, we get that

$$\lim_{t\to\infty} V_t(0) - V_{t-1}(0) = u(\bar{\omega}) \blacksquare$$

Lemma 5.17

$$h(y) + u(\bar{\omega}) = u(c(y)) + Eh(x(y) + \omega) .$$

Proof:

From (5.19) we have

$$h_t(y) + V_t(0) - V_{t-1}(0) = u(c_t(y)) + Eh_{t-1}(x_t(y) + \omega) .$$

Now, taking the limit, using Lemma 5.16 and the uniform convergence of $h_t(y)$ in any finite interval we get

$$h(y) + u(\bar{\omega}) = u(c(y)) + Eh(x(y) + \omega) \blacksquare$$

Theorem 5.18.

The limiting policy $c(y)$ is always positive for $y > 0$.

Proof:

By contradiction suppose not, i.e., $c(y) = 0$ for some $y > 0$. Hence, from Lemma 5.6 $c(y) \equiv 0$. Now, from Lemma 5.15 we have

$$u(\bar{\omega}) = Eh(\omega) \quad (5.21)$$

But

$$p(y) = u'(0) \quad \text{for all } y ,$$

since $c(y) \equiv 0$. Hence,

$$h(y) = u'(0)y . \quad (5.22)$$

Finally, from (5.21) and (5.22) we get

$$u(\bar{\omega}) = u'(o)\bar{\omega} .$$

Which is a contradiction, since $u(c)$ is strictly concave ■

Lemma 5.19.

The function $h(y)$ is strictly concave.

Proof:

From Theorem 5.18 $c(y) > 0$ for $y > 0$, hence from Theorem 5.7 $p(y) = h'(y)$ is strictly decreasing and consequently $h(y)$ is strictly concave ■

Now, we are able to show the surprising fact that $c(y) < \bar{\omega}$ for $0 < y < +\infty$.

Theorem 5.20.

The limiting policy satisfies the following inequalities

$$0 < c(y) < \bar{\omega} \quad \text{for} \quad 0 < y < +\infty .$$

Proof:

From Lemma 5.17 we have

$$h(y) + u(\bar{\omega}) = u(c(y)) + Eh(x(y) + \omega) \ .$$

Now, suppose by contradiction that for some $y < +\infty$, $c(y) = \bar{\omega}$. Hence

$$h(y) + u(\bar{\omega}) = u(\bar{\omega}) + Eh(y - \bar{\omega} + \omega) \ .$$

Thus,

$$\begin{aligned} h(y) &= Eh(y - \bar{\omega} + \omega) \\ &< h(y) \end{aligned}$$

by strictly concavity of $h(y)$. Which is a contradiction ■

Theorem 5.21.

$$\lim_{y\to\infty} c(y) = \bar{\omega}$$

Proof:

Suppose not, i.e., $\lim_{y\to\infty} c(y) = \delta < \bar{\omega}$. Hence $\lim_{y\to\infty} h'(y) = \lim_{y\to\infty} u'(c(y)) = u'(\delta)$.
From Lemma 5.17 we have

$$u(\bar{\omega}) - u(c(y)) = Eh(y - c(y) + \omega) - h(y) \ . \tag{5.23}$$

Now, from the mean value theorem we get

$$h(y - c(y) + \omega) - h(y) = h'(y_\omega(y))(\omega - c(y)) \ , \tag{5.24}$$

where $y_\omega(y) \in [y - c(y) + \omega, y]$ if $c(y) \geq \omega$ and $y_\omega(y) \in [y, y - c(y) + \omega]$ if $c(y) \geq \omega$. Replacing (5.24) in (5.23) we get

$$u(\bar{\omega}) - u(c(y)) = Eh'(y_\omega(y))(\omega - c(y)) .$$

Now, taking the limit as $y \to \infty$ and using the fact that $y_\omega(y) \to \infty$ when $y \to \infty$* we get

$$u(\bar{\omega}) - u(\delta) = Eu'(\delta)(\omega - \delta)$$

$$= u'(\delta)(\bar{\omega} - \delta)$$

which is a contradiction since $u(c)$ is strictly concave and $\bar{\omega} < \delta$.

Now we will obtain some properties of the capital y^t accumulated in t periods.

<u>Theorem</u>.

The sequence of capitals $\{y^t\}$ converges almost surely to $+\infty$.**

<u>Proof</u>:

It suffices to show that y^t converges almost surely, since either $\{y^t\}$ does not converge or it converges almost surely to $+\infty$. From the optimality condition (5.7) we have that

$$p(y^t) \geq E[p(y^{t+1}) \mid y^t] . \tag{5.25}$$

Now if we define $p^t \equiv p(y^t)$ we will have from (5.25) that

$$p^t \geq E[p^{t+1} \mid p^t] \tag{5.26}$$

where we have used the fact that $p(y)$ is a strictly decreasing function and hence, for a given p^t , y^t is uniquely defined. From (5.26) we have that $\{p^t\}$ is a positive Supermartingale and hence it converges almost surely to a random variable $\bar{p}$. Since $p(y)$ is a continuous strictly decreasing function it is possible to define a continuous function $g(p)$ such that

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* $-\bar{\omega} < -c(y) + \omega < A$ since $0 \leq c(y) < \bar{\omega}$ and $\omega \leq A$.

** I am in debt to Professor R. Radner for insisting on this point.

$$y^t = g(p^t) \ .$$

The Theorem now follows from the fact that $\{p^t\}$ converges almost surely and $g(p)$ is a continuous function ∎

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APPENDIX

Without loss of generality we will assume that $u(0) = 0$ whenever $u'(0) < +\infty$.

Lemma A.1.

If $u'(0) < +\infty$ and $1 \leq \rho \leq +\infty$ then

$$V_t(y) - V_t(0) \leq tu\left(\frac{\rho^t y}{t}\right) . \tag{A.1}$$

Where ρ is the $\sup_{\omega} \{f'(o;\omega)\}$.

Proof:

The proof will be done by induction.

I. For $t = 1$ we have

$$V_1(y) = u(y) .$$

Hence,

$$V_1(y) \leq u(\rho y) ,$$

since $u(c)$ is increasing, and $\rho \geq 1$.

II. Assume that it is true for $t - 1$, i.e.,

$$V_{t-1}(y) - V_{t-1}(0) \leq (t - 1)u\left(\frac{\rho^{t-1} y}{t - 1}\right) .$$

Now,

$$V_t(y) - V_t(0) = u(c_t(y)) + \delta EV_{t-1}(f(x_t(y);\omega)) - \delta EV_{t-1}(f(o;\omega)) \tag{A.2}$$

Also,

(A.3) $$f(x_t(y);\omega) \leqq f(0;\omega) + \rho x_t(y) \quad \text{for all } \omega .$$

From (A.2) and (A.3), and using the fact that $V_t(y)$ is nondecreasing we get

(A.4) $$\begin{aligned} V_t(y) - V_t(0) &\leqq u(c_t(y)) + \delta EV_{t-1}(f(0;\omega) + \rho x_t(y)) - \delta EV_{t-1}(f(0;\omega)) \\ &\leqq u(c_t(y)) + \delta(V_{t-1}(\rho x_t(y)) - V_{t-1}(0)) . \end{aligned}$$

Hence,

$$V_t(y) - V_t(0) \leqq u(c_t(y)) + (t-1)u\left(\frac{\rho^{t-1}x_t(y)}{t-1}\right)$$

by the induction hypothesis. So,

$$\begin{aligned} V_t(y) - V_t(0) &\leqq t\left[\left(\frac{1}{t}\, u(c_t(y)\right) + \frac{t-1}{t}\, u\left(\frac{\rho^{t-1}x_t(y)}{t-1}\right)\right] \\ &\leqq tu\left(\frac{1}{t}\, c_t(y) + \frac{\rho^t}{t}\, x_t(y)\right) \end{aligned}$$

by concavity. Hence,

$$\begin{aligned} V_t(y) - V_t(0) &\leqq tu\left(\frac{\rho^t}{t}\,(c_t(y) + x_t(y))\right) \\ &= tu\left(\frac{\rho^t}{t}\, y\right). \blacksquare \end{aligned}$$

<u>Theorem A.1</u>.

If $u'(0) < +\infty$ and $1 \leqq \rho \leqq +\infty$ then

$$V_t'(0) \leqq \rho^t u'(0) .$$

<u>Proof</u>:

Dividing (A.1) by $y > 0$ we have

$$\frac{V_t(y) - V_t(0)}{y} \leqq \frac{t}{y}\, u\left(\frac{\rho^t}{t}\, y\right).$$

Now, taking the limit as $y \to 0+$ we get

$$V_t'(0) \leqq \rho^t u'(0) \ . \blacksquare$$

<u>Lemma A.2</u>.

If $u'(0) < +\infty$ and $\rho \leqq 1$ then

(A.5)
$$V_t(y) - V_t(0) \leqq tu\left(\frac{y}{t}\right) .$$

<u>Proof</u>:

For $t = 1$ we have

$$V_1(y) = u(y) .$$

Hence,

$$V_1(y) \leqq u(y) .$$

Assume that it is true for $t - 1$, i.e.,

$$V_{t-1}(y) = V_{t-1}(0) \leqq (t - 1)u\left(\frac{y}{t - 1}\right) .$$

Now, (A.4) still holds, hence

(A.6)
$$V_t(y) - V_t(0) \leqq u(c_t(y)) + \delta(V_{t-1}(\rho x_t(y)) - V_{t-1}(0))$$

$$\leqq u(c_t(y) + (V_{t-1}(x_t(y)) - V_{t-1}(0)) .$$

Where the last inequality follows from the fact that $V_{t-1}(y)$ is increasing and $\rho \leqq 1$.

From (A.6) using the induction hypothesis we get

$$V_t(y) - V_t(0) \leqq u(c_t(y)) + (t - 1)u\left(\frac{1}{t - 1} x_t(y)\right)$$

$$= t\left(\frac{1}{t} u(c_t(y)) + \frac{t - 1}{t} u\left(\frac{1}{t - 1} x_t(y)\right)\right)$$

$$tu\left(\frac{1}{t} c_t(y) + \frac{1}{t} x_t(y)\right)$$

$$= tu\left(\frac{1}{t} y\right) . \blacksquare$$

Theorem A.2.

If $u'(0) < \infty$ and $\rho \leqq 1$ then

$$V_t'(0) = u'(0) \ .$$

Proof:

First we have immediately that

$$V_t(y) - V_t(0) \geqq u(y) \ . \tag{A.7}$$

Then, from (A.5) and (A.7) we get

$$u(y) \leqq V_t(y) - V_t(0) \leqq tu\left(\frac{1}{t}\, y\right) \ .$$

Dividing by $y > 0$ we have

$$\frac{u(y)}{y} \leqq \frac{V_t(y) - V_t(0)}{y} \leqq \frac{t}{y}\, u\left(\frac{1}{y}\, y\right).$$

Now, taking the limit as $y \to 0^+$ we finally get

$$V_t'(0) = u'(0) \ . \blacksquare$$