AD-756 973

HOMING GUIDANCE (A TUTORIAL REPORT)

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Aerospace Corporation

Prepared for:

Space and Missiles Systems Organization

10 December 1964

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SAMSO TR 73-100

Report No.
TOR-469(9990)-1

AD756973

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El Segundo Technical Operations
AEROSPACE CORPORATION
El Segundo, California

Contract No. AF 04(695)-469

10 December 1964

Prepared for

COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

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The information in a Technical Operating Report is developed for a particular program and is therefore not necessarily of broader technical applicability.

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FOREWORD

This report was originally published on 14 September 1962 and numbered A-62-1732. 3-68. For this printing, it has been renumbered as TOR-469(9990)-1 to satisfy contractual requirements.

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) AEROSPACE CORPORATION, El Segundo, California	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED
	2b. GROUP
3. REPORT TITLE HOMING GUIDANCE (A Tutorial Report)	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Tutorial Report	
5. AUTHOR(S) (First name, middle initial, last name) John P. Janus	
6. REPORT DATE 10 December 1964	7a. TOTAL NO. OF PAGES ~~25~~ 32 / 7b. NO. OF REFS 6
8a. CONTRACT OR GRANT NO. AF 04(695)-469	9a. ORIGINATOR'S REPORT NUMBER(S) TOR 469(9990)-1
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	SAMSO TR 73-100
10. DISTRIBUTION STATEMENT Distribution Statement A: Approved for public release; distribution unlimited.	
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY SPACE & MISSILES SYSTEMS ORGN. (SAMSO) P.O. Box 92960, Worldway Postal Center Los Angeles, CA 90009

13. ABSTRACT

The theory of "Homing Guidance" for space application is developed for both accelerating and non-accelerating targets. The analytical formulation is developed in a relative, rotating reference frame and consists of a linearization of the exact equations of motion.

An intuitive desirability of Proportional Navigation is discussed, and the results are verified analytically for non-accelerating targets. The investigation, which uses fuel consumption as a criterion for evaluation because application to space missions is the chief concern of the paper, shows that the guidance scheme leads to the optimum in the limiting case. Biased Proportional Navigation is examined in an analogous manner, and a similar optimum is obtained for accelerating targets.

The report also includes a brief discussion and formulation of imperfect interceptor dynamics. A few possible methods of solving this problem are mentioned briefly.

DD FORM 1 NOV 65 1473

I-a

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Homing Guidance						
Proportional Navigation						
Biased Proportional Navigation						
non-accelerating targets						
accelerating targets						
imperfect interceptor dynamics						

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ILLUSTRATIONS

I. INTRODUCTION

The term "Homing Guidance" is generally associated with the terminal and mid-course phases of missile guidance; whereby, one vehicle is maneuvered into the general proximity of some other vehicle or location with the purpose of interception or rendezvous. The manner in which the mission is accomplished depends not only on the specific objective, but it also relies upon the types of information (e.g., range, line of sight, etc.) that are available.

The use of "Homing Guidance" for atmospheric interceptors has been very extensive. However, because the maneuvering of the vehicle was accomplished with aerodynamic surfaces, the early analysis was chiefly concerned with minimizing the miss at intercept. Although this consideration is also very important in the space application of "Homing Guidance," the problem of weight, which is closely related to the fuel requirement of the interceptor, becomes another important area of concern. Throughout this report the fuel requirement will always be used as a criterion to measure the value of the particular guidance scheme involved; and, therefore, the vehicle is assumed to have the capability of obtaining the required accuracy.

In this report, only the problem of interception without regard for terminal velocity will be considered. The problem is also restricted to the use of information regarding the line of sight, or its rate. This type of information can be obtained from radar, infrared trackers, optical seekers, and various other sensors.

The assumption of restrictive sensor information requires that a guidance law, which uses only line of sight information, must be developed. This can be accomplished in several ways; the simplest is "Pure Pursuit."

In the "Pure Pursuit" type of navigation, the interceptor is aimed at the target at all times. The analytic analysis of this problem using differential geometry (Reference 5) has been very extensive; and the results, which are typified by Figure I-1, show that even for a near head-on launch a tail chase usually results. This type trajectory indicates that extensive maneuvering, which implies excess fuel consumption, is taking place. Since this is a highly

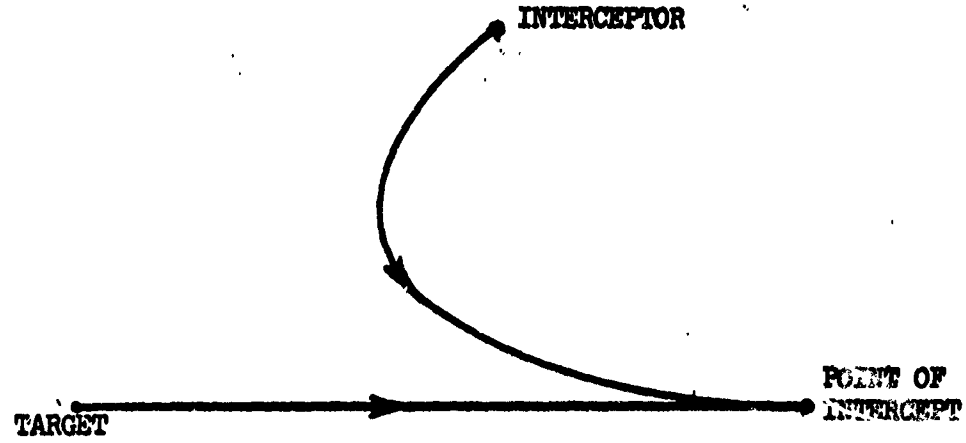

Figure I-1 Typical Pure Pursuit Collision Trajectory

undesirable situation, a more economical method of navigation is needed.

A scheme that requires less manueverability is that of "leading" the target or the so called "Constant-bearing Collision." (See Reference 1.) This method consists of aiming at a point, on the target's trajectory, that is ahead of the immediate position of the target and adjusting the interceptor velocity to cause intercept at the predetermined point. A graphical example of this type of navigation for a constant velocity target is given in Figure I-2. This diagram also shows that the line of sight remains at the same orientation in inertial space throughout the flight. If this idea is generalized, it can be shown that, whenever the time rate of change of the line of sight is

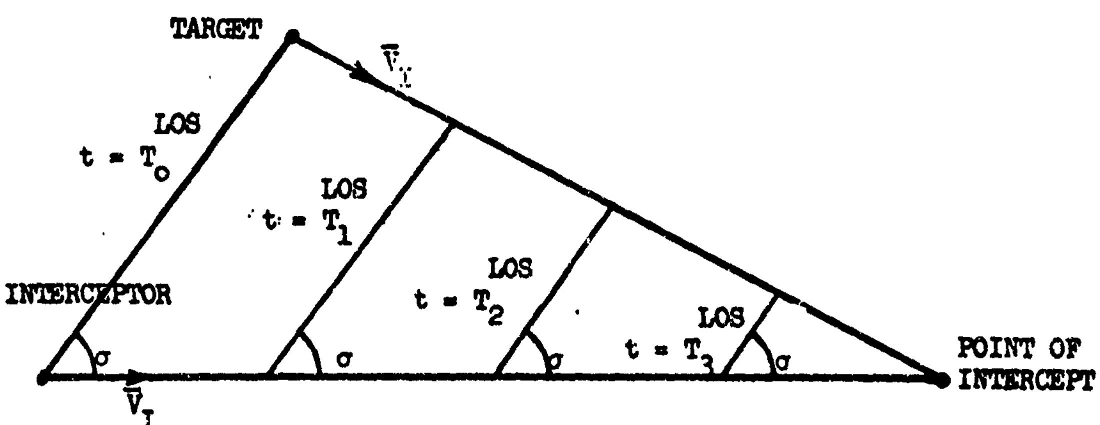

Figure I-2 Typical Constant Bearing Collision Course zero, the interceptor is on a collision course. Thus, if the thrust direction is correct, a guidance law that thrusts proportional to the line of sight rate and drives this rate to zero can be established. This is called Proportional Navigation and can be expressed mathematically as:

$$A_I = N\dot{\sigma} \tag{1.1}$$

where

A_I = interceptor acceleration

$\dot{\sigma} = \frac{d\sigma}{dt}$ = line of sight rate

N = Scale factor

Since the above relation does not specify the direction of the acceleration, it is insufficient as a guidance law. Because this is a space application, it is unlike the atmospheric interceptor which was limited to accelerations normal to its velocity vector. Therefore, the acceleration can have any orientation, and the direction of the thrust should be choosen so that fuel consumption is minimized.

The fuel expenditure, ΔV, can be minimized in the problem if the proper acceleration impulse or step velocity change is made at the initial point, $t = 0$. If the orientation or direction of the acceleration which minimizes ΔV is obtained, the result can be applied to Equation (1.1) to complete the guidance law.

The problem can be most easily formulated in the relative interceptor configuration which is shown in Figure I-3 . Because of its simplicity, this two-dimensional representation will be used throughout the report. However, the ideas and concepts expressed are applicable to the three dimensional case.

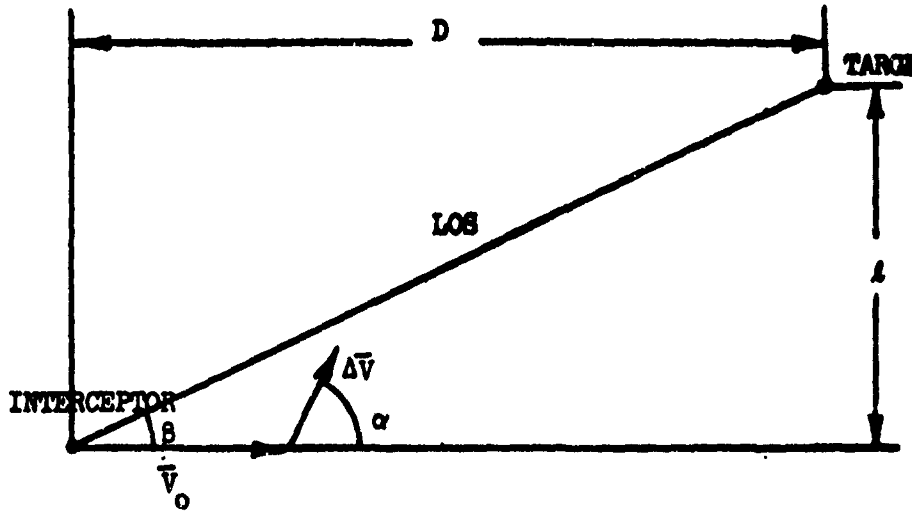

Figure I-3 Relative Interceptor Geometry

The notation in the above diagram can be interpreted in the following manner.

$\bar{V}_o = \bar{V}_{I_o} - \bar{V}_{T_o}$ = initial relative velocity vector of interceptor with respect to the target

$\bar{V}_{I_o}$ = initial interceptor velocity vector

$\bar{V}_{T_o}$ = initial target velocity vector

$\overline{\Delta V}$ = step velocity increment applied to the interceptor

If the following constraints are applied to the problem, α can be determined for minimum ΔV in the following manner.

Constraints: $-\frac{\Pi}{2} < \beta < \frac{\Pi}{2}$, $\Delta V > 0$, $\ell > 0$, $D > 0$

The equation of motion which leads to intercept is.

$$\tan \beta = \frac{\ell}{D} = \frac{\Delta V \sin \alpha \, T_{go}}{(V_o + \Delta V \cos \alpha) \, T_{go}} \tag{1.2}$$

or

$$\Delta V = \frac{V_o \ell}{D \sin \alpha - \ell \cos \alpha} \tag{1.3}$$

The constraints are satisfied if $D \sin \alpha - \ell \cos \alpha > 0$. The fuel consumption, ΔV, can be minimized by maximizing the denominator of Equation (1.3):

$$f(\alpha) = D \sin \alpha - \ell \cos \alpha$$

$$f'(\alpha) = D \cos \alpha + \ell \sin \alpha = 0 \tag{1.4}$$

or

$$\tan \alpha = -\frac{D}{\ell}$$

Since the value of α obtained from Equation (1.4) can be shown to minimize Equation (1.3) under the above constraints, the minimum fuel is expended if the thrust is applied normal to the line of sight.

Although this method minimizes fuel consumption, it also implies a pulse or infinite acceleration requirement. Since this is not practical, the expression in Equation (1.1) is used in conjunction with the direction obtained for minimum fuel consumption to obtain Equation (1.5):

$$\bar{A}_I = N \dot{\bar{\sigma}} \times \bar{e}_R \tag{1.5}$$

where

$\bar{e}_R$ = unit vector along the line of sight directed toward the target

Equation (1.5) completely defines the guidance law for "Proportional Navigation." The magnitude of the commanded acceleration in this scheme is proportional to the line of sight rate, and its direction is normal to the line of sight. Therefore, this guidance law uses only line of sight information, and it is consistent with the previous assumptions.

It is important to note here that this type of homing guidance assumes that the interceptor has been launched toward the target with some initial velocity, and this velocity is sufficient to over-take the target if the direction of flight is controlled properly.

II. TRAJECTORY EQUATIONS

Before the problem of interceptor navigation can be further investigated, it is necessary to derive the equations which describe the interaction of the interceptor maneuvering and the relative kinematics of the system. An elementary illustration of the closed loop behavior of these equations is given in Figure II-1

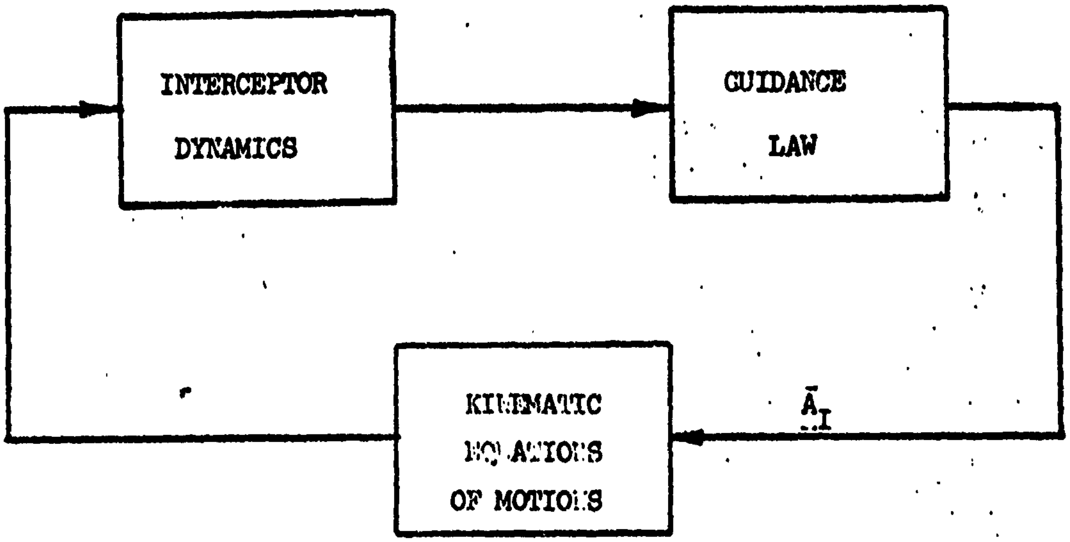

Figure II-1 Block Diagram of Interceptor Trajectory

The use of line of sight rate, $\dot{\sigma}$, in the diagram is merely to fit the needs of this problem. In general, the signal would be any sensor input or information source that satisfied the requirements of the mission. The interceptor dynamics generally include leads and lags in the system, but perfect dynamics will be assumed in this section, as the block can be considered to have unit gain. The extension of the formulation to include lags and leads will appear later in the report (Section V).

The kinematic equations of motion can be derived in several ways, including: perturbation techniques (Reference 3), small angle approximation (Reference 1, 2 & 4), and assumptions and linearization of the exact equations of motion (Reference 6). In this section the latter technique will be used on the two-dimensional case, but the derivation in three-dimensions is possible since it is shown in Reference 1.

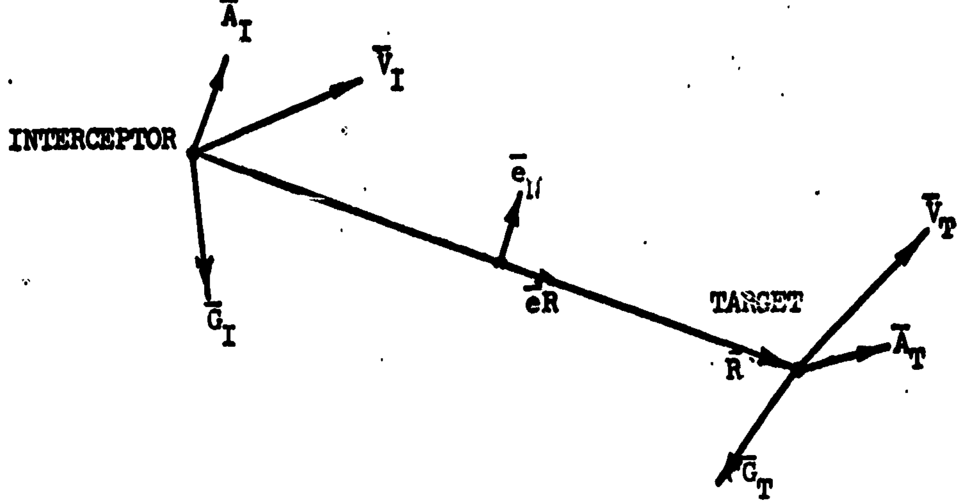

Figure II-2 General Interceptor-Target Geometry

The target-interceptor geometry is shown in Figure II-2. The equations of motion are written in a two-dimensional, rotating, orthogonal coordinate system $(\bar{e}_R, \bar{e}_N)$. The reference frame rotates with an angular rate, $\dot{\sigma}$, with respect to inertial space (positive sense is given to $\dot{\sigma}$ if it projects up out of the page). The exact equations of motion are:

$$\ddot{R} - \dot{\sigma}^2 R = (A_{TR} - G_{TR}) - (A_{IR} - G_{IR}) \tag{2.1}$$

$$\ddot{\sigma} R + 2\dot{\sigma}\dot{R} = (A_{TN} - G_{TN}) - (A_{IN} - G_{IN}) \tag{2.2}$$

where

$\bar{A}_T$ = target acceleration due to thrust

$\bar{A}_I$ = interceptor acceleration due to thrust

$\bar{G}_T$ = gravitational acceleration at the target

$\bar{G}_I$ = gravitational acceleration at the interceptor

$\bar{e}_R$ = unit vector along the line of sight directed toward the target

$\bar{e}_N$ = unit vector normal to the line of sight

$$A_{TR} = \bar{A}_T \cdot \bar{e}_R \qquad A_{IR} = \bar{A}_I \cdot \bar{e}_R$$

$$A_{TN} = \bar{A}_T \cdot \bar{e}_N \qquad A_{IN} = \bar{A}_I \cdot \bar{e}_N$$

$$G_{TR} = \bar{G}_T \cdot \bar{e}_R \qquad G_{IR} = \bar{G}_I \cdot \bar{e}_R$$

$$G_{TN} = \bar{G}_T \cdot \bar{e}_N \qquad G_{IN} = \bar{G}_I \cdot \bar{e}_N$$

$\bar{V}_I$ = velocity of the interceptor

$\bar{V}_T$ = velocity of the target

Since most of the quantities in Equations (2.1) and (2.2) are functions of time, a completely general solution of these equations cannot be found. Therefore, simplifying assumptions that use a knowledge of the physical problem and the type of navigation will be made to linearize the kinematic equations of motion. These assumptions are:

1. $\bar{G}_I - \bar{G}_T \approx 0$ or $G_{IR} - G_{TR} \approx 0$ and $G_{IN} - G_{TN} \approx 0$

These quantities are neglected because the gravity differential over the ranges involved in most interceptor problems is only a few feet per second squared, and the thrust motors used involve accelerations of several g's.

2. $A_{IR} = 0$

Because proportional navigation is used, it has been shown in Equation (1.5) that the guidance accelerations will only be applied normal to the line of sight.

3. $\dot{\sigma}^2 \approx 0$

Since the guidance law tends to drive the line of sight rate to zero, $\dot{\sigma}$ can be assumed to be small in the terminal phase; and, therefore, $\dot{\sigma}^2$ is a second order term and can be neglected.

4. $\dot{R} = -V_c$ = constant closing velocity

If the interceptor is assumed to have sufficient initial velocity to overtake the target regardless of the target acceleration along the line of sight and if $V_c \gg \int_0^T A_{TR}\, dt$ is true for the terminal phase of interception, the assumption of constant closing velocity follows from Equation (2.1) and the first three assumptions. From this assumption it is follows that

$$R = R_o - V_c t \tag{2.3}$$

$$T_f = \frac{R_o}{V_c} \tag{2.4}$$

where

R_o = initial distance between the target and interceptor along the line of sight

T_f = total guidance time

Since Equation (2.1) has already been linearized to obtain Assumption 4, the motion can be completely described by applying Assumption 4 and Equations (2.3) and (2.4) to Equation (2.2). The resulting linearized kinematic equation of motion is:

$$(R_o - V_c t)\,\ddot{\sigma} - 2V_c\,\dot{\sigma} = A_{TN} - A_{IN} \tag{2.5}$$

The so called "Trajectory Equation" can be obtained from Equation (2.5) by replacing the interceptor acceleration term by the guidance law. In the case of proportional navigation, the trajectory equation becomes:

$$(R_o - V_c t)\,\ddot{\sigma} - 2V_c\,\dot{\sigma} = A_{TN} - N\,\dot{\sigma} \tag{2.6}$$

or

$$(R_o - V_c t)\,\ddot{\sigma} + (N - 2V_c)\,\dot{\sigma} = A_{TN} \tag{2.7}$$

Since this equation is first order in $\dot{\sigma}$, it has a closed form solution.

III. PROPORTIONAL NAVIGATION

Now that the guidance law and trajectory equation have been established for Proportional Navigation, the behavior of the interceptor can be established by solving the trajectory equation. For convenience, Equation (2.7) can be put in the following form:

$$\left(\frac{R_o}{V_c} - t\right) \ddot{\sigma} + \left(\frac{N}{V_c} - 2\right) \dot{\sigma} = \frac{A_{TN}}{V_c} \tag{3.1}$$

or

$$(T_f - t)\, \ddot{\sigma} + (\lambda - 2)\, \dot{\sigma} = \frac{A_{TN}}{V_c} \tag{3.2}$$

where

$$\lambda = \frac{N}{V_c} = \text{navigation constant or navigation gain}$$

From the definition of this constant, the more widely accepted form of the guidance law for Proportional Navigation can be written. It is,

$$\bar{A}_I = -\lambda\, V_c\, \bar{e}_R \times \bar{\dot{\sigma}} \tag{3.3}$$

or

$$\bar{A}_I = \lambda\, V_c\, \bar{\dot{\sigma}} \times \bar{e}_R \tag{3.4}$$

The general closed form solution for Equation (3.2) can be found in the following manner. Multiply Equation (3.2) by $(T_f - t)^{-(\lambda - 1)}$, and it becomes

$$(T_f - t)^{-(\lambda-2)}\, \ddot{\sigma} + (\lambda-2)\, (T_f - t)^{-(\lambda-1)}\, \dot{\sigma} = \frac{A_{TN}}{V_c}\, (T_f - t)^{-(\lambda-1)} \tag{3.5}$$

or

$$\frac{d}{dt}\left[\dot{\sigma}\, (T_f - t)^{-(\lambda-2)}\right] = \frac{A_{TN}}{V_c}\, (T_f - t)^{-(\lambda-1)} \tag{3.6}$$

The solution is

$$\dot{\sigma} = (T_f - 2)^{\lambda-2} \int_o^t \frac{A_{TN}}{V_c}\, (T_f - t)^{-(\lambda-1)}\, dt + \dot{\sigma}_o \left(1 - \frac{t}{T_f}\right)^{\lambda-2} \tag{3.7}$$

where

$\dot{\sigma}_o$ = initial condition on the line of sight rate.

Thus, the line of sight rate is caused by two separate effects. The first term shows the rate due to target acceleration normal to the line of sight, and the second term shows the effect due to initial or launch errors.

Since the commanded interceptor acceleration is proportional to $\dot{\sigma}$, Equation (3.7) can be used to obtain an understanding of the required interceptor capability. Using Equations (3.4) and (3.7) the expression is

$$A_I = \lambda \, (T_f - t)^{\lambda-2} \int_0^t A_{TN} \, (T_f - t)^{-(\lambda-1)} \, dt + \lambda \, V_c \, \dot{\sigma}_o \, (1 - \frac{t}{T_f})^{\lambda-2} \tag{3.8}$$

Since $t \to T_f$ at intercept, it is obvious that the navigation constant, λ, must be greater than 2 if intercept is to be accomplished with a finite acceleration capability. The effect of increasing λ can best be shown by considering the simplified example of a non-accelerating target. In this case Equation (3.8) becomes

$$A_I = \lambda \, V_c \, \dot{\sigma}_o \, (1 - \frac{t}{T_f})^{\lambda-2} \tag{3.9}$$

The approximate fuel consumption is

$$\Delta V = \int_0^{T_f} |A_I| \, dt \tag{3.10}$$

If $\dot{\sigma}_o > 0$, Equation (3.10) becomes

$$\Delta V = \int_0^{T_f} \lambda V_c \, \dot{\sigma}_o \, (1 - \frac{t}{T_f})^{\lambda-2} \, dt \tag{3.11}$$

$$\Delta V = \frac{\lambda}{\lambda-1} \, V_c \, T_f \, \dot{\sigma}_o \tag{3.12}$$

$$= \frac{\lambda}{\lambda-1} \, R_o \, \dot{\sigma}_o \tag{3.13}$$

Thus, the minimum amount of fuel is consumed as $\lambda \to \infty$. This case can be seen to correspond to the example shown in Section I where the entire correction is made by a pulse acceleration at the beginning of flight. Since this implies an infinite acceleration capability, it is impractical. However, the example tends to show that as λ is increased the error is corrected sooner, and less fuel is consumed. This simplified approach does not bring into account other factors which limit the maximum value of λ, such as noise on the sensor output and system limitations. If these are considered, $4 \leq \lambda < 10$ has been found to be a practical range of operation.

Besides lending an intuitive interpretation of the navigation constant, this example has shown that a Proportional Navigation scheme will work very well against non-accelerating targets. This conclusion can be reached because the solution degenerates to the optimum case (from the standpoint of fuel consumption) in the limiting case.

The effectiveness of the standard Proportional Navigation against an accelerating target can best be examined by considering another simple example which is that of the constant accelerating target. With this assumption, Equation (3.8) reduces to

$$A_I = A_{TN} \frac{\lambda}{\lambda-2}\left[1 - (1 - \frac{t}{T_f})^{\lambda-2}\right] + \dot{\sigma}_o \lambda V_c (1 - \frac{t}{T_f})^{\lambda-2} \qquad (3.14)$$

$$= A_{I_1} + A_{I_2}$$

where

$$A_{I_1} = A_{TN} \frac{\lambda}{\lambda-2}\left[1 - (1 - \frac{t}{T_f})^{\lambda-2}\right]$$

= part of the interceptor acceleration due to target acceleration

$$A_{I_2} = \dot{\sigma}_o \lambda V_c (1 - \frac{t}{T_f})^{\lambda-2}$$

= part of the interceptor acceleration due to launch or initial errors.

Since A_{I_1} is monotonically increasing with time and because A_{I_2} is monotonically decreasing with time, the interceptor can be interpreted to first null the error due to launch misalignments and then compensate for target accelerations normal to the line of sight. The adverse effects which may occur in certain situations because of this sequential nulling phenomena are illustrated in Figure III-1 . Although this is an exaggerated trajectory, it typifies the "S-shaped" maneuver which may occur when Proportional Navigation is used in the interception of an accelerating target. The excessive maneuvering tends to imply large fuel expenditures. The verification of this point can be accomplished analytically using Equation (3.10). (See Reference 4.)

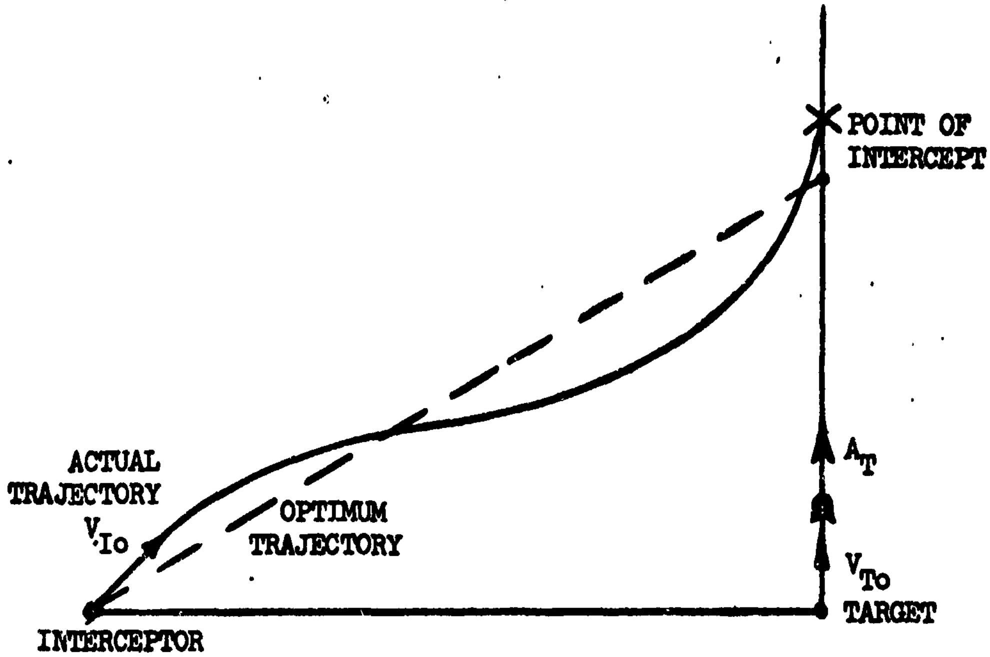

Figure III-1 Interceptor Trajectory for an Accelerating Target[1]

In this case, increasing λ does not lead to the optimum case. As $\lambda \to \infty$, A_{I_2} approaches the impulse acceleration required to correct the launch error that would exist if the target did not accelerate, and A_{I_1} tends to match the component of the target acceleration normal to the line of sight throughout the entire flight. This situation is completely opposed to the basic advantages of Proportional Navigation because it does not guide the interceptor to a lead collision course. Therefore, unlike the non-accelerating target case, the optimum situation is not obtained as $\lambda \to \infty$. It can be concluded that, although an accelerating target can be intercepted with Proportional Navigation, this technique is not desirable from the standpoint of fuel consumption; and a modified guidance law, which is more efficient, is needed.

[1] A graphical technique for determining these trajectories is given in Reference 9.

IV. BIASED PROPORTIONAL NAVIGATION

With the inefficiency of the proportional navigation scheme against accelerating targets, the need for a modified guidance law becomes obvious. Since the initial step change in velocity is the most efficient from the fuel consumption standpoint, a guidance law that corrects the launch errors and compensates for the target acceleration early in flight would be very desirable. This implies that the interceptor would essentially fly a straight line trajectory after the early corrections. Howev..., this type of intercept, which is shown in Figure IV-1, does not have a zero line of sight

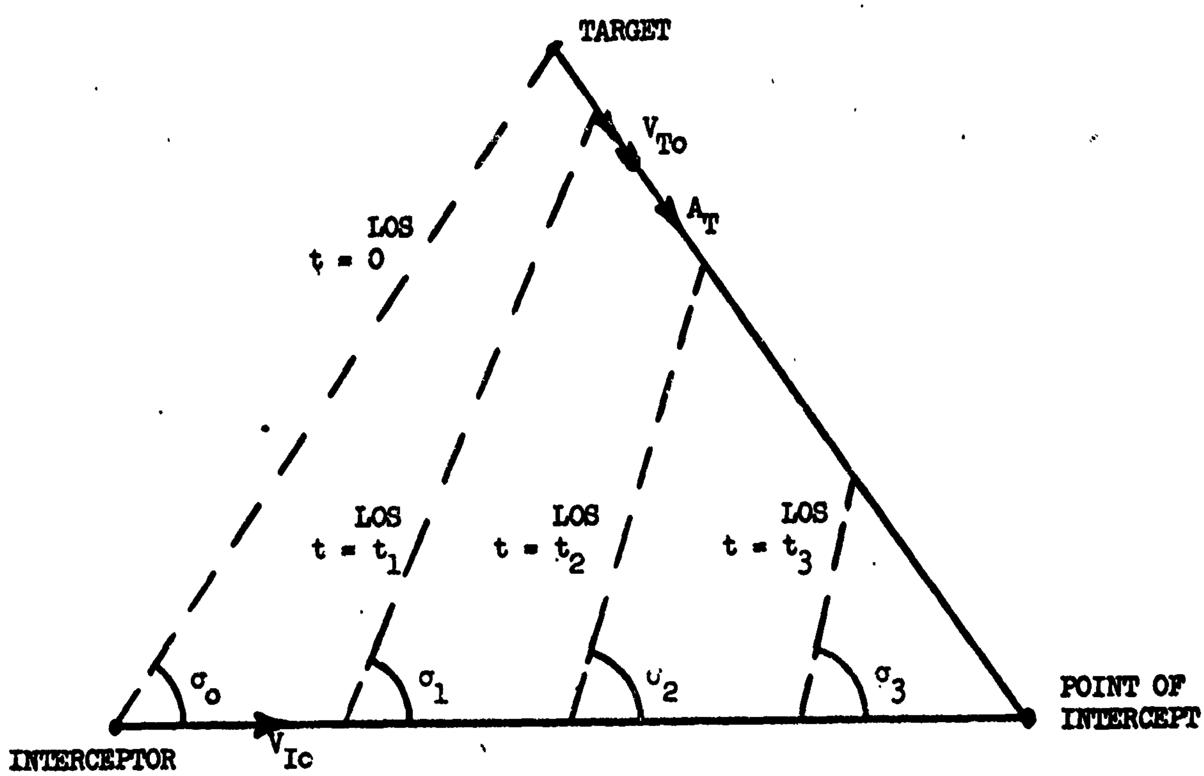

Figure IV-1   Lead Intercept of an Accelerating Target

rate when the interceptor is on a straight line collision course. This factor helps to account for the inefficiency of Proportional Navigation against accelerating targets. If the relative trajectories of the target and interceptor, in particular the target acceleration profile, are known, the interceptor can be guided to the desired line of sight rate; and, therefore, put on a straight line collision course. Thus, rather than nulling the line of sight rate, the difference between the inertial line of sight rate and some desired line of sight rate must be nulled. This concept is more commonly stated as "nulling the biased line of sight rate." With this intuitive approach, the guidance law for "Biased Proportional Navigation" can be stated as

$$A_I = \lambda V_c (\dot{\sigma} - \dot{\sigma}_b) \tag{4.1}$$

where $\dot{\sigma}_b$ - bias or compensating line of sight rate.

Vectorially this is stated as

$$\bar{A}_I = -\lambda V_c \bar{e}_R \times (\bar{\dot{\sigma}} - \bar{\dot{\sigma}}_b) \tag{4.2}$$

or

$$\bar{A}_I = \lambda V_c (\bar{\dot{\sigma}} - \bar{\dot{\sigma}}_b) \times \bar{e}_R \tag{4.3}$$

The trajectory equation in this case can be obtained by starting with the linearized kinematic equation of motion. Equation (2.5) and inserting the guidance law, Equation (4.1). The result is

$$(R_o - V_c t) \ddot{\sigma} - 2V_c \dot{\sigma} = A_{TN} - \lambda V_c (\dot{\sigma} - \dot{\sigma}_b) \tag{4.4}$$

or

$$(T_f - t) \ddot{\sigma} + (\lambda - 2) \dot{\sigma} = \frac{A_{TN}}{V_c} + \lambda \dot{\sigma}_b \tag{4.5}$$

The general solution of Equation (4.5) is

$$\dot{\sigma} = (T_f - t)^{\lambda-2} \int_o^t \frac{A_{TN}}{V_c} + \lambda \dot{\sigma}_b \quad (T_f - t)^{-(\lambda-1)} dt + \dot{\sigma}_o (1 - \frac{t}{T_f})^{\lambda-2} \tag{4.6}$$

The value of the bias that minimizes fuel consumption can be obtained in several ways. One of these is to minimize ΔV in Equation (3.10) (Reference 4); another is to use a modified trajectory equation to specify an ideal kinematic model (Reference 8). In this report, the second approach will be used.

If the interceptor is on a direct or straight line collision trajectory and if the optimum bias is in the system, the interceptor appears to be tracking a non-accelerating target. In other words, the interceptor is essentially flying a Proportional Navigation scheme where it senses a line of sight rate of $(\dot{\sigma} - \dot{\sigma}_b)$ and a target acceleration of $A_{TN} - (R_o - V_c t)\ddot{\sigma}_b + 2V_c\dot{\sigma}_b$

Since Proportional Navigation was shown to be more efficient against non-accelerating targets, the bias should be chosen such that:

$$(R_o - V_c t)\, \ddot{\sigma}_b\,(OPT) - 2\, V_c\, \dot{\sigma}_b\,(OPT) = A_{TN} \tag{4.7}$$

or

$$(T_f - t)\, \ddot{\sigma}_b\,(OPT) - 2\, \dot{\sigma}_b\,(OPT) = \frac{A_{TN}}{V_c} \tag{4.8}$$

Thus, the optimum bias is

$$\dot{\sigma}_b\,(OPT) = (T_f - t)^{-2} \int_o^t (T_f - t) \frac{A_{TN}}{V_c}\, dt + (T_f - t)^{-2}\, T_f^2\, \dot{\sigma}_{bo} \tag{4.9}$$

where $\dot{\sigma}_{bo}$ is the initial bias value

If the bias is optimum at $t = 0$, it will compensate for the accumulative effect of target acceleration over the entire time of flight. Therefore, the expression for the optimum bias is

$$\dot{\sigma}_b\,(OPT) = (T_f - t)^{-2} \int_{T_f}^{t} (T_f - t) \frac{A_{TN}}{V_c}\, dt \tag{4.10}$$

In general, this is time varying function; but in the case of a constant accelerating target, it becomes:

$$\dot{\sigma}_b\,(OPT) = \frac{-A_{TN}}{2V_c} \tag{4.11}$$

To demonstrate the value of this bias, the solution of the trajectory equation must be solved for the same case. Upon integrating Equation (4.6) and substituting in the guidance law, Equation (4.1), the required interceptor acceleration is found to be:

$$A_I = \frac{\lambda}{\lambda-2} \{ [A_{TN} + 2V_c\, \dot{\sigma}_b] - [A_{TN} + \lambda V_c\, \dot{\sigma}_b - (\lambda-2)\, V_c\, \dot{\sigma}_o](1 - \frac{t}{T_f})^{\lambda-2} \} \tag{4.12}$$

If the optimum bias, Equation (4.11), is used in Equation (4.12), the result is

$$A_I = [\lambda \frac{A_{TN}}{2} + \lambda\, V_c\, \dot{\sigma}_o]\, [1 - \frac{t}{T_f}]^{\lambda-2} \tag{4.13}$$

The fuel consumed in this case is

$$\Delta V = \frac{\lambda}{\lambda-1} \left[\frac{A_{TN}}{2} T_f + T_f V_c \dot{\sigma}_o\right] \qquad (4.14)$$

Here, as in the case of Proportional Navigation against a non-accelerating target, the fuel consumption is minimized as the navigation gain is increased without bound. It is also interesting to note that the limiting case fuel consumption, Equation (4.15), is identical to that required by a step velocity change at t = 0 which is the optimum case.

$$\lim_{\lambda \to \infty} \Delta V = \frac{A_{TN}}{2} T_f + T_f V_c \dot{\sigma}_o \qquad (4.15)$$

$$\min \quad \Delta V = \frac{A_{TN}}{2} T_f + R_o \dot{\sigma}_o \qquad (4.16)$$

Because the Biased Proportional Navigation guidance law leads to the physically unrealizeable optimum in the limiting case, it can be concluded to be a very good scheme against accelerating targets.

Although the discussion in the previous paragraphs tends to show the advantage of Biased Proportional Navigation, it neglects the difficulty of target acceleration prediction. If the acceleration profile is not known exactly, it would be impossible to obtain the optimum biased, σ_b (opt). Therefore, an alternate scheme that has been called constant, partial, or crude biasing has been devised. This techniques uses a constant bias which is based on an average or estimated value of the target acceleration. The analytical investigation of this type of bias has been conducted in Reference 4 and 8. The results show that, if the bias falls in a particular interval, a considerable saving in fuel consumption can be realized over the case of ordinary Proportional Navigation. However, if the established bias does not fall within the desired range, extensive fuel expenditures may result. Even near the edges of this allowable bias interval, the weight trade-off between additional equipment needed for bias implementation and actual fuel savings may be such that Proportional Navigation may be desirable.

From the preceding discussions, it is obvious that whether or not constant biasing should be used depends upon the mission of the interceptor and the accuracy of prediction or detection of the target's acceleration profile.

The discussion in the two preceding paragraphs describes only one of the problems associated with Biased Proportional Navigation. Some of the other areas of interest, which must be investigated before an actual guidance system can be realized, are listed below.

1. Does the fuel savings compensate for the weight of the additional equipment required to implement a time varying bias?

2. How can the orientation of the bias be determined and mechanized in three-dimensions?

3. How does a biased scheme effect the miss distance?

4. How can the target's acceleration be predicted and determined in both magnitude and direction?

These and many other questions must be answered before Biased Proportional Navigation can be used as a guidance scheme in any particular mission.

V. TIME LAGS AND POSSIBLE METHODS OF SOLUTION

In Section II, Figure II-1, a block diagram which shows the closed loop nature of the relative trajectory was given. In that section, as in all of the preceding sections, the interceptor was assumed to have perfect dynamics. However, a more realistic situation would be attained if some transfer function were assigned to the interceptor dynamics. Although most work of this nature has been done with computers, a few papers have been written on the analytical analysis. Because the effects of interceptor dynamics cause increased fuel consumption and loss of accuracy, the formulation of a specific problem will be given here. It is hoped that the mathematical model will help to demonstrate how the actual physical situation can be more accurately represented.

For this particular problem, assume that the interceptor dynamics are represented by a transfer function, F (s), and, more specifically, it will be a first order lag. Therefore, the system dynamics can be represented by

$$F(s) = \frac{1}{\tau s + 1} \qquad (5.1)$$

where

s = the differential operator, $\frac{d}{dt}$

τ = time constant

Thus, for Proportional Navigation[1], the combined interceptor dynamics and guidance law have the following form.

$$A_{IN} = \frac{\lambda V_c \dot{\sigma}}{\tau s + 1} \qquad (5.2)$$

Using Equation (5.2) in the linearized kinematic equation of motion, Equation (2.5), the trajectory equation becomes

$$(R_o - V_c t)\, s\dot{\sigma} - 2V_c\, \dot{\sigma} = A_{TN} - \frac{\lambda V_c}{\tau s + 1}\, \dot{\sigma} \qquad (5.3)$$

[1]The formulation for Biased Proportional Navigation is very similar.

or

$$\tau\,(T_f - t)\frac{d^2\sigma}{dt^2} + [\,(T_f - t) - 3\tau\,]\,\frac{d\sigma}{dt} + (\lambda-2)\,\sigma = \frac{\tau}{V_c}\,\frac{dA_{TN}}{dt} + \frac{A_{TN}}{V_c} \qquad (5.4)$$

To simplify the formulation, it is convenient to non-dimensionalize the independent variable, t. This can be accomplished by letting

$$x = \frac{T_f - t}{\tau} \qquad (5.5)$$

Therefore,

$$\frac{d}{dt} = \frac{-1}{\tau}\,\frac{d}{dx} \qquad (5.6)$$

And the trajectory equation takes the following form. (The appropriate variable transformation are assumed to be made in all functions.)

$$x\,\frac{d^2\sigma}{dx^2} - (x-3)\,\frac{d\sigma}{dx} + (\lambda-2)\,\sigma = \frac{-1}{V_c}\,\frac{A_{TN}}{dx} + \frac{A_{TN}}{V_c} \qquad (5.7)$$

This equation can be recognized to be the confluent hypergeometric equation (Reference 7.) which has been thoroughly investigated in the literature. The advantage of arriving at this particular equation is that, although, Equation (5.7) has singular points at $x = 0$ and infinity, the singularity at $x = 0$, which corresponds to $t = T_f$ in the physical situation, is regular. Therefore, a solution which is analytic at $x = 0$ or at the collision point, $t = T_f$, can be obtained. Although, a solution will not be given in the report, a power series solution with a recursion relation for the coefficients would be one possible approach.

A technique similar to that shown above was used by R. C. Booton in Reference 3. Although his solution was not completely general, it was given in a closed form. His results covered miss calculations with both lags and noise.

A more general analysis of miss can be found in Reference 2. There a closed form expression for miss is obtained without actually solving the trajectory equation. The technique used involved the theory of residues, and the actual analysis covered various noise effects.

The survey and description of the possible techniques used to analyze systems with imperfect dynamics shows that although the problem is difficult, it is possible to examine certain systems analytically. This form of analysis could be very beneficial in the quantitative and qualitative investigations of "Homing Guidance" systems.

SUMMARY

By considering the physical situation of the relative interceptor-target geometry, a few approximations can be made so that the trajectory of the interceptor can be described by a first order, linear differential equation. Because of the simplicity of this formulation, various guidance schemes which depend on line of sight information can be easily analyzed.

The investigation of Proportional Navigation shows that, although it is very desirable from the standpoint of fuel consumption against non-accelerating targets, the scheme leads to excessive maneuvering or fuel expenditure against accelerating targets. This situation can be rectified by using Biased Proportional Navigation. Both of these schemes approach an optimum situation in the limiting case.

Although the inclusion of imperfect interceptor dynamics increases the difficulty of analysis, the solution may be possible; and, therefore, results which are better approximations of the actual physical situation may be obtained.

LIST OF SYMBOLS

$\bar{A}_I$	interceptor acceleration vector
A_{IN}	component of the interceptor acceleration normal to the line of sight
A_{IR}	component of the interceptor acceleration along the line of sight
$\bar{A}_T$	target acceleration vector
A_{TN}	component of the target acceleration vector normal to the line of sight
A_{TR}	component of the target acceleration vector along the line of sight
$\bar{e}_N$	unit vector normal to the line of sight
$\bar{e}_R$	unit vector along the line of sight directed toward the target
$F(s)$	interceptor dynamics transfer function
$\bar{G}_I$	gravitational acceleration vector at the interceptor
G_{IN}	component of the gravitational acceleration normal to the line of sight at the interceptor
G_{IR}	component of the gravitational acceleration along the line of sight at the interceptor
$\bar{G}_T$	gravitational acceleration vector at the target
G_{TN}	component of the gravitational acceleration normal to the line of sight at the target
G_{TR}	component of the gravitational acceleration along the line of sight at the target
I	interceptor
LOS	line of sight
N	scale factor
$\bar{R}$	radius vector along the line of sight from the interceptor to the target
$\bar{R}_o$	initial radius ($\bar{R}_o = \bar{R}\ (t = o)$)
s	differential operator $\frac{d}{dt}$
t	time

T_f	total time of flight
T_{go}	time to go to intercept
$\bar{V}_I$	interceptor velocity vector
$\bar{V}_{Io}$	initial interceptor velocity vector
$\bar{V}_T$	target velocity vector
$\bar{V}_{To}$	initial target velocity vector
$\bar{V}_o$	initial velocity vector of the interceptor relative to the target
V_c	closing velocity
x	dimensionless variable that is proportional to time
ΔV	incremental change in velocity
λ	navigation constant or gain
σ	angle of the line of sight
$\dot{\sigma}$	time rate of change of the line of sight
$\dot{\sigma}_b$	line of sight rate bias
$\dot{\sigma}_b$(OPT)	optimum line of sight rate bias
$\dot{\sigma}_{bo}$	initial line of sight rate bias
τ	time constant

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