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Center for Naval Analyses
Arlington, Virginia

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NIRM-13

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Interim Research Memorandum

NAVAL WARFARE ANALYSIS GROUP

Center for Naval Analyses

WASHINGTON 25, D. C.

OFFICE OF THE CHIEF OF NAVAL OPERATIONS
LONG RANGE OBJECTIVES GROUP . . . OP-93

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NAVWAG INTERIM RESEARCH MEMORANDUM

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OPTIMUM ALLOCATION OF EFFORT FOR DETERRENCE

ABSTRACT

Mathematical models of optimum procurement allocation among different types of retaliatory weapons systems are examined. A distinction is drawn between so-called "numerically vulnerable" (NV) systems (which find safety in numbers and are approximated by fixed missile bases) and "percentage vulnerable" (PV) systems (which require the enemy to engage in a search effort to counter them, e. g., because they are mobile). It is shown that it is generally preferable to buy a mix of PV systems and only a single optimal NV system. The models are also applicable to measure-countermeasure problems other than deterrence. The prime value of this paper is believed to be its demonstration that even in the simplest cases cost/effectiveness comparisons may not lead to optimum solutions about allocations of resources among countermeasure susceptible weapons systems.

I. INTRODUCTION

Much has been said qualitatively about the desirability of vesting all-out war deterrence in a "mix" of different weapon systems, in preference to relying on a single system, as is virtually the case at present. The most frequently encountered argument is that although the mix may cost more it is a good buy in terms of insurance against unpredictable enemy technological breakthroughs in countering the single system. The more diversified threat presented by a mix of qualitatively different systems forces the enemy into broader fields of countermeasures research and procurement, dilutes his effort against any single system, and thereby decreases the countermeasures vulnerability of all. In view of the demonstrated shortness of enemy lead-times, which may permit them more rapid reallocations of effort than we can achieve and which could conceivably make any single-system "optimization" on our part obsolete before it was even effected, how (it is asked) can we afford the single-system approach?

While the above argument has merit, there are strong motivations, not solely of historical interest, to "optimize" in the sense of procuring only the single system that is best in terms of cost/effectiveness. If we are severely confined by a budgetary ceiling, how (it is asked) can we afford any system other than the best for the dollar?

We shall show that, in general, the position that a budgetary ceiling implies only the single best system should be purchased is untenable. If we set out to optimize the spending of our deterrent dollar by buying as much "strike second" retaliatory power as possible within a given total funding limitation, it may turn out either that we should buy a mix or that we should buy a single system. And one of the deciding factors, even in the absence of other considerations, is the nature of the systems at our disposal with respect to their countermeasures vulnerability.

It proves profitable to distinguished two broad types of vulnerability which we term percentage vulnerability and numerical vulnerability. These are associated with different ways in which an enemy's countermeasure effort is rewarded.

A numerically vulnerable (NV) system is one for which a fixed investment by the enemy in countermeasures provides him with a fixed numerical counter to the system. That is, for a given investment by the enemy, the number of retaliatory units he can destroy is independent of how many such units we may choose to buy. When a saturation law applies, this may be true only for small countermeasures investments and a NV system is then characterized by the fact that as long as the enemy's investment in countermeasures is proportional to our procurement, he can counter a fixed fraction of the systems.

Examples of NV systems are the ground-alert SAC manned bomber force or fixed MINUTEMAN when the countermeasure to these systems is a preemptive ICBM attack. More generally, any unconcealed immobile force that presents the enemy with a numerical destruction problem when he decides to attack the system at its source is a NV system (Note that it is not the retaliatory system which is NV or PV; it is the combination of the retaliatory system and countermeasures system which can so be characterized).

A percentage vulnerable (PV) system is one for which a fixed investment by the enemy in countermeasures provides him with a fixed percentage threat against the system. That is, for a given investment the enemy can counter a given percentage of the retaliatory units of our PV system, and that percentage is independent of how many units we may choose to buy.

Examples of PV systems are POLARIS submarines or mobile MINUTEMAN when an attempt is made to counter them by blunting. More generally, any system that forces the enemy to a search effort as the major portion of his countermeasures investment can be characterized as being PV.

To understand why search generates a percentage threat, we have only to recall that a given amount of searching produces a given probability of detecting a particular unit searched for. This probability of detection applies to any unit present that can be expected to be found (and destroyed). This fraction (percentage) is independent of the number of units present; but, of course, the number of PV units destroyable does depend on the number present. For NV systems, on the other hand, the number of units destroyable does not depend on the number present; but, of course, the percentage of NV units destroyed does depend on the number present.

It is hoped that the reader has these distinctions clearly in mind for they are crucial to an understanding of the results to be discussed. Systems, are conceivable which would partake of some of the characteristics of each. One example might be the POLARIS system if the enemy develops a moderately effective

AICBM. Another might be the SAC ground-alert force for certain combinations of blunting effort and at-target defenses. When faced by a single countermeasure only, most systems seem to fit quite satisfactorily into either the NV or the PV category.

II. ASSUMPTIONS

The basic mathematical models that treat the problems of optimum choice are presented in the appendicies. In particular, appendix A sets up the equations for the basic models which allow the problem to be formulated as a mathematical game. For the time being we shall only consider the basic models treated in appendicies A to D. These basic models depend upon two fundemental and argueable assumptions.

A. For both the retaliatory and countermeasure systems, there are no prices-of-admission and no learning curve applies to large purchases, i. e., the number of retaliatory units purchased is proportional to the amount spent on a single system and similarly for the number of countermeasure units purchased by the enemy.

B. For each retaliatory system there exists a single and unique countermeasure that the enemy can apply against the system and, moreover, the countermeasure is different for each system.

We shall postpone the discussion of these assumptions and proceed to a discussion of the basic models.

Two opponents, A and B, are considered. A wishes to maintain a "strike second" (retaliatory) capability, while B wishes to counter that capability. (B has similar requirements for a "strike second" capability against A but we ignore that part of the problem). We consider the opponents to have fixed, arbitrary annual spending rates, C_A and C_B. The assumed conditions are:

<u>A's mission</u>: Deterrence through assured retaliatory capability.

<u>A's objectives</u>: Subject to spending-rate ceiling C_A, maximize the total number of weapons deliverable on-target after B's counterforce measures have been applied (strategic and tactical initiative and surprise being conceded to B).

B's mission: Maintain a capability to nullify A's retaliation. B's objective: Subject to spending rate ceiling C_A, minimize the number mentioned above that A seeks to maximize.

Each of the possible weapon systems available to A for accomplishing his mission is described by just two parameters, one that measures cost (per weapon or per unit of destruction) in the absence of enemy countermeasures, the other an "invulnerability index" that measures the outlay required B to reduce (by best available countermeasures) the retaliatory effectiveness of the system by a given amount.

III. RESULTS

A. PV Systems: The discussion that follows is based upon appendix B which examines the allocation problem for a number of PV systems, each of which is subject to a saturation law, i. e., the marginal effectiveness of added countermeasure allocation by the enemy against a given system decreases as the allocation increases. PV systems not subject to a saturation law are not treated although it is pointed out the latter systems can be viewed as special cases of the former and hence that similar results should apply.

If A's choice of retaliatory systems is limited strictly to PV systems, then in general it will pay him to buy a diversified "mix" of several such systems. The determination of the optimum number of systems, n, to be procured depends on the system parameter values (cost and invulnerability) and on C_B, but is independent of C_A. This means that the amount of diversification advisable for A does not depend on whether A is rich or poor, but only on whether his opponent is rich or poor. The richer B is, the larger n, should be, and the greater the penalty A must pay for failure to diversify. Thus, in the case of PV systems, diversification is not a privilege of the rich, but makes good sense for even the most penurious. The hard-to-take implication is that the effect of a strict budget ceiling on A's total expenditures should be to force curtailment not of system diversification but of numerical procurement of any given system. The effect of a very tight budget is properly to prevent procurement of "enough" of any one PV system, but not to prevent procurement of "enough" different PV systems. By the "bundle-of-sticks" philosophy, any single system is (numerically) weak, but all together lend mutual support and offer the nearest attainable approach to adequacy.

Diversification also represents a conservative strategy for A. He can of course, always gamble that B's total countermeasure efforts will be less than B's economic resources make possible. Such an assumption will reduce the diversification required for optimal mix. But it is solely on the basis of such "intention guessing" that A can logically justify the rejection of a diversified mix of PV systems.

Cost enters into A's determination of how many different systems to buy, and which systems to buy. (He should, in fact, buy the cheapest systems available). But once the choice of systems has been made, the relative amounts that should be spent on the various chosen systems are, oddly enough, not influenced by cost considerations at all, but solely by invulnerability considerations. (System procurement expenditure should in fact be proportional to system invulnerability index).

B. NV Systems: The contrast between optimum procurement policies for PV and NV systems could hardly be sharper. In appendix C it is shown that if A's choice is limited strictly to NV systems, then he should buy only a single optimum system. Any diversification or "mixing" of NV systems is undesirable. The reason is easy to see, inasmuch as NV systems are those that challenge the enemy to a numbers race, and therefore find "safety in numbers". Any procurement of NV systems other than the optimum one is done at the expense of the numbers, and hence the safety, of the optimum system. And this expense is greater than any resulting benefits, so the consequence of diversification is a net loss.

The choice of the optimum NV system is governed not only by the system parameters (cost, invulnerability) but also by the spending ratio C_B/C_A. If this ratio is large, A should de-emphasize cost criteria and seek maximum invulnerability.

C. Cross Mixes of NV and PV Systems: The case of a mix of one NV and one PV system, neither subject to a saturation law, is examined in appendix D. The results of the analysis are exhibited in Figure D-3. Some of these results are:

(1) If the resources of both A and B are small, A should choose only the PV system. Moreover, if the cost per unit of retaliation is higher for the NV system and if B's resources are small, then A should choose only the PV system whatever his own resources may be.

(2) If A's resources are fixed and B's resources are large enough, a mix of systems in which the NV system produces no pay-off but acts as a sink down which B pours his countermeasure funds will he optimal.

(3) If B's resources are fixed and if the PV system is more expensive (i. e., higher cost per unit of retaliation), A should choose the NV system alone only if his resources are large.

(4) If the NV system is more expensive, A should never choose the NV system alone.

It should also be noted that a particularly interesting case of a mix of one NV and one PV system has been examined in great detail by J. M. Dobbie.*

D. Multi-purpose Countermeasure: In the models thus far discussed the simplifying assumption has been made that there is a one-one correspondence of measures and countermeasures. If, instead, a given countermeasure can be effective against more than one measure (e. g., as a counter-force ICBM system is effective against both ground-alert bombers and fixed ICBM sites), some modification has been examined and the techniques are exactly analogous to those required for an analysis of the unique countermeasure cases. Since nothing very surprising results, however, the analysis has not been included in this paper.

IV. DISCUSSION

A. Qualifications: The costing procedure used for the basic models is not completely realistic. It has been assumed that the first dollar that A spends on a system will buy a dollar's worth of retaliation and the last dollar invested in the system will buy no more nor no less retaliation than the first.

The "learning effect" cost reductions attendant on massive procurement are believed to have little influence on the main conclusions generated by the basic models. In any case, the learning effect should be of decreasing importance as the age of mass procurement of huge numbers of identical systems seems

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* Operations Research, Vol. 7 No. 3, May-June 1959, Pgs. 335-346.

to have yielded generally to an era of limited procurement in the face of accelerated technological obsolescence.

The effects of "prices-of-admission" are less easily dismissed. It can be argued that R&D costs usually come out of a different pocket and, moreover, that the systems will probably be developed as a matter of insurance. Thus, the question is really one of procuring existing systems and the "price-of-admission" is of no import. This point of view cannot be entirely dismissed but neither is it entirely convincing.

Appendix D considers NV systems and shows that a "price-of-addmission" can either increase or decrease the attractiveness of diversification, even for NV systems which should not be diversified according to the basic model. It is not surprising that prices-of-admission may make diversification less attractive. One need only consider an allocation problem between two PV systems when the sum of the prices-of-admission is greater than the total budget to be allocated. That prices-of-admission may properly increase the tendency to diversification is not so obvious. The reason is not difficult to discern, however, and depends upon the fact that the enemy may also be subject to a price of admission. Thus, for two NV systems, it may be worthwhile to buy as much of the non-optimal system as possible up to the point where it becomes profitable to the enemy to begin to countermeasure the second system.

B. Suggestions for Future Work: A number of objections can be raised to the basic models and various approaches toward more realistic models suggest themselves. Some are listed below and they range from serious shortcomings to merely possibly valid but not too important points. Since the purpose of this paper is to gain an insight into the problems of allocation in the face of countermeasures and to show that a pure cost/effectiveness comparison is insufficient to solve the allocation problem, no attempt has been made to answer the listed objections. Prior to the actual allocation, they must of course be taken into account. For the time being they remain as possible subject for future study. It might be noted, however, that the tendency of these additional considerations is to enhance the value of diversification.

(1) Systems are often of a nature that partake both of numerical and percentage vulnerability, e. g., SAC ground-alert forces faced by both a pre-emptory strike and at-target defenses.

(2) Systems which are not PV may in the future become NV, e. g., POLARIS if an enemy AICBM breakthrough is achieved.

(3) Technological uncertainty, long lead times, and uncertainty of the enemy's technological capabilities make diversifications more attractive than the models imply.

(4) Since C_B is generally not known, a risk analysis should be made. For that matter, even C_A is not well known in advance. All in all, then, the value of diversification is underrated.

(5) Systems and countermeasures are actually phased in and out over time. Thus the two steps process of the models is inappropriate (steady state versus transient state).

(6) The assumptions that the enemy has perfect knowledge of our decisions and that his leadtimes are too short for us to take advantage of are unnecessarily pessimistic.

APPENDIX A

Derivation of the Basic Models

Opponents A and B are considered. We suppose that A has at his disposal an amount of money C_A which he desires to allocate in the most effective manner between n systems. Denoting systems by numerical subscripts, A's problem is to choose an optimal set of numbers $x = \{x_1, x_2 \ldots x_n\}$ subject to the restrictions

$$x_i \geq 0 \qquad i = 1, 2, \ldots n \tag{A-1}$$

and

$$\sum_{i=1}^{n} x_i = C_A \tag{A-2}$$

A's opponent, B, has an analogous problem. Subject to a budget constraint C_2 he must allocate between n countermeasures, each of which affects exactly one of the retaliatory systems. Thus, B must adhere to the constraints

$$y_i \geq 0 \qquad i = 1, 2, \ldots n \tag{A-3}$$

and

$$\sum_{n=1}^{n} y_i = C_B \tag{A-4}$$

in choosing an optimal set $y = \{y_1, y_2, \ldots y_n\}$. We further assume that A chooses first, knowing that B will make his choice with full knowledge of A's choice.

Each of the systems has a purchase cost and a countermeasures cost associated with it. Specifically, we assume that the "cost per unit of retaliation" is $1/\beta_i$ for the i^{th} system. Thus, spending an amount x_i on the i^{th} system by A results (in the absence of countermeasures) in the purchase of

$$M_i = x_i \beta_i \tag{A-5}$$

"units of retaliation", e. g., an expectation of M_i cities destroyed by the $i^{\underline{th}}$ system. If, on the other hand B invests an amount y_i in the $i^{\underline{th}}$ system, the number of "countermeasure units" purchased for use against the $i^{\underline{th}}$ retaliatory system

$$\phi_i = \alpha_i y_i \tag{A-6}$$

where $1/\alpha_i$ is the cost of a "unit countermeasure" against the $i^{\underline{th}}$ system.

We distinguish four basic possibilities for the number of retaliatory units delivered by a system given A's investment in the system and B's investment in countermeasures against the system. The four possibilities depend upon whether the system* is numerically vulnerable (NV) or percentage vulnerable (PV) and whether or not a saturation law applies.

By a PV system, we mean that a specified investment in countermeasures will always result in a fixed <u>percentage</u> of the purchased retaliatory units surviving, i. e., to say that the $i^{\underline{th}}$ system is PV with respect to the $i^{\underline{th}}$ countermeasure is equivalent to

$$\frac{N_i}{M_i} = f_i(\phi_i) \tag{A-7}$$

where N_i is the number of retaliatory units that survive given M_i purchased and ϕ_i countermeasure units purchased and where f_i is a function independent of M_i and hence of x_i.

An NV system, on the other hand is characterized by the condition that the number of "countermeasure units" purchased tends to determine the <u>number</u> of "retaliatory units" that are countered. Actually, this need only be true for small countermeasure investments and a better characterization of NV systems is to say that if B's countermeasure investment is proportional

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* The word "system" in such statements is to be understood as referring to the combination of the retaliatory and the countermeasures mechanism.

to A's investment in the system, the fractional return to A will be constant, i. e.,

$$\frac{N_i}{M_i} = g_i\left(\frac{\phi_i}{M_i}\right) \tag{A-8}$$

g_i being a function that involves only the ratio ϕ_i/M_i.

Whether or not a saturation effect occurs will determine the functions f_i and g_i. We shall assume the usual exponential saturation law which can be applied to systems for which (a) there are diminishing returns from countermeasures procurement and (b) for small counter measures procurement the percentage attrition achieved is linear in the procurement.

Thus, for a PV system subject to a saturation law we have from Equation (A-7)

$$N_i = M_i\, e^{-\phi_i} \tag{A-9}$$

and from Equations (A-5) and (A-6),

$$N_i = \beta_i x_i e^{-\alpha_i y_i} \tag{A-10}$$

For a NV system, a similar expression will apply based on Equation (A-8), specifically

$$N_i = \beta_i x_i e^{\frac{\alpha_i y_i}{x_i}} \tag{A-11}$$

(It should be noted that since "countermeasure units" are not numerically defined, the constants of proportionality have been taken as unity.)

When no saturation law applies, simple proportionality will be assumed and the constant of proportionality will again be assumed equal to one. Thus, for a PV system not subject to a saturation law

$$N_i = M_i(1-\phi_i) = \beta_i x_i(1-\alpha_i y_i) \tag{A-12}$$

and for a NV system

$$N_i = \beta_i x_i \left(1 - \frac{\alpha_i y_i}{x_i}\right) = \beta_i (x_i - \alpha_i y_i) \qquad \text{(A-13)}$$

Finally, we note that the object of the "game" is for A to maximize and B to minimize the pay-off function

$$N(x, y) = \sum_{i=1}^{n} N_i (x_i y_i) \qquad \text{(A-14)}$$

APPENDIX B

The Basic Model for a Number of Percentage Vulnerable Systems

In Appendix A, the basic model for PV systems was derived. In this appendix we use the basic model to derive an expression for the optimal choices of the two opponents. The only case that will be treated is a set of PV systems subject to a saturation law. The case of PV systems not subject to a saturation law can be handled by techniques similar to those that follow and they will also be found to imply the choice of a mix of systems (in general) rather than the choice of a single system. It should also be noted that the lack of a saturation effect makes the problem less difficult to handle as might be expected since the no saturation pay-off function can be derived from the saturation pay-off functions by a series expansion of the exponential and neglect of the higher order terms.

The "game" we are interested in here begins when A chooses a set $x = \{x_1, x_2, \ldots x_n\}$ subject to the conditions $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = C_A$. The next move is B's who, with full knowledge of A's choice, selects a set $y = \{y_1, y_2, \ldots y_m\}$ subject to $y_i \geq 0$ and $\sum_{i=1}^{n} y_i = C_B$. It is A's object to maximize and B's to minimize the pay-off function.

$$N(x, y) = \sum_{i=1}^{n} \beta_i x_i e^{-\alpha_i y_i} \qquad \text{(B-1)}$$

In view of the interpretation of the game (see appendix A), the α_i and β_i will be taken as positive quantities and even as non-zero quantities. While $\alpha_i = 0$ is conceivable, i. e., a non-countermeasureable system, it should be obvious how to treat such a case should it ever arise.* It may also be noted that $x_i = 0$ implies $y_i = 0$, i. e., there is no point in countermeasuring a non-existant system.

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* Just in case it is not obvious, divide the systems into two classes, countermeasurable and non-countermeasurable. Choose the cheapest non-countermeasurable system and the best mix of countermeasurable systems. Treat as a two element system by the methods of this paper.

In order to ease some typographical problems, some notational conventions and abreviations are indicated. First, we shall $\sum_i$, and π_i; for $\sum_{i=1}^{n}$ and $\prod_{i=1}^{n}$ as long as no confusion results. Also we define

$$1/\gamma_i = \alpha_i \tag{B-2a}$$

$$1/K = \sum_{i=1}^{n} \gamma_i \tag{B-2b}$$

To begin, we assume A's choice has been made, say $x^\circ = x_1^\circ, x_2^\circ, \ldots x_n^\circ$ and find B's optimal choice. To do this, we use the method of Lagrangian multipliers and minimize

$$\lambda C_B + N(x^\circ, y) = \sum_i (\beta_i x_i^\circ e^{-\alpha_i y_i} + \lambda y_i) \tag{B-3}$$

where λ is a constant to be determined. Partial differentiation and equation to zero results in

$$-\alpha_i \beta_i x_i^\circ e^{-\alpha_i y_i} + \lambda = 0 \qquad i = 1, 2, \ldots n \tag{B-4}$$

or solving for y_i

$$y_i = \gamma_i \ln \left(\frac{\beta_i x_i^\circ}{\gamma_i}\right) - \gamma_i \ln\lambda \qquad i = 1, 2, \ldots n \tag{B-5}$$

To determine λ, sum equation (B-5) over the index i.

$$C_B = \sum_i \gamma_i \ln\left(\frac{\beta_i x_i^\circ}{\gamma_i}\right) - (\ln \lambda) \sum_i \gamma_i \tag{B-6}$$

Solving equation (B-6) for λ and substituting the result into equation (B-5),

$$y_i^\circ = \gamma_i \ln\left(\frac{\beta_i x_i^\circ}{\gamma_i}\right) + K\gamma_i \left\{ C_B - \sum_j \gamma_j \ln\left(\frac{\beta_i x_i^\circ}{\gamma_i}\right)\right\} \tag{B-7}$$

$$i = 1, 2, \ldots n$$

where the superscript on the y_i° indicates that these represent B's optimal choices given x°. Note that the y_i° may be negative at this stage and that $y_i^\circ = -\infty$ when $x_i^\circ = 0$. For the moment, we accept these anomolies.

While the precedure followed need only result in some sort of inflection point, the form of equation (B-1) certainly indicates that y° is a minimal point. To assure ourselves that this is so, we calculate

$$\frac{\partial^2 N(x, y)}{\partial y_i \, \partial y_i} = 0, \qquad i \doteq j \qquad \text{(B-8a)}$$

and

$$\frac{\partial^2 N(x, y)}{\partial^2 y_i} = \alpha_i^2 \, B_i x_i e^{-\alpha_i y_i} \qquad i = 1, 2, \ldots n \qquad \text{(B-8b)}$$

Thus*, to show that y° is a minimal point we need only show that

$$\prod_{i=1}^{m} \frac{\partial^2 N(x, y)}{\partial^2 y_i} > 0 \qquad m = 1, 2, \ldots n \qquad \text{(B-9)}$$

at y°. That this is so follows immediately from equation (B-8b) as long as all the $x_i > 0$.

To determine A's optimal choice, we first use equation (B-7) to calculate

$$N(x^\circ, y^\circ) = \sum_i \left\{ (B_i x_i^\circ) \left(\frac{B_i x_i^\circ}{\gamma_i} \right)^{-1} e^{-KC_B} \prod_j \left(\frac{B_i x_i^\circ}{\gamma_j} \right)^{K\gamma_j} \right\} =$$

$$\frac{1}{K} e^{-KC_B} \prod_j \left(\frac{B_j x_j^\circ}{\gamma_j} \right)^{k\gamma_j} \qquad \text{(B-10)}$$

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* See, for example, page III of Advanced Calculus, D. V. Widder, Prentice-Hall, Inc., New York, 1947.

Instead of maximizing $N(x^\circ, y^\circ)$, we maximize

$$N(x^\circ, y^\circ) - \lambda C_A e^{-KC_B}\left\{\frac{1}{K}\prod_j \left(\frac{B_j x_j^\circ}{\gamma_j}\right)^{K\gamma_j} - \lambda \sum_j x_j^\circ\right\} \qquad \text{(B-11)}$$

with λ again an undetermined constant. Differentiating the function specified by equation (B-11) with respect to the x_i° and equating to zero,

$$\gamma_i \prod_j \left(\frac{B_j x_j^\circ}{\gamma_j}\right)^{K\gamma_j} - \lambda\, x_i = 0 \qquad \text{(B-12)}$$

Summing equation (B-12) over the index i,

$$\frac{1}{K} \prod_j \left(\frac{B_j x_j}{\gamma_j}\right)^{K\gamma_j} - \lambda\, C_A = 0 \qquad \text{(B-13)}$$

Combining equations (B-12) and (B-13), we find

$$x_i^{\circ\circ} = \gamma_i\, K\, C_A \qquad \text{(B-14)}$$

The double superscript on the $x_i^{\circ\circ}$ indicates that $x^{\circ\circ}$ is now a specific choice of x°, that one we assert is optimal. To find the corresponding choice for B, we use equations (B-7) and (B-14) with the result

$$\begin{aligned} y_i^{\circ\circ} &= \gamma_i \ln(B_i KC_A) + \gamma_i KC_B - \gamma_i K\left\{\sum_j \gamma_j \ln(B_i KC_A)\right\} \\ &= \gamma_i \ln B_i + \gamma_i \ln(K\, C_A) + \gamma_i KC_B = \gamma_i K\left\{\sum_j \gamma_j \ln B_j + \frac{1}{K}\ln(K\, C_A)\right\} \\ &= \gamma_i \ln B_i + \gamma_i K\left\{C_B = \sum_j \gamma_j \ln B_j\right\} \quad i = 1, 2, \ldots n \end{aligned} \qquad \text{(B-15)}$$

We next note that

$$\begin{aligned} N(x, y^{\circ\circ}) &= \sum_i B_i x_i e^{-\left[\ln B_i + K\left\{C_B - \sum_j \gamma_j \ln B_i\right\}\right]} \\ &= \sum_i x_i e^{-K\left\{C_B - \sum_j \gamma_j \ln B_j\right\}} \\ &= C_A e^{-K\left\{C_B - \sum_j \gamma_j \ln B_j\right\}} \qquad \text{(B-16a)} \\ &= C_A e^{-KC_B \prod_j B_j^{K\gamma_j}} \qquad \text{(B-16b)} \end{aligned}$$

and particularly note that $N(x, y^{\circ\circ})$ is independent of x. Thus we have the following

<u>LEMMA 1.</u> For the game under consideration, A's optimal choice is given by equation (B-14), B's optimal choice is given by equation (B-15), and the value of the game is given by equation (B-16) provided that both positive and negative values of the y_i are allowed.

<u>PROOF.</u> We have already shown that if $x = x^{\circ\circ}$ (and thus the $x_i > 0$), that $y^{\circ\circ}$ is B's optimal choice and the value of the game is $N(x^{\circ\circ}, y^{\circ\circ})$. Suppose A makes some other choice, say $x^{\circ\circ\circ}$. Let B's corresponding choice be $y^{\circ\circ\circ}$. Since B's choice is optimal,

$$N(x^{\circ\circ\circ}, y^{\circ\circ\circ}) \leq N(x^{\circ\circ\circ}, y^{\circ\circ\circ}) = N(x^{\circ\circ}, y^{\circ\circ})$$

and hence either $x^{\circ\circ\circ}$ results in the same pay-off as $x^{\circ\circ}$ or it is not optimal for A. Thus, $x^{\circ\circ}$ is an optimal choice for A and the lemma is proved.

We must still account for the condition $y_i \geq 0$. It is more or less obvious that a choice of $x_i^{\circ\circ}$ which results in $y_i^{\circ\circ} < 0$ should probably be altered to $x_i^{\circ\circ} = 0$ and $y_i^{\circ\circ} = 0$. This leads us to first note that the condition $y_i^{\circ\circ} \geq 0$ can be written as

$$\gamma_i \ln\beta_i - \gamma_i K \sum_j \gamma_j \ln\beta_i \geq - \gamma_i K\ C_B \qquad \text{(B-17)}$$

and hence to postulate the following:

<u>THEOREM.</u> Assume that

$$\beta_1 \geq \beta_2 \geq \cdots \beta_n \qquad \text{(B-18)}$$

and let s be the largest integer ($1 \leq s \leq n$) such that

$$Z_s = \sum_{i=1}^{s} \gamma_i \ln\beta_i - \left(\sum_{i=1}^{s} \gamma_i\right) \ln\beta_s \leq C_B \qquad \text{(B-19)}$$

Then for the game under consideration A's o[illegible]ice is given by

$$x_i^* = \frac{\gamma_i C_A}{\sum_{i=1}^{s} \gamma_i} \qquad i \leq s \qquad \text{(B-20a)}$$

and

$$x_i^* = 0 \qquad i > s \qquad \text{(B-20b)}$$

B's optimal choice is given by

$$y_i^* = \gamma_i \ln\beta_i + \frac{\gamma_i}{\sum_{j=1}^{s} \gamma_j} \left\{ C_B - \sum_{j=1}^{s} \gamma_j \ln\beta_j \right\} \qquad i \leq s \qquad \text{(B-21a)}$$

and

$$Y_i^* = 0 \qquad i > s \qquad \text{(B-21b)}$$

Finally, the value of the game is

$$N(x^*, y^*) = C_A \, e^{\frac{1}{\sum_{j=1}^{s} \gamma_j} \left\{ C_B - \sum_{j=1}^{s} \gamma_j \ln\beta_j \right\}} \qquad \text{(B-22)}$$

This is the theorem at which this appendix is aimed.

We first note that if A's choice is made according to the theorem, then LEMMA 1 applies and equations (B-21) and (B-22) do specify B's optimal choice and the value of the pay-off function and, moreover, $y_i \geq 0$ is automatically satisfied. (Compare equations (B-17) and (B-18)). Also, if the theorem is to make sense, Z_s as given by equation (B-19) must be a non-decreasing function. To show that this is so, we need only examine

$$Z_{\tau+1} - Z_\tau = \gamma_{\tau+1} \ln\beta_{\tau+1} - \left(\sum_{i=1}^{\tau+1} \gamma_i\right) \ln\beta_{\tau+1} + \left(\sum_{i=1}^{\tau} \gamma_i\right) \ln\beta_\tau \qquad \text{(B-23)}$$

$$= (\ln\beta_\tau - \ln\beta_{\tau+1}) \left(\sum_{i=1}^{\tau} \gamma_i\right) \geq 0,$$

the inequality depending upon equation (B-18).

To prove the theorem, then, we need only show that equation (B-20) describes A's optimal choice and we start with .

LEMMA 2. If x^* is A's optimal choice and $x_i^* = 0$, then [illegible] implies $x_j^* = 0$.

PROOF. Assume the converse; specifically assume without loss of generality that $x^{*} = \{0, \xi, x_3^*, \ldots x_n^*\}$ with $\xi \pm 0$ is A's optimal choice. Let $x^{**} = \{\xi_1, \xi_2, x_3^*, \ldots x_n^*\}$ with $\xi_1 = \dfrac{\gamma_1 \xi}{\gamma_1+\gamma_2}$, $\xi_2 = \dfrac{\gamma_2 \xi}{\gamma_1+\gamma_2}$ and hence $\xi_1, \xi_2 = \xi$. Let y^* be B's optimal choice corresponding to x^* and let $y^{**} = \{\eta_1, \eta_2, y_3^*, \ldots y_n^{**}\}$ be B's optimal choice in response to x^{**}. We shall show that x^{**} is a better choice for A and the lemma will be proved.

First, suppose that $\beta_2 e^{-\alpha_2(\eta_1+\eta_2)} < \beta_1 e^{-\alpha_1(\eta_1+\eta_2)}$. Then

$$\beta_2 \xi e^{-\alpha_2(\eta_1+\eta_2)} = \beta_2(\xi_1+\xi_2)e^{-\alpha_2(\eta_1+\eta_2)} < \beta_1 \xi_1 e^{-\alpha_1(\eta_1+\eta_2)} \qquad \text{(B-24)}$$

$$+ \beta_2 \xi_2 e^{-\alpha_2(\eta_1+\eta_2)} < \beta_1 \xi_1 e^{-\alpha_1\eta_1} + \beta_2 \xi_2 e^{-\alpha_2\eta_2}$$

and hence since

$$N(x^*, y^*) \leq N(x^*, y^{**}) = \beta_2 \xi e^{-\alpha_2(\eta_1+\eta_2)} + \sum_{i=3}^{n} \beta_1 x_1^* e^{-\alpha_i y_i^{**}}$$

and since

$$N(x^{**}, y^{**}) = \beta_1 \xi_1 e^{-\alpha_1\eta_1} + \beta_2 \xi_2 e^{-\alpha_2\eta_2} + \sum_{i=3}^{n} \beta_i x_i^* e^{-\alpha_i y_i^{**}}$$

we have by equation (B-24)

$$N(x^*, y^*) < N(x^{**}, y^{**})$$

and x^* cannot be an optimal choice for A. Suppose, on the other hand, that $\beta_2 e^{-\alpha_2(\eta_1+\eta_2)} \geq \beta_1 e^{-\alpha_1(\eta_1+\eta_2)}$ or, equivalently

$$\ln\beta_2 - \alpha_2(\eta_1+\eta_2) \geq \ln\beta_1 - \alpha_1(\eta_1+\eta_2). \tag{B-25}$$

Since y** is an optimal choice for B, the values of η_1, and η_2 must represent an optimal division of the quantity $\eta_1+\eta_2$. If we apply LEMMA 1 to the problem we find

$$\eta_1 = \gamma_1\ln\beta_1 + \frac{\gamma_1}{\gamma_1+\gamma_2}\left\{(\eta_1+\eta_2) - \sum_{i=1}^{2} \gamma_i\ln\gamma_i\right\} \tag{B-26a}$$

$$= \frac{\gamma_1}{\gamma_1+\gamma_2} \quad \gamma_2(\ln\beta_1-\ln\beta_2) + (\eta_1+\eta_2)$$

and

$$\eta_2 = \frac{\gamma_2}{\gamma_1+\gamma_2}\left\{\gamma_1(\ln\beta_2-\ln\beta_1) + (\eta_1+\eta_2)\right\} \tag{B-26b}$$

For the lemma to be applicable here, we must check that both $\eta_1\geq 0$ and $\eta_2\geq 0$. For η_1, this is immediate since $\beta_1\geq\beta_2$. For η_2 we use equations (B-25) and (B-26b) with the result

$$\eta_2 \geq \frac{\gamma_2}{\gamma_1+\gamma_2}\ (\eta_1+\eta_2)\left\{\gamma_1(\alpha_2-\alpha_1) + 1\right\} = \frac{\gamma_2}{\gamma_1+\gamma_2}(\eta_1+\eta_2)(\gamma_1\alpha_2)$$

$$= \frac{\gamma_1(\eta_1+\eta_2)}{\gamma_1+\gamma_2} \geq 0$$

Thus lemma 1 is applicable. To show that $N(x^{**}, y^{**}) \geq N(x^*, y^*)$ we must show that

$$N(x^{**}, y^{**}) = \int e^{\frac{-\eta_1+\eta_2}{\gamma_1+\gamma_2}}\left\{\beta_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}}\ \beta_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}}\right\} + \sum_{i=3}^{n} \beta_i x_i^* e^{-\alpha_i y_i^{**}}$$

$$\geq \beta_2 \int e^{\frac{-\eta_1+\eta_2}{\gamma_2}} + \sum_{i=3}^{n} \beta_i x_i^* e^{-\alpha_i y_i^{**}} = N(x^*, \left\{0, \eta_1+\eta_2, y_3^*, \ldots y_n^*\right\})$$

$$\geq N(x^*, y^*)$$

or equivalently

$$-\frac{\eta_1+\eta_2}{\gamma_1+\gamma_2}+\frac{\gamma_1}{\gamma_1+\gamma_2}\ln\beta_1+\frac{\gamma_2}{\gamma_1+\gamma_2}\ln\beta_2\geq\ln\beta_2-\frac{\eta_1+\eta_2}{\gamma_2}$$

or

$$-\frac{\alpha_1\alpha_2(\eta_1+\eta_2)}{\alpha_1+\alpha_2}+\frac{\alpha_2}{\alpha_1+\alpha_2}\ln\beta_1+\frac{\alpha_1}{\alpha_1+\alpha_2}\ln\beta_2\geq\alpha_2(\eta_1+\eta_2)$$

or

$$-\frac{\alpha_2\ln\beta_1}{\alpha_1+\alpha_2}+\frac{\alpha_1}{\alpha_1+\alpha_2}\left\{\ln\beta_2-\alpha_2(\eta_1+\eta_2)\right\}\geq\ln\beta_2-\alpha_2(\eta_1+\eta_2).$$

But since

$$-\frac{\alpha_2\ln\beta_1}{\alpha_1+\alpha_2}+\frac{\alpha_1}{\alpha_1+\alpha_2}\left\{\ln\beta_2-\alpha_2(\eta_1+\eta_2)\right\}\geq\frac{\alpha_2}{\alpha_1+\alpha_2}\left\{\ln\beta_2-\alpha_2(\eta_1+\eta_2)\right\}$$
$$+\frac{\alpha_1}{\alpha_1+\alpha_2}\left\{\ln\beta_2-\alpha_2(\eta_1+\eta_2)\right\}=\ln\beta_2-\alpha_2(\eta_1+\eta_2) \qquad \text{(B-27)}$$

the lemma is proved.

By LEMMA 2 then, if A's choice is optimal, it must be of the form $x^{**}=\left\{\xi_1,\xi_2,\cdots\xi_m,0,\ldots 0\right\}$. We shall postulate a choice for B, y^{**}, and show that these two choices never result in a pay-off larger than that described by the theorem. The theorum will then be proved.

We first define τ as the largest integer ($\leq m$) for which

$$\left(\sum_{i=1}^{\tau}\gamma_i\right)\ln\beta_\tau+\left\{C_B-\sum_{i=1}^{\tau}\gamma_i\ln\beta_i\right\}\geq 0 \qquad \text{(B-28)}$$

and let y^{**} be given by

$$y_i^{**}=\gamma_i\ln\beta_i-\frac{\gamma_i}{\sum_{j=1}^{\tau}\gamma_j}\left\{C_B-\sum_{j=1}^{\tau}\gamma_j\ln\beta_j\right\} \qquad \text{(B-29)}$$

if this value is non-negative and $y_i^{**}=0$ otherwise. If $\tau=m$, the conditions of lemma 1 are met with all the $y_i^{**}\geq 0$ and the theorem is immediate. Suppose on the other hand the $\tau<m$. Then the pay-off is

$$N(x^{**},\ y^{**}) = \left(\sum_{j=1}^{\tau}\xi_j\right)\left\{e^{-\frac{1}{\sum_{j=1}^{\tau}\gamma_j}\left[C_B-\sum_{j=1}^{\tau}\gamma_j\ln\beta_j\right]}\right\} + \sum_{j=\tau+1}^{m}\beta_j\xi_j \tag{B-30}$$

according to lemma 1. If we denote the expression in the curley brackets by Γ_τ, then we can rewrite equation (B-30) as

$$N(x^{**}, y^{**}) = \left(\sum_{j=1}^{\tau}\xi_j\right)\Gamma_\tau+\sum_{j=\tau+1}^{m}\beta_j\xi_j=C_A\Gamma_\tau+\sum_{j=\tau+1}^{m}\xi_j(\beta_j-\Gamma_\tau) \tag{B-31}$$

We now show that for $j\geq\tau+1$ the expression $(\beta_j-\Gamma_\tau)$ is negative. It will suffice to show this for $j=\tau+1$ since β_j is by hypothesis a non-increasing function of j. For suppose $\beta_{\tau+1}\geq\Gamma_\tau$ or equivalently $e^{+\ln\beta_{\tau+1}}\geq\Gamma_\tau$. This leads to an immediate contradiction for it means

$$\left(\sum_{j=1}^{\tau}\gamma_j\right)\ln\beta_{\tau+1}+C_B-\sum_{j=1}^{\tau}\gamma_j\ln\beta_j\geq 0 \tag{B-32}$$

or adding and substracting $\gamma_{\tau+1}\ln\beta_{\tau+1}$

$$\left(\sum_{j=1}^{\tau+1}\gamma_j\right)\ln\beta_{\tau+1}+C_B-\sum_{j=1}^{\tau+1}\gamma_j\ln\beta_j\geq 0 \tag{B-33}$$

which is inconsistant with the definition of τ that it be the largest integer for which an expression of the form of equation (B-33) holds. Thus, we see from equation (B-31) that if x** is to be optimal, we must have $m\leq\tau$.

Assume now that $m<\tau$. Since x_i^{**} implies $y_i^{**}=0$, we modify the definition of y_i^{**} so that m appears in equation (B-29) instead of τ. Then the payoffs are $C_A\Gamma_m$ for the choices (x**, y**) and $C_A\Gamma_\tau$ for the choices defined by the theorem. To complete the proof, we need only show that $C_A\Gamma_\tau\geq C_A\Gamma_m$ or equivalently that $\ln\Gamma_\tau\geq\ln\Gamma_m$. It will suffice to show that for $m<\tau$. $\ln\Gamma_{m+1}\geq\ln\Gamma_m$. From (B-28) we have

$$\left(\sum_{j=1}^{m+1}\gamma_j\right)\ln\beta_{m+1} + \left\{C_B - \sum_{j=1}^{m+1}\gamma_j\ln\beta_j\right\} \geq 0$$

or

$$\left(\sum_{j=1}^{m}\gamma_j\right)\ln\beta_{m+1} + \left\{C_B - \sum_{j=1}^{m}\gamma_j\ln\beta_j\right\} \geq 0$$

or

$$\left(\sum_{j=1}^{m}\gamma_j\right)\gamma_{m+1}\ln\beta_{m+1} + \gamma_{m+1}\left\{C_B - \sum_{j=1}^{m}\gamma_j\ln\beta_j\right\} \geq 0$$

or

$$\left(\sum_{j=1}^{m}\gamma_j\right)\gamma_{m+1}\ln\beta_{m+1} + \left(\sum_{j=1}^{m}\gamma_j\right)\left(\sum_{j=1}^{m}\gamma_j\ln\beta_j\right)$$

$$+ C_B\left(\sum_{j=1}^{m+1}\gamma_j - \sum_{j=1}^{m}\gamma_j\right)$$

$$- \gamma_{m+1}\sum_{j=1}^{m}\ln\beta_j - \left(\sum_{j=1}^{m}\gamma_j\right)$$

$$\left(\sum_{j=1}^{m}\gamma_j\ln\beta_j\right) \geq 0$$

or

$$-\left(\sum_{j=1}^{m}\gamma_j\right)\left\{C_B - \sum_{j=1}^{m+1}\gamma_j\ln\beta_j\right\} \geq -\left(\sum_{j=1}^{m+1}\gamma_j\right)$$

$$\left\{C_B - \sum_{j=1}^{m}\gamma_j\ln\beta_j\right\}$$

or

$$\ln\Gamma_{m+1} \geq \ln\Gamma_m$$

as was asserted. Thus, we must have $m=\tau$. Comparison of (B-28) and (B-19) reveals that $\tau=s$ and the proof of the theorem is complete.

APPENDIX C

The Basic Model for a Number of Numerically Vulnerable Systems

The model for numerically vulnerable systems is exactly analogous to that for percentage vulnerable systems. From equation (A-8) we have

$$N_i = \beta_i x_i g_i\left(\frac{\alpha_i y_i}{\beta_i x_i}\right) \tag{C-1}$$

where g_i is a function that contains neither x_i nor y_i except in the form specified by equation (C-1). The assumed forms for the pay-off functions are

$$N_i = \beta_i x_i e^{-\frac{\alpha_i y_i}{x_i}} \tag{C-2}$$

when a saturation law applies and

$$N_i = \beta_i x_i \left(1 - \frac{\alpha_i y_i}{x_i}\right) \tag{C-3}$$

for no saturation. (From equations (A-11) and (A-13)).

We shall show that the choice of one and only one system is optimal for A when selecting a mix of NV systems. To see this, let τ be an integer for which

$$\beta_\tau g_\tau\left(\frac{\alpha_\tau C_B}{\beta_\tau C_A}\right) \geq \beta_i g_i\left(\frac{\alpha_i C_B}{\beta_i C_A}\right) \qquad i=1, 2, \ldots n \tag{C-4}$$

The pay-off if the τth system is chosen is

$$N^\circ = \beta_\tau C_A g_\tau\left(\frac{\alpha_\tau C_B}{\beta_\tau C_A}\right) \tag{C-5}$$

Suppose A makes some other choice, say $x^* = \{x_1^*, x_2^*, \ldots x_n^*\}$.
Let B's choice be

$$y^* = \left\{x_1^* \frac{C_B}{C_A}, x_2^* \frac{C_B}{C_A}, \ldots x_n^* \frac{C_B}{C_A}\right\}$$

Then the pay-off is

$$N(x^*; y^*) = \sum_{i=1}^{n} \beta_i x_i^* g_i \left(\frac{\alpha_i y_i^*}{\beta_i x_i^*}\right) - \sum_{i=1}^{n} \beta_i x_i^* g_i \left(\frac{\alpha_i C_B}{\beta_i C_A}\right)$$

$$\leq \sum_{i=1}^{n} \beta_\tau x_i^* g_\tau \left(\frac{\alpha_\tau C_B}{\beta_\tau C_A}\right) = \beta_\tau g_\tau \left(\frac{\alpha_\tau C_B}{\beta_\tau C_A}\right) \sum_{i=1}^{n} x_i^* = C_A \beta_\tau g_\tau \left(\frac{\alpha_\tau C_B}{\beta_\tau C_A}\right) = N^\circ \qquad \text{(C-6)}$$

The inequality in equation (C-6) results from equation (C-4). Since x* is an arbitrary choice we have shown that choosing only the τth system is A's optimal choice, even for a mix of NV systems some of which are and others which are not subject to a saturation law.

It might be noted that according to equation (C-3), the N_i can sometimes be negative. In these cases, the N_i must be redefined to be equal to zero, but this does not affect the proof. Its only effect is to sometimes replace inequality signs by equality signs.

APPENDIX D

A Mix of One Percentage Vulnerable and One Numerically Vulnerable System

A mix of PV and NV systems will now be examined. For simplicity we limit ourselves to the case of one system of each type and for variety we assume that neither is subject to a saturation law. It should be recalled, however, that Appendix C made it clear that an optimal mix of PV and NV systems will never contain more than a single NV system.

The model is similar to that of Appendix B, but by equations (A-12) and (A-13) the pay-off function is now

$$N(x,y) = \beta_1 x(1-\alpha_1 y) + \beta_2 \left[(C_A-x) - \alpha_2 (C_B-y)\right] \qquad \text{(D-1)}$$

where the subscript 1 refers to the PV system and the subscript 2 refers to the NV system. The amount allocated to the PV system by A is x and the amount invested in countermeasures to the PV system by B is y.

The pay-off function as it appears in equation (D-1) is not quite correct. For the problem to be meaningful, we must recognize that B's countermeasures cannot result in a negative pay-off for either system. However, before adjusting the pay-off function it will prove convenient to transform the problem into an equivalent but algebraically simpler form.

First, instead of N(x,y) we consider the function

$$M(x,y) = \frac{\alpha_1}{\alpha_2 \beta_1} N(x,y) \qquad \text{(D-2)}$$

Second, we transform the variables x and y to

$$u = \frac{\alpha_1}{\alpha_2} x \quad \text{and} \quad v = \alpha_1 y \qquad \text{(D-3)}$$

Third, we define

$$A = \frac{\alpha_1}{\alpha_2} C_A \,, \quad B = \alpha_1 C_B \,, \quad K = A-B \,, \quad \text{and} \quad \beta = \frac{\beta_2}{\beta_1} \qquad \text{(D-4)}$$

The pay-off function is now

$$M(u,v) = u(1-v) + \beta\left[K + (v-u)\right] \qquad \text{(D-5)}$$

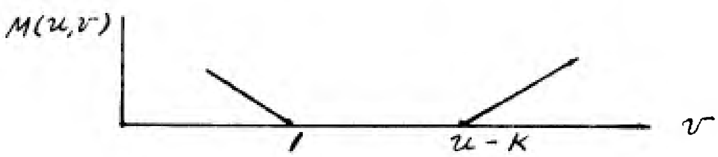
subject to

$$0 \le u \le A, \quad 0 \le v \le B, \quad \text{and } 0 \le \beta \qquad \text{(D-6)}$$

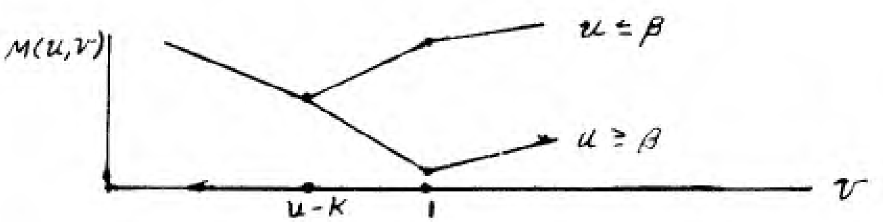
Next, we adjust the pay-off function to eliminate the possibility of either term becoming negative. The correct pay-off is

$$\begin{aligned} M(u,v) &= u(1-v) + \beta\left[K + (v-u)\right] && v \le 1 \text{ and } v \ge u-K && \text{(D-7a)} \\ &= u(1-v) && v \le 1 \text{ and } v \le u-K && \text{(D-7b)} \\ &= \beta\left[K + (v-u)\right] && v \le 1 \text{ and } v \ge u-K && \text{(D-7c)} \\ &= 0 && v \ge 1 \text{ and } v \le u-K && \text{(D-7d)} \end{aligned}$$

For a given value of u, v is chosen so as to minimize M(u,v), i.e., A has the first choice. Using equations (D-7), we see that for $u-K \ge 1$, M(u,v) has the form shown in the diagram below. Since $0 < 1$ and since $B \ge u-K = B(A-u)$, the choice of v can be made anywhere in the interval $[1, u-K]$ with $M(u,v) = 0$ as the result.



Consider now th case u-K - 1 as shown below. If $u \le \beta$, then v should be chosen as $v = u-K$ provided this is an allowable choice, i.e., provided $0 \le u-K$. Otherwise, the optimal choice is $v = 0$. Similarly, if $u \ge \beta$, the optimal choice is $v = 1$ if B - 1 and $v = B$ otherwise.



A's problem is to maximize the function

$$F(u) = \min_{v} M(u,v) \qquad \text{(D-8)}$$

Since $B \geq 1 \longleftrightarrow A \geq 1 + K$, the preceding results allow us to express $F(u)$ in the following manner:

$u < K$ and $u \leq \beta$	$v_1 = 0$	$F_1(u) = u(1-\beta) + \beta K$	(D-9a)
$K \leq u \leq 1 + K$ and $u \leq \beta$	$v_2 = u-K$	$F_2(u) = u(1+K-u)$	(D-9b)
$\beta \leq u \leq 1 + K \leq A$	$v_5 = 1$	$F_3(u) = -u\beta + \beta(1+K)$	(D-9c)
$\beta < u \leq A \leq 1 + K$	$v_4 = B$	$F_4(u) = u(1-B-\beta) + \beta A$	(D-9d)
$1 + K \leq u$	$1 \leq v_5 \leq u-K$	$F_5(u) = 0$	(D-9e)

Since $K < A$ and $(0 \leq \beta)$, the necessary and sufficient conditions for the appearance of these sub-functions are

F_1: $K \geq 0 \longleftrightarrow A \geq B$ (D-10a)

F_2: $K < \beta \longleftrightarrow A \leq B + \beta$ and $1 + K \geq 0 \longleftrightarrow a - B-1$ (D-10b)

F_3: $\beta \leq 1 + K \longleftrightarrow A \geq B + \beta-1$ and $1 + K \leq A \longleftrightarrow B \geq 1$ (D-10c)

F_4: $\beta \leq A$ and $A \leq 1 + K \longleftrightarrow B < 1$ (D-10d)

F_5: $1 + K - A \longleftrightarrow B \geq 1$ (D-10e)

It should be noted that F_1 to F_5 appear from left to right, respectively, i.e. with increasing u. With the exception of F_2, all are linear in u. F_2 on the other hand, can easily be shown to have a maximum at the point $u = \Gamma$ where

$$\Gamma = \frac{1}{2}\left[1 + K\right] = \frac{1}{2}\left[1 + A-B\right]$$

Thus, the slope of F_2 will be negative in the interval of existence of F_2 if $\Gamma \leq 0$ or $\Gamma < K$ and will be positive if $\Gamma \geq 1 + K$ or $\Gamma \geq \beta$ or $\Gamma \geq A$. Otherwise it will have a maximum. If we write $S(F)$ for the slope of F we see from equations (D-9) that

$\beta \geq 1 \longrightarrow S(F_1) \leq 0$ and $\beta < 1 \longrightarrow S(F_1) \geq 0$ (D-11a)

$\{\Gamma \leq 0$ or $\Gamma < K \longleftrightarrow A \leq B-1$ or $A \geq 1 + B\} \longrightarrow S(F_2) < 0$ (D-11b)

$\{\Gamma \geq 1 + K$ or $\Gamma \geq \beta$ or $\Gamma \geq A \longleftrightarrow A < B-1$ or $A \geq 2\beta + B-1$

or $A < 1 - B\} \longrightarrow S(F_2) > 0$ (D-11c)

$S(F_3) \leq 0$ (D-11d)

$$B \geq 1-\beta \longrightarrow S(F_4) \leq 0 \text{ and } B \leq 1-\beta \longrightarrow S(F_4) \geq 0 \qquad \text{(D-11e)}$$

$$S(F_5) = 0 \qquad \text{(D-11f)}$$

We will also find it necessary to know the value of $F_1(0)$ relative to $F_2(\Gamma)$, $F_2(\beta)$, and $F_2(A)$. From equations (D-9) we have:

$$F_1(0) = \beta K \qquad \text{(D-12a)}$$

$$F_2(\Gamma) = \frac{1}{4}(1 + K)^2 \qquad \text{(D-12b)}$$

$$F_2(\beta) = \beta(1 + K) - \beta^2 \qquad \text{(D-12c)}$$

$$F_2(A) = F_4(A) = A(1-B) \qquad \text{(D-12d)}$$

First consider

$$F_1(0) \geq F_2(\beta) \longleftrightarrow \beta K \geq (1 + K) - \beta^2 \longleftrightarrow \beta \geq 1 \qquad \text{(D-13)}$$

Next

$$F_2(A) = F_4(A) \geq F_1(0) \longleftrightarrow A(1-B) \geq \beta(A-B) \longleftrightarrow A \geq \frac{-\beta B}{1-B-\beta} \qquad \text{(D-14)}$$

Also note that

$$B \geq \frac{-\beta B}{1-B-\beta} \longleftrightarrow 1 - B - \beta \geq -\beta \longleftrightarrow 1 \geq B \qquad \text{(D-15)}$$

Combining equations (D-14) and (D-15), we arrive at the following result which will suffice for our purposes:

$$1 \geq B \text{ and } A \geq B \longrightarrow F_2(A) = F_4(A) \geq F_1(0) \qquad \text{(D-16)}$$

Finally, assume $B \geq 1$ and consider the function

$$4\left[F_2(\Gamma) - F_1(0)\right] = 1 + K(2 - 4\beta) 2 K^2 \qquad \text{(D-17)}$$

It is easily shown that the function has a minimum at $K = 2\beta - 1$ and that the function is equal to zero at

$$K = (2\beta - 1) \pm 2\sqrt{\beta^2-\beta} \qquad \text{(D-18)}$$

Since we are only interested in this function when Γ falls within the region of existence F_2, we are only interested in $K - 2\beta - 1$ (i.e. $\Gamma - \beta$) and hence the lesser of the solutions given by equation D-18). Combining the above results we have

$$\text{for } \beta \geq 1, \quad F_2(\Gamma) \geq F_1(0) \longleftrightarrow A \leq B + 2\beta - 1 - 2\sqrt{\beta^2-B} \qquad \text{(D-19)}$$

It should also be noted that

$$B = 1 \longrightarrow 2\beta - 1 - 2\sqrt{\beta^2 - B} = 1 \qquad \text{(D-20a)}$$

$$B = \infty \longrightarrow 2\beta - 1 - 2\sqrt{\beta^2 - B} = 0 \qquad \text{(D-20b)}$$

Now consider Figure D-1, drawn for $\beta \leq 1$. The figure illustrates the regions in which the five sub-functions exist and the regions in which their slopes are negative and positive. For F_2, the region in which neither is true ($A \leq B+2\beta-1$ and $A \geq 1-B$ and $A \geq \beta-1$) is the region in which F_2 exhibits a maximum at $u = \Gamma$. The boundary lines follow immediately from equations (D-10) and (D-11) and examination of their intercepts will quickly show that as long as $\beta \leq 1$ they do not change their relative positions.

Consider first the region $A \geq B+2 -1$ and $B \leq 1-B$. Here F_1, F_2, and F_4 exist, all with slopes ≥ 0. Similarly, in the region $A \leq 2B+2 -1$ and $A \leq 1-B$ only F_1 and F_2 exist, both with non-negative slopes. Thus, in either of these regions $u^o = A$ where u^o is the optimal choice of u. From (D-9b) and (D-9d), $v^o = B$.

Next consider the region $B \geq 1-\beta$ and $A \geq B+2\beta-1$ in which all five sub-functions exist. Those to the left of β (F_1 and F_2) have slopes ≥ 0 and those to the right of β have slopes ≤ 0. Hence $u^o = B$. This value of u makes $M(u^o,v)$ independent of v as long as $\beta-K \leq v \leq 1$ and it follows that $\beta-K \leq v^o \leq 1$.

For the region $A \leq B+2\beta-1$ and $A \geq B-1$ and $A \geq 1-B$, the sub-function to the right of F_2 (F_3, F_4, and F_5) can exist only with slopes ≤ 0 and to the left F_1 can exist only with a slope ≥ 0. Moreover, F_2 exists throughout the region and has a maximum at Γ. Hence $u^o = \frac{1}{2}\left[1+K\right]$ and $v^o = \frac{1}{2}\left[1-K\right]$.

Finally, for $A \leq B-1$, only F_5 exists, u^o is arbitrary and $1 \leq v^o \leq u^o-K$.

Now consider figure D-2 drawn for $\beta \geq 1$. The line $A = B+2\beta-1 - 2\sqrt{\beta^2-B}$ has been drawn in two positions since by equation (D-20) it is hounded by $A = B$ and $A = B+1$.

In the region $A \leq 1-B$, F_2 always exists with $S(F_2) \geq 0$. F_1 may exist with $S(F_1) \leq 0$. F_3, F_4, and F_5 do not exist. Thus there is a maximum at $u = A$ and perhaps a second maximum

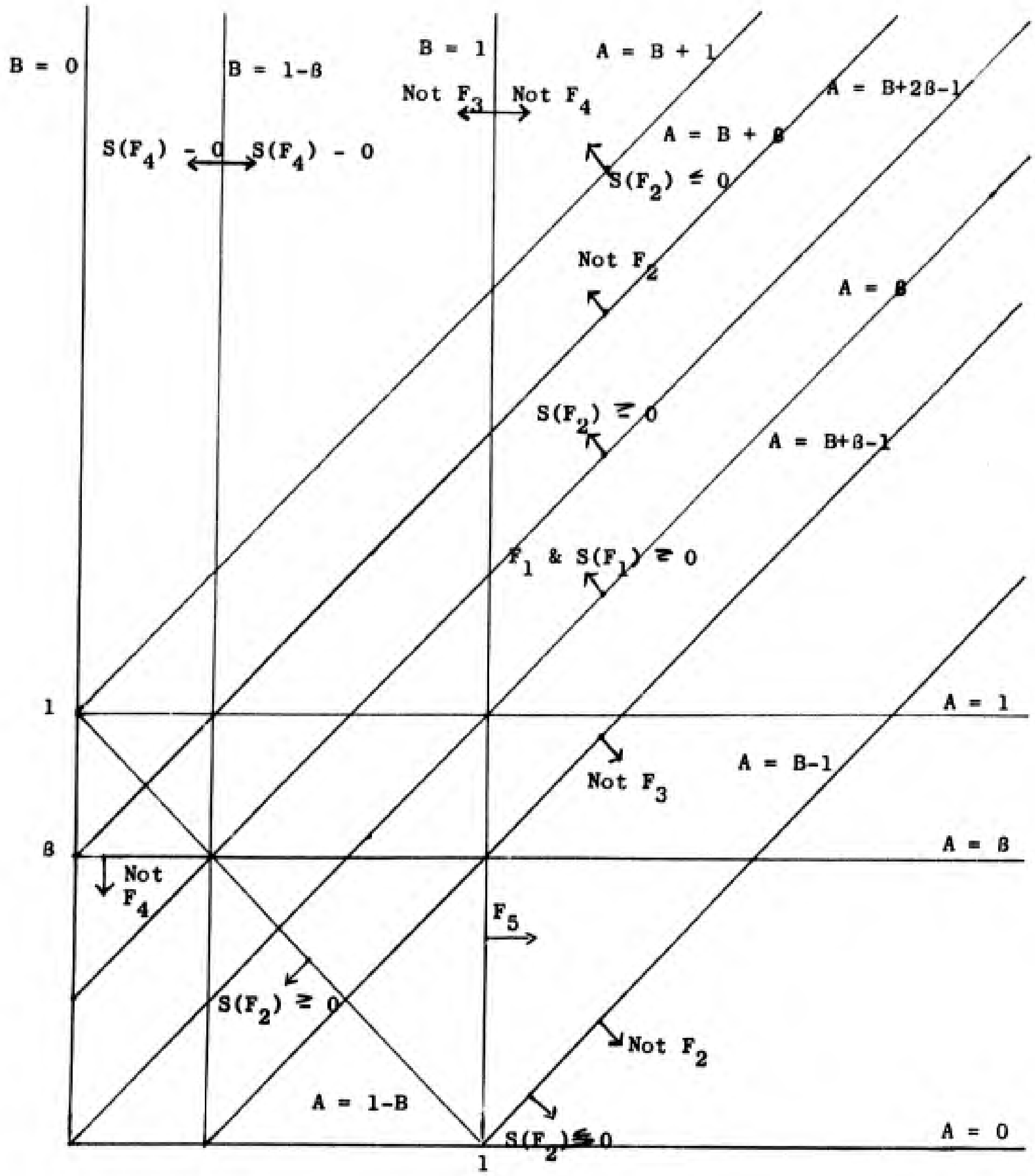


Figure (D-1)
Boundaries for $\beta \leq 1$

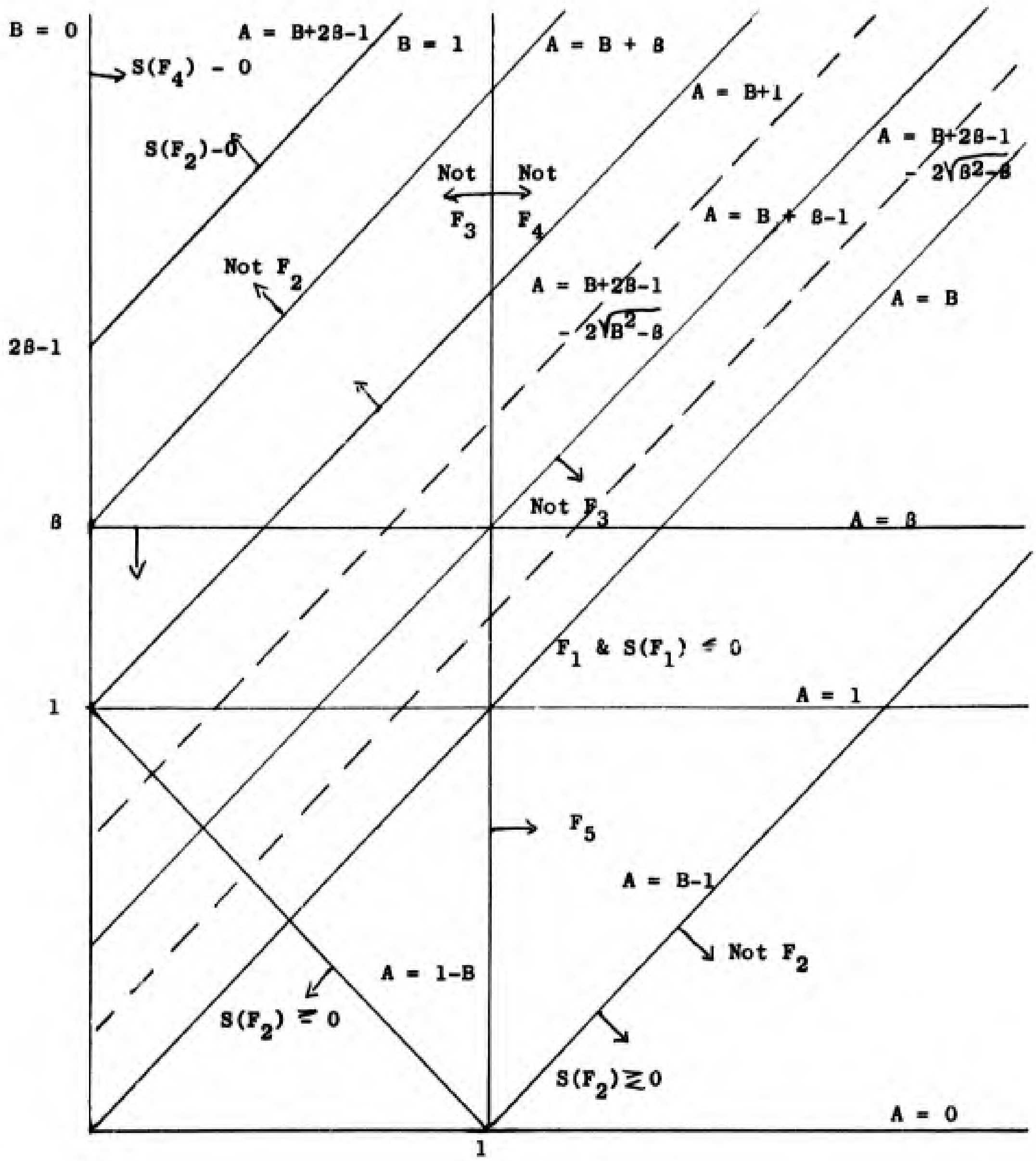


Figure (D-2)

Boundaries for β - 1

at $u = 0$. Since $B \leq 1$ and since F_1 exists only for $A \geq B$, we have $F(A) \geq F(0)$ by equation (D-16). Hence $u^o = A$ and $v^o = B$.

In the region $A \geq 1-B$ and $A \geq B+2\beta-1 - 2\sqrt{\beta^2-B}$, F_3, F_4, and F_5 exist only with slopes ≤ 0. F_1 always exists with $S(F_1) \leq 0$ so that there is a maximum at $u = A$. F_2 may exist but not with $S(F_2) > 0$. Hence if there is a second maximum, it must be at $u = \Gamma$. But by equation (D-19), $F(0) \geq F(\Gamma)$ and thus $u^o = v^o = 0$.

For the region $A \geq 1-\beta$, $A \leq B+2\beta-1 - 2\sqrt{\beta^2-\beta}$, and $A \geq B-1$, F_2 exists with a maximum at $u = \Gamma$. Also F_3 and F_5 exist only with slopes ≤ 0. F_1 may contribute a second maximum at $u = 0$ but by equation (D-19), $F(\Gamma) \geq F(0)$ so that $u^o = \frac{1}{2}\,[1+K]$ and $v^o = \frac{1}{2}\,[1-K]$.

Finally, for $A \leq B-1$ only F_5 exists, u^o is arbitrary and $1 \leq v^o \leq u^o - K$.

Figure D-3 summarizes the solutions. It is of particular interest that for the region in which $u^o = \Gamma = \frac{1}{2}\,[1+K]$ and $v^o = u^o - K = \frac{1}{2}\,[1-K]$, A puts resources into the NV system only to have B completely countermeasure them. The pay-off is from the PV system and the NV system serves no other purpose than to siphon off some of B's countermeasure resources. (See equations D-7).

The other mixed solution, $u^o = B$ is of a more complex nature. Here, A's choice produces a stationary pay-off for B, at least as long as $B-K \leq v^o \leq 1$ (Equations D-7). Thus, if $B \geq 1$, v^o can be chosen so that there is no pay-off from the PV system. If $0 \leq \beta-K$, then v^o can be chosen so that no pay-off from the NV system results.

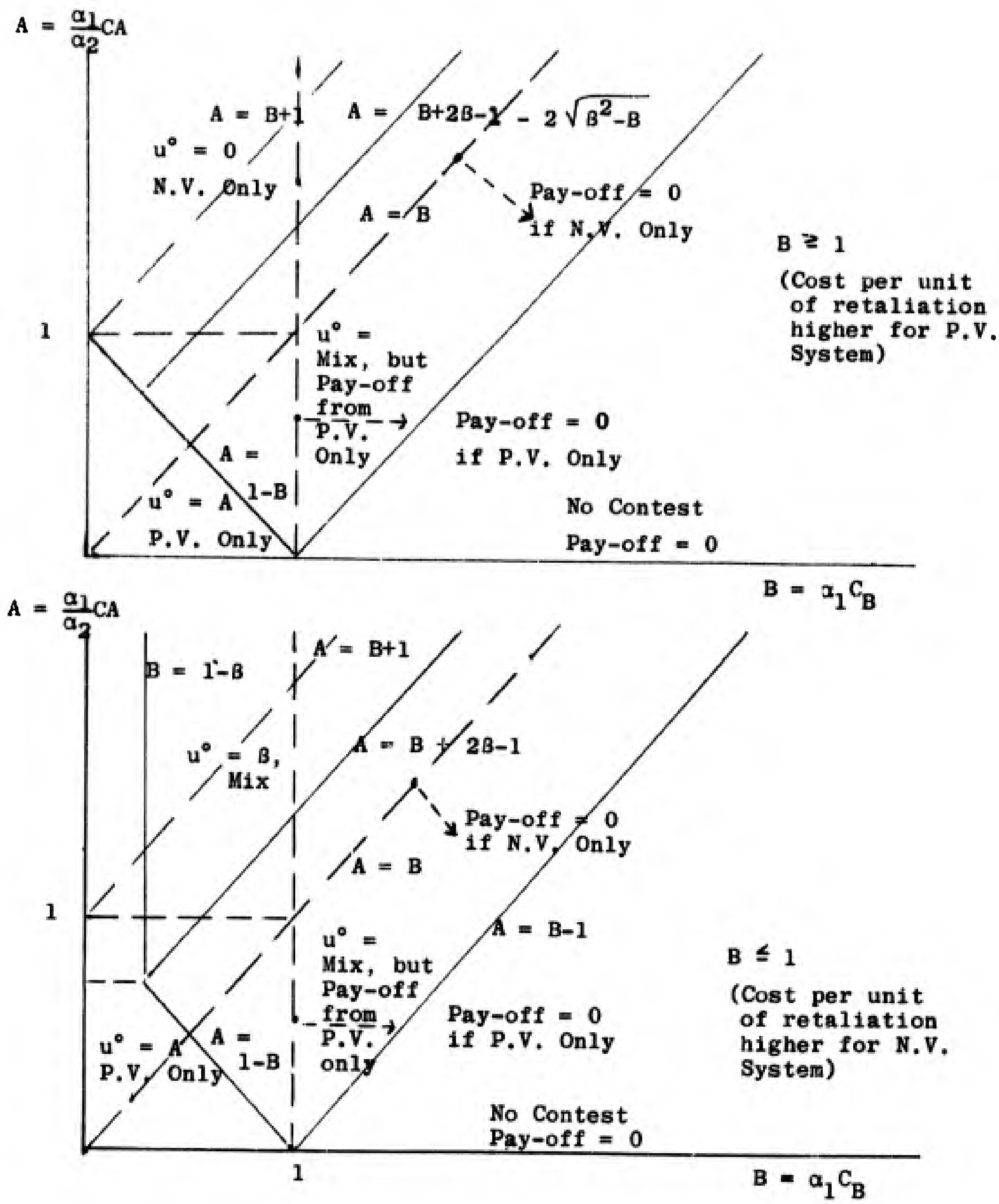


Figure D-3

Solution Diagram for a PV/NV Mix