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EFFECTS OF TURBULENCE INSTABILITIES ON LASER PROPAGATION

David A. de Wolf

RCA Laboratories

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13. ABSTRACT

We report results on Furutsu's contested theory of beam wave irradiance in turbulent air, on our attendance of the XVII-th General Assembly of URSI (Warsaw, Poland), on a simple model for studying feature resolution, on the log-amplitude variance for plane and spherical waves, and on the angle-of-arrival power spectrum for an interferometer. Section 2 deals exhaustively with the spherical-wave extension of the irradiance scintillation theory. The log-amplitude variance for spherical waves is found to be

$$\langle \delta\chi^2 \rangle = 1.25(\kappa_m^{7/3} L^3 C_n^2)^{-1/6}.$$

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FOREWORD

The Quarterly Report was prepared by RCA Laboratories, Princeton, New Jersey, under Contract No. F30602-72-C-0486. It describes work performed from September 19, 1972 to December 15, 1972 in the Communications Research Laboratory, Dr. K. H. Powers, Director. The principal investigator and project scientist is Dr. D. A. de Wolf.

The report was submitted by the author on January 15, 1973. Submission of the report does not constitute Air Force approval of the report's findings or conclusions. It is submitted only for the exchange and stimulation of ideas.

The Air Force Program Monitor is Lt. Darryl P. Greenwood.

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1. SUMMARY OF RESULTS -- June to December 1972

The work performed in the period of June 1972 to December 1972 under Contract No. F30602-72-C-0486 is a logical extension of that performed under Contract No. F30602-71-C-0356 from June 1971 to June 1972. The technical and final reports that have appeared previously under both these efforts are listed under [1] in the References and are cited in abbreviated notation (TRI through TRVI) in the text. A summary of the work reported in TRVI and of subsequent work follows.

1.1 The Furutsu Theory

We have indicated that Furutsu's result for the statistics of the log-irradiance of a beam wave in turbulent air may be incorrect. The error appears to lie in an approximation made to solve the equation for the fourth-order coherence: an approximation that assumes that the permittivity correlation may be replaced everywhere by the square law (which only holds for very small separation distances).

1.2 The XVII-th General Assembly of URSI

We attended the XVII-th General Assembly of the International Union of Radio Science (URSI) in Warsaw, Poland, August 21-29, 1972. A special report on this conference has been issued to all who receive Management Reports. Copies are available upon request.

1.3 Resolution Study

A simple model was constructed for the purpose of estimating when one can no longer distinguish two bright point features on an object in the sky due to intervening layers of turbulent air. The criterion of clear distinction between two such objects separated by a distance ρ_o is determined by the ratio of ρ_o to the distance $C_n^2 \kappa_m^{1/3} L^3$. When this ratio becomes small, the two points smear. Details are worked out for horizontal propagation and for slant-path propagation.

1.4 The Log-Amplitude Variance (Plane Waves)

The plane-wave theory of TRV has been completed, and a calculation of the log-amplitude variance yields $\langle \delta\chi^2 \rangle = 0.41(\kappa_m^{7/3} L^3 C_n^2)^{-1/6}$ in the saturation regime. Comparison with plane-wave data appears to be reasonable.

1.5 Angle-of-Arrival Power Spectrum for an Interferometer

The calculation of the angle-of-arrival power spectrum given in TRIV has been extended for the case of a simple interferometer of two parallel rays. Some closed-form results are obtained, and - specifically - the single-ray formulas are shown to require only slight modification in most cases of interest. An important difference with results of other workers appears to be the prediction that $W(\omega) \propto \omega^{-2/3}$ in the inertial subrange of turbulence, and not $W(\omega) \propto \omega^{-8/3}$!

1.6 Spherical-Wave Irradiance in Turbulent Air

An extension of the theory for plane waves, mentioned above in subsection 1.4, has been worked out for spherical waves. It yields $\langle \delta\chi^2 \rangle = 1.25(\kappa_m^{7/3} L^3 C_n^2)^{-1/6}$ in the saturation regime. It is the topic of Section 2 of this report.

This completes, and even extends somewhat, the required tasks of the effort under Contract No. F30602-72-C-0486. Extensions of this work will be performed under a future contract effort.

2. IRRADIANCE SCINTILLATION FOR SPHERICAL WAVES

The purpose of this section is to expand the plane-wave theory of irradiance scintillation to include spherical waves, and specifically to compute the variance of the log-amplitude in that case. There are several versions of the plane-wave theory - in diverse stages of development - in circulation, and the reader may have difficulty in following the development. A brief guide through previous work will therefore be given.

The first of our reports to deal with the subject is TRIII (the notation for referring to previous reports is explained on p.1). It is chiefly a tutorial work that develops the basic equation (6.10), and then works out all of the theory except that for the saturation regime. Sections 10 through 12 of TRIII are self-contained, and serve to prove a crucial theorem in optical propagation through turbulent air: *the n-point correlation of refractive-index fluctuations always occurs in certain integrals such that it may be factored into <u>one</u> ordered product of two-point correlations, even though it is <u>not</u> a multivariate Gaussian.*

In TRV, a first attempt was made at a saturation-regime theory. As we see it now, there are objections to be raised against the development given in Section III of TRV. Furthermore, the important parts of the theory are really in the Appendix, which has been very abbreviated and which yields Eq.(A9) [unfortunately not labeled in TRV] as a result of a development from Eq.(A4) on, that differs somewhat from that given in this report. Consequently, Sections III through V and the Appendix of TRV have been thoroughly revised for publication in the Journal of the Optical Society of America in February or March 1973.

Therefore, there appears to be a departure from a fully logical development in Section 3 of TRVI, because we do indeed start from Eq.(A4) of TRV - bearing in mind that the development of Eq.(A4) from Eq.(A2) has been greatly expanded in the article to appear in the J. Opt. Soc. Am. - and then ignore the further development in TRV. Section 3 of TRVI does repeat the pertinent development, leading to an expression for the log-amplitude variance, also given in the aforementioned article.

Let us now continue the development for spherical waves. In referring to TRIII, we find that the first question is whether a fundamental equation such as (6.10) can be developed for spherical waves, using the Green's function $G_s(\vec{r},\vec{r}_1)$ of Eq.(2.7). When this Green's function is used in Eq.(3.3), we can define a stationary-phase exponential $\exp(i\Psi_m)$ as in Eq.(6.1), but in this case,

$$\Psi_m = k[(\Delta\rho_m^2 + \Delta z_m^2)^{1/2} - (\rho_{m-1}^2 + z_{m-1}^2)^{1/2} + (\rho_m^2 + z_m^2)^{1/2}] - \vec{K}_m \cdot \vec{\rho}_m, \quad (1)$$

and the points of stationary phase are found by setting the gradient of Ψ_m with respect to $\vec{\rho}_m$ equal to zero. It is easily seen that the stationary-phase points $\vec{\rho}_m$ obey

$$\Delta\vec{\rho}_m/\Delta r_m = -\vec{Q}_m/k + \vec{\rho}_n/r_n, \quad (2)$$

where $\vec{Q}_m \equiv \sum_{j=m}^{n} \vec{K}_j.$

We adhere to the notation developed in previous work: it is quite uniform and consistent. An exhaustive glossary of symbols is given in TRIII. Note that the physical content of Eq.(2) is obvious and practically identical to that for plane waves. The initial propagation angle is not zero but $\vec{\rho}_n/r_n$ for spherical waves because the source is a point at $\vec{\rho} = 0$, $z = 0$, and the first deflection occurs at $(\vec{\rho}_n, z_n)$. The scattering angle at $\vec{r}_m$ is given by $-\vec{Q}_m/k$, exactly as in the case of plane waves. Consequently, the physical small-angle approximations are identical too.

Since this identity permits one to utilize $\Delta\rho_m \ll \Delta z_m$, $\rho_{m-1} \ll z_{m-1}$, and $\rho_m \ll z_m$ in Eq.(1), one can approximate Ψ_m adequately by

$$\Psi_m \approx \frac{k}{2}\,\frac{z_m z_{m-1}}{\Delta z_m}\left(\frac{\vec{\rho}_{m-1}}{z_{m-1}} - \frac{\vec{\rho}_m}{z_m}\right)^2 - \vec{K}_m \cdot \vec{\rho}_m. \quad (3)$$

It requires some algebraic rearranging of terms to obtain this form, but once we have Eq.(3) we can carry out all the $d^2\rho_m$ integrations in Eq.(3.3) of TRIII to obtain

$$B_n = \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{m-1}} dz_m \int d^2K_m \; \delta\tilde{\varepsilon}(\vec{K}_m, z_m) \exp -\left[\frac{i}{2k}\left(\frac{1}{z_m} - \frac{1}{z_{m-1}}\right)\left(\vec{Q}''_m\right)^2\right] \tag{4}$$

where $\vec{Q}''_m = \sum_{j=m}^{n} z_j \vec{K}_j$

This form is analogous to Eqs.(6.10) and Eq.(9.1) of TRIII. As in the transition to (9.3), we prefer to rearrange the exponential terms. The ultimate result is

$$B_n = \prod_{j=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{j-1}} dz_j \int d^2K_j \; \delta\tilde{\varepsilon}(\vec{K}_j, z_j) \cdot$$
$$\times \prod_{m=1}^{n} \exp\left[\frac{-i(L-zm)}{2k} \cdot \frac{z_m}{L} K_m^2\right] \exp\left[-\frac{i}{k}(L-z_m)\vec{K}_m \cdot \vec{Q}'_{m+1}\right], \tag{5}$$

where $\vec{Q}'_m = \sum_{j=m}^{n} \vec{K}_j z_j / L$

This form is the spherical-wave analog of Eq.(14a) of TRV, which is the point of departure for nearly all results (except the saturation regime). Note the occurrence of the extra geometrical factors z_m/L which make these exponentials slightly less significant than in the plane-wave case. All known results for spherical waves follow from Eq.(5). For example, the extremely important intermediate steps discussed in Section 13 of TRIII yield the same result as Eq.(13.5) for the bead contributions, i.e., for correlations between two $\delta\tilde{\varepsilon}(K,z)$ factors of one and the same Born term B_n. However, correlations between an i-th B factor and a j-th B* factor of $\langle B(1)---B(N)B^*(1)---B^*(N)\rangle$, or between two unasterisked or two asterisked factors of this 2N-point correlation ultimately yield Eq.(16.8) with new definitions,

$$C_m \equiv \cos[K_m^2(z_m/L)(L-z_m)/k]$$
$$C_m(ij) \equiv \cos\;[\vec{K}_m \cdot (\vec{Q}'^{(i)}_{m+1} - \vec{Q}'^{(j)}_{m+1})(L-z_m)/k] \tag{6}$$

Note that the $\vec{Q}'^{(i)}_m$ are defined in Eq.(5) above. In fact, with these definitions, the entire development of TRIII from Section 16 on can be continued. Likewise, Eq.(18a) of TRV also can be taken over unchanged, provided C_1 and $C_1(ij)$ are given by the above Eq.(6).

The modified-Rytov result for spherical waves follows as before by setting $C_m(ij) = 1$. The error estimate is identical to that developed in Eqs.(17.12) and (17.13) of TRIII. The result for $\langle\delta\chi^2\rangle$, the log-amplitude variance, is identical to Tatarski's[2] Eqs.(49.25) and (49.36). Equation(49.36) in our notation is

$$\langle\delta\chi^2\rangle = 0.124C_n^2k^{7/6}L^{11/6}, \tag{7}$$

different only from Tatarski's expression because $C_\epsilon^2 = 4C_n^2$. The Rayleigh result is unchanged from the plane-wave case because the only important contributions require $\vec{Q}'^{(j)}_m \approx \vec{Q}'^{(i)}_m$ to within errors of order ℓ^2/L^2 in deviations of $C_m(ij)$ from unity.

However, the most important matter is to derive a saturation-regime corollary to Eq.(7). One can start - once again - from the parabolic equation for the normalized field $B = rE\exp(-ikr)$:

$$\Delta_T B + 2ik(\hat{r}\cdot\vec{\nabla} - \frac{1}{r})B + k^2\delta\varepsilon B = 0, \tag{8}$$

where $\hat{r}$ is a unit vector of the radial coordinate, which is not going to deviate strongly from the z-direction. It yields for $\psi = \ln B$:

$$\Delta_T\psi + 2ik\ \hat{r}\cdot\vec{\nabla}\psi + (k^2\delta\varepsilon - 2ik/r) + (\vec{\nabla}_T\psi)^2 = 0, \tag{9}$$

which yields the spherical-wave Rytov approximation $\psi_R(L)$ when the last nonlinear term is dropped. I.e.,

$$\psi_R(L) = \frac{ik}{2}\int_0^L dz\ \exp[i(2kL)^{-1}\ z(L-z)\Delta_T]\delta\varepsilon(0,z). \tag{10}$$

Returning to Eq.(9), we note that the plane-wave procedure for eliminating $(\vec{\nabla}_T\psi)^2$ can be repeated verbatim. This procedure was not given explicitly in TRV, and it will therefore be included here, based on the J. Opt. Soc. Am. manuscript.

We integrate Eq.(9) along geometrical-optical rays with coordinates $\underset{\sim}{r} = [\underset{\sim}{\rho}'(s),s]$ where s is a ray coordinate and $\underset{\sim}{\rho}'(s) = [x'(s),y'(s)]$ are two mutually orthogonal coordinates such that $\underset{\sim}{\rho}'(s)$ and s are the three components of a locally orthogonal coordinate system. Let $\theta(z)$ be the angle between s and z after traveling a distance $\sim z$ along s. Let θ_x and θ_y be the projections of $\theta(z)$ upon the y = 0 and x = 0 planes, respectively. It can be seen that

$$
\begin{aligned}
\partial/\partial x &= (\theta_y/\theta)\partial/\partial x' + (\theta_x/\theta)(1+\theta^2)^{-1/2}\partial/\partial y' + \theta_x(1+\theta^2)^{-1/2}\partial/\partial s \\
\partial/\partial y &= -(\theta_x/\theta)\partial/\partial x' + (\theta_y/\theta)(1+\theta^2)^{-1/2}\partial/\partial y' + \theta_y(1+\theta^2)^{-1/2}\partial/\partial s \\
\partial/\partial z &= \qquad\qquad - (1+\theta^2)^{-1/2}\partial/\partial y' + (1+\theta^2)^{-1/2}\partial/\partial s
\end{aligned}
\tag{11}
$$

In particular, we note from Eq.(11) that

$$
\begin{aligned}
\partial/\partial z &= \partial/\partial s + 0(\theta) \\
\Delta_T &= \Delta_T' + 0(\theta).
\end{aligned}
\tag{12}
$$

Furthermore, it can also be seen from the coordinate transformation that

$$
\begin{aligned}
\partial/\partial x' &= (\theta_y/\theta)\partial/\partial x - (\theta_x/\theta)\partial/\partial y \\
\partial/\partial y' &= (1+\theta^2)^{-1/2}[(\theta_x/\theta)\partial/\partial x + (\theta_y/\theta)\partial/\partial y - \theta\partial/\partial z].
\end{aligned}
\tag{13}
$$

With Eqs.(11-13), we can rewrite Eq.(9) after dividing by k^2,

$$
\begin{aligned}
&[2ik^{-1}\partial/\partial s + 0(\theta)]\psi + [k^{-2}\Delta_T' + 0(\theta)]\psi \\
&-2ik/r + \delta\varepsilon(\underset{\sim}{\rho}'(s),s) + [(k^{-1}\underset{\sim}{\nabla}_T' + 0(\theta))\psi]^2 = 0 \ .
\end{aligned}
\tag{14}
$$

Now we shall choose θ and thus define the ray coordinates. To do so, we define the *complex-angle* vector

$$
\underset{\sim}{\theta} = k^{-1}\underset{\sim}{\nabla}_T\phi - ik^{-1}\underset{\sim}{\nabla}_T\chi = -ik^{-1}\underset{\sim}{\nabla}_T\psi \ .
\tag{15}
$$

Thus, $\underset{\sim}{\theta}$ has two complex components, $\underset{\sim}{\theta}_x$ and $\underset{\sim}{\theta}_y$, defined by replacing $\underset{\sim}{\nabla}_T$ in Eq.(15) by $\hat{x}\partial/\partial x$ and $\hat{y}\partial/\partial y$, respectively. The *magnitude* of $\theta_x(\theta_y)$ is the square root of the sum of the squares of both terms of the $\theta_x(\theta_y)$ component of Eq.(15). Note in particular that $\theta^2 = k^{-2}[(\underset{\sim}{\nabla}_T\phi)^2 + (\underset{\sim}{\nabla}_T\chi)^2]$.

Using Eq.(13) with Eq.(15), we see that $\underline{\nabla}_T{}'\psi = 0$ with this choice. The last term of Eq.(14) is therefore zero, except for corrections of $O(\theta)$ included in the other terms. Consequently, the solution of the thus linearized Eq.(14) becomes

$$\psi(L) = \frac{ik}{2}\int_0^L dz\ \exp[i(2kL)^{-1}z(L-z)\Delta_T]\ \delta\varepsilon[\vec{\rho}(z),z], \tag{16}$$

where we have replaced ds by dz and L-s by L-z, and have written $\delta\varepsilon(s) = \delta\varepsilon[\vec{\rho}(z),z]$ in terms of the old coordinates. The crucial difference from Eq.(10) is the dependence of $\delta\varepsilon$ upon the coordinate $\vec{\rho}(z)$ which is given by the fundamental relationship, for $z \leq L$,

$$d\vec{\rho}/dz = \vec{\theta}(z)\ , \tag{17}$$

with $\vec{\theta}(z)$ given by Eq.(15). The set of three equations (15-17) comprises a nonlinear set of equations for $\psi(L)$. The procedure for solving them is not quite as circular as it may seem. Assume that we know $\psi(z)$ for $z < L$. Equations (15) and (17) then determine $\vec{\rho}(z)$ for $z < L$. The result is inserted into Eq.(16) to yield $\psi(L)$. At the very least, this suggests a numerical procedure. It does more than that, however, because the role of $\vec{\rho}(z)$ appears only to be a description of the *refractive* properties of the radiation, not of the diffractive properties. It is for that reason that one may ignore diffraction in Eq.(15) and utilize simple ray formulas. Thus, Eq.(15) is approximated well by

$$\begin{aligned}\vec{\theta}(z) &= \vec{\theta}_o + \frac{1}{2}\int_0^z dz_1\ \ \vec{\nabla}_T\delta\varepsilon(0,z_1) \\ &= \vec{\theta}_o - \frac{i}{8\pi^2}\int_0^z dz_1 \int d^2K\ \vec{K}\delta\tilde{\varepsilon}(\vec{K},z_1)\ ,\end{aligned} \tag{18}$$

where $\vec{\theta}_o$ is the initial angle of the unscattered ray from source to point of scattering with the z-axis, and $\delta\tilde{\varepsilon}(\vec{K},z)$ is the two-dimensional transverse Fourier transform of $\delta\varepsilon$ as usual.

We note that $<\delta\chi^2(L)> = -<\chi(L)>$ where the mean $<\chi(L)>$ is determined by the real part of Eq.(16). It can be seen that Eq.(16) and the above considerations yield

$$\langle\delta\chi^2(L)\rangle = \frac{k}{8\pi^2}\int_0^L dz\int d^2K\ \sin[K^2(L-z)z/2kL^2]\langle\delta\tilde{\varepsilon}(\vec{K},z)\exp[-i\vec{K}\cdot\vec{\rho}(z)]\rangle \tag{19}$$

The remaining problem is to substitute Eq.(18) into Eq.(19) and perform the calculation. The procedure is quite comparable to the plane-wave case.

The first step is to integrate $\vec{\theta}(z_1)$ from $z_1 = 0$ to $z_1 = z$ in Eq.(18) to obtain $\vec{\rho}(z)$ according to Eq.(17). If this is done with the boundary conditions $\vec{\rho}(0) = \vec{\rho}(L) = 0$ one obtains

$$\begin{aligned}\vec{\rho}(z) &= \frac{i}{8\pi^2}\int_0^L dz_1\ M(L,z,z_1)\int d^2K\vec{K}\delta\tilde{\varepsilon}(\vec{K},z_1)\\ M(L,z,z_1) &= z_1(L-z)/L \quad , \quad z_1 < z\\ &= z(L-z_1)/L \quad , \quad z_1 > z\end{aligned} \tag{20}$$

Two steps are required to reduce Eq.(19) to a form suitable for computing closed-form expressions [see the Appendix]. They are

$$\langle\delta\tilde{\varepsilon}(\vec{K},z)\ \exp[-i\vec{K}\cdot\vec{\rho}(z)]\rangle = \langle -i\delta\tilde{\varepsilon}(\vec{K},z)\vec{K}\cdot\vec{\rho}(z)\rangle\exp[-K^2\langle\rho^2(z)\rangle/4] \tag{21a}$$

$$\langle -i\delta\tilde{\varepsilon}(\vec{K},z)\vec{K}\cdot\vec{\rho}(z)\rangle = (\epsilon^2/4)K^2\Phi(K)(L-z)z/L \tag{21b}$$

A calculation of $\langle\rho^2(z)\rangle$ remains to be performed. We utilize Eq.(20), and the usual approximations as in the Appendix calculation of Eq.(A.8) to obtain from straightforward calculation

$$\begin{aligned}\langle\rho^2(z)\rangle &= (\epsilon^2/24\pi)\int_0^\infty dKK^3\Phi(K)\ z^2(L-z)^2/L\\ &= 4\gamma\Gamma(7/6)C_n^2\kappa_m^{1/3}\ z^2(L-z)^2/L,\end{aligned} \tag{22}$$

where γ is defined below in connection with Eq.(23). The remaining work is completely analogous to the derivation of Eq.(18b) in TRVI. Equations (21a) and (21b) are substituted into Eq.(19), and the steps described beyond Eq.(18a) in TRVI are repeated verbatim except that the above spherical-wave forms are used. The result is

$$\langle\delta\chi^2\rangle = \frac{1}{2}\gamma k^{7/6} L^{11/6} C_n^2 \int_0^1 dy\, y^{5/6}(1-y)^2 \int_0^\infty dx\, x^{1/6} \exp[-x\sigma_1^2 f(y)] \tag{23}$$

where $\gamma = 0.033\pi^2 \approx 0.326$ and $f(y) = \gamma\Gamma(7/6)y(1-y)^2$. After performing the dx integration, and some reordering of factors we obtain

$$\langle\delta\chi^2\rangle = \beta(\kappa_m^{7/3} L^3 C_n^2)^{-1/6} \tag{24}$$

where $\beta = [\gamma\Gamma(7/6)]^{-1/6} \cdot \frac{1}{2}\int_0^1 dy\, y^{-1/3}(1-y)^{-1/3}$,

and after a numerical integration we obtain

$$\langle\delta\chi^2\rangle = 1.25(\kappa_m^{7/3} L^3 C_n^2)^{-1/6} \tag{25}$$

Equation(24) is the final result for spherical waves. It should be noted that Eq.(23) yields the result $\langle\delta\chi^2\rangle \approx 0.005\ \kappa_m^{7/3} L^3 C_n^2$ in the limit $\sigma_1^2 \to 0$, a result that agrees with Tatarski's [ref. 2, page 251] for spherical waves. There is therefore as much measure of confidence in Eq.(25) as in the plane-wave result.

Let $\sigma_\epsilon^2 = 0.124\ k^{7/6} L^{11/6} C_n^2$, the well-known spherical-wave result for the log-amplitude variance in the Rytov approximation [see Eq.(6)]. It is easily seen from Eq.(25) that

$$\langle\delta\chi^2\rangle \approx 0.88(\kappa_m^2 L/k)^{-7/36} (\sigma_\epsilon^2)^{-1/6}, \tag{26}$$

and therefore at $k = 10^7\ m^{-1}$ and $\kappa_m = 1000$ that

$$\begin{aligned} \langle\delta\chi^2\rangle &\approx 0.36(\sigma_\epsilon^2)^{-1/6} \quad \text{at } L = 1 \text{ km} \\ &\approx 0.41(\sigma_\epsilon^2)^{-1/6} \quad \text{at } L = 0.5 \text{ km} \end{aligned} \tag{27}$$

We have utilized Eq.(27) as a basis for comparing the available published data. In Fig. 1 we have plotted data by Kerr[4], Ochs[5], and Gracheva et al. [6]. Two lines in the saturation regime represent the mean data of these two sources. The standard deviation around the mean is of the order of 50%; thus, the theoretical curve [Eq.(27)] appears to be low but still within the standard deviation.

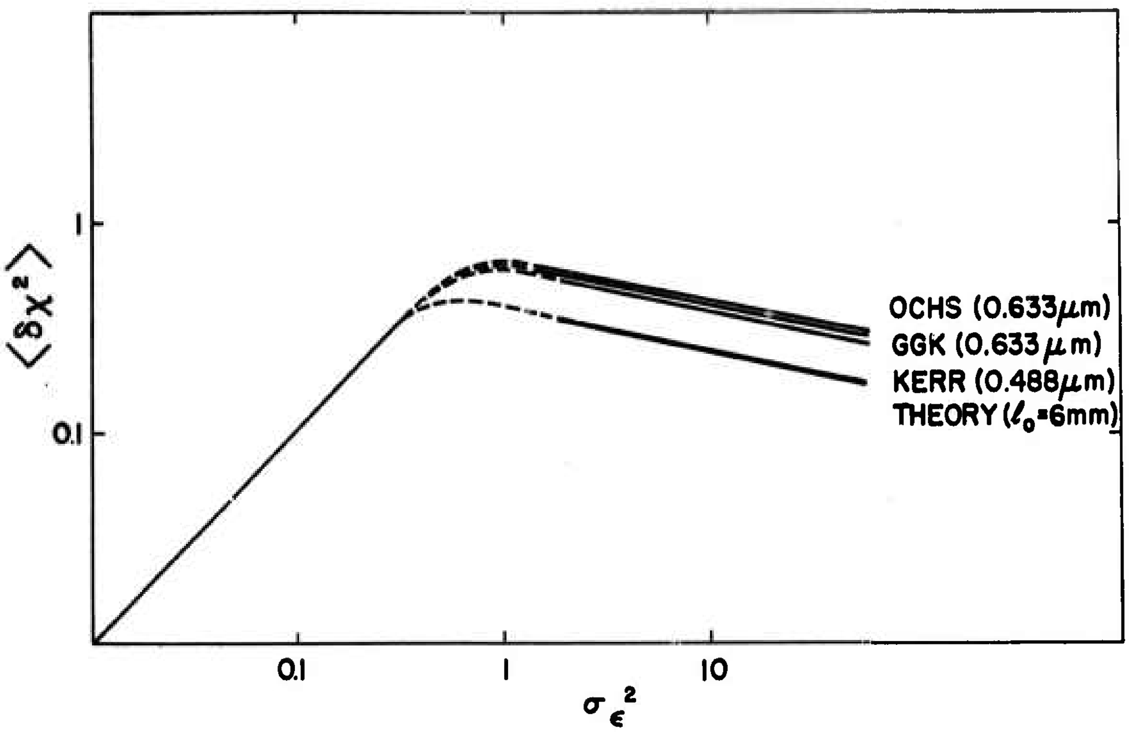

Figure 1. Log-amplitude variance $\langle\delta\chi^2\rangle$ vs. parameter $\sigma_\epsilon^2 = 0.124\, k^{7/6} L^{11/6} C_n^2$ for spherical waves: data and theory.

We have also reexamined plane-wave data by Gracheva et al.[6], and by Mevers et al.[7], replotted in Fig. 2. Data spreads are of the order of 50% around the mean. The theoretical curve is based on Eq.(21) of TRVI, and it appears to be systematically on the low side (spherical-wave data appear to agree better with the analytical prediction).

The spherical-wave curve appears to fit the data slightly better than the plane-wave curve. It is interesting to note, particularly from Fig. 1, that the data do not exhibit an outspoken frequency dependence - in accord with the saturation-regime prediction. It should also be noted that the theoretical curve is proportional to $\kappa_m^{-7/18}$. We have arbitrarily chosen ℓ_o = 6 mm, but a choice of ℓ_o = 12 mm will move the curves up by a factor 1.3.

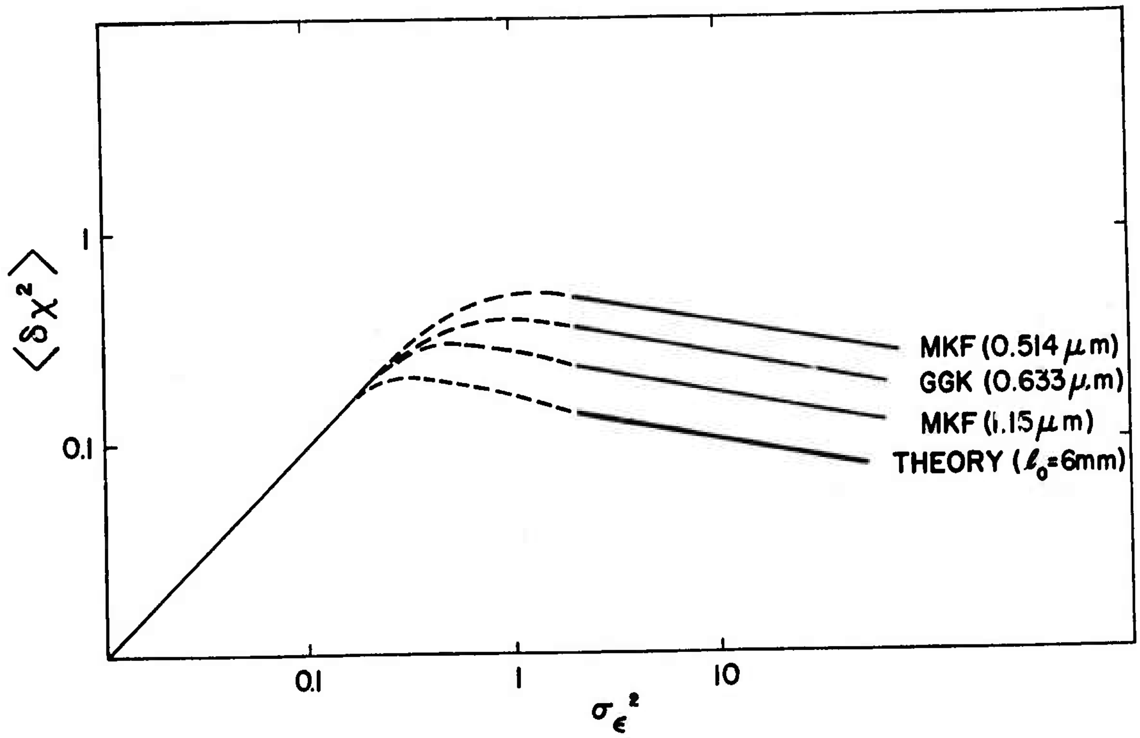

Figure 2. Log-amplitude variance $\langle\delta\chi^2\rangle$ vs. parameter $\sigma_\epsilon^2 = 0.31\ k^{7/6}L^{11/6}C_n^2$ for plane waves: data and theory.

Even so, it appears that both theoretical curves are slightly low. We do not understand why this is the case at present, but hope to study this problem later.

APPENDIX

Derivation of Equations(21)

The derivation of Eqs.(21) from Eq.(19) is not quite trivial. The correlation $<\delta\tilde{\varepsilon}(\vec{K},z')\exp[-i\vec{K}\cdot\vec{\rho}(z)]>$ must be computed at the point $z' = z$. Tatarski has used variational (or functional) derivatives to yield a result for plane waves[3], but it appears possible to obtain similar results more simply:

First of all, we note from Eq.(19) that $\vec{\rho}(z)$ is a Gaussian random variable with zero mean. Although a rigorous proof of this statement is not given, it can be verified heuristically by applying a central-limit theorem to the sum of sections of the integral, each of which is several macroscales long.

The next step is to note that

$$<\delta\tilde{\varepsilon}(\vec{K},z')[\rho(z)]^m> = m<\delta\tilde{\varepsilon}(\vec{K},z')\vec{\rho}(z)><[\rho(z)]^{m-1}> \tag{A.1}$$

where m is an integer. The easiest way to see this, is to write out the left-hand side as a product of m integrals dz_1---dz_m after substituting Eq.(20) for $\vec{\rho}(z)$. The correlation is then essentially

$$<\delta\tilde{\varepsilon}(\vec{K},z')\delta\tilde{\varepsilon}(\vec{K}_1,z_1)----\delta\tilde{\varepsilon}(\vec{K}_m,z_m)>, \tag{A.2}$$

a product of m + 1 factors. Now, whenever z' and z_j $(1 \leq j \leq m)$ are both at least several macroscales L_0 distant from each and everyone of the m-1 remaining z coordinates one can factorize Eq.(A.2) into

$$<\delta\tilde{\varepsilon}(\vec{K},z')\delta\tilde{\varepsilon}(\vec{K}_j,z_j)><\delta\tilde{\varepsilon}(\vec{K}_1,z_1)----/----\delta\tilde{\varepsilon}(\vec{K}_m,z_m)>\ , \tag{A.3}$$

where the slash indicates that $\delta\tilde{\varepsilon}(\vec{K}_j,z_j)$ is missing in the second correlation. Of course, it is also true that we could have separated just z' and then factorized (A.2) into $<\delta\tilde{\varepsilon}(\vec{K},z')>$ and an m-point correlation, but $<\delta\tilde{\varepsilon}(\vec{K},z')> = 0$ so the next non-zero factorization to look at is Eq.(A.3). The factorization, Eq.(A.3), is not possible when both z' and z_j are "near" (with respect to length L_o) to one or more of the remaining m-1 points. However, that situation restricts the dz_1---dz_m integration to an effective multiple-integration region that is less by a factor of order $(L_o/L)^2$ than in the case of Eq.(A.3). Of course, this is only a heuristic description of a theorem that we have proved more rigorously in Sections 11 and 12 of TRIII. The net result is that we may replace Eq.(A.2) by

$$\sum_{j=1}^{m} <\delta\tilde{\varepsilon}(\vec{K},z')\delta\tilde{\varepsilon}(\vec{K}_j,z_j)><\delta\tilde{\varepsilon}(\vec{K}_1,z_1)----/----\delta\tilde{\varepsilon}(\vec{K}_m,z_m)> \tag{A.4}$$

It is easily seen that Eq.(A.4) yields Eq.(A.1), which is now used in the following sequence of steps:

$$\begin{aligned} <\delta\tilde{\varepsilon}(\vec{K},z')e^{-i\vec{K}\cdot\vec{\rho}(z)}> &= \sum_{m=0}^{\infty}\frac{(i)^m}{m!} <\delta\tilde{\varepsilon}(\vec{K},z')[\vec{K}\cdot\vec{\rho}(z)]^m> \\ &= \sum_{m=1}^{\infty}\frac{(-i)^m}{(m-1)!} <\delta\tilde{\varepsilon}(\vec{K},z')\vec{K}\cdot\vec{\rho}(z)><[\vec{K}\cdot\vec{\rho}(z)]^{m-1}> \qquad \text{(A.5)} \\ &= <-i\delta\tilde{\varepsilon}(\vec{K},z')\vec{K}\cdot\vec{\rho}(z)><\exp[-i\vec{K}\cdot\vec{\rho}(z)]> \end{aligned}$$

We obtain Eq.(21a) from this last step by noting that $<\exp\xi> = \exp<\xi^2/2>$ for a Gaussian random variable ξ with zero mean, and by noting that x and y components of $\vec{\rho}(z)$ are equal and uncorrelated.

Now we derive Eq.(21b) by substituting Eq.(20) into $<-i\delta\tilde{\varepsilon}(\vec{K},z')\vec{K}\cdot\vec{\rho}(z)>$ to obtain

$$-(8\pi^2)^{-1}\int_0^L dz_1\, M(L,z,z_1)\int d^2K_1\vec{K}\cdot\vec{K}_1<\delta\tilde{\varepsilon}\ (\vec{K},z')\delta\tilde{\varepsilon}(\vec{K}_1,z_1)>, \tag{A.6}$$

and apply Eq.(17) of TRV [or Eq.(5.2) of TRIII] to reduce this to

$$(\epsilon^2K^2/2)[(1-z/L)\int_0^L dz_1 z_1\Phi_2(\vec{K},z'-z_1) + \int_z^L dz_1(z-z_1)\Phi_2(\vec{K},z'-z_1)] \tag{A.7}$$

where $\Phi_2(\vec{K},\Delta z)$ is a *partial* spectrum, namely the two-dimensional Fourier transform of the stationary correlation function $<\delta\varepsilon(\vec{\rho}+\Delta\vec{\rho},z+\Delta z)\delta\varepsilon(\vec{\rho},z)>/<(\delta\varepsilon)^2>$ with respect to $\Delta\vec{\rho}$. By ignoring some boundary effects of $O(L_o/L)$ we can reduce Eq.(A.7) and obtain

$$<-i\delta\tilde{\varepsilon}(\vec{K},z')\vec{K}\cdot\vec{\rho}(z)> = (\epsilon^2K^2/2)\Phi(K)z'(L-z)/L \quad , \quad z'<z$$

$$= (\epsilon^2K^2/2)\Phi(K)z(L-z')/L \quad , \quad z'>z \qquad \text{(A.8)}$$

Note that the two results in Eq.(A.8) become equal at $z' = z$. However, Eq.(A.8) does not lead to the right answer for $<\delta\chi^2>$ in the limit $\vec{\rho}(z) \to 0$. The limit then yields a value for $<\delta\chi^2>$ that exceeds the geometrical-optics approximation by exactly a factor 2. Apparently, there is a factor 1/2 missing in Eq.(A.8). The same discrepancy holds in the plane-wave case. We have included an extra factor 1/2 in Eq.(21b), as we did in Section 3 of TRVI for plane waves. The following argument is suggested: When $z' > z$ it cannot be, physically, that $\vec{\rho}(z)$ is correlated with $\delta\tilde{\varepsilon}(\vec{K},z')$ because $\vec{\rho}(z)$ is the transverse coordinate of a ray traveling from $z_1 = 0$ to $z_1 = z$, and the medium at $z' > z$ does not influence the medium at $z_1 < z$ beyond several macroscales L_o. It is true that the boundary condition $\vec{\rho}(L) = 0$ forces the above *formal* result for $z' > z$ - which seems to imply correlation of $\vec{\rho}(z)$ with $\delta\tilde{\varepsilon}(\vec{K},z')$ for $z' > z$ - but the ray can only arrive at $\vec{\rho}(L) = 0$ because the medium *beyond* $z_1 = z$ makes the ray bend into $\vec{\rho}(L) = 0$. In fact, we could physically change the medium beyond $z_1 = z$ and it appears obvious that $\vec{\rho}(z)$ will be unchanged. The above result, Eq.(A.8) for $z' > z$, appears therefore to apply to the reciprocal situation: a ray traveling from $z = L$ to $z = 0$. This situation cannot be distinguished in this formalism, but it is physically different. Somehow, the information about the *direction* in which the rays travel has been lost. If a time-dependent formalism were pursued more carefully, we would obtain a Green's function in $\psi(z)$, hence in $\vec{\rho}(z)$, that would be zero for "advanced" times, i.e., for a field at $z_1 = z$ due to a signal emanated from $z_1 = z'$ when $z' > z$, and that will rule out correlation in Eq.(A.8) for $z' > z$ if this correlation is defined more carefully. Note that *reciprocity* is not in question: the result for $z' > z$ is perfectly symmetric with that for $z' < z$ provided the ray direction is considered. The suggested form of Eq.(A.8) for rays traveling from $z_1 = 0$ to $z_1 = L$ is

$$
\begin{aligned}
<-i\delta\tilde{\varepsilon}(\vec{K},z')\vec{K}\cdot\vec{\rho}(z)> &\approx (\epsilon^2K^2/2)\Phi(K)z'(L-z)/L \quad , \quad z' < z \\
&\approx 0 \quad , \quad z' > z
\end{aligned}
\tag{A.9}
$$

and a reciprocal version with zero for $z' < z$ when rays travel in the other direction. There is a jump discontinuity at $z' = z$ where half the sum of the values for $z' < z$ and $z' > z$ in Eq.(A.9) should be taken. That finally yields Eq.(21b).

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