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THE FAR FIELD ZONE BEAM PATTERN OF A PARAMETRIC ACOUSTIC ARRAY

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NAVAL UNDERWATER SYSTEMS CENTER

Technical Memorandum

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ABSTRACT

Mellen and Moffett (1, 3) have derived models for parametric sonar radiator design following Westervelt's (2) theory. Their models analyze both absorption and saturation limits in both the near field and far field. This memorandum is concerned with the far field zone expansion of the far field beam pattern and pressure into a series of eigenfunctions which can be integrated term by term. After summing over the terms, ~~we~~ can show that the far field beam pattern is a product of the two primary beam patterns. The conversion efficiency reduces to an experimental integral which appears in Reference (b).

ADMINISTRATIVE INFORMATION

This memorandum was prepared under Project No. A65405, Task No. ZR000101, "Parametric Sonar Research," Principal Investigator Dr. R. H. Mellen, Code PA41. The sponsoring activity is CNM/MAT 033, Dr. J. H. Huth.

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INTRODUCTION

The purpose of this memo is to present in some detail a derivation of the far field zone beam pattern of a parametric sonar. Moffett[1] has derived from Westervelt's theory[2] the two fold expression

$$P(r) = \frac{\beta P_o^2 \omega^2 e^{i\omega t}}{4\pi\rho_o c_o^4} \iiint dv' \; \frac{e^{-ik(\zeta + r')}}{\zeta} \; S^2(\vec{r'}) \qquad r' < R_o$$

$$= \frac{\beta P_o^2 \omega^2 e^{i\omega t}}{4\pi\rho_o c_o^4} \iiint dv' \; \frac{R_o^2}{(r')^2} \; \frac{e^{-ik(\zeta + r')}}{\zeta} \; S^2(\vec{r'}) \qquad r' > R_o \tag{1}$$

for the pressure of the secondary wave where R_o is the primary wave collimation distance. The volume integral is to be performed over the region occupied by the primary wave. In this expression ζ is the distance $(\vec{r}-\vec{r}')$ between the field point and the integration point. The other terms are defined in the section "Glossary of Terms."

The first integral in Equation (1) represents the contribution to the far field from near field interactions. It is evaluated in references (1) and (3). The essence of these papers is that the secondary beam pattern is a product of the two primary beam patterns. The parametric efficiency $\left|\frac{rP}{R_o P_o}\right|$ is derived in both references and the directivity index DI is derived in reference (3). Curves of the directivity index and parametric efficiency are presented in reference (3).

The second integral represents the contribution to the far field from the spherically spreading region. It is the intent of this memorandum to evaluate this integral.

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FAR FIELD INTERACTION ZONE

We rewrite the second integral in Equation (1) in the form

$$P(\vec{r}) = \frac{\beta P_o^2 R^2 k^2 e^{i\omega t}}{4\pi\rho_o^2 c^2} \iiint dv' \, \frac{R_o^2}{r'^2} \, \frac{e^{-ik[r' + |\vec{r}-\vec{r}'|]}}{|r-r'|} \, S^2(\vec{r}') \tag{2}$$

where ζ has been replaced by $|\vec{r}-\vec{r}'|$ the distance between the field point $\vec{r}$ and the integration point r'. $S^2(\vec{r}')$ is the source function. The volume element dv' is $r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$ and r' extends from R_2 to ∞. The distance R_2 is defined to be $\frac{k_o R_o}{k}$ and is called the spherical wave distance.

The justification for extending the range of integration from R_o to R_2 was that most of the contribution in the first integral in Equation (1) came from on the axis and the integrand was non oscillatory up to $r' = R_2$. Thus the near field contribution was extended out to the spherical wave distance and the far field or spherically spreading form was assumed to begin at R_2.

The quantity $\frac{1}{|r-r|}$ exp $(-ik|r-r'|)$ is the Green's function for the Helmholtz operator. It has the expansion

$$\frac{e^{-ik(\vec{r}-r')}}{|\vec{r}-r'|} = \begin{cases} -ik \sum_{n=o}^{\infty} (2n+1) \, P_n(\cos\theta) \, j_n(kr') \, h_n^{(2)}(kr'), & r'>r> \\ -ik \sum_{n=o}^{\infty} (2n+1) \, P_n(\cos\theta) \, j_n(kr') \, h_n^{(2)}(kr), & r>r'> \end{cases} \tag{3}$$

where $P_n(\cos\theta) = P_n(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi')$

$$= P_n(\cos\theta) \, P_n(\cos\theta') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \, P_n^m(\cos\theta') \cos\phi'. \tag{4}$$

The terms $P_n^m(\cos\theta)$ are the asssociated Legendre polynomials of order n. If we integrate over the azimuthal angle ϕ, we have from (4)

$$\int_0^{2\pi} P_n(\cos\theta)\, d\phi = 2\pi P_n(\cos\theta)\, P_n(\cos\theta'). \tag{5}$$

In the far field we use the second of the expansions in Equation (3). Combining the results of Equation (5) with the expansion in Equation (3) we rewrite Equation (2) as

$$p = \frac{\beta P_o^2 k^2 R_o^2 e^{i\omega t}}{2\rho c_o^2} \int_{R_2}^{\infty}\int_0^{\pi} (-ik) \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\theta)\, P_n(\cos\theta')\, x$$

$$j_n(kr')\quad h_n^{(2)}(kr)\; T^2(r')\, e^{-ikr'}\, |F(\theta')|^2 \sin\theta'\, dr' d\theta'. \tag{6}$$

The source function $S^2(\vec{r'})$ has been resolved into a product $T^2(r')|F(\theta')|^2$. where $T^2(r')$ is a linear taper function and $|F(\theta')|^2$ is the product of the two primary beam patterns. In the far field Hankel function $h_n^{(2)}(kr)$ can be written as

$$h_n^{(2)}(kr) = \frac{e^{-i[kr-\frac{1}{2}(n+1)\pi]}}{kr} = i^{n+1}\,\frac{e^{-ikr}}{kr}$$

We then have

$$p = \frac{\beta P_o^2 k^2 R_o^2 e^{i\omega t-ikr}}{2\rho c_o^2 kr} \int_{R_2}^{\infty}\int_0^{\pi} \sum_{n=0}^{\infty} (2n+1)\, i^n\, P_n(\cos\theta)\, P_n(\cos\theta')\, x$$

$$j_n(kr')\; T^2(r')\, e^{-ikr'}|F(\theta')|^2 \sin\theta'\, dr'\, d\theta'. \tag{7}$$

This expression can be written symbolically as

$$P = K\frac{e^{-ikr}}{kr} \sum_n (2n+1)\, i^n a_n b_n P_n (\cos\theta), \tag{8}$$

$$\text{where} \quad a_n = \int_{R_2}^{\infty} T^2(r') \, j_n(kr') \, e^{-ikr'} \, dr' \tag{9}$$

$$b_n = \int_0^{\pi} |F(\theta')|^2 \, P_n(\cos\theta') \sin\theta' \, d\theta'. \tag{10}$$

The integrand has been resolved into a series of eigenfunctions which can be integrated term by term. We proceed first with the integration over the angle θ.

Each of the primary beams is assumed to have the beam pattern

$$F(\theta') = \frac{2J_1(k_o a \sin\theta)}{k_o a \sin\theta'} \tag{11}$$

where a is the radius of the transducer and k_o is the average primary wave number. The aperture a is related to the primary wave collimation distance R_o by

$$R_o = \frac{k_o a^2}{2} \tag{12}$$

for a circular pistons the coefficients b_n becomes in Equation (10)

$$b_n = \int_0^{\pi} 4 \left[\frac{J_1(k_o a \sin \theta')}{k_o a \sin \theta'}\right]^2 \sin \theta' \, d\theta'. \qquad (13)$$

The square of the Bessel function has the representation

$$J_1^{\,2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2)! \, (\tfrac{1}{2}x)^{2m+2}}{m! \, (m+2)! \, [(m+1)!]^2} .$$

Thus b_n becomes

$$b_n = 4 \sum_{m=0}^{\infty} \frac{(-1)^m \, (2m+2)! \, (1/2 \, k_o a)}{m! \, (m+2)! \, (m+1)! \, (m+1)!} \int_0^{\pi} \sin^{2m+1} \theta' P_n(\cos \theta') \, d\theta'. \qquad (14)$$

Let $x = \cos \theta$. Then

$$b_n = 4 \sum_{m=0}^{\infty} \frac{(-1)^m \, (2m+2)! \, (1/2 \, k_o a)^{2m}}{m! \, (m+2)! \, (m+1)! \, (m+1)!} \int_{-1}^{1} (1-x^2)^m \, P_n(x) \, dx. \qquad (15)$$

It is obvious that $b_n = 0$ for n odd since $P_n(x)$ is odd for n odd. By induction it can be shown that

$$\int_{-1}^{1} (1-x^2)^m \; P_{2n}(x)\, dx$$

$$= \frac{(-1)^n\,(2n)!}{(2m+2n+1)!}\;\frac{2^{2m+1}\,(m!)^2}{(m-n)!}\;\frac{(m+n)!}{(n!)^2} \tag{16}$$

Using this result we have

$$b_{2n} = 4\sum_{m=0}^{\infty} \frac{(-1)^{m+n}\,(1/2\;k_o a)^{2m}\,2^{2m+1}\,(2n)!\,(m+n)!\,(2m+2)!}{n!\,(2m+2n+1)!\,(m+2)!\,(m+1)!\,(m+1)!}\;\frac{m!}{n!\,(m-n)!} \tag{17}$$

When $n = 0$ $\quad b_o = 4\sum_{m=0}^{\infty} \frac{(-1)^m\,(1/2\;k_o a)^{2m}\,2^{2m+2}}{(m+1)!\,(m+2)!} = \frac{16}{(k_o a)^2}\left[1-\frac{J_1(2k_o a)}{2k_o a}\right]$ (18)

When $n = 1$ $\quad b_2 = 4\sum_{m=0}^{\infty} \frac{(-1)^{m+1}\,(1/2\;k_o a)^{2m}\,2^{2m+2}m}{(2m+3)\,(m+1)!\,(m+2)!}$

$$= \frac{16}{(k_o a)^2}\left\{\left[1-J_1\,\frac{(2k_o a)}{2k_o a}\right] + 3\,J_2\,(2k_o a) - 3\,\pi/2\left[J_1(2k_o a)\,H_o\,(2k_o a) - J_o(2k_o a)\,H_o\,(2k_o a)\right]\right\} \tag{19}$$

In this case $H_o(2k_oa)$ and $H_1(2k_oa)$ are Struve functions of orders 0 and 1 and argument $2k_oa$. Using the Lagrange Duplication formula, we can represent b_{2n} as follows:

$$b_{2n} = \frac{(-1)^n \, 2^{1-2n} \, (2n)!}{(n!)^2 \, \Gamma(1-n)} \, \frac{\Gamma(3/2)}{\Gamma(n+3/2)} \; {}_3F_4(3/2, 1, 1; 3, 2, n+3/2, 1-n; -4k_o{}^2a^2) \tag{20}$$

where ${}_3F_4(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4; z)$ is a generalized hypergeometric function.

Returning to Equation (9) we calculate

$$a_n = \int_{R_2}^{\infty} T^2(r') \, jn\,(hr') \, e^{-ikr'} \, dr',$$

where n is even since $b_{2n+1}=0$. Following the work of Mellen and Moffett[3], we choose

$$T^2(r) = \frac{e^{-2\alpha r'}}{1+(\frac{X}{2} \sinh^{-1} r'/R_o)^2} \tag{21}$$

The term $e^{-\alpha r'}$ represents the absorbtion of the primary wave and the term $[1+(X/2 \sinh^{-1} r'/R_o)^2]^{-1/2}$ is due to the propagation of the sawtooth wave, (see Equation (19) of reference (1)).

If we let $r' = R_oP$, we have

$$a_{2n} = \int_{1/\sigma}^{\infty} \frac{e^{-2\alpha R_o \rho}}{1+\frac{x}{2}\,\sinh^{-1}\rho} \quad e^{-i\sigma k_o R_o \rho}\ j_{2n}(\sigma k_o R_o \rho)\ d\rho, \tag{22}$$

where $\frac{1}{\sigma}$ represents the downshaft ratio $\frac{f_o}{f} = \frac{k_o}{k}$. The first two spherical Bessel functions are

$$j_o(z) = \frac{\sin z}{z} \qquad z = \sigma k_o R_o \rho$$

$$j_2(z) = \left(\frac{3}{z^2} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z \tag{23}$$

$$= -\frac{1}{z} \sin z + 3\left(\frac{\sin z}{z^3} - \frac{\cos z}{z^2}\right).$$

It is expected that because the variable of integration is large to start with that the term in parenthesis in j_2 will contribute very little to the integral since each of the terms is oscillatory and cancels each other out. It is expected that $j_{2n}(z)$ can be set approximately to $(-1)^n\ j_o(z)$.

When the saturation parameter x in Equation (22) is non zero, the integral cannot be evaluated in closed form. A series of machine computations was made by Mr. Marvin Goldstein of the New London computer laboratory for the following sets of parameters as shown in Table I.

TABLE I

Parameters Used in Evaluating Equation (22)

x	$\frac{1}{\sigma}$	$\frac{1}{2aR_c}$	k_oR_o
0	5	1	100
.02	10	10	
.04	20	100	
.06	50	1000	
.08			
.10			

For the case $1/\sigma = 10$ and $\frac{1}{2aR_o} = 100$, the following results are shown in Table II.

TABLE II

Values of $a_{2n}{}^2$

x	$a_o{}^2$	$a_2{}^2$	$a_4{}^2$	$a_6{}^2$
0	$.8271\times10^{-2}$	$.8271\times10^{-2}$	$.8271\times10^{-2}$	$.8256\times10^{-2}$
.02	$.8240\times20^{-2}$	$.8241\times10^{-2}$	$.8241\times10^{-2}$	$.8226\times10^{-2}$
.04	$.8150\times10^{-2}$	$.8151\times10^{-2}$	$.8151\times10^{-2}$	$.8136\times10^{-2}$
.06	$.8004\times10^{-2}$	$.8004\times10^{-2}$	$.800[illegible]\times10^{-2}$	$.7990\times10^{-2}$

The values of $|a_o|^2$, $|a_2|^2$, and $|a_4|^2$ agree to 1 part in 10^4 and the value of $|a_6|^2$ agrees with the others to 15 parts in 10^4. However, the series for $|a_o|^2$ was converging so slowly that the agreement could be closer. As a very good approximation (and one which was expected) all values of $|a_{2r}|^2$ can be expected to be the same as $|a_o|^2$. We now return to Equation (8). The fact that a_1, a_3, ... a_{2n+1} was not calculated is not important since the b_{2n+1} is identically 0. We can write Equation (8) as

$$P = \frac{Ke^{-ikr}}{kr} \quad a_o \sum_n (2n+1)\ i^n\ (-1)^n\ b_n P_n(\cos\theta) \tag{23}$$

since $a_{2n} = (-1)^n\ a_o$. This becomes

$$P = \frac{Ke^{-ikr}}{kr} \quad 2a_o \sum_n \int_0^\pi |F(\theta')|^2\ \frac{(2n+1)}{2}\ P_n(\cos\theta)\ P_n(\cos') \sin\theta' d\theta', \tag{24}$$

The terms $\frac{(2n+1)}{2}$ $P_n(\cos\theta)$ $P_n(\cos\theta')$ are normalized signfunctions and by the closure property

$$\sum_{n=0}^{\infty} u_n(x)\ u_n^*(x') = \delta(x-x') \tag{25}$$

Thus

$$P = \frac{Ke^{-ikr}}{kr} \quad 2a_o \int_0^\pi |F(\theta')|^2\ \delta(\cos\theta' - \cos\theta') \sin\theta' d\theta' \tag{26}$$

$$= \frac{Ke^{-ikr}}{kr} \quad 2a_o |F(\theta)|^2 .$$

Strictly speaking, the beam pattern in the far field is a very weak convolution of the two primary beam patterns but to a very good approximation as shown in Equation (27), the beam patterns in the far field is a product of the two primary beam patterns.

If one considers a value of the saturation parameter x to be zero, Equation (22) can be integrated in closed form. Let us write the pressure distribution as

$$P = \frac{\beta P_o^2 R_o^2 k^2}{\rho c^2} e^{iwt} \frac{e^{-ikr}}{kr} |F(\theta)|^2 kR_o \int_{1/\sigma}^{\infty} e^{-2aR_o\rho - i k_oR_o\rho x}$$

$$j_o(\sigma k_o R_o \rho)\, d\rho,$$

and consider

$$\int_{1/\sigma}^{\infty} e^{-2aR_o\rho - i\sigma R_o\rho} \quad j_o(\sigma k_o R_o \rho)\, d\rho$$

$$= \int_{1/\sigma}^{\infty} \frac{e^{-2aR_o\rho}}{2i\sigma k_o R_o \rho}\, d\rho - \int_{1/\sigma}^{\infty} e^{\frac{-2aR_o\rho - 2i\sigma k_o R_o\rho}{2i\sigma R_o \rho}}\, d\rho,$$

$$= \frac{1}{2i\sigma k_o R_o} E_1\left(\frac{2aR_o}{\sigma}\right) - \frac{1}{2i\sigma k_o R_o} E_1\left(\frac{2aR_o}{\sigma} + 2ik_oR_o\right).$$

For imaginary $k_oR_o = 200$, $z\, e^{z}\, E_1(z) \cong 1.$

Thus $E_1\left(\frac{2aR_o}{\sigma} + 2ik_oR_o\right) \cong \frac{\exp\left(-\frac{2aR_o}{\sigma} - 2ik_oR_o\right)}{\frac{2aR_o}{\sigma} - 2ik_oR_o} \rightarrow o$

Hence

$$P = \frac{\beta P_o^{\,2} R_o^{\,2} k^2}{\rho c^2} \; e^{iwt} \; \frac{e^{-ikr}}{kr} \; |F(\theta)|^2 \; kR_o \; \frac{1}{2i\sigma\, k_o R_o} \; \frac{E_1(2aR_o)}{\sigma}$$

Taking the value of the parametric efficiency $\left|\frac{rP}{R_o P_o}\right|$, we have

$$\left|\frac{rP}{R_o P_o}\right| = \frac{x}{2} \; \frac{k}{k_o} \; E_1(2aR_o) \; |F(\theta)|^2,$$

which is the same value as Equation 35 of Reference 1. Equation (35) of Reference 1 is the on axis value of the far field parametric efficiency.

CONCLUSIONS

This memorandum has shown that the secondary beam pattern of the far field interaction zone is a very weak convolution of the two primary beam patterns which to a very good approximation is a product of the two primary beam patterns. The pressure amplitude of the secondary wave was calculated using all the off axis contributions and agrees with that calculation made on the basis of the on axis contributions in References 1 and 3. This gives great credence to the approximations made in these two references so that all the quantities and curves appearing in these two references can be considered to have much validity.

GLOSSARY OF TERMS

a_o	Coefficient of radial integral
a_o	Coefficient of angular integral
c_o	Speed of Sound in Water
f_o	Average primary frequency
f	Secondary frequency
k_o	Average primary wave number
k	Secondary primary wave number
K	$\frac{\beta P_o^2 k^2}{2\rho c^2} e^{i\omega t}$
P_o	Primary peak pressure amplitude
P	Secondary peak pressure amplitude
R_o	Primary wave collimation distance
R_2	$\frac{k_o R_o}{k}$
r	Field Point
r'	Integration point
$S(r)$	Source Function
$T(r)$	Amplitude tapes function
α	Absorption coefficient
β	Non linear number for water $= 1 + \frac{\beta}{2A} = 4$
θ	Angle off main beam axis
θ', ϕ'	Angular integration variables

τ	$\lvert r-r' \rvert$ = Distance between field point and integration point
ρ	Density of water
σ	Reciprocal of downshift ratio = $\frac{f}{f_o} = \frac{k}{k_o}$
x	Saturation number = $\frac{\beta P_o k_o R_o}{\rho c^2}$
ω	Angular frequency of secondary = $2\pi f$

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