

AD-756 647

THE CONTROL DYNAMICS OF A LOOP IN R^2 AND
AND RELATED DIFFERENTIAL GAME-THEORETIC
PROBLEMS

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Prepared for:

Office of Naval Research

1973

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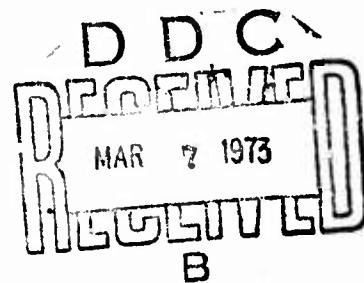
by

D. L. Lukes *

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Applied Mathematics & Computer Science University of Virginia, Charlottesville, VA		2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE THE CONTROL DYNAMICS OF A LOOP IN R^2 AND RELATED DIFFERENTIAL GAME-THEORETIC PROBLEMS		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report		
5. AUTHOR(S) (First name, middle initial, last name) Dahlard L. Lukes		
6. REPORT DATE		7a. TOTAL NO. OF PAGES - 5
8a. CONTRACT OR GRANT NO. N00014-69-A-0060-0010		9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.		
c.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.		
10. DISTRIBUTION STATEMENT Distribution of this document is unlimited.		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Office of Naval Research
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Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Control Theory Differential games Search Evasion Distributed model						

DD FORM 1 NOV 1968 1473 (BACK)

(PAGE 2)

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Security Classification

Abstract

A distributed parameter model for a class of search and capture problems in the plane is derived. The output is a time-varying simple loop in R^2 and the input a moving line segment. The applications discussed include destroyer search, oil spill clearance, forest fire control and fishing. Circular motion is shown to produce a non-trivial steady-state region whenever the velocity ratio between searcher and evader attains a computed critical value.

Introduction

Suppose that we wish to carry out a search in the plane, R^2 , with the visible points at time t being a line segment of fixed length d attached to a moveable origin O . (See Figure 1-a.) The search might begin with a rotation of "the line of sight" as indicated in Figure 1-b.

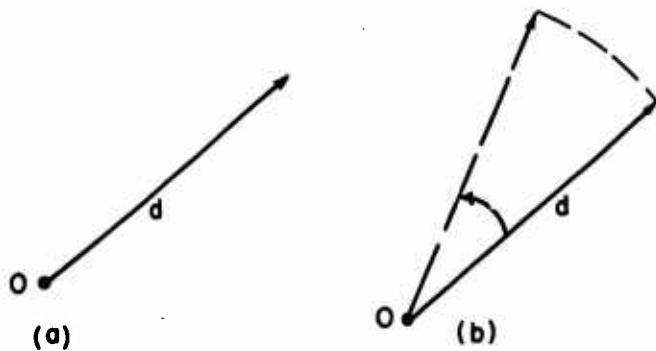


Fig. 1 Starting the search

The objective is to search for evasive objects in the vicinity of O which have speed known to be bounded by a given number $\beta > 0$. If after a steady rotation of the line of sight by 360° no evaders have been detected then we can be assured there are no evaders present in the clear region indicated in Figure 2. Notice that the clear region is a proper subset of the searched region since the objects sought can move about.

If the search were now terminated the clear region would tend to contract. To counter this tendency with the hope of eventually expanding the clear region the searcher might proceed with some combined motion of O and the line of sight. The partial differential equation derived in this paper provides a mathematical foundation for studying the relative merits of such search procedures.

Other game-theoretic problems to which the model also applies are discussed in the later sections. A problem in which the searcher moves along a circular path is studied in some detail.

Some reflection upon the nature of the search situation presenting itself in Figure 2 suggests that the appropriate choice for the state of the system might be a time-varying loop bounding the searched-and-clear region. The control input will be the moving

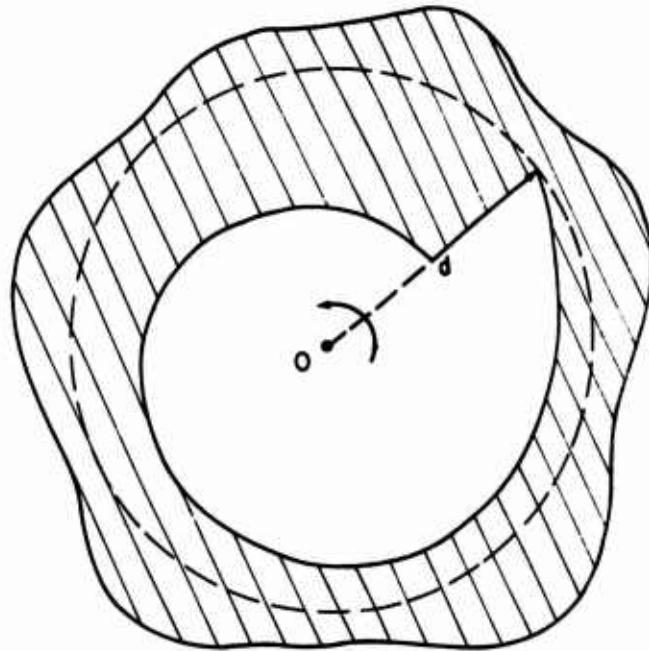


Fig. 2 The searched-and-clear region assuming no detection has occurred

straight line segment extracted from the line of sight by intersection with the loop. This segment will hereafter be called the control or input segment.

Derivation of the model

To each simple loop of the type eluded to we attach coordinates in the manner indicated in Figure 3.

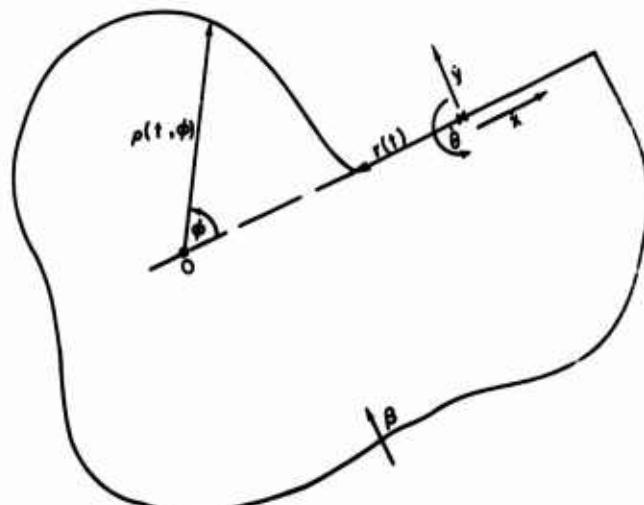


Fig. 3 Assignment of system coordinates

Looking at the loop at time t first locate the control segment which, when extended, penetrates the interior. Along this extended line select an origin O . The

state of the system, i.e. the loop, can then be described by a function $\rho(t, \phi)$ where ϕ is the indicated angular coordinate. To describe the motion of the control segment a control point such as the center is selected along it. The radial distance to the end of the loop nearest O is denoted by $r(t)$ and $\dot{\theta}(t)$ is the segment's angular velocity. The respective speeds of the control point along the segment and orthogonal to it are indicated as \dot{x} and \dot{y} . At any time that a point on the loop is not being driven by the control segment it will be assumed to be moving inward along a normal to the loop at speed β .

Looking ahead, it turns out to be convenient to describe the system in terms of the following canonical control variables.

$$u_1 = \dot{\theta} \quad (1)$$

$$u_2 = \dot{y} - \rho(t, 0)\dot{\theta} - r\dot{\theta} \quad (2)$$

$$u_3 = \dot{x} - r\dot{\theta} - \rho_t(t, 0) \quad (3)$$

$$u_4 = \dot{x} \quad (4)$$

$$u_5 = \dot{y} - r \quad (5)$$

and to position the control point at the outer end as indicated in Figure 4. There u_2 and u_3 are identified as the respective speeds of O orthogonal to and parallel to the moving segment. The variables u_4 and u_5 give the speeds of the ends of the control segment along the line of sight.

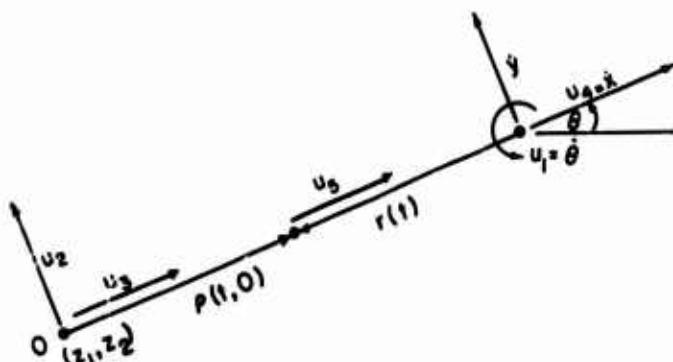


Fig. 4 The canonical control variables

Observe that the change of control variables (1)-(5) can be inverted by integrating that system of linear ordinary differential equations. Upon assigning rectangular coordinates (z_1, z_2) to the origin of the loop it is clear that

$$z_1 = u_3 \cos \theta - u_2 \sin \theta \quad (6)$$

$$z_2 = u_3 \sin \theta + u_2 \cos \theta \quad (7)$$

which points out that the motion of O is strictly governed by u_1 , u_2 and u_3 .

In Figure 5 we study the change in the geometry of the system during a small time displacement Δt . To each point on the loop at time t with coordinates $\phi, \rho(t, \phi)$ there is a corresponding point with coordinates $\phi + \Delta\phi, \rho(t+\Delta t, \phi)$ on the displaced loop. This correspondence produces the induced angles $\Delta\phi$, γ and η indicated in the diagram along with the magnitude of the loop displacement parallel to itself, $\beta\Delta t$.

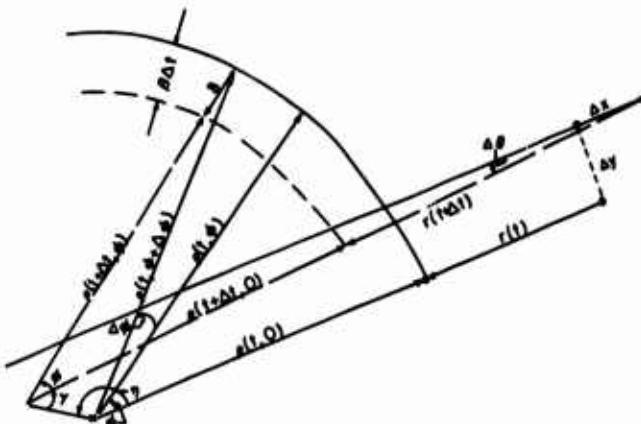


Fig. 5 The change in geometry during Δt

First we compute δ , the induced change in ρ indicated in Figure 5. Consider the infinitesimal right triangle generated by the displacements δ , $\beta\Delta t$ and replacement of the loop at the point $\phi, \rho(t+\Delta t, \phi)$ by its tangent line. (See Figure 6.)

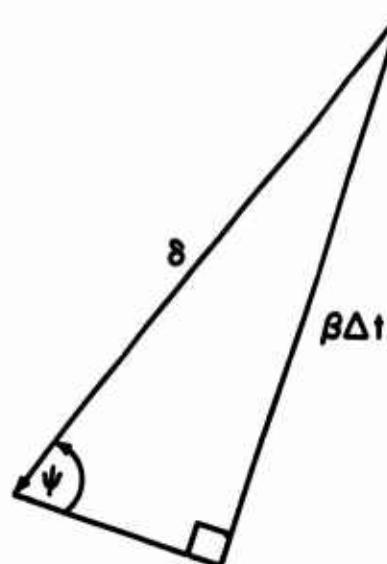


Fig. 6 The infinitesimal triangle for computing δ

Recall that in the elementary differential geometry of a curve with polar coordinates $\theta, r(\theta)$ (not to be confused with the control variables) the standard angle ψ between the radial and tangent line is given by

$$\tan \psi = \frac{r}{r_\theta}. \text{ Thus we compute}$$

$$\delta = \sqrt{(\beta\Delta t)^2 + \frac{(\beta\Delta t)^2}{\tan^2 \psi}} = \frac{\beta\Delta t}{\tan \psi} \sqrt{\rho^2(t, \phi) + \rho_\phi^2(t, \phi)}. \quad (8)$$

Next we apply the law of sines in Figure 5 to get

$$\frac{\rho(t+\Delta t, \phi) + \delta}{\rho(t, \phi + \Delta\phi)} = \frac{\sin(\eta - \phi - \Delta\phi)}{\sin(\gamma + \phi)}. \quad (9)$$

By adding and subtracting the appropriate terms this last equation can be written as

$$\frac{\delta}{\Delta t} = \frac{\sin(n-\phi-\Delta\phi)}{\sin(\gamma+\phi)} \frac{\Delta\phi}{\Delta t} \left[\frac{p(t, \phi + \Delta\phi) - p(t, \phi)}{\Delta\phi} \right] \quad (10)$$

$$+ \frac{1}{\Delta t} \left[\frac{\sin(n-\phi-\Delta\phi)}{\sin(\gamma+\phi)} - 1 \right] u_1(t, \phi) - \left[\frac{p(t + \Delta t, \phi) - p(t, \phi)}{\Delta t} \right]$$

From Figure 5 it is clear that

$$\frac{\sin(n-\phi-\Delta\phi)}{\sin(\gamma+\phi)} \rightarrow 1 \text{ as } \Delta t \rightarrow 0. \quad (11)$$

Notice that $\gamma = \Delta\theta + n = \pi$. Hence $\gamma + n \rightarrow \pi$ as $\Delta t \rightarrow 0$ and by the employment of standard trigonometric identities

$$\frac{1}{\Delta t} \left[\frac{\sin(n-\phi-\Delta\phi)}{\sin(\gamma+\phi)} - 1 \right] = \frac{1}{\Delta t} \left[\frac{\sin(n-\phi-\Delta\phi)}{\sin(n-\phi-\Delta\theta)} - 1 \right]$$

$$= \frac{2}{\Delta t} \sin\left(\frac{\Delta\theta-\Delta\phi}{2}\right) \left[\frac{\cos(n-\phi-\frac{\Delta\theta+\Delta\phi}{2})}{\sin(n-\phi-\Delta\theta)} \right] \quad (12)$$

$$\rightarrow \frac{\lim(\frac{\Delta\theta}{\Delta t} - \frac{\Delta\phi}{\Delta t})}{\tan(\lim(n-\phi))} = \left[\frac{1 + \tan(\lim(n-\phi)) \tan \phi}{\tan(\lim(n-\phi)) - \tan \phi} \right] \lim(\frac{\Delta\theta}{\Delta t} - \frac{\Delta\phi}{\Delta t}).$$

But $\tan(\lim(n-\phi)) = \frac{u_2}{u_3}$ and

$$\lim(\frac{\Delta\phi}{\Delta t}) = u_1 + \frac{u_2 \cos \phi - u_3 \sin \phi}{p(t, \phi)}, \quad (13)$$

both direct conclusions drawn from Figure 5. Substitution into (12) then shows

$$\frac{1}{\Delta t} \left[\frac{\sin(n-\phi-\Delta\phi)}{\sin(\gamma+\phi)} - 1 \right] \rightarrow - \frac{[u_2 \sin \phi + u_3 \cos \phi]}{p(t, \phi)} \quad (14)$$

as $\Delta t \rightarrow 0$.

By substituting (8) into (10) and then using (11), (13) and (14) to take limits as $\Delta t \rightarrow 0$ the partial differential equation in the following model is derived.

The distributed model

$$r_t = \left[u_1 + \frac{u_2 \cos \phi - u_3 \sin \phi}{p} \right] p_\phi \quad (15)$$

$$- u_2 \sin \phi - u_3 \cos \phi - \frac{B}{p} \sqrt{p^2 + p_\phi^2}$$

$$p(0, \phi) = p_0(\phi) \quad (16)$$

$$p(t, 0) = p_0(0) + \int_0^t [u_5(\sigma) - u_3(\sigma)] d\sigma \quad (17)$$

$$p(t, 2\pi) = p_0(2\pi) + \int_0^t [u_4(\sigma) - u_3(\sigma)] d\sigma \quad (18)$$

The initial function $p_0(\cdot)$ in (16) is prescribed. The last two boundary conditions arise from the choice of positioning the control point (x, y) at the end of the control segment; i.e. requiring that $r(t) = p(t, 2\pi) - p(t, 0)$ in the integrated form of $r = u_4 - u_5$ and $r_t(t, 0) = u_5 - u_3$ which follow from (3), (4) and (5).

Remarks about the model

The partial differential equation (15) contains a nonlinear term which reflects the evasion being taken

into account. For zero evader speed ($B = 0$) that term drops out leaving a quasi-linear equation. Its role in the damping of the solution will become apparent.

In the model u_1, u_2 are distributed controls with u_4, u_5 strictly boundary controls and u_3 mixed. The u_i variables might well be called input-output variables (distinguished from the state-the loop) since from the geometric nature of the model it is clear they can not be chosen arbitrarily. The input-output nature of u_4, u_5 depends upon whether u_1, u_2, u_3 are in the mode tending to stretch or compress the ends of the loop.

The model will exhibit memory-loss if the control segment is "backed-up" over the interior of the loop. This is a sacrifice made to maintain the relative simplicity of the model. It is also capable of exhibiting shock and bifurcation phenomena.

Game-theoretic applications

The originally stated goal underlying the derivation of (15)-(18) was a model for generating the boundary loop of the searched-and-clear set for search in which the field of visibility at any given time is a line segment and the objects being sought have evasive speed bounded by a parameter B . We shall point out some other conflict situations to which the model applies as well. The control theory of this distributed system will be presented in another article.¹

Destroyer search

Suppose that an airplane flying about an aircraft carrier has two observers on board each watching out one of the side windows for the purpose of detecting any enemy destroyers before the destroyers can bring the carrier within range of their missiles. If we wish we can assume that the destroyers have superior radar or perhaps satellite information so they would maneuver in an attempt to approach within range of the carrier while remaining outside the detection range of the radar on the carrier and sight of the airplane. Using the search path selected by the airplane as input (15)-(18) would generate the time-varying region which the carrier could be assured contains no destroyers as long as the airplane reports no sightings. Clearly the airplane would want to choose a path which would expand this region into a comfortable area about the carrier (say larger than the destroyer missile range if possible).

Here we will consider only circular search and look for the path with the largest steady state disk. That is we select $u_1(t) = c_1$, constant, and $u_i(t) = 0$ ($i = 2, 3$) identically zero in (15). This gives rise to the ordinary differential equation for the steady state

$$c_1 p_\phi - \frac{B}{p} \sqrt{p^2 + p_\phi^2} = 0 \quad (19)$$

which can be integrated to give

$$\phi = \int_{p_0}^p \sqrt{\left(\frac{c_1}{B}\right)^2 - \left(\frac{1}{r}\right)^2} dr. \quad (20)$$

Equation (20) determines a one-parameter family of monotone loops $p = p(\phi)$ with parameter $c_1 = p_0 \geq \frac{B}{c_1}$.

(See Figure 7.) Define $\Delta = p(2\pi) - p_0$, the length of the control segment, and let the airplane move

along at the center of the control segment with linear speed v .

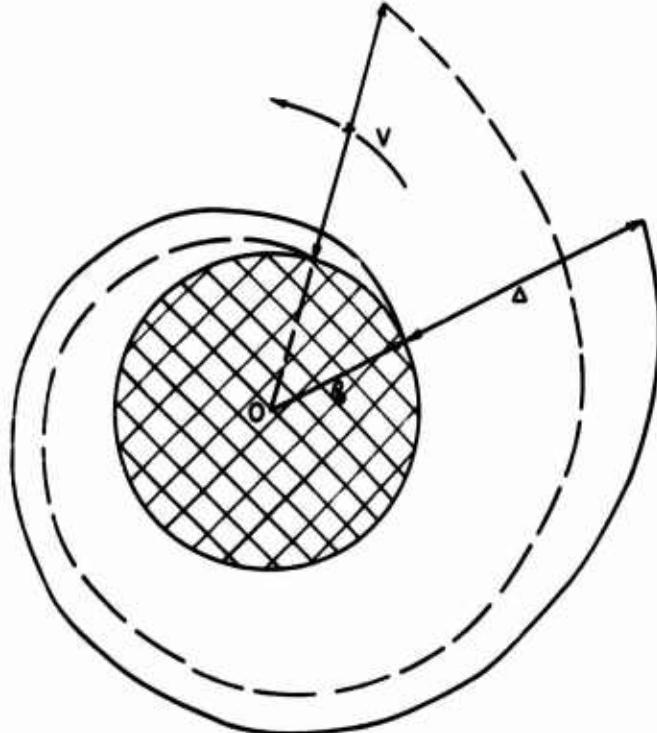


Fig. 7 The steady state disk

In view of the airplane's circular path

$$c_1 = \frac{v}{\rho_0 + \frac{\Delta}{2}} \quad (21)$$

which can be substituted into (20) along with $\phi = 2\pi$ and $\rho = \rho_0 + \Delta$ to get

$$2\pi = \int_{\frac{v}{\rho_0 + \frac{\Delta}{2}}}^{\frac{v}{\rho_0 + \Delta}} \sqrt{1 - (\frac{1}{w})^2} dw. \quad (22)$$

The following theorem states that (22) implicitly defines $\rho_0 = \rho_0(\Delta, \frac{v}{\beta})$ as a function of the parameters Δ and $\frac{v}{\beta}$.

Theorem 1. If $\frac{v}{\beta}$ is sufficiently large so that

$$\sqrt{\frac{(2v)^2}{\beta}} - 1 + \sin^{-1}\left(\frac{\beta}{2v}\right) \geq \frac{5\pi}{2} \quad (23)$$

then:

(a) for each $\Delta > 0$ (22) has a unique solution $\rho_0 > 0$. Moreover (23) is necessary for such a solution to exist.

$$(b) \rho_0(\Delta, \frac{v}{\beta}) \rightarrow \infty \text{ as } \Delta \rightarrow \infty.$$

$$(c) \rho_0(\Delta, \frac{v}{\beta}) \rightarrow 0 \text{ as } \Delta \rightarrow 0.$$

Proof. One can easily verify that (23) implies that $\frac{v}{\beta} > 1$. Hence the integral in (22) converges to zero as $\rho_0 \rightarrow \infty$. On the other hand if we set

$$\rho_0 = \frac{\Delta}{2(\frac{v}{\beta} - 1)} \text{ in the integral and apply (23) then}$$

we see the integral takes on a value at least as great as 2π . The existence of the required solution ρ_0 to (22) then follows from the intermediate value theorem.² The uniqueness follows from the fact that the integral is monotone decreasing in ρ_0 . The necessity of (23) is clear from the monotonicity and the observation that if ρ_0 were chosen any smaller than the above choice then the integral would not be defined since the lower limit of integration would be less than one.

To prove (b) note that if for some sequence $\Delta_k \rightarrow \infty$ the corresponding values of ρ_0 remained bounded then (22) would be violated since the lower limit of integration would again become less than one. Part (c) follows from the observation that if for some sequence Δ_k converging to zero the corresponding values of ρ_0 remained bounded away from zero then the limits of integration in (22) would both converge to $\frac{v}{\beta}$ and that equation would be violated.

Remark. In the destroyer search problem $\frac{\Delta}{2}$ is the observers' range of visibility. Theorem 1 implies that if the search plane can attain or exceed the critical speed ratio (about 3.9) then once the disk of radius ρ_0 computed from (22) has been cleared about the carrier² the carrier can rest assured that there are no destroyers within distance ρ_0 from it as long as the search plane reports no sightings. Moreover this is the largest such disk for the given search path.

Oil spill clearance

Many seemingly different conflict situations have essentially the same underlying mathematics. For example the model (15)-(18) would apply as well to the problem of cleaning up an oil spill on the surface of the ocean as it tends to dissipate in all directions at a speed β or less due to wave action, wind, etc. Here we are thinking of a clean-up device which absorbs a swath of the oil as it is steered around the perimeter of the spill. In this problem β would be a negative parameter. The results given for the circular search problem would apply directly to give the radius of the largest circular spill that could be contained and eventually cleaned up by a piece of equipment of speed v and swath width Δ .

Forest fire control

The distributed model is well suited for analyzing a variety of forest fire problems. For example consider the situation wherein an airplane is dropping fire fighting chemicals about a town to protect it from a surrounding fire. If the fire had not yet contacted the treated region we would set $\beta = 0$ in (15)-(18) to predict the extent of the protection which could be developed by any available equipment. The control segment in this problem would be the cross sectional line of the chemical trail as it struck the surface. Once the fire reached the treated region a

positive value of β would be used to activate the non-linear term accounting for the spread of fire across the treated area. The spread of an initially localized fire would be modeled using β negative.

Netting fish

Suppose that a fishing vessel towing a net sights a large school of fish swimming near the surface but the school quickly submerges from sight upon being pursued. If a bound upon the swimming speed of a fish is known can the fishing boat catch all the fish in its net? This is another interesting question which can be asked of our model. Of course we are making the assumption that the fish remain at a sufficiently shallow depth so that we can keep the problem two dimensional. The answer will depend upon the speed of the vessel, the size of its net and the initial distribution of fish.

Example

Consider the model with $\beta = 1$, $u_1(t) = u_2(t) = u_3(t) \equiv 0$, $u_4(t) = u_5(t) \equiv -1$ and initial function

$$\rho_0(\phi) = \begin{cases} 1 & , 0 \leq \phi \leq \pi \\ \cos \phi + \sqrt{4 - \sin^2 \phi} & , \pi \leq \phi \leq 2\pi \end{cases} \quad (24)$$

shown in Figure 8.

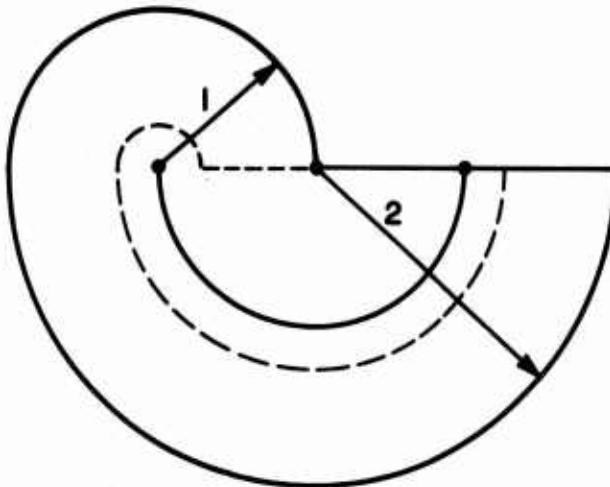


Fig. 8 The initial loop and the state at switching

The partial differential equation then is

$$\rho_t = \frac{-1}{\rho} \sqrt{\rho^2 - \rho_\phi^2} . \quad (25)$$

Its solution could be obtained using the method of characteristics although it is much easier to obtain the following solution directly from the initial loop and the geometric nature of the problem.

$$\rho(t, \phi) = \begin{cases} 1 - t & , 0 \leq \phi \leq \pi \\ \cos \phi + \sqrt{(2-t)^2 - \sin^2 \phi} & , \pi \leq \phi \leq 2\pi. \end{cases} \quad (26)$$

The solution exists for $0 \leq t \leq 1$, $0 \leq \phi \leq 2\pi$. At

time $t = 1$ the initial loop has been steered down to the semi-circular loop indicated in Figure 8. That loop can be further shrunk down to a point according to the solution

$$\rho(t, \phi) = \begin{cases} 0 & , 0 \leq \phi \leq \frac{3\pi}{2} \\ 2(2-t) \cos \phi & , \frac{3\pi}{2} \leq \phi \leq 2\pi \end{cases} \quad (27)$$

during $1 \leq t \leq 2$ by switching the controls to $u_3(t) = u_5(t) = 1$, $u_4(t) = -1$ on $1 \leq t \leq 2$. Hence the model continues to hold in spite of the fact that the origin of the loop strikes the loop itself and then sticks there.

Acknowledgement

This work was sponsored by the U.S. Office of Naval Research under Contract N00014-69-A-0060-0010.

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Reprinted from the Proceedings of the 1972 IEEE Conference on Decision and Control and 11th Symposium on Adaptive Processes, December 13-15, 1972, New Orleans