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GEOMETRICAL CHARACTERISTICS OF NOSES AND TAILS FOR PARALLEL MIDDLE BODIES

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOTATION

a_i	Polynomial coefficients
C	Constant of integration
C_p	Prismatic coefficient of nose or tail
D	Diameter or thickness of body
D[x]	Polynomial associated with d
d	General adjustable parameter at end
f	"Quadratic" polynomial
g	"Square root" polynomial
K_0	Polynomial associated with k_0
$\tilde{K}_1[x]$	Polynomial associated with $\tilde{k}_1$
k	Curvature
k_0	Curvature at $x = 0$
$\tilde{k}_1$	Rate of change of curvature at $x = 1$
L	Length of nose or tail
ℓ	Arc length
n	Degree of polynomial
P_n	Polynomial
Q[x]	Polynomial for restraining conditions
R[x]	Polynomial associated with r
r	End radius
S[x]	Polynomial associated with s
s	End slope
λ	Axial coordinate
x	Normalized axial coordinate
Y	Radius or offset
y	Normalized radius or offset
z	General function of offset of least-squares fitted body
z_1	General function of offset of body to be fitted
α	Unspecified constant
α_i	Adjustable conditions
β	Unspecified constant

β_j	Conditions of restraint
γ	Unspecified constant
I	Single differentiation with respect to x
II	Double differentiation with respect to x
III	Triple differentiation with respect to x
IV	Quadruple differentiation with respect to x

ABSTRACT

Two-parameter geometric systems are developed for noses and tails attached to parallel middle bodies in terms of independent polynomials. A new parameter is introduced--the rate of change of curvature with arc length at the juncture with the parallel middle body. "Quadratic" polynomials are considered for bodies of revolution and "square root" polynomials for symmetric two-dimensional bodies. Permissible ranges of the two adjustable parameters are obtained for selected geometrical constraints such as the presence of inflection points.

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INTRODUCTION

The geometrical characteristics of streamlined bodies of revolution and symmetrical two-dimensional bodies have been developed in a previous report.[1] There, a body is divided into a forebody and an afterbody at the maximum section, and the curvature at the junction is one of two adjustable parameters controlling the shape. These forebodies or afterbodies may also be attached to parallel middle bodies. In this case, it is considered more desirable in hydrodynamic applications if the junction curvature is made zero to match that of the parallel middle body in order to avoid discontinuities in curvature. The use of a junction curvature of zero, however, eliminates an adjustable parameter and leaves only one adjustable parameter for the polynomials involved.

It is the purpose of this report to return to a two-parameter system for forebodies or noses and afterbodies or tails to be attached to parallel middle bodies in order to have a more extensive group of shapes. A new adjustable parameter has been introduced, namely, the rate of change of

[1]Granville, P. S., "Geometrical Characteristics of Streamlined Shapes," NSRDC Report 2962 (Mar 1969); Journal of Ship Research, Vol. 13, No. 4 (Dec 1969).

curvature at the junction with the parallel middle body. The requirement of zero curvature here becomes an additional restraint.

The method developed in Reference 1 is followed closely. The so-called "quadratic" and "square root" polynomials are used in a system of independent polynomials controlled by the two adjustable parameters. The analytical description of the polynomials is obtained by a factorial method where feasible. Rounded, pointed, and cusped ends are considered. Permissible ranges of the adjustable parameters are considered in terms of geometric criteria such as the presence of inflection points. Regions of well-behaved shapes are delimited by boundary curves obtained by an envelope analysis. The least squares fit of graphically or analytically delineated shapes is also considered. The fullness of the shapes is obtained in terms of a prismatic coefficient.

GENERAL ANALYSIS

The shapes of families of noses or tails of bodies of revolution and of two-dimensional symmetrical bodies may be stated functionally as

$$Y = f\,[X;\ \alpha_i,\ \beta_j] \qquad \begin{matrix} i = 1, 2, \ldots \\ j = 1, 2, \ldots \end{matrix} \tag{1}$$

where Y is the radius of the body of revolution or the offset of the two dimensional body,

X is the axial distance of the body measured from the end of the nose or tail,

α_i are the parameters to be varied which specify the family, and

β_j are the boundary conditions or restraints.

The analytical analysis is more useful in a normalized coordinate system $[x, y]$: $y = 0$ at $x = 0$, and $y = 1$ at $x = 1$.

If D is the diameter or maximum thickness of the parallel middle body, and L is the length of the nose or tail, then the normalized coordinates become

$$y = \frac{2Y}{D} \tag{2}$$

and

$$x = \frac{X}{L} \tag{3}$$

as shown in Figure 1.

To achieve "hydrodynamic continuity" as contrasted with mathematical continuity, it is considered necessary that the position, slope, and curvature match at the junction of the nose or tail with the parallel middle body. In this case the slope and curvature at the junction are zero.

In normalized coordinates the contour is given by

$$y = f\,[x;\ \alpha_i,\ \beta_j] \qquad \begin{matrix} i = 1, 2, \ldots \\ j = 1, 2, \ldots \end{matrix} \tag{4}$$

where α_i and β_j are now defined in normalized coordinates. In this study two adjustable parameters α_1 and α_2 are to be considered for simplicity of analysis.

If a functional form like that of a polynomial is selected as

$$y = \sum_{n=0}^{n=n} a_n\, x^n = P_n[x] \tag{5}$$

a resolution into linearly independent polynomials may be obtained like that of linearly independent vectors multiplied by scalars, such as

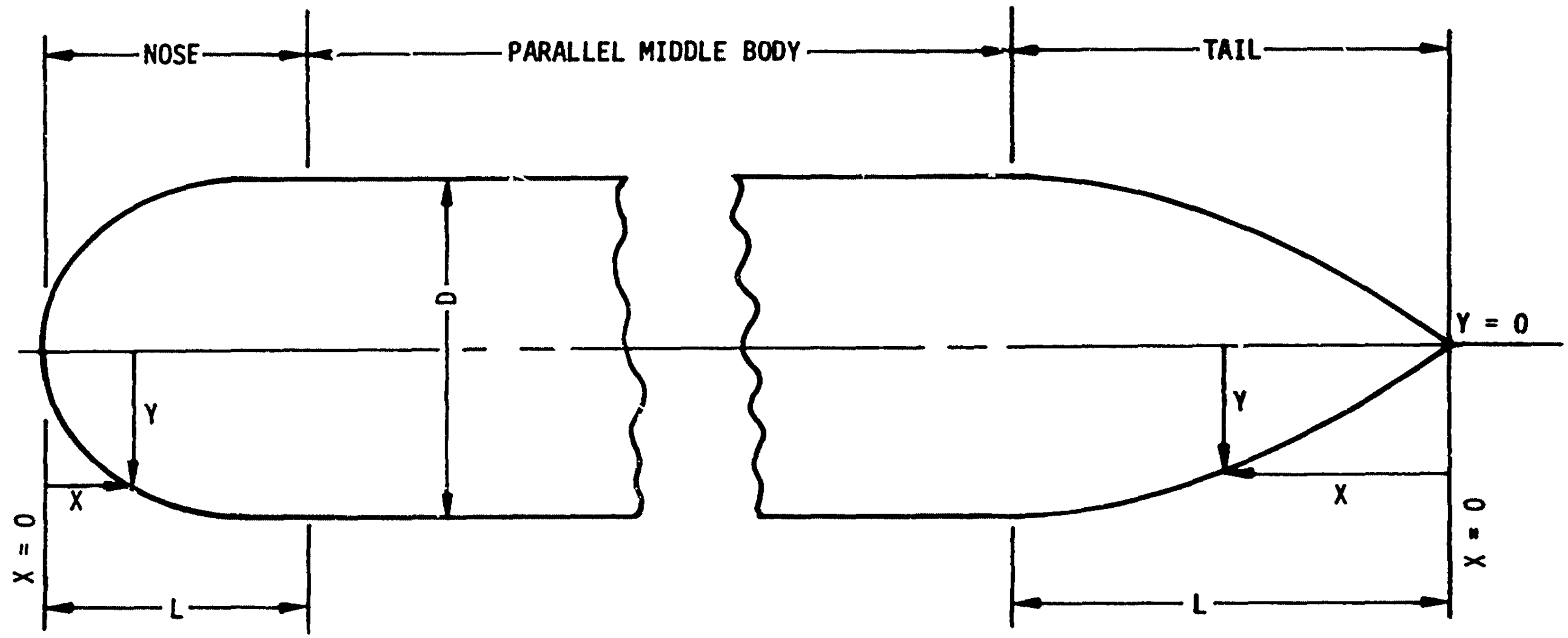
$$y = \sum_i f[\alpha_i]\ P_{n,i}\ [x] + \sum_j f\ [\beta_j]\ P_{n,j}\ [x] \tag{6}$$

or since the β_j are constant

$$y = \sum_i f\ [\alpha_i]\ P_{n,i}\ [x] + Q[x] \tag{7}$$

where

$$Q[x] = \sum_j f\ [\beta_j]\ P_{n,i}\ [x] \tag{8}$$



NORMALIZED COORDINATES $x = X/L \quad 0 \leq x \leq 1$

$y = 2Y/D \quad 0 \leq x \leq 1$

Figure 1 - Basic Geometry

This permits the effect of the controllable parameters α_i to be obtained independently of each other.

The independent polynomials may be determined by substituting, one by one, conditions α_i and β_j into the general polynomial, by evaluating the polynomial coefficients by a solution of the resulting simultaneous algebraic equations, and then by a gathering of terms corresponding to each α_i. Another method is to use the factorial properties of polynomials in considering conditions α_i and β_j, which is illustrated in the specific cases to follow.

Not all combinations of α_i give desirable shapes. It will be of value to analyze the possible limitations in terms of simple criteria:

1. Zero condition--$y = 0$ for $0 \leq x \leq 1$; negative values of y would be meaningless.

2. Unity condition--$y = 1$ for $0 \leq x \leq 1$; bulges above $y = 1$ are undesirable.

3. Maximum or minimum condition--$dy/dx = 0$ for $0 \leq x \leq 1$; no other maximum or minimum is to be permitted than at $x = 1$.

4. Inflection-point condition--$d^2y/dx^2 = 0$ for $0 \leq x \leq 1$; inflection points are undesirable on noses.

For example, the condition for zero values of y

$$y\,[x;\,\alpha_1,\,\alpha_2] = 0 \qquad 0 \leq x \leq 1 \tag{9}$$

may be studied as follows.

If α_1 and α_2 are now considered as variables, and x is considered as an adjustable parameter, a line may be defined for each x-value. An envelope to these lines may be developed which represents the boundary of regions for values of α_1 and α_2 with values of $y = 0$ for different values of x. The envelope condition is given by

$$\frac{\partial}{\partial x}\, y\,[x;\,\alpha_1,\,\alpha_2] = 0 \tag{10}$$

From Equations (9) and (10)

$$\alpha_1 = f_1\,[x] \tag{11}$$

$$\alpha_2 = f_2\,[x] \tag{12}$$

A plot of α_1 against α_2 for the range of values of x, $0 \leq x \leq 1$ gives the envelope curve. Other conditions may be handled in a similar way.

For the shapes to be considered, a common adjustable parameter to be used is the rate of change of curvature with arc length at the junction of the nose or tail with the parallel middle body at x = 1, given by

$$\tilde{k}_1 = \left(\frac{dk}{d\ell}\right)_{x\,=\,1} \tag{13}$$

where k is curvature, and ℓ is arc length.

In general

$$k = \left(\frac{d^2y}{dx^2}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2} \tag{14}$$

and

$$\frac{dk}{d\ell} = \left(\frac{d^3y}{dx^3}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-2} - 3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3} \tag{15}$$

For

$$x = 1,\ \frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$$

and then

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x\,=\,1} \tag{16}$$

"QUADRATIC" POLYNOMIAL REPRESENTATION

GENERAL

The functional relation

$$y^2 = \sum_{n=0}^{n=N} a_n x^n \tag{17}$$

is called the "quadratic" polynomial for want of a better name. It is very suitable for describing bodies of revolution for which volume is an important consideration, since it represents the axial distribution of the cross-sectional area. It has the additional advantage of providing a means of accommodating the analytical description of bodies with rounded ends, something the ordinary polynomial cannot do because of the requirement of infinite slope at x = 0. In addition to bodies with rounded ends, the "quadratic" polynomial may also be applied to bodies with pointed ends and cusped ends.

Although any number of adjustable parameters α_i may be used, the analysis here is to be limited to two. This is sufficiently general for describing a wide variety of geometrical shapes.

ROUNDED ENDS

The adjustable parameters α_i to be considered at each end of the curve are

α_1: r = radius of curvature at x = 0

$$r = \frac{1}{\left(\dfrac{d^2x}{dy^2}\right)_{x=0}} \tag{18}$$

α_2: $\tilde{k}_1$ = rate of change of curvature at x = 1

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x = 1} \tag{19}$$

The boundary conditions β_j are

$$\begin{aligned} &\beta_1: \quad x = 0,\ y = 0 \\ &\beta_2: \quad x = 1,\ y = 1 \\ &\beta_3: \quad x = 1,\ \frac{dy}{dx} = 0 \\ &\beta_4: \quad x = 1,\ \frac{d^2y}{dx^2} = 0 \end{aligned} \tag{20}$$

Since there are six conditions in all, n = 5.

The α_i and β_j are substituted into the polynomial. Differentiating Equation (17) successively with respect to y gives

$$2y = (a_1 + 2\ a_2x + 3\ a_3x^2 + 4\ a_4x^3 + 5\ a_5x^4)\ \frac{dx}{dy} \tag{21}$$

and

$$\begin{aligned} 2 = {} & (a_1 + 2\ a_2x + 3\ a_3x^2 + 4\ a_4x^3 + 5\ a_5x^4)\ \frac{d^2x}{dy^2} \\ & + (2\ a_2 + 6\ a_3x + 12\ a_4x^2 + 20\ a_5x^3)\left(\frac{dx}{dy}\right)^2 \end{aligned} \tag{22}$$

Since $a_1 \neq 0$, dx/dy = 0 at x = 0. This automatically provides a rounded end. Then

$$\alpha_1: \quad a_1 = 2r$$

The other substitutions yield

$$\alpha_2: \quad 6\, a_3 + 24\, a_4 + 60\, a_5 = 2\, \tilde{k}_1$$

$$\beta_1: \quad a_0 = 0$$

$$\beta_2: \quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1 \tag{23}$$

$$\beta_3: \quad a_1 + 2\, a_2 + 3\, a_3 + 4\, a_4 + 5\, a_5 = 0$$

$$\beta_4: \quad 2\, a_2 + 6\, a_3 + 12\, a_4 + 20\, a_5 = 0$$

The solution of Equations (23) by determinants shows that the a's are linear functions of r and $\tilde{k}_1$. Hence y^2 is also a linear function of r and $\tilde{k}_1$ and may be written as

$$y^2 = r\, R[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \tag{24}$$

where R[x], $\tilde{K}_1[x]$, and Q[x] are also polynomials of the fifth degree in x. It is possible to determine R[x], $\tilde{K}_1[x]$, and Q[x] by first solving for the a's from the simultaneous equations and then regrouping terms applicable to R[x], $\tilde{K}_1[x]$, and Q[x]. Another method is to use polynomials as follows.

It is evident that the relations for α_i and β_j correspond to

$$\alpha_1: \quad \frac{d}{dx}\, y^2[0] = 2r$$

$$\alpha_2: \quad \frac{d^3}{dx^3}\, y^2[1] = 2\, \tilde{k}_1$$

$$\beta_1: \quad y^2[0] = 0$$

$$\beta_2: \quad y^2[1] = 1$$

$$\beta_3: \ \frac{d}{dx} y^2[1] = 0$$

$$\beta_4: \ \frac{d^2}{dx^2} y^2[1] = 0 \tag{25}$$

Since the foregoing apply identically to r and $\tilde{k}_1$ because of linearity, it is further evident that

$$\alpha_1: \ R^I[0] = 2, \ \tilde{K}_1{}^I[0] = Q^I[0] = 0$$

$$\alpha_2: \ \tilde{K}_1{}^{III}[1] = 2, \ R^{III}[1] = Q^{III}[1] = 0$$

$$\beta_1: \ R[0] = \tilde{K}_1[0] = Q[0] = 0$$

$$\beta_2: \ Q[1] = 1, \ R[1] = \tilde{K}_1[1] = 0 \tag{26}$$

$$\beta_3: \ R^I[1] = \tilde{K}_1{}^I[1] = Q^I[1] = 0$$

$$\beta_4: \ R^{II}[1] = \tilde{K}_1{}^{II}[1] = Q^{II}[1] = 0$$

where $R^I = dR/dx$, $R^{II} = d^2R/dx^2$, etc.

Evaluation of R[x]

Since $R[0] = R[1] = R^I[1] = R^{II}[1] = R^{III}[1] = 0$, and R[x] is a polynomial of the fifth degree, R[x] may be written as

$$R[x] = \alpha \ x \ (x - 1)^4 \tag{27}$$

Since $R^I[0] = 2$, $\alpha = 2$.
Then

$$R[x] = 2 \ x \ (x - 1)^4 \tag{28}$$

Evaluation of $\tilde{K}_1[x]$

Since $\tilde{K}_1[0] = \tilde{K}_1{}^{I}[0] = \tilde{K}_1[1] = \tilde{K}_1{}^{I}[1] = \tilde{K}_1{}^{II} = 0$, $\tilde{K}_1[x]$ may be written factorially as

$$\tilde{K}_1[x] = \beta\, x^2\, (x - 1)^3 \tag{29}$$

Since $\tilde{K}_1{}^{III}[1] = 2$, $\beta = 1/3$, then

$$\tilde{K}_1[x] = \frac{1}{3} x^2\, (x - 1)^3 \tag{30}$$

Evaluation of Q[x]

Since $Q^{I}[0] = Q^{I}[1] = Q^{II}[1] = Q^{III}[1] = 0$, $Q^{I}[x]$ may be written factorially as

$$Q^{I}[x] = \gamma\, x\, (x - 1)^3 \tag{31}$$

Then integrating produces

$$Q[x] = \gamma \left(\frac{x^5}{5} - \frac{3x^4}{4} + x^3 - \frac{x^2}{2} \right) + C \tag{32}$$

With $Q[0] = 0$ and $Q[1] = 1$, $C = 0$, and $\gamma = 20$. Then

$$Q[x] = - x^2\, (4\, x^3 - 15\, x^2 + 20\, x - 10) = 1 - (x - 1)^4\, (4\, x + 1) \tag{33}$$

For Rounded Ends in Summary

$$y^2 = r\, R[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \tag{24}$$

with

$$R[x] = 2\, x\, (x - 1)^4 \tag{28}$$

$$\tilde{k}_1[x] = \frac{1}{3} x^2 \ (x - 1)^3 \qquad (30)$$

$$Q[x] = 1 - (x - 1)^4 \ (4 x + 1) \qquad (33)$$

PERMISSIBLE RANGES OF PARAMETERS r AND $\tilde{k}_1$

Zero Condition

$$y^2 = f[x; r, \tilde{k}_1] = 0 \qquad 0 \leq x \leq 1 \qquad (34)$$

The envelope in r and $\tilde{k}_1$ with x as the variable parameter is given by

$$f^I = \frac{\partial f}{\partial x} = 0 \qquad (35)$$

The two envelope conditions, Equations (34) and (35), provide two simultaneous equations in r and $\tilde{k}_1$ which are solved by the Cramer rule to give r[x] and $\tilde{k}_1[x]$:

$$r = \frac{x^2 \ (3 x^2 - 10 x + 10)}{2 \ (x - 1)^4} \qquad (36)$$

$$\tilde{k}_1 = \frac{3 \ (x^3 - 5 x^2 + 10 x - 10)}{(x - 1)^3} \qquad (37)$$

The envelope curve is shown in Figure 2. Since the tangent lines representing values of r and $\tilde{k}_1$ for y = 0 are outside the envelope curve, the inside region contains values of r and $\tilde{k}_1$ which do not have values of y = 0 (except at x = 0).

Unity Condition

The unity condition is that

$$y^2 = f[x; r, \tilde{k}_1] = 1 \qquad 0 \leq x \leq 1 \qquad (38)$$

The envelope in r and $\tilde{k}_1$ with x as the variable parameter is given by

$$\frac{\partial}{\partial x} (f - 1) = 0 \tag{39}$$

The two envelope conditions, Equations (38) and (39), provide two simultaneous equations in r and $\tilde{k}_1$, which are solved by the Cramer rule to give

$$r = \frac{3\ x + 2}{2\ x} \tag{40}$$

$$\tilde{k}_1 = \frac{3\ (x - 1)^2}{x^2} \tag{41}$$

The envelope curve is shown in Figure 2. Desirable values of r and $\tilde{k}_1$, that is, without bulges, are on the "inside curved" side of the envelope curve.

Maximum or Minimum Condition

The maximum or minimum condition is given by

$$\frac{dy}{dx} = f^{I} = 0 \tag{42}$$

The envelope curve in r and $\tilde{k}_1$ with x as the variable parameter is given by

$$f^{II} = 0 \tag{43}$$

The envelope curve is shown in Figure 2. A better understanding of the envelope curve is developed in Figure 3. Each point on the envelope curve represents a tangent, giving the locus of values of r and $\tilde{k}_1$ which provide a maximum or minimum at each value of x other than the maximum at x = 1 which prevails at all times. Two such loci are represented. Their point of intersection provides a value of r and $\tilde{k}_1$, representing maxima or minima at two values of x. Evidently from any point in the region outside the envelope curve, two tangents may be drawn to the envelope curve. Thus

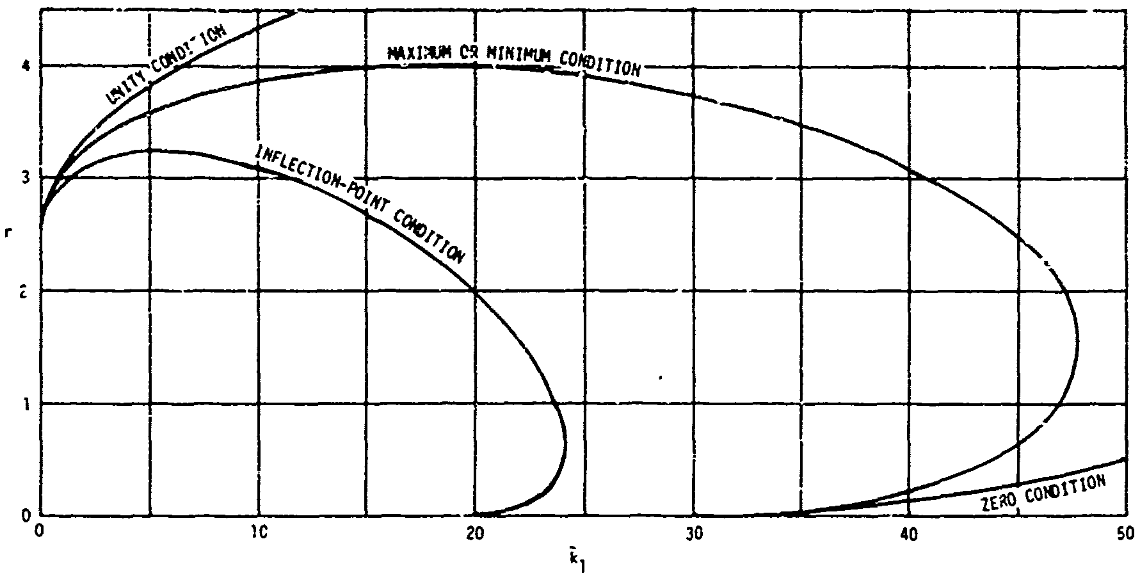


Figure 2 - "Quadratic" Polynomial: Rounded End, Permissible Range of Parameters r and $\tilde{k}_1$

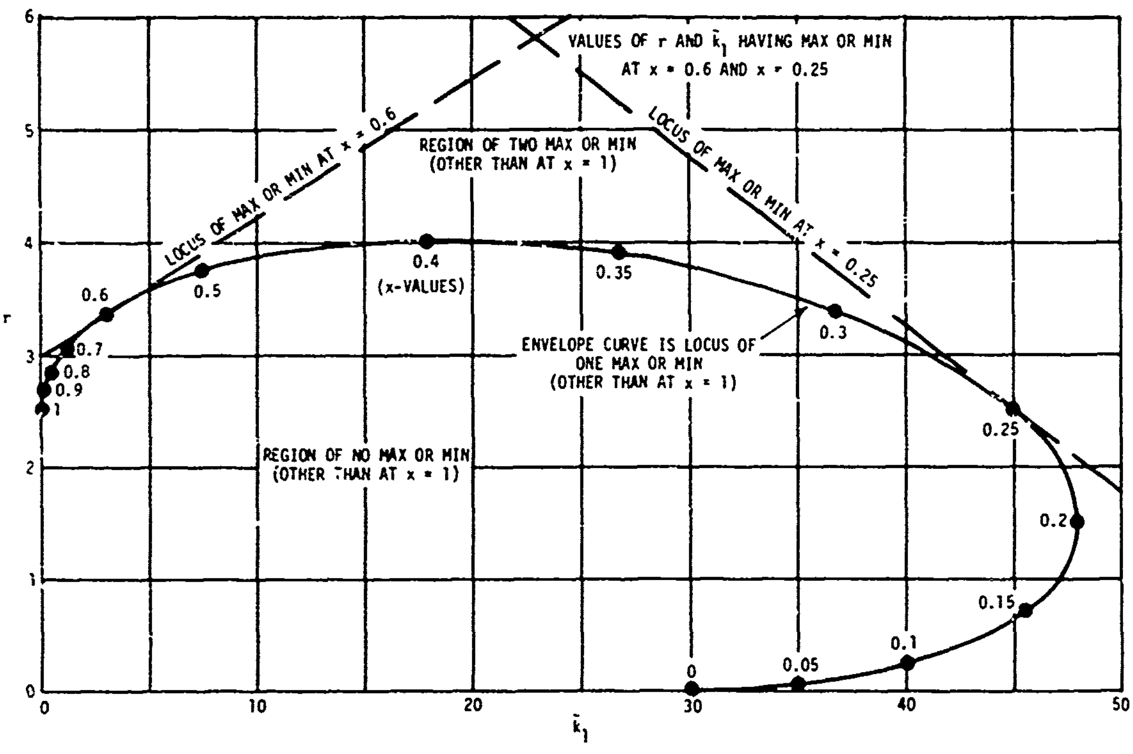


Figure 3 - "Quadratic" Polynomial: Rounded End, Delineation of Regions by Envelope Curve for Maximum or Minimum Condition

the region outside the envelope curve represents values of r and $\tilde{k}_1$, giving two maxima or minima. The region inside the envelope curve provides no maximum or minimum. Finally there is only one maximum or minimum specified by the envelope curve.

The two envelope conditions, Equations (42) and (43), provide two simultaneous equations in r and $\tilde{k}_1$, which are solved by the Cramer rule to give $r[x]$ and $\tilde{k}_1[x]$ as

$$r = \frac{15\,x^2}{10\,x^2 - 5\,x + 1} \tag{44}$$

$$\tilde{k}_1 = \frac{30\,(x-1)^2}{10\,x^2 - 5\,x + 1} \tag{45}$$

Inflection-Point Condition

The inflection-point condition is given by

$$\frac{d^2y}{dx^2} = 0$$

For $y^2 = f[x]$

$$2\,f\,f^{II} - f^{I^2} = 0 \tag{46}$$

and the envelope condition

$$f^{III} = 0 \tag{47}$$

The two conditions provide two simultaneous equations in r and $\tilde{k}_1$ in terms of x. Since the boundary condition leads to a quadratic relation, the Cramer rule does not apply. For specified values of x the two simultaneous equations may be numerically solved by direct substitution of one equation into the other. The results are shown in Figures 2 and 4. The results for the rounded end are summarized in Table 1.

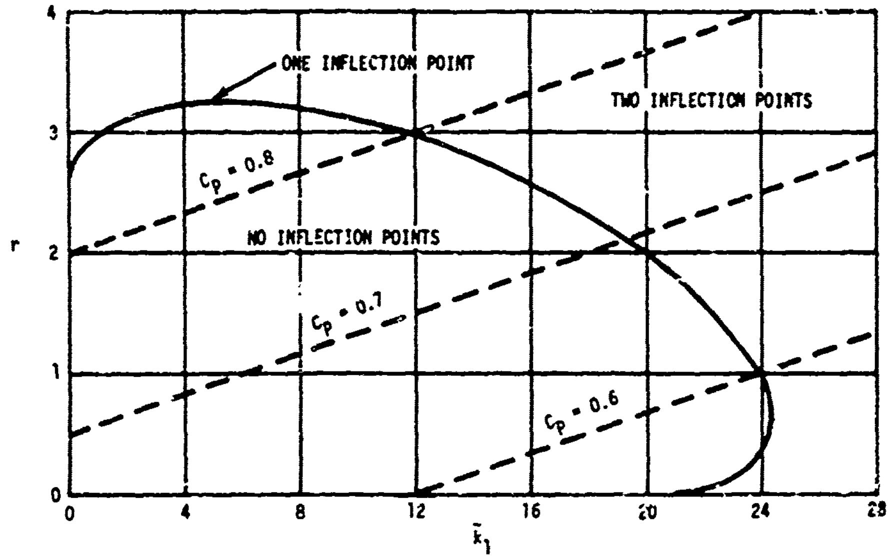


Figure 4 - "Quadratic" Polynomial: Rounded End, Inflection-Point Condition

TABLE 1 - SUMMARY OF "QUADRATIC" POLYNOMIALS FOR ROUNDED END

Polynomial Equation:

$$y^2 = r\,R[x] + \tilde{k}_1\,\tilde{K}_1[x] + Q[x] \qquad 0 \le x \le 1$$

where

$$R[x] = 2\,x\,(x - 1)^4$$

$$\tilde{K}_1[x] = \frac{1}{3}\,x^2\,(x - 1)^3$$

$$Q[x] = 1 - (x - 1)^4\,(4\,x + 1)$$

Envelope Equations:

Zero Condition:

$$r = \frac{x^2\,(3\,x^2 - 10\,x + 10)}{2\,(x - 1)^4} \quad ; \quad \tilde{k}_1 = \frac{3\,(x^3 - 5\,x^2 + 10\,x + 10)}{(x - 1)^3}$$

Unity Condition:

$$r = \frac{3\,x + 2}{2\,x} \quad ; \quad \tilde{k}_1 = \frac{3\,(x - 1)^2}{x^2}$$

Maximum or Minimum Condition:

$$r = \frac{15\,x^2}{10\,x^2 - 5\,x + 1} \quad ; \quad \tilde{k}_1 = \frac{30\,(x - 1)^2}{10\,x^2 - 5\,x + 1}$$

Inflection-Point Condition: Numerical calculation

POINTED ENDS

The adjustable parameters α_i are

$$\alpha_1: \quad s = \text{slope at } x = 0$$

$$s = \left(\frac{dy}{dx}\right)_{x = 0} \tag{48}$$

$$\alpha_2: \quad \tilde{k}_1 = \text{rate of change of curvature at } x = 1$$

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x = 1} \tag{49}$$

The boundary conditions β_j are the same as those given in Equation (20)

$$\beta_1: \quad x = 0,\ y = 0$$

$$\beta_2: \quad x = 1,\ y = 1$$

$$\beta_3: \quad x = 1,\ \frac{dy}{dx} = 0 \tag{50}$$

$$\beta_4: \quad x = 1,\ \frac{d^2y}{dx^2} = 0$$

Since the "quadratic" polynomial as previously used gives infinite slope at x = 0, an additional condition, indicated as follows, is necessary to give controlled slopes at x = 0. Hence, the degree of the polynomial becomes six.

For α_1

$$\frac{dy}{dx} = \frac{a_1 + 2\ a_2\ x + \ldots + n\ a_n\ x^{n-1}}{2\ y} \tag{51}$$

Since $y = 0$ at $x = 0$, $dy/dx \to \infty$, unless $a_1 = 0$. For $a_1 = 0$, dy/dx is indeterminate at $x = 0$. Then by the L'Hôpital rule

$$\underset{x \to \infty}{L} \frac{dy}{dx} = \left(\frac{dy}{dx}\right)_{x = 0} = s = \frac{a_2}{s} \tag{52}$$

or

$$s^2 = a_2$$

α_1 then requires that

$$\begin{aligned} a_1 &= 0 \\ a_2 &= s^2 \end{aligned} \tag{53}$$

The other substitutions yield

$$\begin{aligned}
&\alpha_2: \quad 6\,a_3 + 24\,a_4 + 60\,a_5 + 120\,a_6 = 2\,\tilde{k}_1 \\
&\beta_1: \quad a_0 = 0 \\
&\beta_2: \quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 \\
&\beta_3: \quad a_1 + 2\,a_2 + 3\,a_3 + 4\,a_4 + 5\,a_5 + 6\,a_6 = 0 \\
&\beta_4: \quad 2\,a_2 + 6\,a_3 + 12\,a_4 + 20\,a_5 + 30\,a_6 = 0
\end{aligned} \tag{54}$$

y^2 is then a linear function of s^2 and $\tilde{k}_1$ or

$$y^2 = s^2\,S[x] + \tilde{k}_1\,\tilde{K}_1[x] + Q[x] \tag{55}$$

The analysis proceeds as before, and the results are shown in Table 2 and Figures 5 and 6.

TABLE 2 - SUMMARY OF "QUADRATIC" POLYNOMIALS FOR POINTED END

Polynomial Equation:

$$y^2 = s^2 \, S[x] + \tilde{k}_1 \, \tilde{K}_1[x] + Q[x] \qquad 0 \leq x \leq 1$$

where

$$S[x] = x^2 \, (x - 1)^4$$

$$\tilde{K}_1[x] = \frac{1}{3} x^3 \, (x - 1)^3$$

$$Q[x] = 1 - (x - 1)^4 \, (10 \, x^2 + 4 \, x + 1)$$

Envelope Equations:

Zero Condition:

$$s^2 = \frac{3 \, x^2 \, (2 \, x^2 - 6 \, x + 5)}{(x - 1)^4} \quad ; \quad \tilde{k}_1 = \frac{6 \, (2 \, x^3 - 9 \, x^2 + 15 \, x - 10)}{(x - 1)^3}$$

Unity Condition:

$$s^2 = \frac{3 \, (2 \, x^2 + 2 \, x + 1)}{x^2} \quad ; \quad \tilde{k}_1 = \frac{6 \, (x - 1)^2 \, (2 \, x + 1)}{x^3}$$

Maximum or Minimum Condition:

$$s^2 = \frac{30 \, x^2}{5 \, x^2 - 4 \, x + 1} \quad ; \quad \tilde{k}_1 = \frac{60 \, (x - 1)^2}{5 \, x^2 - 4 \, x + 1}$$

Inflection-Point Condition: Numerical calculation

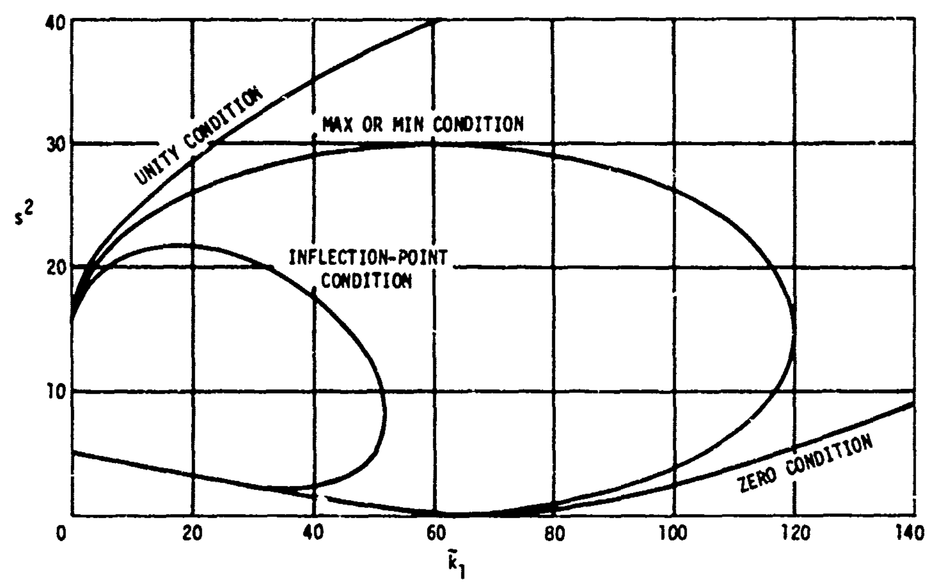


Figure 5 - "Quadratic" Polynomial: Pointed End, Permissible Range of Parameters s^2 and $\tilde{k}_1$

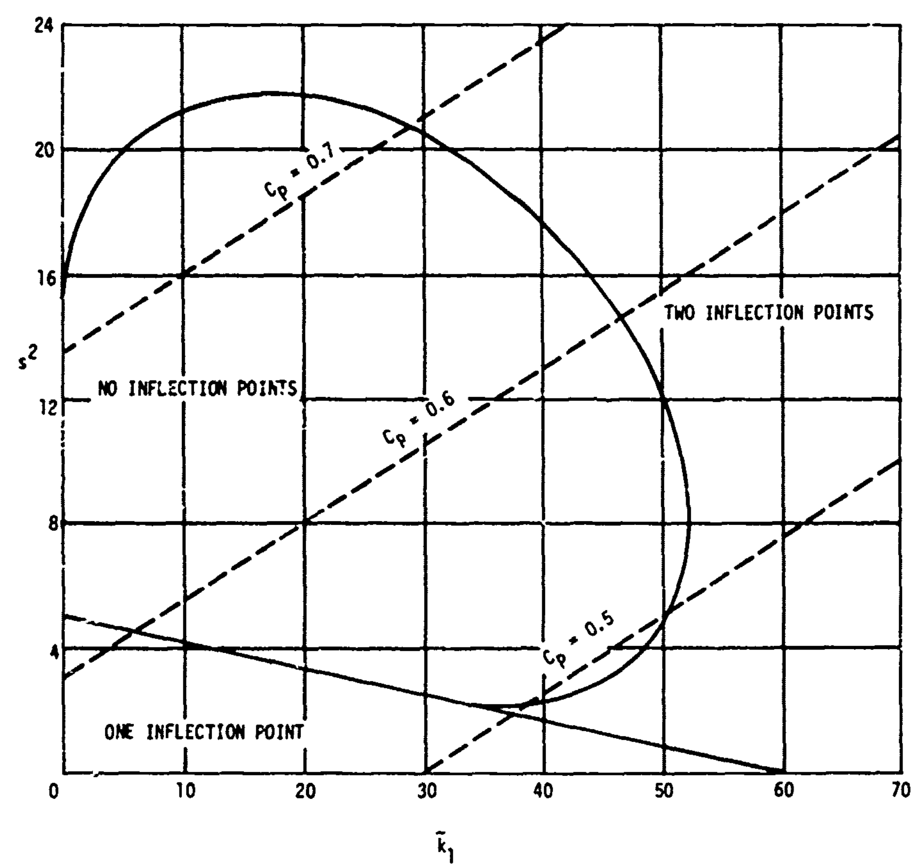


Figure 6 - "Quadratic" Polynomial: Pointed End, Inflection-Point Condition

In analyzing the inflection-point condition, a problem arises for $x = 0$, where an indeterminate condition exists. By the L'Hôpital rule

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = \frac{-24\,s^2 - 2\,\tilde{k}_i + 120}{6\,s} \tag{56}$$

The boundary curve at $x = 0$ is then

$$-12\,s^2 - \tilde{k}_1 + 60 = 0 \tag{57}$$

CUSPED ENDS

The adjustable parameters α_i are

$$\alpha_1:\quad k_o = \text{curvature at } x = 0$$

$$k_o = \left(\frac{d^2y}{dx^2}\right)_{x=0} \tag{58}$$

$$\alpha_2:\quad \tilde{k}_1 = \text{rate of change of curvature at } x = 1$$

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x=1} \tag{59}$$

The boundary conditions β_j are the same as those of Equation (20), except for the additional condition of zero slope at $x = 0$.

$$\beta_1:\quad x = 0,\ y = 0$$

$$\beta_2:\quad x = 0,\ \frac{dy}{dx} = 0$$

$$\beta_3:\quad x = 1,\ y = 1$$

$$\beta_4: \quad x = 1, \frac{dy}{dx} = 0$$

$$\beta_5: \quad x = 1, \frac{d^2y}{dx^2} = 0 \tag{60}$$

As will be shown the cusped end requires two additional conditions which makes n = 8.

For

$$\beta_1: a_0 = 0$$

For

$$\beta_2: \quad \frac{dy}{dx} = \frac{a_1 + 2\, a_2\, x + \ldots + n\, a_n\, x^{n-1}}{2\,(a_1\, x + \ldots + a_n\, x^n)^{1/2}} \tag{61}$$

For

$$\frac{dy}{dx} = 0 \text{ at } x = 0,\ a_1 = 0$$

Then

$$\frac{dy}{dx} = \frac{2\, a_2 + 3\, a_3\, x + \ldots + n\, a_n^{n-2}}{2\,(a_2 + a_3\, x + \ldots + a_n\, x^{n-2})^{1/2}} \tag{62}$$

$$\left(\frac{dy}{dx}\right)_{x=0} = a_2^{1/2} \tag{63}$$

For

$$\left(\frac{dy}{dx}\right)_{x=0} = 0\ ,\ a_2 = 0 \tag{64}$$

For

$$\frac{d^2y}{dx^2} = \frac{6\ a_3\ x + \ldots + n(n-1)\ a_n^{\ n-2} - 2\left(\frac{dy}{dx}\right)^2}{2\ y} \tag{65}$$

Then

$$\frac{d^2y}{dx^2} = \frac{2[6a_3+\ldots n(n-1)a_nx^{n-3}](a_3+\ldots+a_nx^{n-3}) - \left(3a_3+\ldots+na_nx^{n-2}\right)^2}{4\left(a_3x^{1/3}+\ldots+a_nx^{n-8/3}\right)^{3/2}} \tag{66}$$

Equation (66) gives $d^2y/dx^2 \to \infty$ at $x = 0$.
To prevent this, let $a_3 = 0$.
Now

$$\frac{d^2y}{dx^2} = \frac{2[12a_4+\ldots+n(n-1)a_nx^{n-4}]\ (a_4+\ldots+a_nx^{n-4}) - \left(4a_4+\ldots+na_nx^{n-3}\right)^2}{4\left(a_4+\ldots+a_nx^{n-4}\right)^{3/2}} \tag{67}$$

At $x = 0$

$$\frac{d^2y}{dx^2} = 2\ a_4^{\ 1/2} \tag{68}$$

$$4\ a_4 = k_o^{\ 2} \tag{69}$$

The other substitutions yield

$$\alpha_2:\ \ 6\ a_3 + \ldots + n\ (n-1)\ (n-2)\ a_n = 2\ \tilde{k}_1$$

$$\beta_3:\ \ a_4 + \ldots + a_n = 1$$

$$\beta_4: \quad 4\, a_4 + \ldots + n\, a_n = 0$$

$$\beta_5: \quad 12\, a_4 + \ldots + n\,(n - 1)\, a_n = 0 \tag{70}$$

y^2 is then a linear function of $k_o^{\,2}$ and $\tilde{k}_1$

$$y^2 = k_o^{\,2}\, K_o[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \tag{71}$$

The analysis proceeds as before, and the results are shown in Table 3 and Figures 7 and 8.

"SQUARE ROOT" POLYNOMIAL REPRESENTATION

GENERAL

The functional relation

$$y = a_{1/2}\, x^{1/2} \sum_{n=0}^{n=n} a_n\, x^n \tag{72}$$

is to be called the "square root" polynomial for want of a better name. It is suitable for describing two-dimensional shapes with rounded ends, since the square-root term gives infinite slope at x = 0. Of course, without the square-root term an ordinary polynomial remains.

The same analysis procedure used for the quadratic polynomial is to be applied where possible to the square-root polynomial for the same cases: rounded ends, pointed ends, and cusped ends.

ROUNDED ENDS

The adjustable parameters α_i are the same as those for the quadratic polynomial

α_1: r = radius of curvature at x = 0

$$r = \frac{1}{\left(\dfrac{d^2x}{dy^2}\right)_{x\,=\,0}} \tag{73}$$

TABLE 3 - SUMMARY OF "QUADRATIC" POLYNOMIALS FOR CUSPED END

Polynomial Equation:

$$y^2 = k_o^2\, K_o[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \qquad 0 \le x \le 1$$

where

$$K_o[x] = \frac{1}{4} x^4 (x - 1)^4$$

$$\tilde{K}_1[x] = \frac{1}{3} x^5 (x - 1)^3$$

$$Q[x] = x^5 (35 x^3 - 120 x^2 + 140 x - 56)$$

Envelope Equations:

Zero Condition:

$$k_o^2 = \frac{4 x^2 (15 x^2 - 40 x + 28)}{(x - 1)^4} \quad ; \quad \tilde{k}_1 = \frac{12 (5 x^3 - 20 x^2 + 28 x - 14)}{(x - 1)^3}$$

Unity Condition:

$$k_o^2 = \frac{4 (15 x^4 + 20 x^3 + 18 x^2 + 12 x + 5)}{x^4}$$

$$\tilde{k}_1 = \frac{12 (x - 1)^2 (5 x^3 + 5 x^2 + 3 x + 1)}{x^5}$$

Maximum or Minimum Condition:

$$k_o^2 = \frac{840 x^2}{14 x^2 - 16 x + 5} \quad ; \quad \tilde{k}_1 = \frac{840 (x - 1)^2}{14 x^2 - 16 x + 5}$$

Inflection-Point Condition: Numerical calculation

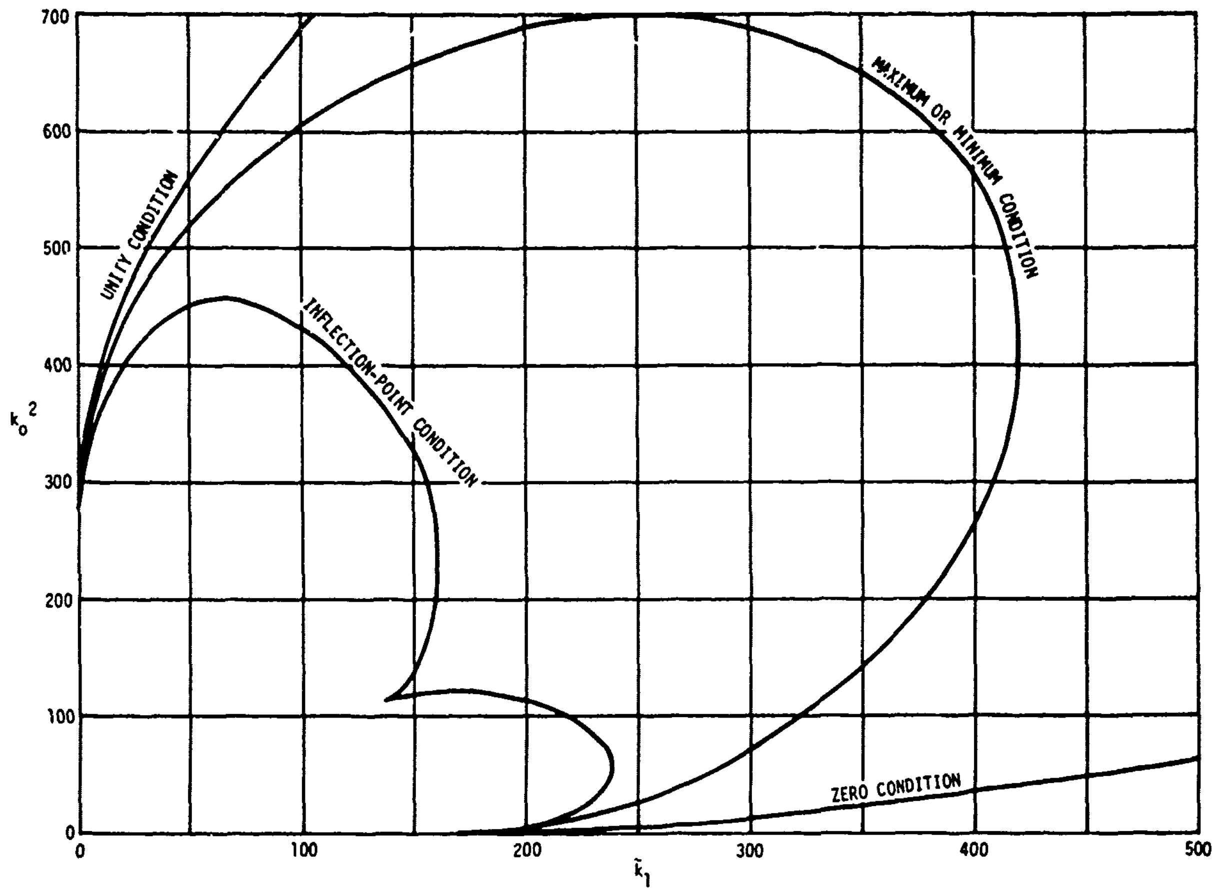


Figure 7 - "Quadratic" Polynomial: Cusped End, Permissible Range of Parameters k_0^2 and $\tilde{k}_1$

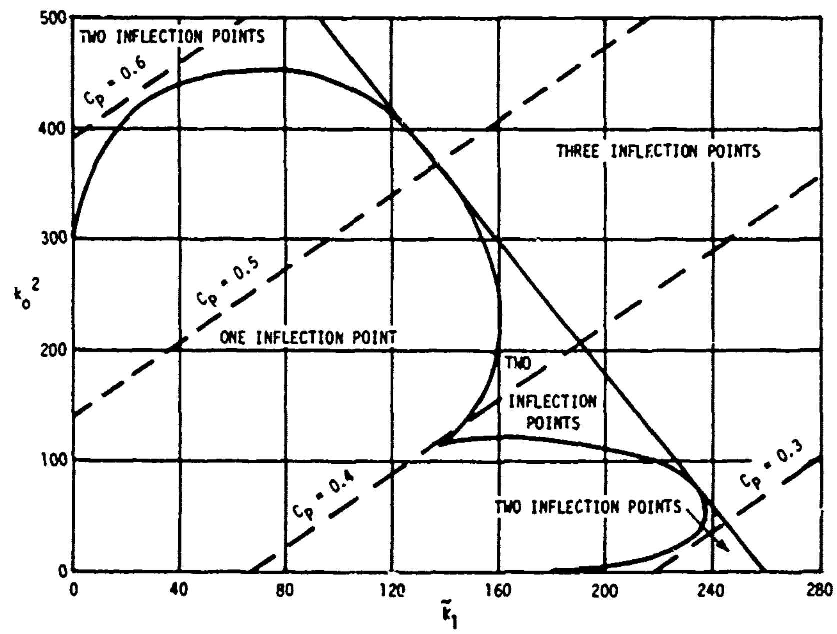


Figure 8 - "Quadratic" Polynomial: Cusped End, Inflection-Point Condition

α_2: $\tilde{k}_1$ = rate of change of curvature at x = 1

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3} \right)_{x = 1} \tag{74}$$

The boundary conditions β_j are the same as those for the quadratic polynomial

$$\beta_1: \quad x = 0, \; y = 0$$

$$\beta_2: \quad x = 1, \; y = 1$$

$$\beta_3: \quad x = 1, \; \frac{dy}{dx} = 0 \tag{75}$$

$$\beta_4: \quad x = 1, \; \frac{d^2y}{dx^2} = 0$$

Since there are six conditions in all, n = 4.

The α_i and β_j are substituted into the polynomial. For α_1: Differentiating Equation (72) with respect to y gives

$$1 = \left(\frac{1}{2} a_{1/2} x^{-1/2} + a_1 + \ldots n a_n x^{n - 1}\right) \frac{dx}{dy} \tag{76}$$

or

$$\frac{dx}{dy} = \frac{x^{1/2}}{\frac{1}{2} a_{1/2} + a_1 x^{1/2} + \ldots + n a_n x^{2n - 1/2}} \tag{77}$$

At x = 0, dx/dy = 0 which ensures a rounded end. Differentiating Equation (76) with respect to y gives

$$\frac{1}{\frac{d^2x}{dy^2}} = \frac{\left(\frac{1}{2} a_{1/2} + \ldots + n a_n x^{2n - 1/2}\right)\left(\frac{1}{2} a_{1/2} + \ldots + n a_n x^{2n - 1/2}\right)^2}{\frac{1}{4} a_{1/2} - \ldots - n (n - 1) x^{2n - 1/2}} \tag{78}$$

At $x = 0$, $r = 1/2\ a^2_{1/2}$

$$a_{1/2} = \sqrt{2\ r} \tag{79}$$

or

$$\alpha_2:\ \frac{3}{8} a_{1/2} + 6\ a_3 + 24\ a_4 = \tilde{k}_1$$

$$\beta_1:\ a_o = 0$$

$$\beta_2:\ a_{1/2} + a_1 + a_2 + a_3 + a_4 = 1 \tag{80}$$

$$\beta_3:\ \frac{1}{2} a_{1/2} + a_1 + 2\ a_2 + 3\ a_3 + 4\ a_4 = 0$$

$$\beta_4:\ -\frac{1}{4} a_{1/2} + 2\ a_2 + 6\ a_3 + 12\ a_4 = 0$$

The presence of the square-root term prevents the use of the factorial analysis.

The solution as simultaneous equations in the a's produces

$$y = \sqrt{2\ r}\ R[x] + \tilde{k}_1\ \tilde{K}_1[x] + Q[x] \tag{81}$$

with

$$R[x] = x^{1/2} + \frac{x}{16} (5\ x^3 - 21\ x^2 + 35\ x - 35) \tag{82}$$

$$\tilde{K}_1[x] = \frac{x}{6} (x - 1)^3 \tag{83}$$

$$Q[x] = 1 - (x - 1)^4 \tag{84}$$

The permissible ranges of parameters $\sqrt{2\ r}$ and $\tilde{k}_1$ are studied as before, and the results are shown in Table 4 and Figures 9 and 10.

TABLE 4 - SUMMARY OF "SQUARE ROOT" POLYNOMIALS FOR ROUNDED END

Polynomial Equation:

$$y = \sqrt{2\,r}\; R[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \qquad 0 \le x \le 1$$

where

$$R[x] = \sqrt{x} + \frac{x}{16}\,(5\,x^3 - 21\,x^2 + 35\,x - 35)$$

$$\tilde{K}_1[x] = \frac{1}{6}\,x\,(x - 1)^3$$

$$Q[x] = 1 - (x - 1)^4$$

Envelope Equations:

Zero Condition:

$$\sqrt{2\,r} = \frac{-\,8\,x^{3/2}\,(x^2 - 4\,x + 6)}{28\,x - 4 - x^{3/2}\,(3\,x^2 - 14\,x + 35)}$$

$$\tilde{k}_1 = \frac{24\,(\,7\,x^3 - 20\,x^2 + 18\,x - 4) - 3\,x^{3/2}\,(x^4 - 10\,x^3 + 59\,x^2 - 112\,x + 70)}{(x - 1)^2\,[28\,x - 4 - x^{3/2}\,(3\,x^2 - 14\,x + 35)]}$$

Unity Condition:

$$\sqrt{2\,r} = \frac{-\,8\,(x - 1)^4\,x^{3/2}}{x^2\,[28\,x - 4 - x^{3/2}\,(3\,x^2 - 14\,x + 35)]}$$

$$\tilde{k}_1 = \frac{3\,(x - 1)\,[56\,x^2 + 8\,x - x^{3/2}\,(x^3 - 7\,x^2 + 35\,x + 35)]}{x^2\,[28\,x - 4 - x^{3/2}\,(3\,x^2 - 14\,x + 35)]}$$

Maximum or Minimum Condition:

$$\sqrt{2\,r} = \frac{-\,48\,(x - 1)^3\,x^{3/2}}{28\,x^2 - 17\,x + 1 - x^{3/2}\,(18\,x^3 - 62\,x^2 + 91\,x - 35)}$$

$$\tilde{k}_1 = \frac{6\,(x - 1)\,[28\,x - 4 - x^{3/2}\,(3\,x^2 - 14\,x + 35)]}{28\,x^2 - 17\,x + 1 - x^{3/2}\,(18\,x^2 - 62\,x^2 + 91\,x - 35)}$$

Inflection-Point Condition:

$$\sqrt{2\,r} = \frac{96\,(x - 1)^2\,x^{5/2}}{14\,x^2 - 15\,x + 3 + 2\,x^{5/2}\,(18\,x^2 - 40\,x + 21)}$$

$$\tilde{k}_1 = \frac{12\,(x - 1)\,[7\,x - 3 + x^{5/2}\,(3\,x - 7)}{14\,x^2 - 15\,x + 3 + 2\,x^{5/2}\,(18\,x^2 - 40\,x + 21)}$$

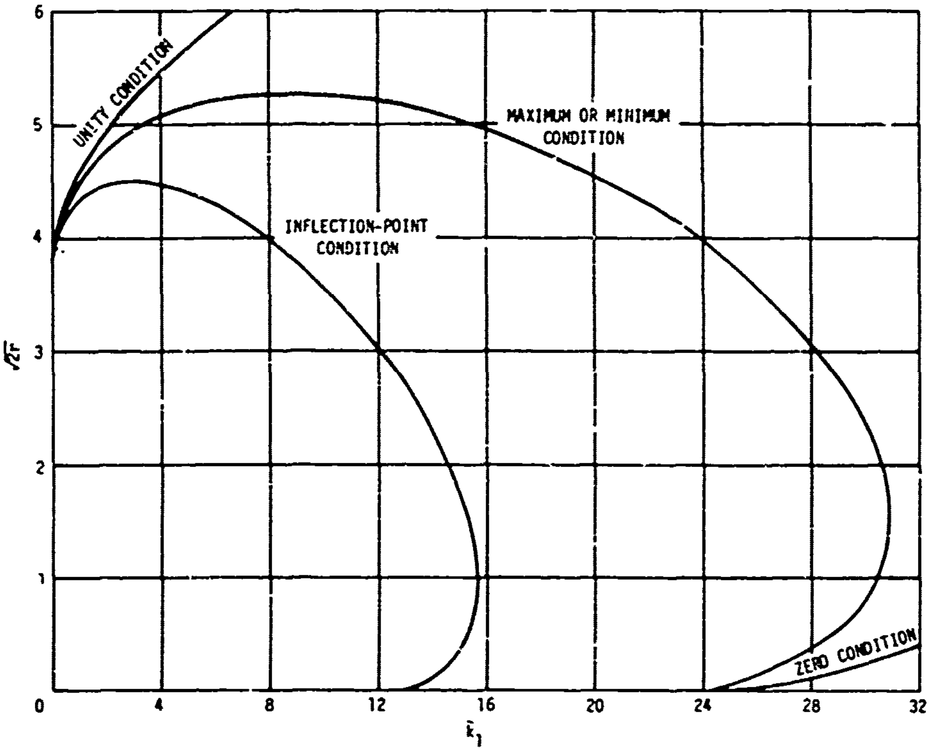


Figure 9 - "Square Root" Polynomial: Rounded End, Permissible Range of Parameters $\sqrt{2r}$ and $\tilde{k}_1$

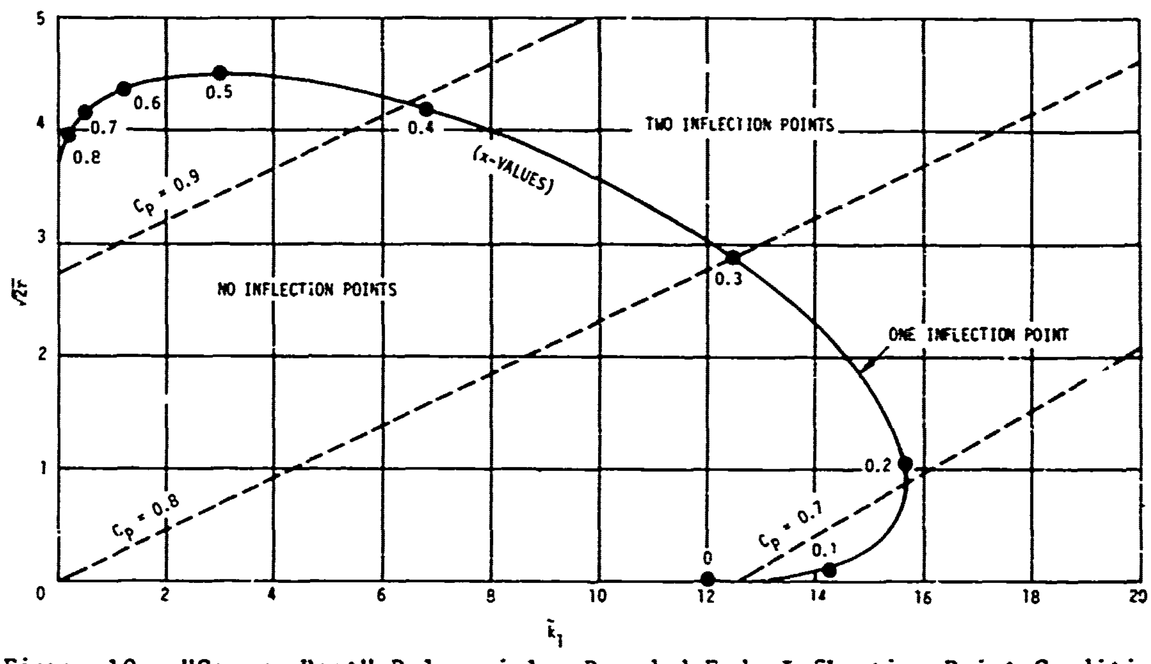


Figure 10 - "Square Root" Polynomial: Rounded End, Inflection-Point Condition

POINTED ENDS

The ordinary polynomial is utilized, namely

$$y = \sum_{n=0}^{n=n} a_n x^n \tag{85}$$

The adjustable parameters α_i are the same as those for the quadratic polynomial

$$\alpha_1: \quad s = \text{slope at } x = 0$$
$$s = \left(\frac{dy}{dx}\right)_{x=0} \tag{86}$$

$$\alpha_2: \quad \tilde{k}_1 = \text{rate of change of curvature at } x = 1$$

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x=1} \tag{87}$$

The boundary conditions β_j are the same as those for the quadratic polynomial

$$\beta_1: \quad x = 0,\ y = 0$$
$$\beta_2: \quad x = 1,\ y = 1$$
$$\beta_3: \quad x = 1,\ \frac{dy}{dx} = 0 \tag{88}$$
$$\beta_4: \quad x = 1,\ \frac{d^2y}{dx^2} = 0$$

Since there are six conditions in all, $n = 5$.

Substitution of α_i and β_j into the polynomial produces

$$
\begin{aligned}
&\alpha_1: \quad a_1 = s \\
&\alpha_2: \quad 6\, a_3 + 24\, a_4 + 60\, a_5 = \tilde{k}_1 \\
&\beta_1: \quad a_0 = 0 \\
&\beta_2: \quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1 \\
&\beta_3: \quad a_1 + 2\, a_2 + 3\, a_3 + 4\, a_4 + 5\, a_5 = 0 \\
&\beta_4: \quad 2\, a_2 + 6\, a_3 + 12\, a_4 + 20\, a_5 = 0
\end{aligned}
\tag{89}
$$

y is a linear function of s and $\tilde{k}_1$, or

$$y = s\, S[x] + \tilde{k}_1\, \tilde{K}_1[x] + Q[x] \tag{90}$$

The analysis proceeds as before, and the results are shown in Table 5 and Figures 11 and 12.

CUSPED ENDS

The ordinary polynomial is utilized

$$y = \sum_{n=0}^{n=n} a_n\, x^n \tag{91}$$

The adjustable parameters α_i are the same as those for the quadratic polynomial

$$
\begin{aligned}
&\alpha_1: \quad k_o = \text{curvature at } x = 0 \\
&k_o = \left(\frac{d^2y}{dx^2}\right)_{x\,=\,0}
\end{aligned}
\tag{92}
$$

TABLE 5 - SUMMARY OF ORDINARY POLYNOMIALS FOR POINTED END

Polynomial Equation:

$$y = s\,S[x] + \tilde{k}_1\,\tilde{K}_1[x] + Q[x] \qquad 0 \leq x \leq 1$$

where

$$S[x] = x\,(x-1)^4$$

$$\tilde{K}_1[x] = \frac{1}{6}x^2\,(x-1)^3$$

$$Q[x] = 1 - (x-1)^4\,(4x+1)$$

Envelope Equations:

Zero Condition:

$$s = \frac{x^2\,(3x^2 - 10x + 10)}{(x-1)^4} \quad ; \quad \tilde{k}_1 = \frac{6\,(x^3 - 5x^2 + 10x - 10)}{(x-1)^3}$$

Unity Condition:

$$s = \frac{3x + 2}{x} \quad ; \quad \tilde{k}_1 = \frac{6\,(x-1)^2}{x^2}$$

Maximum or Minimum Condition:

$$s = \frac{30\,x^2}{10x^2 - 5x + 1} \quad ; \quad \tilde{k}_1 = \frac{12\,(x-1)^2}{6x^2 - 4x + 1}$$

Inflection-Point Condition:

$$s = \frac{5\,(6x^2 - 4x + 1)}{10x^2 - 10x + 3} \quad ; \quad \tilde{k}_1 = \frac{60\,(x-1)^2}{10x^2 - 10x + 3}$$

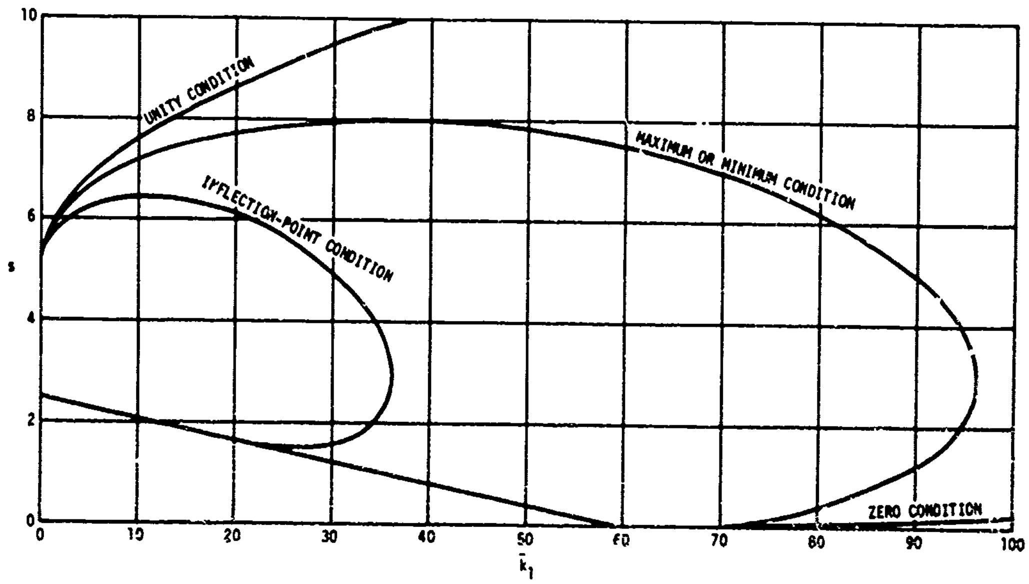


Figure 11 - Ordinary Polynomial: Pointed End, Permissible Range of Parameters s and $\tilde{k}_1$

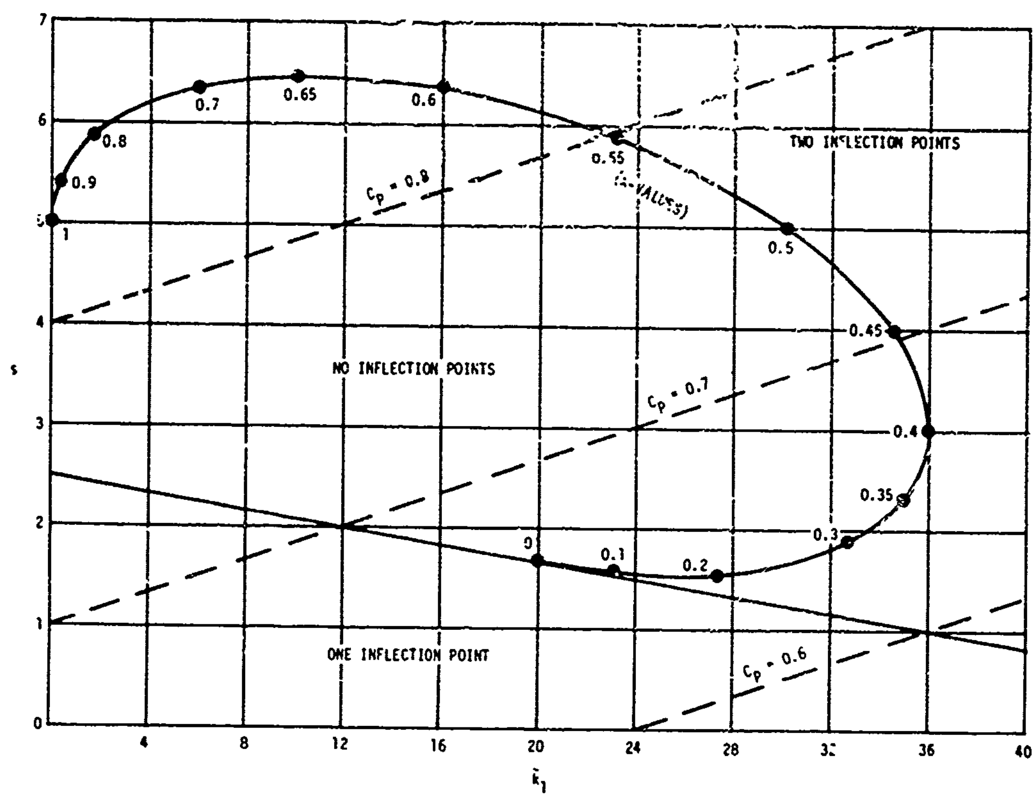


Figure 12 - Ordinary Polynomial: Pointed End, Inflection-Point Condition

$\tilde{k}_1$ = rate of change of curvature at $x = 1$

$$\tilde{k}_1 = \left(\frac{d^3y}{dx^3}\right)_{x=1} \tag{93}$$

The boundary conditions β_j are the same as those for the quadratic polynomial

$$\begin{aligned}
\beta_1&: \quad x = 0,\ y = 0 \\
\beta_2&: \quad x = 0,\ \frac{dy}{dx} = 0 \\
\beta_3&: \quad x = 1,\ y = 1 \\
\beta_4&: \quad x = 1,\ \frac{dy}{dx} = 0 \\
\beta_5&: \quad x = 1,\ \frac{d^2y}{dx^2} = 0
\end{aligned} \tag{94}$$

Since there are seven conditions in all, $n = 6$.

Substitution of α_i and β_j into the polynomial produces

$$\begin{aligned}
\alpha_1&: \quad 2\,a_2 = k_o \\
\alpha_2&: \quad 6\,a_3 + 24\,a_4 + 60\,a_5 + 120\,a_6 = \tilde{k}_1 \\
\beta_1&: \quad a_0 = 0 \\
\beta_2&: \quad a_1 = 0 \\
\beta_3&: \quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 \\
\beta_4&: \quad a_1 + 2\,a_2 + 3\,a_3 + 4\,a_4 + 5\,a_5 + 6\,a_6 = 0 \\
\beta_5&: \quad 2\,a_2 + 6\,a_3 + 12\,a_4 + 20\,a_5 + 30\,a_6 = 0
\end{aligned} \tag{95}$$

y is a linear function of k_0 and $\tilde{k}_1$ in

$$y = k_0 \, K_0[x] + \tilde{k}_1 \, \tilde{K}_1[x] + Q[x] \tag{96}$$

The analysis proceeds as before and the results are shown in Table 6 and Figures 13 and 14.

LEAST-SQUARES FIT

Given shapes either in graphical form or in analytical form may be fitted to the polynomials of this report by a least-squares fit. If

$$z = y^2$$

for quadratic polynomials, or

$$z = y$$

for square-root polynomials; then, in general

$$z = d \, D[x] + \tilde{k}_1 \, \tilde{K}_1[x] + Q[x] \tag{97}$$

where d and $D[x]$ refer to the appropriate type of shape. For example, $d = r$, and $D[x] = R[x]$ for the rounded nose.

In general, a least-squares fit requires that $\int_0^1 (z - z_1)^2 \, dx$ be minimized where $z_1[x]$ represents the body shape to be fitted. Consequently, by differentiating with respect to the coefficients to be determined, there result two simultaneous equations

$$d \int_0^1 D^2 \, dx + \tilde{k}_1 \int_0^1 D \, \tilde{K}_1 \, dx + \int_0^1 D \, Q \, dx - \int_0^1 z_1 \, D \, dx = 0 \tag{98}$$

$$d \int_0^1 D \, \tilde{K}_1 \, dx + \tilde{k}_1 \int_0^1 \tilde{K}_1^{\,2} \, dx + \int_0^1 \tilde{K}_1 \, Q \, dx - \int_0^1 z_1 \, \tilde{K}_1 \, dx = 0 \tag{99}$$

The values of d and $\tilde{K}_1$ are then determined by the Cramer rule. Values of the indicated integrals for the various polynomials may be found in Table 7.

TABLE 6 - SUMMARY OF ORDINARY POLYNOMIALS FOR CUSPED END

Polynomial Equation:

$$y = k_0 \, K_0[x] + \tilde{k}_1 \, \tilde{K}_1[x] + Q[x] \qquad 0 \leq x \leq 1$$

where

$$K_0[x] = \frac{1}{2} x^2 (x - 1)^4$$

$$\tilde{K}_1[x] = \frac{1}{6} x^3 (x - 1)^3$$

$$Q[x] = 1 - (x - 1)^4 (10 x^2 + 4 x + 1)$$

Envelope Equations:

Zero Condition:

$$k_0 = \frac{6 x^2 (2 x^2 - 6 x + 5)}{(x - 1)^4} \; ; \; \tilde{k}_1 = \frac{12 (2 x^3 - 9 x^2 + 15 x - 10)}{(x - 1)^3}$$

Unity Condition:

$$k_0 = \frac{6 (2 x^2 + 2 x + 1)}{x^2} \; ; \; \tilde{k}_1 = \frac{12 (x - 1)^2 (2 x + 1)}{x^3}$$

Maximum or Minimum Condition:

$$k_0 = \frac{60 x^2}{5 x^2 - 4 x + 1} \; ; \; \tilde{k}_1 = \frac{120 (x - 1)^2}{5 x^2 - 4 x + 1}$$

Inflection-Point Condition:

$$k_0 = \frac{60 x^2 (10 x^2 - 10 x + 3)}{50 x^4 - 80 x^3 + 45 x^2 - 10 x + 1}$$

$$\tilde{k}_1 = \frac{120 (x - 1)^2 (10 x^2 - 5 x + 1)}{50 x^4 - 80 x^3 + 45 x^2 - 10 x + 1}$$

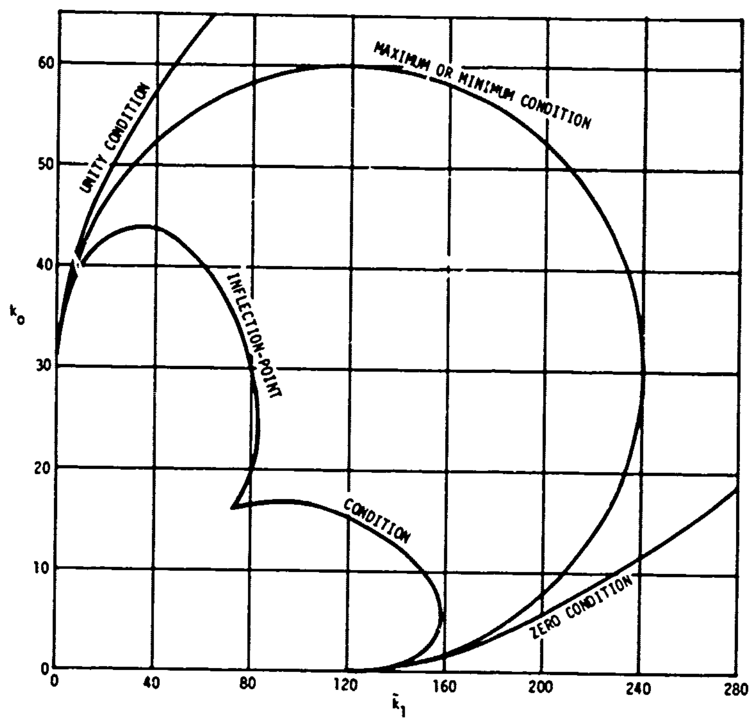


Figure 13 - Ordinary Polynomial: Cusped End, Permissible Range of Parameters k_0 and $\tilde{k}_1$

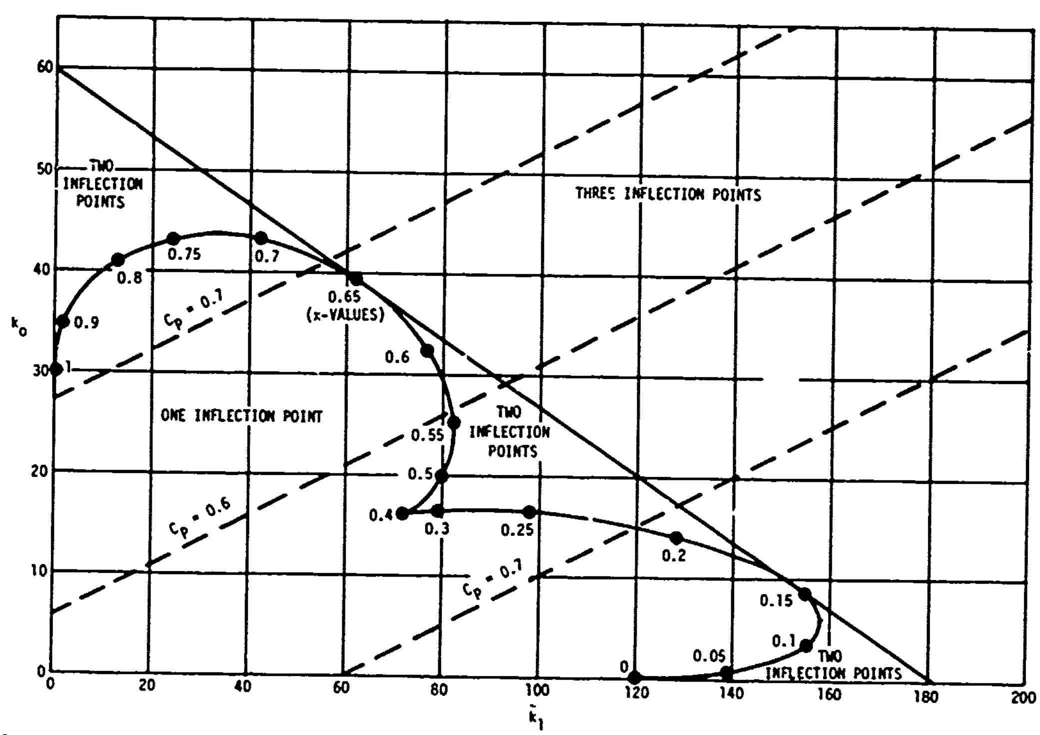


Figure 14 - Ordinary Polynomial: Cusped End, Inflection-Point Condition

TABLE 7 - INTEGRATED VALUES FOR LEAST-SQUARES FIT

Polynomial		D	$\int_0^1 D^2\,dx$	$\int_0^1 D\,\tilde{K}_1\,dx$	$\int_0^1 D\,Q\,dx$	$\int_0^1 \tilde{K}_1^{\,2}\,dx$	$\int_0^1 \tilde{K}_1\,Q\,dx$
"QUADRATIC"	Rounded	R	$\frac{4}{495}$	$-\frac{1}{1980}$	$\frac{14}{495}$	$\frac{1}{20,790}$	$-\frac{43}{11,880}$
	Pointed	S	$\frac{1}{6435}$	$-\frac{1}{30,888}$	$\frac{202}{15,015}$	$\frac{1}{108,108}$	$-\frac{529}{360,360}$
	Cusped	K_0	$\frac{1}{3,500,640}$	$-\frac{1}{2,333,760}$	$\frac{647}{4,084,080}$	$\frac{1}{1,225,224}$	$-\frac{941}{2,450,448}$
"SQUARE ROOT"	Rounded	R	$\frac{39,911}{101,376}$	$-\frac{947}{2,661,120}$	$\frac{278.7}{1,138,368}$	$\frac{1}{9072}$	$-\frac{13}{2160}$
	Pointed	S	$\frac{1}{495}$	$-\frac{1}{7920}$	$\frac{7}{495}$	$\frac{1}{83,160}$	$-\frac{43}{23,760}$
	Cusped	K_0	$\frac{1}{25,740}$	$-\frac{1}{123,552}$	$\frac{59}{30,030}$	$\frac{1}{432,432}$	$-\frac{529}{720,720}$

PRISMATIC COEFFICIENT

The fullness of a shape is given by the prismatic coefficient, which is the ratio of the volume of the body to the volume of a prism having the maximum cross-sectional area and the length of the body.

The prismatic coefficient C_p for the shapes of this report is then

$$C_p = \int_0^1 y^2 \, dx \tag{100}$$

for bodies of revolution, and

$$C_p = \int_0^1 y \, dx \tag{101}$$

for two-dimensional bodies.

For the polynomials of this report

$$C_p = d \int_0^1 D \, dx + \tilde{k}_1 \int_0^1 \tilde{K}_1 \, dx \int_0^1 Q \, dx \tag{102}$$

Consequently for bodies of revolution described by quadratic polynomials

rounded end

$$C_p = \frac{r}{15} - \frac{\tilde{k}_1}{180} + \frac{2}{3} \tag{103}$$

pointed end

$$C_p = \frac{s^2}{105} - \frac{\tilde{k}_1}{420} + \frac{4}{7} \tag{104}$$

cusped end

$$C_p = \frac{k_o^{\ 2}}{2520} + \frac{\tilde{k}_1}{1512} + \frac{4}{9} \tag{105}$$

and for two-dimensional bodies described by square-root polynomials

rounded end

$$C_p = \frac{7}{192}\sqrt{2\ r} - \frac{\tilde{k}_1}{120} + \frac{4}{5} \tag{106}$$

pointed end

$$C_p = \frac{s}{30} - \frac{\tilde{k}_1}{360} + \frac{2}{3} \tag{107}$$

cusped end

$$C_p = \frac{k_o}{210} - \frac{\tilde{k}_1}{840} + \frac{4}{7} \tag{108}$$

Lines of constant C_p are plotted in Figures 4, 6, 8, 10, 12, and 14.