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SOLVING STAIRCASE LINEAR PROGRAMS BY A NESTED BLOCK-ANGULAR METHOD

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BY

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The objective is to have a compact inverse representation of the basis of a staircase structure. Every other step in the staircase is assigned to a subsystem partition and the remaining to a "master" partition. This permits an extension of the generalized upper-bounding technique to be applied. After a column elimination, the resulting working basis associated with the "master" partition turns out to also have a staircase formate but with half the number of steps. This permits reapplication of the same technique recursively until the number of steps of the p^{th} working basis has only one step. An interesting aspect of the procedure is that a number of operations can be performed in parallel and are not affected by a change in basis.

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III

SOLVING STAIRCASE LINEAR PROGRAMS BY A

NESTED BLOCK-ANGULAR METHOD

by

George B. Dantzig

Block angular systems can be solved by the simplex method using a compact inverse representation of the basis. The method is a direct extension of the generalized upper-bound technique. We wish to apply this result to staircase systems. To do this, we assign every other "step" in a staircase basis structure (1) to a subsystem by rearranging the rows, the result is as in (2).

$$(1) \qquad \begin{array}{cccccc|c}
\mathrm{I} &&&&&& b^1\\
\mathrm{I}' & \mathrm{II} &&&&& b^2\\
& \mathrm{II}' & \mathrm{III} &&&& b^3\\
&& \mathrm{III}' & \mathrm{IV} &&& b^4\\
&&& \mathrm{IV}' & \mathrm{V} && b^5\\
&&&& \mathrm{V}' & \mathrm{VI} & b^6
\end{array}$$

INITIAL STAIRCASE BASIS STRUCTURE R.H.S.

$$(2) \qquad \begin{array}{cccccc|cl}
\mathrm{I} &&&&&& b^1 & \\
& \mathrm{II}' & \mathrm{III} &&&& b^3 & \Bigg\} \text{MASTER}\\
&&& \mathrm{IV}' & \mathrm{V} && b^5 & \\
\hline
\mathrm{I}' & \mathrm{II} &&&&& b^2 & \\
&& \mathrm{III}' & \mathrm{IV} &&& b^4 & \Bigg\} \text{SUBSYSTEM}\\
&&&& \mathrm{V}' & \mathrm{VI} & b^6 &
\end{array}$$

SAME BASIS AFTER REARRANGING ROWS R.H.S.

Note that the subsystem consists of independent sets of equations e.g., [I, II], $[III^1, IV]$ and [V, VI], so that it is in the proper form for applying block-angular techniques. In the extension of the generalized upper-bounding technique for this class of problems, a column operation is performed to reduce the columns of each set, like $[I', II]$ and its corresponding right handside (R.H.S.) to the form $[B_1, 0]$ where B_1 is any non-singular subset of columns of $[I', II]$ which we will refer to as "key" columns. The same column operations are performed on the corresponding columns of the master. After the operations, the master part has a new format. For system (2) the result is

$$
\begin{array}{cccccc|c}
A_1 & \bar{I} & & & & & \bar{b}^1 \\
A_1' & \bar{I}' & A_2 & \overline{II} & & & \bar{b}^3 \\
 & & A_2' & \overline{II}' & A_3 & \overline{III} & \bar{b}^5 \\
\hline
B_1 & 0 & & & & & 0 \\
 & & B_2 & 0 & & & 0 \\
 & & & & B_3 & 0 & 0
\end{array} \tag{3}
$$

BASIS AFTER COLUMN OPERATIONS R.H.S.

The important point to note that the inverse of matrix (3) can be obtained from the knowledge of B_1^{-1}, B_2^{-1}, B_3^{-1} and the inverse of

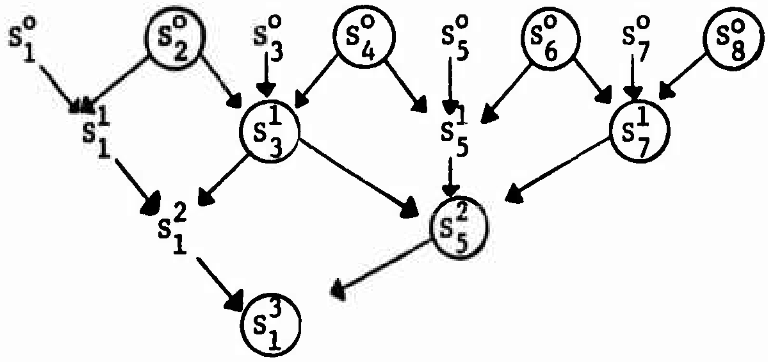
$$(4) \qquad \begin{bmatrix} \bar{\mathrm{I}} & & \\ \bar{\mathrm{I}}' & \overline{\mathrm{II}} & \\ & \overline{\mathrm{II}}' & \overline{\mathrm{III}} \end{bmatrix} \quad .$$

But (4) is again staircase form with <u>half</u> the number of rows. Thus one is now in a position to recursively re-apply the technique. A 32 time period model, for example, would require 5 recursions.

<u>Effect of Change of Basis</u>. Let us consider a $T = 8$ period staircase structure (5). Let p be defined by $2^p = T$, i.e., $p = \log_2 T$.

$$(5) \qquad \begin{array}{ccccc} \boxed{S_1^o} & & & & \\ & \boxed{S_2^o} & & & \\ & & \ddots & & \\ & & & \boxed{S_7^o} & \\ & & & & \boxed{S_8^o} \end{array}$$

We can associate with the various rearrangements and column operations a heiarchy of staircase systems:



Initial Staircase System

Circles indicate that on the initial recursion we place s_2^o, s_4^o, s_6^o, s_8^o in the subsystem and as a result of the column operations s_1^o, s_3^o, s_5^o, s_7^o are modified to s_1^1, s_3^1, s_5^1, s_7^1. The directed arcs pointing to s_3^1, for example, indicate that s_3^1 only depends on s_2^o, s_3^o, s_4^o, i.e., only on those with adjacent subscripts in the previous recursion. Suppose now in a basis change only s_8^o is modified. It will cause a change in s_7^1, s_5^2 and s_1^3 but the remaining s_t^ℓ will be unmodified. In general, if there are 2^p steps a "local" change of basis can be shown to affect $2(p+1)$ of the s_t^ℓ. However, a non-local change of basis (such as the introduction into the basis of a column associated with an early period t_1 and dropping a column of a later period $t_2 > t_1$) could effect less than $4(p+1)$ of the s_t^ℓ.

Information to be stored. This has not (at this writing) been completely analysed but is less than twice the number of elements in the arrays s_t^ℓ (assuming s_t^ℓ are composed of only non-zero coefficients) plus the storage of the inverses of T matrices of size $m_t \times m_t$ where m_t is the number of equations corresponding to the t^{th} step of the staircase.

Storage-wise the method appears to be competitive with the best of alternative solution techniques that have been proposed so far.

<u>A Comparison for Square-Block Staircase Systems</u>: Suppose we have system (1) and the diagonal submatrices I, II, ... , VI are each square and non-singular. It is clear in this case we could efficiently solve the system by using I to eliminate I', use II to eliminate II', etc. In many applications of staircase systems it can be shown that non-singular square blocks hold along the diagonal for "almost all" periods. It is of interest therefore to see how our proposed partitioning of the rows into a subsystem and a master would fare as an alternative solution procedure. Note that in (2) we are assuming II, IV, VI are each non-singular, hence if we use these as key submatrices for the elimination of columns, the result corresponding to (4) in this case would be

$$(7)\qquad \left.\begin{array}{ccc} \mathrm{I} & & \\ \bar{\mathrm{I}} & \mathrm{III} & \\ & \overline{\mathrm{III}} & \mathrm{V} \\ \hline \end{array}\right\} \quad \text{Master}$$

Rearrangement yields

$$
(8)\qquad
\begin{array}{ccc}
\overline{I}^{1} & III & \\
\hdashline
I & & \\
 & \overline{III}^{1} & V
\end{array}
\qquad
\begin{array}{l}
\text{Master within the master} \\
\\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Subsystem within the master}
\end{array}
$$

Column Elimination using non-singular V yields finally

$$
(9)\qquad III
$$

Thus there is no disadvantage using the proposed technique even for the case of square block systems. Indeed there is an advantage, because a change of basis generates only local changes affecting only $2(p+1)$ out of $2 \cdot 2^p$ of the submatrices generated by the process for the case where the number of steps in the staircase is $T = 2^p$.

<u>Parallel Processing</u>. Some of the recent designs for computers possess extensive parallel processing features. The proposed solution method could be efficiently handled on such machines because they could do parallel processing of every other step and then recursively in parallel for a system half the size etc. Updating would proceed in an analogous parallel manner except (as indicated in our analysis of (5) and (6)) many of the sub-matrices being processed in parallel would require no change in the updating. Thus a parallel process machine would be very efficient for obtaining the initial compact inverse representation or a subsequent complete update but would probably

have little advantage for the iterative updating of the inverse representation when there is only a simple one column change in the basis.