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NEW ORBITAL ELEMENTS FOR MOON AND PLANETS

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Naval Weapons Laboratory
Dahlgren, Virginia

March 1972

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Warfare Analysis Department

I-b

FOREWORD

The work described in this report was undertaken as a Naval Weapons Laboratory Independent Research project during fiscal years 1966 to 1970.

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13. ABSTRACT

A simultaneous solution is made for the orbital elements of moon and planets. A modern Cowell integrator is used for orbit computations, and least-squares fits are made to some 40,000 optical observations taken since 1913. The model includes relativistic terms, the leading zonal harmonics of earth and moon, the precession of the lunar equator, and the tidal couple between earth and moon. The tidal term in the moon's mean longitude is found to be -19" ± 4" per century squared. The solution also yields an extrapolation of the atomic time scale back to 1912.5. At that time, the difference between atomic and ephemeris time is about 6 ± 2 seconds. It is found that solar oblateness cannot quite be determined with optical data covering about 50 years, but J_2 is unlikely to be much larger than 10^{-5}.

This solution is believed to be the only simultaneous improvement of the orbits of moon and planets. The simultaneity is found to be an essential feature in separating the moon's mean motion, the lunar tidal deceleration, and the corrections to the earth rotation rate. It is now possible to refer all astronomical events of the past 60 years to a time with uniform rate, namely the atomic clock system. Considering the long baseline, this model should facilitate the prediction of fast variables, such as the lunar longitude, with considerably increased confidence. The planetary orbital elements compete with efforts of similar scope and accuracy at the Massachusetts Institute of Technology and the Jet Propulsion Laboratory.

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I-a

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CONTENTS

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CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

The principal source of information on the positions of planets and moon are the various national ephemerides. These publications enjoy a reputation of great scientific competence. Rightfully so. Most of the tables that are published today are based on the life work of many great men in dynamical astronomy. There is abundant evidence of the painstaking and loving care that went into various theories of sun, moon, and planets.

This recognition must not impede our desire to make progress when there is a need to do so. And, in spite of the renown of the classical theories, there is considerable room for improvement. A major reason for present defects in the ephemerides lies in the fact that a substantial part of the theories was done by hand, and that quite some time ago. There is only so much algebra a man can do in forty years, so that all series have to be truncated somewhere. Complications arise since it is very difficult to find all terms greater than a given threshold. Moreover, the determination of constants in such a theory cannot go much beyond the orbital elements of the body under investigation. Although it is no secret that changes in one constant require the simultaneous adjustment of many others, it is simply not possible to treat 50 or more parameters by hand. Such a partial solution usually leads to formal variances that are too small. At times they are so bad that completely unrealistic error estimates result. Finally, not every theory is fitted directly to observations. No one doubts that this is definitely the best way to proceed, but it is also known to be considerably more laborious. Hence, some theories made use of a previous investigator's residuals instead of actual observations. It is clear that this can and will produce further complications.

As a consequence of the deficiencies just discussed, is there any direct evidence for the shortcomings of the presently published ephemerides? Yes, there are some large discrepancies between observations and published places. But it is only fair to note that there are cases in which agreement is still excellent. It is difficult, for example, to find fault with Newcomb's motion of the sun, even though the theory is 75 years old. We had occasion to compare his position predictions for the present time with our own solution and found the differences to be much less than 1″. However, on the other end of the spectrum are Pluto and the moon. The discrepancies in Pluto's published places are now about 10″. This cannot come as a surprise since the tables are based on a set of orbital elements determined in 1932! The residuals in the transit circle observations of the moon, as published in several series, are also quite large and presently exceed 20″. However, to quote this number may possibly be unfair to Brown's lunar theory. Since empirical terms and other corrections are involved, it is undoubtedly possible to explain or remove such large

planning, the moral support and encouragement of Gerald M. Clemence and Paul Herget is gratefully acknowledged. At the Naval Observatory, quite a number of people were most helpful. In the Six-Inch Division, we would like to express special thanks to David K. Scott, who, with his profound knowledge of practical astronomy, has set us straight on several occasions. B. L. Klock and A. N. Adams were equally willing to listen to our problems and to provide immediate assistance. Klock also was kind enough to let us use the newly reduced lunar observations before they were released for publication. In the Nautical Almanac Office, Raynor L. Duncombe and his staff assisted us on many occasions. They were particularly helpful in locating much of the valuable observational material. Thomas C. Van Flandern brought his knowledge of lunar theory to bear on several problems. Since Douglas A. O'Handley at JPL is engaged in work similar to ours, it was natural that we had a large number of discussions with him on many aspect of this study. Although work of J. Derral Mulholland on the lunar orbit follows a somewhat different philosophy, there was some overlap and exchange of information. On a few occasions we compared notes with Irwin I. Shapiro of MIT. His keen observations always provide much food for thought. It should also be remembered that Ash, Shapiro, and Smith (1967) were first to publish a modern solution for planetary orbits and masses. If any of our friends were left out, it was unintentional, and we apologize for the oversight.

II. OBJECTIVES, RESULTS OBTAINED AND MISSED

Our original plans called for a solution of the orbital elements and masses of the major planets based on optical observations back to 1750 or so and on radar ranges to Mercury, Venus, and Mars.

We soon found that it was relatively easy to obtain meridian circle observations for about the last 55 years. Older data are more difficult to resurrect. We decided to begin our work with the 55 year span and to extend to older observations at a later time.

In the early phases of the study we were not particularly interested in the motion of the moon. It was clear that it had to be considered but only inasmuch as it affects the motion of the earth. Available knowledge of the orbit of the moon was sufficient to fill this need.

The next two decisions changed the course of this study drastically. Since we had written an n-body program, it was only too easy to numerically integrate the moon's heliocentric coordinates as if it were a planet. Also, having accumulated optical observations of the moon along with those for the planets, it seemed like another small step to utilize all these data and differentially correct the lunar orbit, too.

But this move turned out to be more of a giant leap. The lunar orbit proved so much more difficult to treat than the planetary ones that we spent almost all of our time on the moon. However, we will not bore the reader with a list of difficulties that were encountered.

The moon cannot be held responsible for all the trouble. While there were a number of new features to be learned, we also made some mistakes. But none of these errors showed up in the planetary residuals and, hence, they were mostly inconsequential for that part of the study. The moon is much more demanding. Its orbit is exceedingly sensitive to even minute changes in the algorithm as well as to slight deviations from the proper procedure of running the program. The latter statement will be clarified later.

So all our attention was focused on the moon for a while. Fortunately, there were some dividends, predictable and unexpected. First of all, we think that our lunar orbit is probably as good as can be obtained today. We feel strongly about our approach to the problem. At the present time, the best and most desirable procedure is a fit of a precise numerical integration orbit directly to observations.

Secondly, some of the problems encountered prompted us to attempt an improvement of the ephemeris time clock, used in the early stages of this study. Upon switching to atomic time, this program feature allowed us to extrapolate the atomic time scale back to 1912. Relations are given between atomic and universal as well as atomic and ephemeris time. This result alone should prove of considerable interest. To the best of our knowledge, it represents the first determination of a uniform time directly from observations.

Finally, we obtained the coefficient for the tidal coupling between the orbital motion of the moon and the rotation of the earth. We find that this phenomenon produces $-19'' T^2$ in the moon's mean longitude with T measured in centuries.

A casual examination of the problem may lead to the conclusion that the six lunar elements, tidal coefficient, and the clock corrections cannot all be solved for from a given set of normal equations because any signal in the longitude can always be removed by a suitable adjustment of the clock. This would undoubtedly be true if we used lunar observations only and imposed no other restrictions. But we also have the observations of sun and planets and we have atomic time from 1955 on. Since the accuracy of atomic time far exceeds anything we could do with observations, we do not attempt clock corrections past 1954. Heuristically, atomic time from 1955 to 1968 serves as the clock with respect to which the planetary ephemerides are constructed. Prior to 1955, the planetary observations strengthen the consistency of the planetary motions with respect to one another and with respect to the atomic clock data of 1955 to 1968. The ephemerides of the inner planets, of course, furnish the more sensitive backward extensions of the atomic clock. The lunar mean motion and change in mean motion are fitted to the atomic clock and to its backward extension through the planetary ephemerides. Any discrepancies with UT, which would be largest for the lunar data, are then identified as earth rotation inequalities relative to atomic time.

We think that the solution for the planetary orbits is also a considerable improvement over the classical ephemerides, certainly for some planets. After all, the residuals have been reduced to an absolute minimum in the least-squares sense. No biases removable within the present model remain.

Among other non-orbital parameters, we tried to solve for the leading zonal harmonic of the sun. Results were as follows. The formal standard deviation for J_2 was 1.4×10^{-5}, and J_2 itself about the same and with the wrong sign. Just about the only information which can be extracted from that kind of statistics is that J_2 is not likely to be much larger than 1.4×10^{-5}. A fair amount of effort went into this result. Since we did not need the relatively expensive numerically integrated partial derivatives, we derived analytical partials from Brouwer's (1959) satellite theory. This proved not as simple as one might expect. We left the fairly lengthy derivation in this paper for the sake of documentation. The solar J_2 was a suppressed parameter in our final iterations.

As stated earlier, original plans called for the addition of electronic data. They were made available to us through the courtesy of Shapiro and his colleagues at the Lincoln Laboratory and O'Handley at the Jet Propulsion Laboratory. However, since we had devoted so much time to the moon, we did not want to delay the study any further and decided to go for a purely optical solution; at least for the time being.

We did not attempt to solve for mass corrections in the present solution. There were reasons for this postponement, over and above the lack of time. We had already decided from the beginning that numerically integrated partial derivatives would be used for this purpose. As will be discussed in a later section, inadequate partials in an ill-conditioned normal matrix may lead to disaster. Since it is clear that some masses would be difficult to determine, the need for precise differential coefficients was obvious and we knew they would be expensive. However, as the computer program grew, it began taxing the storage capabilities of our computer. It became inefficient to operate, and we priced our mass partials out of reach. A new program needs to be written for this purpose.

Recent experiments by other researchers using various types of partials tend to support our conjecture. While there is fair agreement for some masses, there is none at all for others. As one would expect, the marginal masses of Pluto and Mercury are particularly troublesome. In one such set of experiments Pluto's mass fluctuates wildly due to only minor program changes. These results strengthen our desire to employ very accurate partials. It is clear that the partials for the orbital elements, if adjusted simultaneously, must be obtained to the same degree of accuracy.

III. SPECIAL FEATURES OF DESIGN AND ANALYSIS

The entire program was designed for and run on our STRETCH computer, the IBM 7030. The 48 bit binary mantissa yields 14 decimal places precision. Although such a word length is more than adequate for most calculations, certain operations in the integration routine are done in double precision. They will be pointed out later. Overall design precision is 0.″01. Practically all calculations meet this criterion. However, the truncation error in the lunar orbit slightly exceeds this threshold. As will be explained later, the maximum effective truncation error here is 0.″013.

The starting time of the integration is JD 242 0000.5 which corresponds to 1913 August 21.0. The reasons for this choice are given in the section on observational material. Observations span the 20,000 days to JD 244 0000.5. In case of the moon, the data were separated into two groups, those before and after 1925. As it were, all lunar Six-Inch observations have recently been uniformly reduced by Adams, Klock, and Scott (1969). They were put on the FK4 system and have all received limb corrections. Since no such uniform treatment exists for the data before 1925, the two sets should receive different weights. The weights calculated and employed are found in Table IX.

Among the various perturbing forces studied, but not discussed elsewhere in this paper, is the solar radiation pressure. Although small, this acceleration is known to cause problems in the case of some artificial satellites. In our problem, the effects are entirely negligible. Peters (1964) has pointed out that radiation pressure on any given planet acts so as to decrease the apparent mass. Even for Mercury, for which the effect would be largest, solar radiation pressure corresponds to a change of its mass by only 2×10^{-7}.

The partial derivatives used in our program were obtained partly by numerical integration but mostly analytically. They are discussed in later sections. For some experiments we also approximated partials of the form $\partial D/\partial P$ by finite difference ratios $\Delta D/\Delta P$.

We made some effort to include the lunar tesseral harmonics of degree and order two in our model. All runs with such terms resulted in some peculiar signals. Since we were unable to find an error in either formulation or computer program, these terms were eventually suppressed by putting the coefficients c_{22} and s_{22} to zero. The derivation of the corresponding accelerations is retained in this report.

It may seem that the documentation of derivations is not at all uniformly complete. There is a reason for this. We saw no need to record anything that is well known and easily found. We did, however, try to retain the essential steps of new developments or those derivations which we always have difficulties finding.

The reader will see in a later part of this paper that not all 73 parameters can be solved for simultaneously. They are divided into two subsets, one containing the planetary elements and the other the lunar elements and associated parameters. These sets are alternately solved for in consecutive least-squares solutions. The formal standard deviations produced with each such partial solution are ignored. After final convergence we made a global solution only in order to obtain a more realistic variance-covariance matrix. All standard deviations are taken from this matrix.

The same matrix was used to get the unitized correlation coefficients. Some of these coefficients are tabulated near the end of the paper. It is clear that only limited information can be obtained from such figures.

IV. OBSERVATIONS

As remarked earlier, we had plans of collecting observations of sun, moon, and planets back to 1750 or so. A preliminary library search revealed that this task was probably even more formidable than expected. The major difficulty was due to the fact that in older publications the observations of the bodies of interest to us were listed among those taken on stars. Starting about 1911, though, results of the U. S. Naval Observatory transit circle programs were published separately for each planet. It seemed to us that the data from 1911 on were a good starting set for our work. Shapiro (1970) recently pointed out that the USNO transit circle results are of higher quality than any other series.

In the meantime, Shapiro has actually gone back about 200 years and collected meridan circle observatories of the solar system bodies taken at several major observatories. The product of this tremendous and commendable effort are something like 350,000 or 400,000 observations in right ascension and declination.

Parts of the initial data set punched by us were eventually replaced with updated reductions done at the USNO. However, we presently plan to retain all of the observational material in machine readable form for a few years. Hence, the data as originally published as well as those eventually used by us in our orbit improvements will be available to other interested researchers.

The original observations were found in five volumes of the Publications of the United States Naval Observatory, Second Series (1927, 1933, 1948, 1949, 1952). Table VII lists the volumes we punched. All these data were taken with the Six-Inch or Nine-Inch transit circle. The "source code", given in the last column of Table VII, is a ready identification of the particular observation series, and it is included on every observation card punched.

Table VIII represents an attempt of listing the sources of all the observations used in our final solution. As may be seen, three agencies cooperated in punching the data onto cards. Sections of our master tape were updated a number of times when "better" observation, usually the results of a more definitive reduction, became available. Such changes were made for all planets in a given time span or, as in the case of V XIX Pt III, for one body at a time. After so many revisions, minor discrepancies between the publications and our master tape are likely. The only major discrepancy we are aware of is an apparent loss of 340 days of data from Circular No. 115(C115). It is not known how this occured and no attempt will be made at this time to recover the missing points. No data was lost from C127 as the cutoff was planned.

All observation cards are now available in one unique format. This was not so at first. In punching from the USNO publications, some 15 to 20 differen formats were encountered. It was clear that all cards had to be converted to a uniform format. In checking with JPL in the fall of 1967, we found that they had just devised a card format for in-house usage. With a few minor modifications, the format was made to fit the needs of JPL, NWL, and USNO. Other interested parti., were asked for inputs, and a standard format emerged. The latter is explained in detail by O'Handley (1968).

A few additional comments have to be made about Pluto. Since this object is too faint to be observed with a transit instrument, all observations are photographic. The entire material used in our Pluto orbit correction (Cohen, Hubbard, Oesterwinter, 1967) was employed here again. However, all Sharaf-Budnikova normal places were treated as individual observations. We believe that this simplification is of little consequence in the present investigation. Added to this data set were two Russian observations taken in 1965 (Chernykh and Chernykh, 1967). The source code D XX for Pluto in Table VIII is only the general form of seven different codes used. They are explained in Table IX.

Our master tape also contains all the observations made on Ceres, Pallas, Juno, and Vesta as listed in D7, D9, D1, D2, D3 and D4. These data were not used.

Observation times given in any of the publications were ignored. We calculated our own observation times from the observed right ascensions using a procedure described below. Hence, any observation of declination alone or incomplete in right ascension is useless and was disregarded. On the other hand, the master tape contains observations with missing or incomplete information in declination. Although still useful in right ascension, such data points were eliminated at a later stage in an effort to simplify the program.

Most observations showing large residuals on preliminary runs were also removed from the master tape. In view of the large amount of data available no effort was made to trace the discrepancies. Simple keypunch errors are the likely cause in many cases.

First runs showed a distinct bias in the right ascension residuals for early observations of sun, Mercury, and Venus. The problem was quickly traced to a difference in procedure employed in the reduction of source code 6 data in contrast to later volumes. The so-called equinox correction of $-1\overset{''}{.}218$ was not applied to the observed right ascension in those days. We simply subtracted $0\overset{s}{.}081$ from all right ascensions in source code 6. The data on our present master tape are thus corrected.

This modification was the only change made in any of the original observations. Since the master tape is saved, the data are preserved in this form.

There are a number of other corrections that have to be applied but none of these have been added to the original master tape. The old astronomical tradition of leaving observed places alone is a very sensible one. The exception discussed above, however, is where it belongs, we think, since it removes an inconsistency in the original data reduction.

The reader may wish to check the section on residuals for additional corrections. Among others, we made a bias correction just like the one above. The important difference is that we were unable to find a cause for the discrepancies. Hence, we did not wish to alter the original data.

V. DATA PREPARATION

The observations on the master tape cannot yet be compared with computed places. A number of modifications are still to be made. Unless otherwise stated, the following discussions do not pertain to Pluto observations.

Older catalogs did not use universal time. Since approximate times are needed, as will be seen in a moment, we first converted all published observations times to UT. Table I shows the scheme. The reader will recognize immediately that $0^{d}.5$ of these corrections is due to the fact that the astronomical day before 1925 began at noon. The value $0^{d}.2$ is simply the reduction from EST to UT. These numbers are required only to the nearest $0^{d}.1$.

TABLE I

Correcting Catalog Times to UT

Source Code	Add to t
6	$0^{d}.7$
7(t $\leqslant$ 1924)	0.7
7(t $\geqslant$ 1925)	0.2
8, 9, 0	0.2

The derivation of observation time from the recorded right ascension, subsequently labelled α, is of utmost importance to the success of the entire operation. Hence, we will record here the essential steps.

Consider any calendar date. Let I represent the year, J the month, K the integral part of the day, and d its fractional part. Then the Julian Date at 0^{h}UT can be obtained from:

$$\begin{aligned} JD_0 = {} & 1721074 + K + 1461 * (I + (J - 14)/12)/4 + 367 * (J - 2 - 12 * \\ & ((J - 14)/12))/12 + (24002 - 12 * I - J)/1200 - 0.5. \end{aligned} \tag{1}$$

Except for the last term, 0.5, this is obviously a Fortran statement employing integer arithmetic. It can be used at once on almost any computer. This formula is based on one originally supplied by Seidelmann (1967) at the USNO. We modified the relation so that it is now valid in the open interval 1600 to 2100.

Next we need Greenwich apparent sidereal time at 0^hUT:

$$GAST_0 = 0^d.2764\ 9045 + 0.002\ 737\ 909\ 298\ (JD_0 - 243\ 1090.5) + 0^d.000\ 000\ 7716\ \Delta\Psi \cos\epsilon . \quad (2)$$

This number must be between 0 and 1 which can be done by adding and subtracting multiples of 1. The last term is usually called equation of the equinoxes. Here $\Delta\Psi$ is the nutation in longitude, expressed in seconds of arc. As in other places, ϵ is the obliquity of the ecliptic.

With this precomputation, the fractional part of the day follows from

$$fd = 0.997\ 269\ 566\ (\alpha^d + 0^d.2140\ 7233 - GAST_0) . \quad (3)$$

If fd does not fall into the interval from 0 to 1, the expression in parentheses is increased or decreased by one. In this formula, α is the observed right ascension, and the next term represents the longitude of the USNO. Finally, the observation time in UT is

$$t_{UT} = JD_0 + fd . \quad (4)$$

The procedure does not always yield a unique time inasmuch as two observation times, about 24^h apart, are possible. The ambiguity is easily resolved as long as the original observation time is recorded to the nearest $0^d.1$ or so. Comparing t_{UT} with the approximate $JD_0 + d$ eliminates the spurious solution. However, in source code 6 times are given to the nearest day only. In this case we wrote both possible t_{UT} on the master tape and rejected the erroneous one only after inspection of the residuals.

It should be emphasized that we obtained observations times for all data, except Pluto, in the manner described above. In other words, this applies not only to data punched here, but also to the more recent data sets provided by JPL and USNO.

At this stage, only the Pluto observations are still in calendar time. Conversion to the Julian data is accomplished using (1) and, since the exact fractional day is available,

$$t_{UT} = JD_0 + d\,. \qquad (5)$$

The discussion up to this point explains the data written on the "master tape". The latter is now fed into our data conversion program (DCP) which produces the "observation tape" actually used by the main program.

All transit circle observations are referred to the true equator and equinox of date. Since the numerical integration yields results in a fixed frame, the observations must be corrected for nutation and precession. We selected the mean frame of 1950.0 to which the Pluto data are already referred.

Most of the algorithm for calculating nutation and precession was lifted directly from the Explanatory Supplement (1961). The required elements of sun and moon are given there on p. 44. The trigonometric series for $\Delta\Psi$ and $\Delta\epsilon$, nutation in longitude and obliquity, pp. 44 and 45, were truncated after terms with coefficient $0''.0050$. We also copied the equation for ϵ, the mean obliquity of the ecliptic of date, written as a polynomical in time, from p. 98. With these preparations, we calculate the effects of nutation as follows. Adding $\Delta\epsilon$ to ϵ gives the true obliquity of date. Together with the available α and δ, we now obtain the true longitude and latitude from the spherical trigonometry relating equator and ecliptic. It is only necessary to subtract $\Delta\Psi$ from the true longitude, and one has mean longitude and latitude of date. Together with ϵ, this time, one solves for α_M and δ_M, where the subscript M indicates values referred to the mean frame of date. In order to be generally applicable, such angles are always obtained from inverse tangent relations.

Also copied from the Explanatory Supplement are the expressions for precession. The quantities ζ_0, z, θ are given on p. 30. We rewrote the spherical relations on p. 31 a bit. Evidently one can combine two equations to obtain tan $(\alpha - z)$, solve for $(\alpha - z)$, and then for α. Subsequently one can also get δ from an arc tan.

All times are still referred to the UT clock. The last function of our DCP is the conversion of the observation times to the adopted clock. For this, ephemeris time was chosen in our earlier solution. The ΔT's to be added to t_{UT} were then obtained from the American Ephemeris and from Brouwer's (1952) original paper on the subject. Later we switched to atomic time. Here the quantity added from 1955.5 on was $32^{s}.15 + A.1 - U.T.2$, given in the American Ephemeris. All values before 1955.5 were obtained as part of our solution. More about this will be said in a later section.

VI. ALGORITHM AND DERIVATIONS

1. PRECOMPUTATIONS

One feature of the program, exercised only in special tests, permits to designate which bodies gravitationally affect any given planet. This option could be used not only for very small asteriods or artificial bodies, but also in cases where, for example, Pluto's effect on Mercury is negligible. Using the list of perturbing planets, separately for each body being integrated, the program selects the appropriate equation of motion, with the minimum number of terms required, from available series of such equations.

In case the initial conditions of any planet are referred to the mean equator and equinox of T different from the selected T_0, the updating is quickly made. The angles ζ_0, z, and θ are obtained as discussed above. Then the precession matrix

$$P = \begin{pmatrix} \cos\zeta_0\cos\theta\cos z - \sin\zeta_0\sin z & -\sin\zeta_0\cos\theta\cos z - \cos\zeta_0\sin z & -\sin\theta\cos z \\ \cos\zeta_0\cos\theta\sin z + \sin\zeta_0\cos z & -\sin\zeta_0\cos\theta\sin z + \cos\zeta_0\cos z & -\sin\theta\sin z \\ \cos\zeta_0\sin\theta & -\sin\zeta_0\sin\theta & \cos\theta \end{pmatrix}$$

Finally

$$\bar{r}(T_0) = P\,\bar{r}(T)$$

and

$$\dot{\bar{r}}(T_0) = P\dot{\bar{r}}(T)$$

Note that T_0 = JD 2433282.423 = B.Y. 19500 in all our production work.

The obliquity of the ecliptic is required in various places. The mean value if given by

$$\epsilon = 23^\circ.452294 - 0^\circ.0130125\Delta T - 0^\circ.00000164\Delta T^2 + 0^\circ.0000005030\,\Delta T^3$$

where

$$\Delta T = \frac{T_0 - 2415020.0}{36525}$$

2. NUMERICAL INTEGRATION ROUTINE

The differential equations described in the next section were numerically integrated using our present "Cowell" routine. This program is the latest modification of a series of such integrators developed and improved over the years by Cohen and Hubbard. Some versions are described by Hubbard and Broadwater in the open literature and in various local reports. We used them in several studies of planetary motion, and they are being employed extensively in satellite geodesy.

The program begins by deriving a number of auxiliary coefficients using the Cowell and Adams coefficients c_i and a_i given in Table X.

$$\gamma'_j = \sum_{i=0}^{[j+1,b]} (-1)^i \binom{b}{i} \left(a_{j+1-i} - c_{j+1-i} \right) \qquad j = 0,1,2,\cdots,a+b$$

where $[j+1,b]$ is the smaller of $j+1$ and b,

$$\gamma_i = (-1)^i \sum_{j=i}^{a+b} \binom{j}{i} \gamma'_j \qquad i = 0,1,2,\cdots,a+b$$

$$b_j = (-1)^j \binom{a+b+1}{j+1} \qquad j = 0,1,2,\cdots,a+b$$

$$\alpha^*_j = (-1)^j \sum_{i=j}^{c} \binom{i}{j} c_i \qquad j = 0,1,2,\cdots,c$$

$$\beta^*_j = (-1)^j \sum_{i=j}^{c} \binom{i}{j} a_1 \qquad j = 0,1,2,\cdots,c$$

$$c'_i = \sum_{j=0}^{i} c_j \qquad i = 0,1,2\cdots,m$$

$$\alpha'_j = (-1)^j \sum_{i=j}^{m} \binom{i}{j} c'_i \qquad j = 0,1,2,\cdots,m$$

$$a_i' = \sum_{j=0}^{i} a_j \qquad i = 0,1,2,\cdots,m$$

$$\beta_i' = (-1)^j \sum_{i=j}^{m} \binom{i}{j} a_i' \qquad j = 0,1,2,\cdots,m$$

$$\alpha_j = (-1)^j \sum_{i=j}^{m} \binom{i}{j} c_i \qquad j = 0,1,2,\cdots,m$$

$$\beta_j = (-1)^j \sum_{i=j}^{m} \binom{i}{j} a_i \qquad j = 0,1,2,\cdots,m$$

Here c and m are the order of starting and running routine, respectively. The controls a and b are better explained farther below.

The starting sequence begins by evaluating the differential equations at epoch, that is, $\ddot{\bar{r}}_0$. The starting table is initialized by putting

$$\ddot{\bar{r}}_n = \ddot{\bar{r}}_0 \text{ for } -a \leqslant n \leqslant b$$

Figure 1 depicts the scheme. In words, the unknown accelerations are set equal to the initial value in the range from −a to b. The numbers a and b cannot exceed c. Next the position on the time line before epoch is given by

$$\bar{r}_{-1} = \bar{r}_0 - h\,\dot{\bar{r}}_0 + h^2 \sum_{i=0}^{a+b} \gamma_i\, \ddot{\bar{r}}_{b-1} \qquad (6)$$

where h is the integration step size.

The accelerations $\ddot{\bar{r}}$ are extrapolated beyond −a and b to c lines before and after epoch by

$$\ddot{\bar{r}}_{-n} = \sum_{j=0}^{a+b} b_j\, \ddot{\bar{r}}_{-n+1+j} \qquad n = a+1, a+2,\cdots,c$$

$$\ddot{\bar{r}}_n = \sum_{j=0}^{a+b} b_j\, \ddot{\bar{r}}_{n-1-j} \qquad n = b+1, b+2,\cdots,c$$

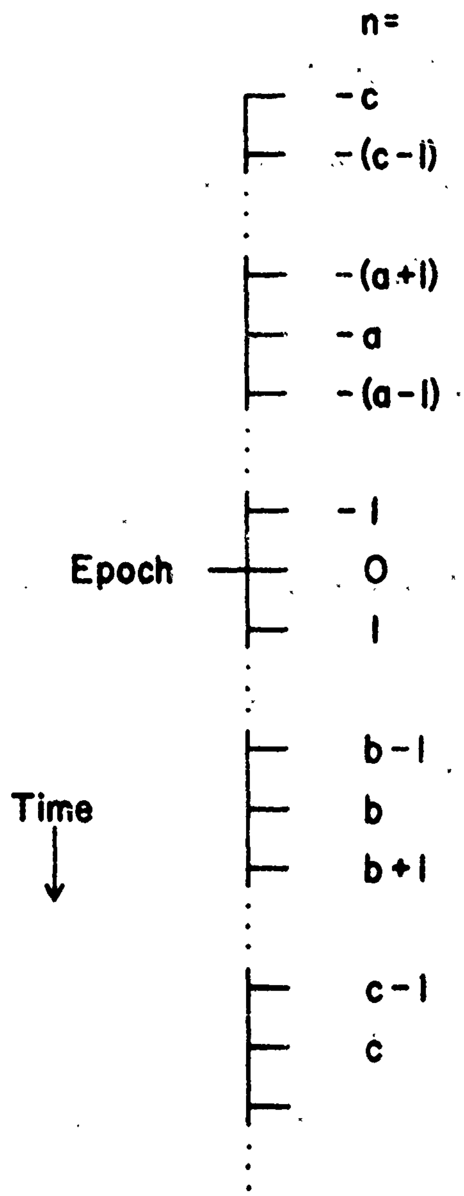

FIGURE 1

Scheme for Generating Starting Table

Position and velocity, the latter always required in our model, are made in the range a to b only:

$$\bar{r}_n = 2\bar{r}_{n-1} - \bar{r}_{n-2} + h^2 \sum_{j=0}^{c} \alpha_j^* \ddot{\bar{r}}_{n-j} \qquad n = 1,2\cdots,b$$

$$\bar{r}_{-n} = 2\bar{r}_{-n+1} - \bar{r}_{-n+2} + h^2 \sum_{j=0}^{c} \alpha_j^* \ddot{\bar{r}}_{-n+j} \qquad n = 2,3,\cdots,a$$

$$\dot{\bar{r}}_n = \dot{\bar{r}}_{n-1} + h \sum_{j=0}^{c} \beta_j^* \ddot{\bar{r}}_{n-j} \qquad n = 1,2,\cdots,b$$

$$\dot{\bar{r}}_{-n} = \dot{\bar{r}}_{-n+1} - h \sum_{j=0}^{c} \beta_j^* \ddot{\bar{r}}_{-n+j} \qquad n = 1,2,\cdots,a$$

Finally, the accelerations in the same interval are calculated by evaluating the differential equations using the coordinates just generated.

At this point the program tests whether K cycles through the starting routine have been completed. We used K = 4 for almost all of our runs based on previous experience. If K is not yet reached, the computer returns to (6) for a new cycle. When K cycles are finished, a convergence test is made. For this purpose, the absolute difference in the accelerations on two successive sweeps is monitored for each differential equation. When the maximum of these numbers reaches a minimum, the starting tables are considered converged.

The running procedure, of course, advances one step at a time. Coordinates on the next time line are made by the predictor formulae

$$\bar{r}_n = 2\,\bar{r}_{n-1} - \bar{r}_{n-2} + h^2 \sum_{j=0}^{m} \alpha_j' \ddot{\bar{r}}_{n-i-j}$$

$$\dot{\bar{r}}_n = \ddot{\bar{r}}_{n-1} + h \sum_{j=0}^{m} \beta_j' \ddot{\bar{r}}_{n-1-j}$$

where m is the order, in general different from c. The corresponding accelerations are calculated from the differential equations. The routine may be exited at this point or coordinates can be refined with the corrector formulae

$$\bar{r}_n = 2\bar{r}_{n-1} - \bar{r}_{n-2} + h^2 \sum_{j=0}^{m} \alpha_j \ddot{\bar{r}}_{n-j}$$

$$\dot{\bar{r}}_n = \dot{\bar{r}}_{n-1} + h \sum_{j=0}^{m} \beta_j \ddot{\bar{r}}_{n-j}$$

Again, new accelerations are computed. In order to enhance stability, the last step is always the evaluation of the accelerations.

After extensive testing, we used the following constants for all our production work:

$$h = 0\overset{d}{.}4$$

$$c = 14$$

$$a = b = 7$$

$$m = 12$$

For the planets Mars through Pluto, it was found that above predictor formulae sufficed. For all other differential equations, including the various variational equations, the predictor-corrector loop was required.

In order to minimize roundoff, coordinates were always computed and stored as double precision numbers. We examined this device quite recently and found that in many applications single precision calculations will give identical results.

3. EQUATIONS OF MOTION

There is no need to reproduce the particle terms of the accelerations. As indicated before, the moon is treated like any other planet, that is, it has its own heliocentric orbit. This choice makes for a simpler program, and no numerical disadvantages result.

Evaluating the differential equations is usually the most time-consuming operation in a numerical integration routine such as this. Hence, we took pains to rewrite the principal equations of motion in a form advantageous to the computer. Also, if a body of negligible mass was integrated, its effects on the other planets was not simply suppressed by putting the mass to zero, but the terms containing its mass were altogether absent.

Relativistic terms could be computed or by-passed separately for each planet through a simple input control. Except for experiments, they were always calculated for every planet. The equations, as quoted by Brouwer and Clemence (1961), are based on the standard Schwarzschild metric. Experts assure us that these approximations are more than adequate to process optical observations, as Brouwear and Clemence stated. Rewritten a bit, the additional terms due to general relatively are

$$\Delta\ddot{\bar{r}} = \frac{2\mu}{c^2 r^5}\left[\mu r - r^2 v^2 + (\bar{r}\cdot\dot{\bar{r}})^2\left(1 + \frac{c^2 r}{2(c^2 r - 2\mu)}\right)\right]\bar{r} + \frac{2\mu}{r^2}\,\frac{\bar{r}\cdot\dot{\bar{r}}}{c^2 r - 2\mu}\,\dot{\bar{r}} \tag{7}$$

where

$$\mu = k^2(1+m)$$

and

$$v = |\dot{\bar{r}}|$$

Spherical harmonics of earth and moon

It is clear that some of the deviations of earth and moon from spherical symmetry play a role in our work. The effects of zonal harmonics can easily be assessed from existing artificial satellite theory, such as Brouwer's (1959). For some of the other terms we made a series of computer runs, with and without the pertinent accelerations, and compared results.

The forces due to the various gravitational terms were derived as follows. In inertial space, the force potential F for a spherical sun and nonspherical earth and moon can be written as

$$\frac{1}{k^2} F = \frac{m_1 m_2}{r_{12}} \left[1 - U_{12}(\bar{P}_1, \bar{r}_{12}) - U_{21}(\bar{P}_2, \bar{r}_{12})\right]$$

$$+ \frac{m_0 m_1}{r_{01}} \left[1 - U_{10}(\bar{P}_1, \bar{r}_{01})\right] + \frac{m_0 m_2}{r_{02}} \left[1 - U_{20}(\bar{P}_2, \bar{r}_{02})\right] + 0(U \times U) \tag{8}$$

The subscripts 0, 1, and 2 refer to sun, earth, and moon. The U_{ij} is the perturbing potential of body i affecting body j, and $\bar{P}_i$ is a vector along the spin axis of body i. The meaning of the other symbols seems clear.

There is no need to consider the oblate figure of the sun in this development. It will be treated when required in a later section. In fact, if the sun were left out altogether, the resulting equations would still be found to provide very good approximations.

We will employ the usual development of (8) using essentially the notation adopted by the IAU (Hagihara, 1962). However, we shall dispense with the commas in subscripts. Moreover, gravitational coefficients belonging to the earth will be denoted by capital letters, those of the moon by lower case letters. Equation (8) then becomes

$$\frac{1}{k^2} F = \frac{m_1 m_2}{r_{12}} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_1}{r_{12}}\right)^n P_n^m(\sin \beta_{12}) \times (C_{nm} \cos m\lambda_{12} + S_{nm} \sin m\lambda_{12})\right.$$

$$\left. + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_2}{r_{12}}\right)^n P_n^m(\sin \beta_{21}) \times (c_{nm} \cos m\lambda_{21} + s_{nm} \sin m\lambda_{21})\right] \tag{9}$$

$$+ \frac{m_0 m_1}{r_{01}} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_1}{r_{01}}\right)^n P_n^m(\sin \beta_{10}) \times (C_{nm} \cos m\lambda_{10} + S_{nm} \sin m\lambda_{10})\right]$$

$$+ \frac{m_0 m_2}{r_{02}} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R_2}{r_{02}}\right)^n P_n^m(\sin \beta_{20}) \times (c_{nm} \cos m\lambda_{20} + s_{nm} \sin m\lambda_{20})\right]$$

In this formula, R stands for the equatorial radius and P for the associated Lengendre polynomials. The symbols β_{ij} and λ_{ij} are the latitude and longitude of body j in an equatorial frame of body i. Since the origin of the customary rectangular frame is, hopefully, at the center of mass, terms with n = 1 vanish. Also, $C_{21} = S_{21} = 0$ and $c_{21} = s_{21} = 0$ since the z-axis is taken along the principal axis of inertia. From various considerations we know that we need the terms with C_{20} and C_{30}, but those factors by C_{22} and S_{22} are negligible. In case of the moon, we need to go to degree two only, but c_{22} and s_{22} will be retained. Recall that the earth is a synchronous satellite over the first meridian of the moon. Finally, the central terms are already in the equations of motion so that only the distrubing part ΔF of (9) is required. There remains

$$\frac{1}{k^2}\Delta F = m_1 m_2 \frac{R_1^2}{r_{12}^3}\left[P_2^0(\sin\beta_{12}) \times C_{20} + \frac{R_1}{r_{12}} P_3^0(\sin\beta_{12}) \times C_{30}\right]$$

$$+ m_1 m_2 \frac{R_2^2}{r_{12}^3}\left[P_2^0(\sin\beta_{21}) \times c_{20} + P_2^2(\sin\beta_{21}) \times \; c_{22}\cos 2\lambda_{21} + s_{22}\sin 2\lambda_{21})\right]$$

$$(10)$$

$$+ m_0 m_1 \frac{R_1^2}{r_{01}^3}\left[P_2^0(\sin\beta_{10}) \times C_{20} + \frac{R_1}{r_{01}} P_3^0(\sin\beta_{10}) \times C_{30}\right]$$

$$+ m_0 m_2 \frac{R_2^2}{r_{02}^3}\left[P_2^0(\sin\beta_{20}) \times c_{20} + P_2^2(\sin\beta_{20}) \times (c_{22}\cos 2\lambda_{20} + s_{22}\sin 2\lambda_{20}\right]$$

The equations of motion in inertial space would be of the form

$$m_j \ddot{\bar{\rho}}_j = \tilde{\nabla}_j \; F \qquad\qquad i = 0,1,2$$

where the tilde indicates differentation with respect to inertial coordinates. Since we want the heliocentric coordinates

$$\ddot{\bar{r}}_j = \ddot{\bar{\rho}}_j - \ddot{\bar{\rho}}_0 \qquad\qquad j = 1,2$$

the equations of motion in the latter frame are found to be

$$\Delta\ddot{\bar{r}}_1 = \left(\frac{1}{m_1} + \frac{1}{m_0}\right)\nabla_1 \Delta F + \frac{1}{m_0}\nabla_2 \Delta F$$

$$\Delta\ddot{\bar{r}}_2 = \left(\frac{1}{m_2} + \frac{1}{m_0}\right)\nabla_2 \Delta F + \frac{1}{m_0}\nabla_1 \Delta F \qquad (11)$$

The gradient now indicates differentation w.r.t. heliocentric coordinates.

Execution of (11) is a bit lengthy but fairly straight-forward. In fact, $\nabla_1 \Delta F$ is available from many different sources. In doing $\nabla_2 \Delta F$, the rapid precessional rate of the lunar equator has to be taken into account, and the transformation into the adopted frame, the earth's equator, has to be made. There are only a few details in the development we wish to record here.

Now that we are in the heliocentric frame, we will write

$$\bar{r}_1 \text{ for } \bar{r}_{01}$$

and

$$\bar{r}_2 \text{ for } \bar{r}_{02}$$

We will also use the geocentric vector to the moon

$$\bar{r}_G = \bar{r}_2 - \bar{r}_1$$

It is then easily seen that

$$\sin\beta_{12} = \frac{z_G}{r_G}$$

and

$$\sin\beta_{10} = \frac{-z_1}{r_1}$$

Both these relations are needed in (10).

Let us now turn our attention to the task of relating coordinates referred to the equator of the moon to that of the earth. For a moment, let any vector without subscript be presented in the latter frame, then the familiar rotation

$$\bar{r}_{ECL} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \bar{r} \tag{12}$$

gives the same vector in the frame of the ecliptic. We will use $\epsilon = 23^\circ\!.446$. With the aid of Figure 2, in the frame of the lunar equator this vector becomes

$$\bar{r}_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{r}_{ECL} \tag{13}$$

Here I is the inclination of the lunar equator on the ecliptic for which we take the mean value

$$I = 1^\circ\!.535.$$

H stands for the ascending node of the lunar equator on the ecliptic. Again, for our purpose we need not be concerned with periodic perturbations. Hence, we find, appealing to lunar theory and Cassinis laws,

$$H = 79^\circ\!.183275 - 0^\circ\!.05295\ 39222\ (t - 241\ 5020.0)$$

where t is expressed as a Julian date. Putting (12) into (13), we have a relation of the form

$$\bar{r}_M = M\ \bar{r}. \tag{14}$$

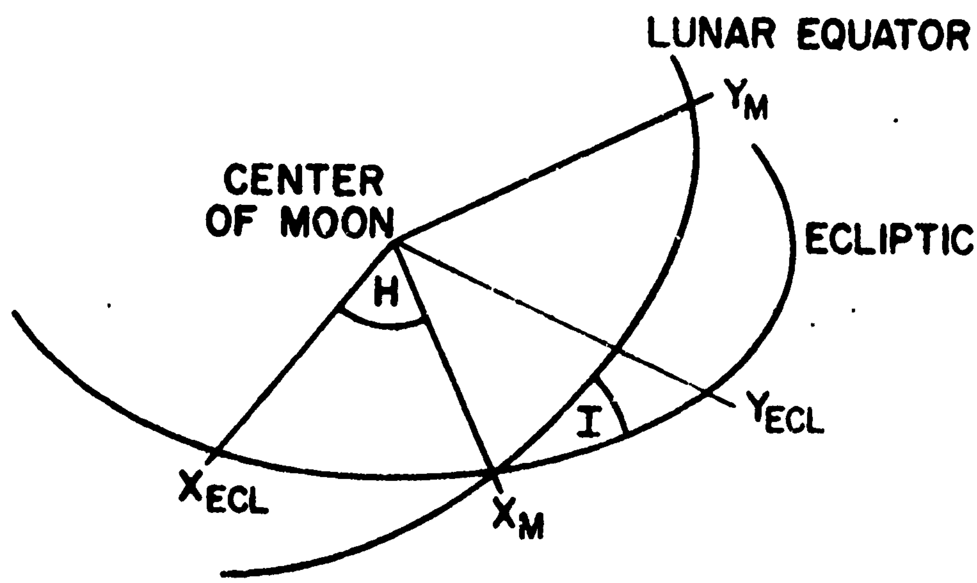

FIGURE 2

Relation of Lunar Equator to Ecliptic

Upon substitution of the numerical values given, one finds

$$M = \begin{pmatrix} \cos H & 0.91744 \sin H & 0.39788 \sin H \\ -.99964 \sin H & 0.91711 \cos H - .01066 & 0.39774 \cos H + .02458 \\ 0.02679 \sin H & -.02458 \cos H - .39774 & -.01066 \cos H + .91711 \end{pmatrix} \quad (15)$$

Now we are ready to consider

$$\sin \beta_{21} = \frac{z_{21}}{r_{21}} = \frac{z_{21}}{r_G}$$

and (16)

$$\sin \beta_{20} = \frac{z_{20}}{r_{20}} = \frac{z_{20}}{r_2}$$

Because of (14), we can write

$$\bar{r}_{21} = M(\bar{r}_1 - \bar{r}_2) = -M\bar{r}_G \quad (17)$$

$$\bar{r}_{20} = M(\bar{0} - \bar{r}_2) = -M\bar{r}_2$$

Let now the last row of M be designated $\overline{M}_3$, that is

$$\overline{M}_3 = \begin{pmatrix} M_{31} \\ M_{32} \\ M_{33} \end{pmatrix} \quad (18)$$

Then, because of (17),

$$z_{21} = -\overline{M}_3 \cdot \bar{r}_G$$

and

$$z_{20} = -\overline{M}_3 \cdot \overline{r}_2$$

Putting this into (16), we have the desired result

$$\sin \beta_{21} = \frac{-\overline{M}_3 \cdot \overline{r}_G}{r_G}$$

$$\sin \beta_{20} = \frac{-\overline{M}_3 \cdot \overline{r}_2}{r_2}$$

In order to ıelate the two selenocentric longitudes in (10) to available coordinates, we begin with Figure 3. The unprimed coordinates are already familiar. The primed ones refer to a frame rotating with the moon oriented such that x_M goes through the first meridian in the equator from the latter's ascending node on the ecliptic. The angle λ is the selenocentric longitude of an object at S. The relation between any object's polar and rectangular coordinates are obviously

$$\begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} = r \begin{pmatrix} \cos\beta \cos(\lambda + \theta) \\ \cos\beta \sin(\lambda + \theta) \\ \sin\beta \end{pmatrix}$$

Hence

$$\lambda_{21} = \tan^{-1} \frac{y_{21}}{x_{21}} - \theta$$

$$\lambda_{20} = \tan^{-1} \frac{y_{20}}{x_{20}} - \theta \qquad (19)$$

Now consider (17). If we designate the first two rows of M by $\overline{M}_1$ and $\overline{M}_2$, analagous to (18), there follows

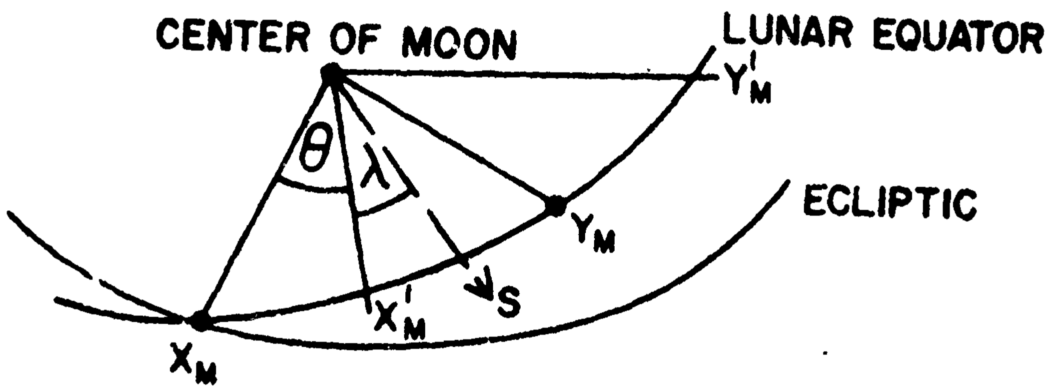

FIGURE 3

Relation of Selenocentric Longitude to Ecliptic

$$x_{21} = -\overline{M}_1 \cdot \bar{r}_G \qquad y_{21} = -\overline{M}_2 \cdot \bar{r}_G$$

$$x_{20} = -\overline{M}_1 \cdot \bar{r}_2 \qquad y_{20} = -\overline{M}_2 \cdot \bar{r}_2$$

With these relations (19) becomes

$$\lambda_{21} = \tan^{-1} \frac{-\overline{M}_2 \cdot \bar{r}_G}{-\overline{M}_1 \cdot \bar{r}_G}$$

$$\lambda_{20} = \tan^{-1} \frac{-\overline{M}_2 \cdot \bar{r}_2}{-\overline{M}_1 \cdot \bar{r}_2} \qquad (20)$$

All quantities in (10) are now expressed in terms of available coordinates so that the differentiations indicated in (11) can be performed. The final results together with an outline of the sequence of calculations follows.

Notation is often a problem, and this case is no exception. While it is convenient in this section to let the subscript 1 refer to the earth and 2 to the moon, the subscripts actually used in the computer program were 3 and 11. Since it will be necessary to use the latter elsewhere in this paper, this minor inconvenience should be noted.

Calculations begin with (14) followed by the various M_{ij} in (15). Next we find

$$\theta = \theta_0 + \omega(t - t_0)$$

Here ω is the draconic rate of the moon. The numerical values employed by us were

$$\dot{\omega} = 13\overset{\circ}{.}2293\ 5045\ /\ \text{day}$$

and

$$\theta_0 = 25\overset{\circ}{.}531$$

The longitudes λ_{21} and λ_{20} follow from (20). Let

$$\hat{P} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and calculate

$$S_1 = \frac{3}{2} k^2 C_{20} R_1^2 \frac{1}{r_G^5} \left[\left(5 \frac{z_G^2}{r_G^2} - 1 \right) \bar{r}_G - 2 z_G \hat{P} \right]$$

$$S_2 = \frac{3}{2} k^2 C_{20} R_1^2 \frac{1}{r_1^5} \left[\left(5 \frac{z_1^2}{r_1^2} - 1 \right) \bar{r}_1 - 2 z_1 \hat{P} \right]$$

$$S_3 = \frac{1}{2} k^2 C_{30} R_1^2 \frac{1}{r_G^5} \left[5 \frac{z_G}{r^2} \left(7 \frac{z_G^2}{r_G^2} - 3 \right) \bar{r}_G - 3 \left(5 \frac{z_G^2}{r_G^2} - 1 \right) \hat{P} \right]$$

$$S_4 = \frac{1}{2} k^2 C_{30} R_1^3 \frac{1}{r_1^5} \left[5 \frac{z_1}{r_1^2} \left(7 \frac{z_1^2}{r_1^2} - 3 \right) \bar{r}_1 - 3 \left(5 \frac{z_1^2}{r_1^2} - 1 \right) \hat{P} \right]$$

$$S_5 = \frac{3}{2} k^2 R_2^2 \frac{\bar{r}_G}{r_G^5} \left\{ c_{20} \left[3 \frac{(\bar{M}_3 \cdot \bar{r}_G)^2}{r_G^2} - 1 \right] - 6 \left[\frac{(\bar{M}_3 \cdot \bar{r}_G)^2}{r_G^2} - 1 \right] (c_{22} \cos 2\lambda_{21} + s_{22} \sin 2\lambda_{21}) \right\}$$

$$S_6 = \frac{3}{2} k^2 R_2^2 \frac{\bar{r}_2}{r_G^5} \left\{ c_{20} \left[3 \frac{(\bar{M}_3 \cdot \bar{r}_2)^2}{r_2^2} - 1 \right] \right.$$

$$\left. - 6 \left[\frac{(\bar{M}_3 \cdot \bar{r}_2)^2}{r_2^2} - 1 \right] \left(c_{22} \cos 2\lambda_{20} + s_{22} \sin 2\lambda_{20} \right) \right\}$$

$$S_7 = 3k^2 R_2^2 \frac{\bar{M}_3 \cdot \bar{r}_G}{r_G^7} \left(r_G^2 \bar{M}_3 - \bar{M}_3 \cdot \bar{r}_G \, \bar{r}_G \right)$$

$$\times \left[c_{20} - 2 \left(c_{22} \cos 2\lambda_{21} + s_{22} \sin 2\lambda_{21} \right) \right]$$

$$S_8 = 3k^2 R_2^2 \frac{\bar{M}_3 \cdot \bar{r}_2}{r_2^7} \left(r_2^2 \bar{M}_3 - \bar{M}_3 \cdot \bar{r}_2 \bar{r}_2 \right)$$

$$\times \left[c_{20} - 2 \left(c_{22} \cos 2\lambda_{20} + s_{22} \sin 2\lambda_{20} \right) \right]$$

$$S_9 = 6k^2 R_2^2 \frac{1}{r_G^3} \left[\frac{(\bar{M}_3 \cdot \bar{r}_G)^2}{r_G^2} - 1 \right]$$

$$\times \frac{\bar{M}_1 \cdot \bar{r}_G \bar{M}_2 - \bar{M}_2 \cdot \bar{r}_G \bar{M}_1}{(\bar{M}_1 \cdot \bar{r}_G)^2 + (\bar{M}_2 \cdot \bar{r}_G)^2} \left(c_{22} \sin 2\lambda_{21} - s_{22} \cos 2\lambda_{21} \right)$$

$$S_{10} = 6k^2 R_2^2 \frac{1}{r_2^3} \left[\frac{(\bar{M}_3 \cdot \bar{r}_2)^2}{r_2^2} - 1 \right]$$

$$\times \frac{\bar{M}_1 \cdot \bar{r}_2 \bar{M}_2 - \bar{M}_2 \cdot \bar{r}_2 \bar{M}_1}{(\bar{M}_1 \cdot \bar{r}_2)^2 + (\bar{M}_2 \cdot \bar{r}_2)^2} \left(c_{22} \sin 2\lambda_{20} - s_{22} \cos 2\lambda_{20} \right)$$

Finally, the additions to the accelerations of earth and moon become

$$\Delta\ddot{\bar{r}}_1 = m_2\left(S_1 + S_3 + S_5 - S_6 - S_7 + S_8 - S_9 + S_{10}\right) + \left(1 + m_1\right)\left(- S_2 + S_4\right)$$

$$\Delta\ddot{\bar{r}}_2 = m_1\left(- S_1 - S_2 - S_3 + S_4 - S_5 + S_7 + S_9\right) + \left(1 + m_2\right)\left(- S_6 + S_8 + S_{10}\right)$$

Acceleration terms due to precession of the earth's equator were developed but found to be well below required precision.

Lunar and solar tide coupling effects

The principal component of orbital acceleration due to tidal forces is undoubtedly in the transverse direction. We will assume, without validation, that any radial and normal components of such acceleration produce only negligible effects. Then the moon, in its orbit around the earth, would have an additional term of the form

$$\Delta\ddot{\bar{r}}_G = C\frac{\bar{h}_G \times \bar{r}_G}{h_G\, r_G}$$

The semimajor axis of the moon is a good approximation to r_G, so that we can use

$$\Delta\ddot{\bar{r}}_G = C\frac{\bar{h}_g \times \bar{r}_G}{0.00256\, h_G} \qquad (21)$$

This is how we felt when this force was first introduced. Now we are no longer sure that putting a_{ζ} for r_G is a satisfactory approximation, especially when (21) is employed later to obtain partial derivatives. No changes in the model were made, though.

In (21), as in many other parts of this paper, the angular momentum vector

$$\bar{h} = \bar{r} \times \dot{\bar{r}}.$$

For the reader familiar with the treatment of tidal effects in general theory, the following remarks should be made. Taking Gauss' equation for the time derivative of the semimajor axis and considering (21), we have

$$\dot{a} = \frac{2}{n} C$$

Double integration gives, in mean anomaly or longitude, the term

$$\bar{\Delta} l = \frac{-3C}{2a} (t - t_0)^2$$

Back to the problem at hand. Using (21), we need to determine the corresponding heliocentric accelerations. The barycenter of the earth-moon system is

$$\bar{r}_B = \frac{m_3 \bar{r}_3 + m_{11} \bar{r}_{11}}{m_3 + m_{11}}$$

The same holds for the accelerations and, in particular, for a small increment:

$$(m_3 + m_{11}) \Delta \ddot{\bar{r}}_B = m_3 \Delta \ddot{\bar{r}}_3 + m_{11} \Delta \ddot{\bar{r}}_{11}$$

Since the perturbations of tidal forces on this barycenter are insensibly small, it is quite safe to put

$$\Delta \ddot{\bar{r}}_B = \bar{0}$$

There follows

$$\Delta \ddot{\bar{r}}_3 = \frac{-m_{11}}{m_3} \Delta \ddot{\bar{r}}_{11} \tag{22}$$

Recall that

$$\bar{r}_G = \bar{r}_{11} - \bar{r}_3$$

or (23)

$$\Delta \ddot{\bar{r}}_G = \Delta \ddot{\bar{r}}_{11} - \Delta \ddot{\bar{r}}_3$$

Combining (22) and (23) gives

$$\Delta \ddot{\bar{r}}_{11} = \frac{m_3}{m_3 + m_{11}} \Delta \ddot{\bar{r}}_G$$

Upon substituting (21) into this, we have

$$\Delta \ddot{\bar{r}}_{11} = \frac{m_3}{m_3 + m_{11}} C \frac{\bar{h}_G \times \bar{r}_G}{0.00256\, h_G} \tag{24}$$

The equations actually programmed are (24) and (22), in that order.

Earlier versions of the program assumed the acceleration to be along the velocity vector, that is, instead of $\hat{h} \times \hat{r}$ we used $\dot{\hat{r}}$. The change to our present formulation produced only minute differences.

Since the sun gives rise to a tide of similar proportions, we decided to add a solar tide coupling term and to see whether its coefficient could be determined. The addition to the model is even simplier than in case of lunar tides. It is seen, by inspecting (21), that the earth will have the additional term

$$\Delta \ddot{\bar{r}}_3 = C_{\odot} \frac{\bar{h}_3 \times \bar{r}_3}{0.9994\, h_3} \tag{25}$$

There is no contribution to the motion of the moon. Indirect effects are automatically applied through the equations of motion. In this case, replacing r_3 by a_3 is probably an adequate approximation.

The Oblate Sun

It will be assumed that only the second zonal harmonic is needed. Letting z_s be a z-coordinate in the equatorial frame of the sun, the disturbing potential is

$$\Delta U = -\frac{\mu}{r} J_{\odot} \left(\frac{R_{\odot}}{r}\right)^2 P_2\left(\frac{z_s}{r}\right) \tag{26}$$

The meaning of the various symbols is believed to be clear. Since only J_2 is considered, the subscript 2 was left off. We used the number

$$R_{\odot} = 0.0046524 \text{ a.u.}$$

$J_{\odot}$ will be solved for, and an initial estimate of zero will serve nicely.

Before computing the accelerations due to (26), we only have to relate z_s to the available coordinates. The procedure is quite analogous to that employed in reducing the lunar equatorial elements. The transformation matrix here will be called N, but it has the form of M in the earlier section. Since only z_s is needed, only the third row of N will be required, labeled N_3. Hence

$$\bar{N}_3 = \begin{pmatrix} \sin\Omega \sin I \\ -\cos\Omega \sin I \cos\epsilon - \cos I \sin\epsilon \\ -\cos\Omega \sin I \sin\epsilon + \cos I \cos\epsilon \end{pmatrix}$$

where

Ω is the longitude of the ascending node of the solar equator on the ecliptic and

I is the inclination of this equator to the ecliptic.

In the Explanatory Supplement on page 307 we find, for our frame of 1950:

$$\Omega = 75°4'$$

$$I = 7°15'$$

There follows that

$$\overline{N}_3 = \begin{pmatrix} 0.12194 \\ -0.42454 \\ 0.89716 \end{pmatrix}$$

with much more precision than required. We can now put

$$z_s = \overline{N}_3 \cdot \overline{r}$$

into (26) and start differentiating. The resulting acceleration terms are

$$\Delta\ddot{\overline{r}} = -\frac{3}{2}\frac{\mu J_\odot R_\odot^2}{r^5}\left\{\overline{r} + \overline{N}_3 \cdot \overline{r}\left[2\overline{N}_3 - 5\left(\overline{N}_3 \cdot \overline{r}\right)\frac{\overline{r}}{r^2}\right]\right\} \qquad (27)$$

Note that $\mu = k^2$ because of our units.

Above equation is valid for all planets. However, since only the observations of Mercury and Venus are to be used to determine $J_\odot$, (27) was added only to the equations of motion of these two planets. This is more than adequate, as may be seen later.

4. INTERPOLATION AND LIGHT TIME

The coordinates and numerically integrated partials will have to be interpolated to the observation time. We are using the Lagrangian interpolation polynomical of degree

$$b = \begin{cases} m+4 & m \text{ even} \\ m+3 & m \text{ odd} \end{cases}$$

This m is meant to be the order of the running routine. The reasons for the relatively high degree are as follows. Since the accelerations are integrated with a polynomical of degree m, the corresponding coordinates require $m+2$ if to be interpolated without loss of accuracy. But since our formulation accepts only even b, and because of considerations involving light transmission time corrections, we adopted above scheme.

Until now, all observation times were light front arrival times at the earth. However, the coordinates of all bodies have to be found at the instant the solar illumination reaches them. This statement also implies that solar observations are the only ones not to be adjusted. Hence, for moon and planets one first calculates

$$t' = t - \frac{|\bar{r}_i - \bar{r}_3|}{c}$$

where c is the speed of light.

One now goes back to the interpolation routine and finds new coordinates $\bar{r}_i$ at t', but always retaining $\bar{r}_3$ at t. The process is continued to convergence, that is, until

$$|t'_{n+1} - t'_n| < \tau$$

For us $\tau = 10^{-7}$ seemed quite satisfactory. The program finally interpolates also for velocities for the last t′.

As the program iterates toward a least-squares solution, t′ will also change slightly and converge to a final value. The latter, of course, is not known in the beginning. We investigated this problem and found the consequences negligible, as intuition would suggest.

5. RESIDUALS AND CORRECTIONS

For each observation we make the computed place

$$\alpha = \tan^{-1} \frac{y - y_3}{x - x_3}$$
$$\delta = \tan^{-1} \frac{z - z_3}{\sqrt{(x - x_3)^2 + (y - y_3)^2}} \qquad (29)$$

and then the residuals

$$\Delta\alpha = \left(\alpha_{OBS} - \alpha_{COMP}\right) \cos\delta_{COMP}$$
$$\Delta\delta = \delta_{OBS} - \delta_{COMP} \qquad (30)$$

Early results on the moon indicated a distinct break in the declination residuals at about JD 242 3859.0 (1924 March). We were unable to find a probable cause for this bias. After a careful determination of the magnitude, we added the correction

$$\delta\delta = +\,0''.57$$

to all $\Delta\delta_{11}$ before above date. Note that this adjustment, together with the declination bias parameter solved for of about 0".33, adds a large 0".90 to those early observations.

All observations except those of Pluto must be corrected for the effects of annual aberration. Hence, we add to their residuals

$$\delta\alpha \cos\delta = \frac{1}{c}(\dot{x}_3 \sin\alpha - \dot{y}_3 \cos\alpha)$$

$$\delta\delta = \frac{1}{c}(\dot{x}_3 \cos\alpha \sin\delta + \dot{y}_2 \sin\alpha \sin\delta - \dot{z}_3 \cos\delta)$$

where c is the speed of light.

In keeping with standard procedures of reducing photographic observations, Pluto's observations had not been corrected for the eliptic terms of aberration. We make this adjustment now by adding to the residuals of Pluto

$$\delta\alpha \cos\delta = 0''.061 \cos\alpha - 0''.336 \sin\alpha$$

$$\delta\delta = -\ 0''.061 \sin\alpha \sin\delta - 0''.336 \cos\alpha \sin\ \delta + 0''.027 \cos\delta$$

As mentioned elsewhere in this paper, there exists a pronounced effect of the phase of Venus on both α and δ . Corrections are made at this point. To the above residuals we add

$$\delta\alpha \cos\ \delta = -0''.9159 - 0''.02401 \times 10^{-2}\varphi + 0''.22745 \times 10^{-4}\varphi^2$$

$$\delta\delta = -0''.6789 + 0''.81930 \times 10^{-2}\varphi - 0''.46772 \times 10^{-4}\varphi^2 + 0''.93085 \times 10^{-7}\varphi^3$$

The angle φ is the difference of orbital longitudes in the sense Venus minus Earth. It is given by

$$\varphi = 74^{\circ}.37 + 0^{\circ}.616515\ [t(JD) - 242\ 0000.5] \qquad 0 \leqslant \varphi < 360^{\circ}$$

In a final step, a test is made to ascertain that the residuals are very small numbers. They could conceivably be in the vicinity of -360° or $+360^{\circ}$. If so, 360° is added or subtracted in (29) before scaling by $\cos\delta$.

6. PARTIAL DERIVATIVES

Partials for Planetary Orbits

The orbital parameters originally chosen for the major planets are the nonsingular elements devised by Cohen and Hubbard (1962). At the time we constructed our computer program this seemed like a sensible choice mostly because various subroutines, involving these variables, were available and carefully checked. The obvious drawbacks of this approach are a more complicated program since the nonsingular elements need to be related to the classical ones or position and velocity. The frequently used partials by Eckert and Brouwer would serve as well.

We will not repeat anything that is easily found in the paper by Cohen and Hubbard. In order to simplify both formulation and programming, their variables were renamed as follows:

$$E_i = q_{i-1}, \qquad i = 1,2,3,4$$

$$E_5 = e_X$$

$$E_6 = e_y$$

As a precomputation, the initial conditions of the planets have to be converted to the E_i at t_0. All necessary equations are given in the Cohen and Hubbard paper. Also, their equation (9) has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where the denominator $E_1^2 + E_2^2 + E_3^2 + E_4^2$ has been absorbed in our a_{ij}. We compute and store away for future use the first six a_{ij}. The third row is not needed.

With these preparations, we are ready to calculate the partials

$$\frac{\partial D}{\partial \bar{E}^T}$$

where the vector $\bar{D}$ contains the observables α and δ and $\bar{E}$ our six nonsingular elements E_i. It is clear that the values of these elements at t_0 are to be improved, but we will skip the additional subscript zero. Since the observed α and δ of any planet are affected by the positions of planet and earth, we need to allow for this dependence in our normal equations. Hence, we calculate both $\partial \bar{D}_i / \partial \bar{E}_i^T$ and $\partial \bar{D}_i / \partial \bar{E}_3^T$. The relation between $\bar{D}$ and $\bar{E}$ is easily established using the rectangular coordinates as an intermediary, that is

$$\bar{D}_i = \bar{D}_i \left[\bar{r}_i(\bar{E}_i), \; \bar{r}_3(\bar{E}_3) \right]$$

Therefore

$$\frac{\partial \bar{D}_i}{\partial \bar{E}^T} = \frac{\partial \bar{D}_i}{\partial \bar{r}^T} \frac{\partial \bar{r}_i}{\partial \bar{E}^T} \tag{30}$$

$$\frac{\partial \bar{D}_i}{\partial \bar{E}_3^T} = \frac{\partial \bar{D}_i}{\partial \bar{r}_3^T} \frac{\partial \bar{r}_3}{\partial \bar{E}_3^T} \tag{31}$$

Equation (31) is not used in this form. Because of (28)

$$\frac{\partial \bar{D}_i}{\partial \bar{r}_3^T} = \frac{-\partial \bar{D}_i}{\partial \bar{r}_i^T}$$

Hence, (31) can be written

$$\frac{\partial \bar{D}_i}{\partial \bar{E}_3^T} = - \frac{\partial \bar{D}_i}{\partial \bar{r}_i^T} \frac{\partial \bar{r}_3}{\partial \bar{E}_3^T} \tag{32}$$

Equations (30) and (32) are actually employed, and evaluated for every observation of moon and planets. In case i = 10, observations of the sun, only formula (32) is calculated.

The first member on the right of (30) and (32) can be obtained from (28). The results are

$$\frac{\partial \alpha}{\partial \bar{r}^T} = \frac{1}{(\Delta x)^2 + (\Delta y)^2} \left(-\Delta y, \Delta x, 0\right)$$

$$\frac{\partial \delta}{\partial \bar{r}^T} = \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]} \left(-\Delta z \Delta x, -\Delta z \Delta y, (\Delta x)^2 + (\Delta y)^2\right)$$

where

$$\Delta \bar{r} = \bar{r}_i - \bar{r}_3$$

The other two matrices, namely $\partial \bar{r}_i / \partial \bar{E}_i^T$ and $\partial \bar{r}_3 / \partial \bar{E}_3^T$, are given by Cohen and Hubbard. We cannot consider the second matrix a special case of the first since their treatment differs, as we will see in a moment. One first computes

$$\varphi = n(t - t_0) + \frac{1}{a^2 n} \bar{r} \cdot \dot{\bar{r}}$$

Next comes $\bar{p}$, C_X, and C_Y, given by Cohen and Hubbard in equations (49), (50), and (51). Then we need their quantities X and Y which are best obtained from

$$X = a_{11}\, x + a_{12} y + a_{13}\, z$$

$$Y = a_{21}\, x + a_{22} y + a_{23}\, z$$

Wherever the time appears in φ, $\bar{\rho}$, C_X, and C_Y, it is to be taken as the reduced time t′. When dealing with the partials for the elements of the earth, recall that t′ is identical to t. Finally, the partials $\partial\bar{r}/\partial\bar{E}^T$ are given by the six matrices in the Cohen and Hubbard equation (48).

The above equations seem to suggest that all elements or coordinates to be evaluated at the running time t or t′, as opposed to initial values at t_0, can be taken from the numerical integration results. This must not be done. The difficulties encountered when evaluating Keplerian partials using ingredients from the actual, perturbed orbit were discussed in detail in an earlier paper (Cohen, Hubbard, and Oesterwinter, 1967). Hence, we proceed as we did in the 1967 study. For the purpose of calculating the partials, a Keplerian orbit was carried for each planet which coincided with the actual orbit at epoch. This is simple enough since only the mean anomaly needs to be updated for each observation time. Subsequent conversion to coordinates is done by a subroutine already available.

Partials for the Lunar Orbit

A fair amount of time was spent in experimenting with various sets of partial derivatives for correcting the lunar orbit. It is clear that Keplerian partials are inadequate because of the very large perturbations in the lunar orbit. We did, however, try to use partials that took account of the secular perturbations in l, g, and h. This approach failed miserably, evidently because of the large periodic perturbations. We then appealed to lunar theory and used partials considerably more sophisticated. Again, no reasonable solution was obtained. Since such analytical partials are used successfully elsewhere, perhaps we carried an insufficient number of terms. On the other hand, it is probable that the addition of clock corrections and tidal coefficients made for a fairly ill-conditioned normal matrix. In that case, partials of a given accuracy might fail which would be perfectly good when solving for orbital parameters only.

By this time we had decided to pay the price for partials obtained by numerical integration. We have used this device for many years in application where analytical theories do not exist or where great precision is required. The latter is the case in our work in satellite geodesy and various other studies involving artificial satellites. Many significant figures are lost in inverting the normal matrix which, obviously, requires that the partials have several more figures to start with.

Since we are dealing with additional differential equations, usually lengthy, that are to be integrated numerically, this approach is always costly. In the case at hand, we have 30 differential equations for the motion of the 10 bodies involved. It will be seen that the partials for the lunar orbit corrections require 18 more equations. Hence, computer time is increased by 50% or so.

The differential equations in question are usually called variational or perturbation equations. The procedure depends on the fact that

$$\ddot{\left(\frac{\partial \bar{r}}{\partial P}\right)} = \frac{\partial}{\partial P}\ddot{\bar{r}} \tag{33}$$

The derivatives exist and all functions involved are continuous. P is any parameter $\bar{r}$ depends on. The right hand side of (33) can always be obtained since the equations of motion are known. Cowell integration of this expression then yields the desired partials $\partial \bar{r} / \partial P$. In our problem, we need the matrix $\partial \bar{r} / \partial \bar{E}^T$, where $\bar{E}$ contains any six elements describing the lunar orbit. Hence the 18 different partials mentioned earlier.

We decided, somewhat arbitrarily, to differentially correct the geocentric orbit of the moon. It is possible that a set of heliocentric parameters would work equally well, especially when rigorous partial derivatives are used. So we need the geocentric acceleration of the moon which is obtained from the heliocentric equations of motion and

$$\ddot{\bar{r}}_G = \ddot{\bar{r}}_{11} - \ddot{\bar{r}}_3 .$$

The result is short enough to be recorded comfortably:

$$\ddot{\bar{r}}_G = -k^2(m_3 + m_{11})\frac{\bar{r}_3}{r_G^3} - k^2\left(\frac{\bar{r}_{11}}{r_{11}^3} - \frac{\bar{r}_3}{r_3^3}\right) - k^2 \sum_{\substack{j=1 \\ j \neq 3}}^{9} m_j\left(\frac{\bar{r}_j - \bar{r}_3}{|\bar{r}_j - \bar{r}_3|^3} - \frac{\bar{r}_j - \bar{r}_{11}}{|\bar{r}_j - \bar{r}_{11}|^3}\right)$$

$$- \frac{3}{2}k^2(m_3 + m_{11})C_{20}R_3^2\frac{1}{r_G^5}\left[\left(5\frac{z_G^2}{r_G^2} - 1\right)\bar{r}_G - 2z_G P\right] \tag{34}$$

This equation is seen to be incomplete. It does not contain relativistic or tidal terms nor spherical harmonics except the second zonal harmonic of the earth. We made a calculation to determine the contribution of relativistic effects on certain partial derivatives for Mercury and found figures like 1.0×10^{-7} in units of the leading term. Hence, we concluded that it would be safe to ignore relativity in partials for the moon. The other neglected terms were not checked carefully. They were assumed to be negligible because of their relative size in the equations of motion. In any event, with partial derivatives based on (34) the least-squares solutions converge rapidly, making any gross errors in this area unlikely.

The effects of small changes in the geocentric orbit of the moon on the orbit of the earth-moon barycenter around the sun are minute. We can safely assume that the heliocentric coordinates are invariant when correcting the geocentric orbit of the moon. This simplification can be used to our advantage immediately. It is clear that we can replace $\bar{r}_3$ and $\bar{r}_{11}$ in (34) by

$$\bar{r}_3 = \bar{r}_B - \frac{m_{11}}{m_3 + m_{11}} \bar{r}_G$$

$$\bar{r}_{11} = \bar{r}_B + \frac{m_3}{m_3 + m_{11}} \bar{r}_G$$

where $\bar{r}_B$ is the barycenter mentioned. The latter will be treated as constant when differentiating (34). Since the planetary coordinates can also be treated as constants, the only variable left in (34) will be $\bar{r}_G$.

We frequently use the abbreviation

$$\bar{\xi} = \frac{\partial \bar{r}}{\partial P}$$

where P is any parameter $\bar{r}$ depends on. In this particular section P is any one of the moon's orbital elements, but it will later be seen to be advantageous to perform the formal differentiation of (34) without assigning a specific meaning to P. The final result of the operation is

$$\xi_k = \frac{\partial \ddot{\bar{r}}_G}{\partial P_k} = \frac{\partial \ddot{\bar{r}}_G}{\partial \bar{r}_G}\frac{\partial \bar{r}_G}{\partial P_k} k = \left\{ k^2(m_3 + m_{11})\frac{1}{r_G^3}\left(\frac{3}{r_G^2}\bar{r}_G\bar{r}_G^T - I_3\right)\right.$$

$$+\frac{k^2}{m_3 + m_{11}}\left[\frac{m_3}{r_{11}^3}\left(\frac{3}{r_{11}^2}\bar{r}_{11}\bar{r}_{11}^T - I_3\right) + \frac{m_{11}}{r_3^3}\left(\frac{3}{r_3^2}\bar{r}_3\bar{r}_3^T - I_3\right)\right]$$

$$-\frac{3}{2}k^2(m_3 + m_{11})\frac{C_{20}R_3^2}{r_G^5}\left[\left(5\frac{z_G^2}{r_G^2} - 1\right)I_3 - \frac{5}{r_G^2}\left(7\frac{z_G^2}{r_G^2} - 1\right)\bar{r}_G\bar{r}_G^T + 10\frac{z_G}{r_G^2}\left(P\bar{r}_G^T + \bar{r}_G P^T\right) - 2PP^T\right] \tag{35}$$

$$+\frac{k^2 m_{11}}{m_3 + m_{11}}\sum_{\substack{j=1\\ j\neq 3}}^{9}\frac{m_j}{|\bar{r}_j - \bar{r}_3|^3}\left[\frac{3}{|\bar{r}_j - \bar{r}_3|^2}(\bar{r}_j - \bar{r}_3)(\bar{r}_j - \bar{r}_3)^T - I_3\right]$$

$$\left. +\frac{k^2 m_3}{m_3 + m_{11}}\sum_{\substack{j=1\\ j\neq 3}}^{9}\frac{m_j}{|\bar{r}_i - \bar{r}_{11}|^3}\left[\frac{3}{|\bar{r}_j - \bar{r}_{11}|^2}(\bar{r}_j - \bar{r}_{11})(\bar{r}_j - \bar{r}_{11})^T - I_3\right]\right\}\ddot{\xi}_k$$

What cannot be seen is that (35) is incomplete. The addition will be treated where it logically arises, namely under tidal effects. The unit vector $\hat{P}$ has nothing to do with the parameter P_k. As elsewhere in this paper, $\hat{P} = (0,0,1)^T$. I_3 is the identity matrix. An expression such as $\bar{r}\bar{r}^T$ is shorthand for the matrix

$$\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$

It is important to remember that all coordinates in (35) are presented in the equatorial frame of the earth.

Except for an additional application to be discussed in the next section, equation (35) is used for the orbital elements of the moon. Hence P_k stands for such elements, and we found it more convenient to deal with equatorial elements rather than the customary ecliptic ones. What remains to be done before (35) can be integrated numerically is to find the initial values of the various $\bar{\xi}_k$. This is easily done using the Keplerian relations between elements and coordinates. They are given here in a form we like to use, but many variants can be found in the literature.

$$\frac{\partial \bar{r}}{\partial a} = \frac{1}{a}\left[\bar{r} - \frac{3}{2}\dot{\bar{r}}(t - t_0)\right]$$

$$\frac{\partial \bar{r}}{\partial e} = -\frac{\cos E + e}{1 - e^2}\bar{r} + \left(1 + \frac{r}{p}\right)\frac{\sin E}{n}\dot{\bar{r}}$$

$$\frac{\partial \bar{r}}{\partial I} = \hat{N} \times \bar{r} = \begin{pmatrix} z \sin h \\ -z \cos h \\ y \cos h - x \sin h \end{pmatrix}$$

$$\frac{\partial \bar{r}}{\partial l} = \frac{\dot{\bar{r}}}{n} \tag{36}$$

$$\frac{\partial \bar{r}}{\partial g} = h \times \bar{r} = \frac{-1}{\sqrt{1 - e^2}}\left(e \sin E\bar{r} - \frac{r^2}{a^2 n}\dot{\bar{r}}\right)$$

$$\frac{\partial \bar{r}}{\partial h} = P \times \bar{r} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$\frac{\partial \dot{\bar{r}}}{\partial a} = \frac{1}{2a}\left[3\left(\frac{a}{r}\right)^3 n^2(t - t_0)\bar{r} - \dot{\bar{r}}\right]$$

$$\frac{\partial \dot{\bar{r}}}{\partial e} = -n\left(\frac{a}{r}\right)^3 \left(1 + \frac{r}{p}\, e \cos E\right) \sin E\bar{r} + \frac{1}{1 - e^2} \cos E\dot{\bar{r}}$$

$$\frac{\partial \dot{\bar{r}}}{\partial I} = \hat{N} \times \dot{\bar{r}} = \begin{pmatrix} \dot{z} \sinh \\ - \dot{z} \cosh \\ \dot{y} \cosh - \dot{x} \sinh \end{pmatrix}.$$

$$\frac{\partial \dot{\bar{r}}}{\partial l} = - n\left(\frac{a}{r}\right)^3 \bar{r} \qquad (37)$$

$$\frac{\partial \dot{\bar{r}}}{\partial g} = \hat{h} \times \dot{\bar{r}} = - \frac{an}{r\sqrt{1 - e^2}} (1 + e \cos E)\,\bar{r} + \frac{e \sin E}{\sqrt{1 - e^2}} \dot{\bar{r}}$$

$$\frac{\partial \dot{\bar{r}}}{\partial h} = \hat{P} \times \dot{\bar{r}} = \begin{pmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{pmatrix}$$

In trying to use standard notation, some problems have been introduced. While h is the longitude of the ascending node, $\hat{h}$ is the unitized angular momentum vector $\bar{r} \times \dot{\bar{r}}$. E in the above equations stands for the eccentric anomaly. $\hat{P} = (0,0,1)^T$ has been used before, while $\hat{N} = (\cos h, \sin h, 0)^T$ is the unit vector to the ascending node. The equations (36) and (37) are evidently valid for any time t. In our present application they are to be evaluated for $t = t_0$.

As seen in a previous section, the partial derivatives we are trying to obtain in case of the moon should be

$$\frac{\partial \bar{D}_{11}}{\partial \bar{E}_G^T} = \frac{\partial \bar{D}_{11}}{\partial \bar{r}_G^T} \frac{\partial \bar{r}_G}{\partial \bar{E}_G^T} \qquad (38)$$

$$\frac{\partial \bar{D}_{11}}{\partial \bar{E}_3^T} = - \frac{\partial \bar{D}_{11}}{\partial \bar{r}_{11}^T} \frac{\partial \bar{r}_3}{\partial \bar{E}_3^T} \tag{39}$$

It is cle: that we will not use (39). Geocentric observations of the moon cannot be used directly to correct the heliocentric orbit of the earth. There is an indirect adjustment, but this discussion belongs in a different section. Hence, we will evaluate only (38). The first member was described earlier, and the second is the result of the numerical integration just discussed.

Note again that we have chosen to deal with equatorial elements for the moon in contrast to ecliptic ones for the planets. The algorithm for the moon was developed later, and it was noticed that going to ecliptic elements would introduce unnecessary complications.

Partials for Lunar Tide Coupling

An earlier version of our program contained analytical partial derivatives for the tidal coefficients which proved to be inadquate. There is probably a simple explanation. It was seen before that very accurate partials were needed for the lunar orbit. Since these elements and the lunar tidal coefficient are solved for simultaneously, partial derivatives for the latter must be of comparable quality. Hence, we turned to "rigorous" partials, again obtained by numerical integration. The derivation required some care, and we felt the essential steps should be recorded here.

What we need is $\partial \bar{D}_{11} / \partial C$ which can immediately be expanded into

$$\frac{\partial \bar{D}_{11}}{\partial C} = \frac{\partial \bar{D}_{11}}{\partial \bar{r}_3^T} \frac{\partial \bar{r}_3}{\partial C} + \frac{\partial \bar{D}_{11}}{\partial \bar{r}_{11}^T} \frac{\partial \bar{r}_{11}}{\partial C} \tag{40}$$

We can take advantage of a relation used previously, namely

$$\frac{\partial \bar{D}_{11}}{\partial \bar{r}_3^T} = - \frac{\partial \bar{D}_{11}}{\partial \bar{r}_{11}^T} \tag{41}$$

In order to put this into terms of geocentric coordinates, we recall

$$\bar{r}_B = \bar{r}_3 + \frac{m_{11}}{m_3 + m_{11}} \bar{r}_G = \bar{r}_{11} - \frac{m_3}{m_3 + m_{11}} \bar{r}_G \tag{42}$$

Since we assume that the barycenter is not affected, differentiation of (42) gives

$$\frac{\partial \bar{r}_3}{\partial C} = \frac{-m_{11}}{m_3 + m_{11}} \frac{\partial \bar{r}_G}{\partial C} \tag{43}$$

$$\frac{\partial \bar{r}_{11}}{\partial C} = \frac{-m_3}{m_3 + m_{11}} \frac{\partial \bar{r}_G}{\partial C}$$

Upon substitution of (41) and (43) into (40), our required partial becomes

$$\frac{\partial \bar{D}_{11}}{\partial C} = \frac{\partial \bar{D}_{11}}{\partial \bar{r}_{11}^T} \frac{\partial \bar{r}_G}{\partial C} \tag{44}$$

The factor $\partial \bar{D}_{11} / \partial \bar{r}_{11}^T$ is quite familiar by now. The second factor is to be obtained by numerical integration. Hence, we will derive the corresponding variational equation next.

Consider the acceleration of an orbiting body in the presence of an additional perturbation such as tidal forces. It is clear that

$$\ddot{\bar{r}} = \ddot{\bar{r}}[\bar{r}(\bar{r}_0, \dot{\bar{r}}_0, C, t), \dot{\bar{r}}(\bar{r}_0, \dot{\bar{r}}_0, C, t), C]$$

Formally, then,

$$\frac{\partial \ddot{\bar{r}}(\bar{r}_0, \dot{\bar{r}}_0, C, t)}{\partial C} = \frac{\partial \ddot{\bar{r}}(\bar{r}, \dot{\bar{r}}, C)}{\partial \bar{r}^T} \frac{\partial \bar{r}(\bar{r}_0, \dot{\bar{r}}_0, C, t)}{\partial C}$$

$$+\frac{\partial\ddot{\bar{r}}(\bar{r},\dot{\bar{r}};C)}{\partial\dot{\bar{r}}^T}\frac{\partial\dot{\bar{r}}(\bar{r}_0,\dot{\bar{r}}_0,C,t)}{\partial C}+\frac{\partial\ddot{\bar{r}}(\bar{r},\dot{\bar{r}},C)}{\partial C} \tag{45}$$

On the left hand side we indicate that a partial with respect to C is needed holding the initial conditions and, of course, the time fixed.

In the following, we will use a shorthand notation for (45). Also, we will apply above relation to the geocentric orbit of the moon. Then (45) becomes

$$\frac{\partial\ddot{\bar{r}}_G(t)}{\partial C}=\frac{\partial\ddot{\bar{r}}_G}{\partial\bar{r}_G^T}\frac{\partial\bar{r}_G}{\partial C}+\frac{\partial\ddot{\bar{r}}_G}{\partial\dot{\bar{r}}_G^T}\frac{\partial\dot{\bar{r}}_G}{\partial C}+\frac{\partial\ddot{\bar{r}}_G}{\partial C} \tag{46}$$

A large part of the matrix $\partial\ddot{\bar{r}}_G / \partial\bar{r}_G^T$ is already available and given by (35). Hence, we now have to supply the contribution due to our tidal forces. The acceleration is given elsewhere in this paper as

$$\Delta\ddot{\bar{r}}_G=\frac{C}{0.00256}\frac{\bar{h}_G\times\bar{r}_G}{h_G} \tag{47}$$

The result of differentiating this relation is

$$\frac{\partial\Delta\ddot{\bar{r}}_G}{\partial\bar{r}_G^T}=\frac{C}{0.00256h_G}\left[(\dot{\bar{r}}_G\bar{r}_G^T-\bar{r}_G\cdot\dot{\bar{r}}_G I_3)+\bar{h}_G\times I_3+\frac{1}{h_G^2}(\bar{r}_G\times\bar{h}_G)(\dot{\bar{r}}_G\times,\bar{h}_G)^T\right] \tag{48}$$

Note that

$$\bar{h}\times I_3=\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}\times\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&-h_1&h_2\\h_3&0&-h_1\\-h_2&\dagger h_1&0\end{pmatrix}$$

The next matrix in (46), namely $\partial \ddot{\bar{r}}_G / \partial \dot{\bar{r}}_G^T$, stems entirely from (47). The result of this operation is

$$\frac{\partial \ddot{\bar{r}}_G}{\partial \dot{\bar{r}}_G^T} = \frac{C}{0.00256 h_G} \left[(r_G^2 I_3 - \bar{r}_G \bar{r}_G^T) \quad \frac{1}{h_G^2} (\bar{r}_G \times \bar{h}_G) (\bar{r}_G \times \bar{h}_G)^T \right] \tag{49}$$

Thirdly, the last vector in (46) is the explicit derivative of (47), which yields

$$\frac{\partial \ddot{\bar{r}}_G}{\partial C} = \frac{\bar{h}_G \times \bar{r}_G}{0.00256} \tag{50}$$

Adopting the ξ-notation introduced earlier, equation (46) now becomes

$$\ddot{\xi}_c = \left(\frac{\partial \ddot{\bar{r}}_G}{\partial \bar{r}_G^T} + \frac{\partial \Delta \ddot{\bar{r}}_G}{\partial \bar{r}_G^T} \right) \bar{\xi}_c + \frac{\partial \ddot{\bar{r}}_G}{\partial \dot{\bar{r}}_G^T} \dot{\xi}_c + \frac{\partial \ddot{\bar{r}}_G}{\partial C} \tag{51}$$

These are the three new differential equations to be integrated numerically. The vector $\dot{\xi}_c$ is generated in the process much like the velocity. The other factors are:

$\dfrac{\partial \ddot{\bar{r}}_G}{\partial \bar{r}_G^T}$ is given by (35)

$\dfrac{\partial \Delta \ddot{\bar{r}}_G}{\partial \bar{r}_G^T}$ is given by (48)

$\dfrac{\partial \ddot{\bar{r}}_G}{\partial \dot{\bar{r}}_G^T}$ is given by (49)

$\dfrac{\partial \ddot{\bar{r}}_G}{\partial C}$ is given by (50)

The initial values of $\bar{\xi}_c$ and $\dot{\bar{\xi}}_c$ are both $\bar{0}$. Results of the numerical integration are substituted into (44), and this, in turn, goes into the normal equations.

It seems that this is the logical place to update equation (35). Presently, it is of the form

$$\ddot{\bar{\xi}}_k = \frac{\partial \ddot{\bar{r}}_G}{\partial \bar{r}_G^T} \bar{\xi}_k \tag{52}$$

Because of the introduction of tidal terms and the latter's dependence on the velocity, (52) must be augmented to read

$$\ddot{\bar{\xi}}_k = \left(\frac{\partial \ddot{\bar{r}}_G}{\partial \bar{r}_G^T} + \frac{\partial \Delta \ddot{\bar{r}}_G}{\partial \bar{r}_G^T} \right) \bar{\xi}_k + \frac{\partial \ddot{\bar{r}}_G}{\partial \dot{\bar{r}}_G^T} \dot{\bar{\xi}}_k \tag{53}$$

The matrix coefficients are, of course, the same as in (51). The initial conditions of $\bar{\xi}_k$ are the same as before. For the new vector $\dot{\xi}_k$ initial values are obtained from (37), again putting $t = t_0$.

Partials for Solar Tide Coupling

As we saw in an earlier section, the result of a constant acceleration in the transverse direction is a quadratic term in the orbital longitude or, to be more precise, in the mean anomaly. Since we will ignore the eccentricity of the earth orbit, this perturbation would also appear in the argument of latitude, that is

$$u = u_0 + n(t - t_0) - \frac{3}{2} \frac{C_\odot}{a_3} (t - t_0)^2 \tag{54}$$

Figure 4 relates u to the coordinates and to α and δ. After some obvious intermediate steps, the necessary partials are found to be

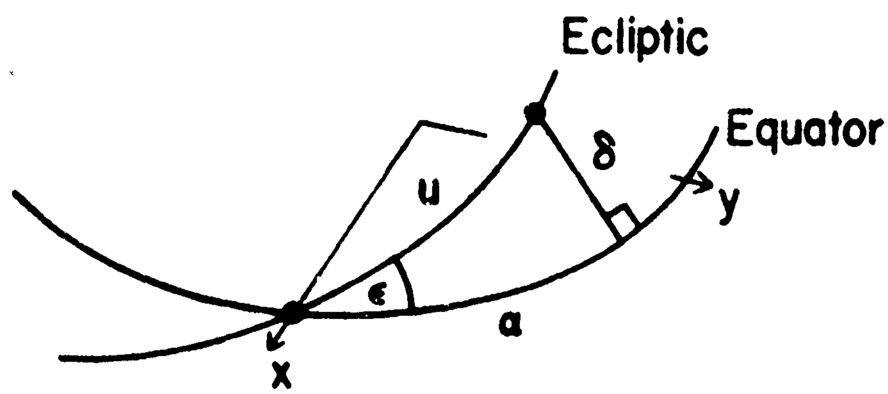

FIGURE 4

Relation of Argument of Latitude μ to Polar and Rectangular Coordinates.

$$\frac{\partial\alpha}{\partial C_\odot} = \frac{3}{2}\,\frac{(t-t_0)^2}{a_3\cos\epsilon}\,\frac{\cos^2\epsilon x_3\Delta x + y_3\Delta y}{\Delta x^2+\Delta y^2}$$

(55)

$$\frac{\partial\delta}{\partial C_\odot} = \frac{3}{2}\,\frac{(t-t_0)^2}{a_3\cos\epsilon}\,\frac{\sin\epsilon\cos\epsilon x_3(\Delta x^2+\Delta y^2) - (\cos^2\epsilon x_3\Delta y - y_3\Delta x)\Delta z}{\sqrt{\Delta x^2+\Delta y^2}\left(\Delta x^2+\Delta y^2+\Delta z^2\right)}$$

where again

$$\Delta\bar{r} = \bar{r}_i - \bar{r}_3$$

These relations are probably more compact in terms of α and δ, but our computer program, at this stage, has more convenient access to the rectangular coordinates.

If any perturbations in the earth's orbit can be obtained from observations of the sun, they must also be visible in the apparent positions of some of the nearer planets. Hence, (55) is executed for observations of the sun and the major planets. The computer program treats these bodies identically except that $\bar{r}_{10} = \bar{0}$ always.

Partials for Solar Oblateness

In a later part of this paper we will show that the solution for the solar oblateness coefficient is attempted simultaneously with the planetary orbital elements. Hence, approximate analytical partial derivatives suffice. We selected Brouwer's artificial satellite oblateness theory (1959) to supply the required relations. As stated earlier, we will consider the leading zonal harmonic only and label its coefficient $J_\odot$.

Required for our purpose are the partials $\partial\bar{D}/\partial J_\odot$. This can be expanded into

$$\frac{\partial\bar{D}}{\partial J} = \frac{\partial\bar{D}}{\partial\bar{r}^T}\,\frac{\partial\bar{r}}{\partial\bar{r}_s^T}\,\frac{\partial\bar{r}_s}{\partial\bar{E}_s^T}\,\frac{\partial\bar{E}_s}{\partial J} \tag{56}$$

The first matrix on the right is well known by now. The subscript s designates coordinates and elements referred to the plane of the solar equator.

It is the last vector in (56) which we want to obtain by Brouwer's theory. For this purpose, let

E = osculating elements at any time

E_0 = osculating elements at t_0

E_0'' = Brouwer's mean elements at t_0

The mean elements at running time are not needed. We can now write

$$\bar{E} = \bar{E}\left[\bar{E}_0(\bar{E}_0'', J), t, J\right] \tag{57}$$

Even though $\bar{E}(\bar{E}_0, t, J)$ is not explicity given by Brouwer, this is no stumbling block. Using (57), we can compute the partial derivative

$$\frac{\partial \bar{E}(\bar{E}_0'', t, J)}{\partial J} = \frac{\partial \bar{E}(\bar{E}_0, t, J)}{\partial J} + \frac{\partial \bar{E}(\bar{E}_0, t_1, J)}{\partial \bar{E}_0^T} \frac{\partial \bar{E}_0(E_0'', J)}{\partial J} \tag{58}$$

Inspection will show that the partial on the left of (58) is obtained by differentiating the theory since the constants of the Brouwer orbit, $\bar{E}_0''$, appear explicitely. We, however, require a partial in which the osculating elements at epoch are held fixed. Hence, we solve (58) for the first term on the right hand side. Again in shorthand, this means

$$\frac{\partial \bar{E}}{\partial J} = \frac{\partial \bar{E}(\bar{E}_0'')}{\partial J} - \frac{\partial \bar{E}}{\partial \bar{E}_0^T} \frac{\partial \bar{E}_0}{\partial J} \tag{59}$$

Since we are presently working in the plane of the solar equator,

$$E = E_s.$$

The first member on the right of (59) is quickly obtained. It can be shown that only secular terms need be considered. We looked at the coefficients of most periodic terms and found the largest about 10^{-4} times the size of the secular effects over 50 years. Hence, all periodic terms are ignored at this step. Also, secular terms can safely be restricted to first order. Copying directly form Brouwer, we have

$$\Delta l = -\frac{3}{2}\gamma_2' \; \eta n\left(1 - 3\theta^2\right)(t - t_0)$$

$$\Delta g = -\frac{3}{2}\gamma_2' n \left(1 - 5\theta^2\right) \left(t - t_0\right) \tag{60}$$

$$\Delta h = -3\gamma_2' n\theta \left(t - t_0\right)$$

where

$$\gamma_2' = \frac{J_\odot R_\odot^2}{2\eta^4 a^2}$$

$$\eta = \sqrt{1 - e^2}$$

$$\theta = \cos I$$

The difference between osculating and mean elements is of no consequence here. With this information we find

$$\frac{\partial \overline{E}\left(\overline{E}_0''\right)}{\partial J} = -\frac{3}{4}\frac{nR_\odot^2(t - t_0)}{a^2\eta^4}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta\left(1 - 3\theta^2\right) \\ 1 - 5\theta^2 \\ 2\theta \end{pmatrix} \tag{61}$$

The second term on the right hand side of (59) requires some care. It is helpful to write this out in detail:

$$\frac{\partial \bar{E}}{\partial \bar{E}_0^T} \frac{\partial \bar{E}_0}{\partial J} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial l}{\partial a_0} & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial a_0}{\partial J} \\ \frac{\partial e_0}{\partial J} \\ \frac{\partial I_0}{\partial J} \\ \frac{\partial l_0}{\partial J} \\ \frac{\partial g_0}{\partial J} \\ \frac{\partial h_0}{\partial J} \end{pmatrix} \qquad (62)$$

Since

$$\frac{\partial l}{\partial a_0} = -\frac{3}{2}\frac{n}{a}(t - t_0)$$

it seems that this secular term will soon exceed all the periodic functions in (62). Hence we proceed to reduce (62) to

$$\frac{\partial \bar{E}}{\partial \bar{E}_0^T} \frac{\partial \bar{E}_0}{\partial J} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial l}{\partial a_0} \frac{\partial a_0}{\partial J} \\ 0 \\ 0 \end{pmatrix} \qquad (63)$$

Second thoughts prompted us to go back to this assumption after final results were obtained. A careful numerical evaluation of (62) showed that for Mercury, with our particular set of initial conditions, the term retained in (63) is about 10^3 times the largest one neglected at the end of the data span. In the case of Venus, however, the term retained is of the same size as those neglected. Fortunately we find that here $(\partial\bar{E}/\partial\bar{E}_0^T)(\partial\bar{E}_0/\partial J)$ is about 10^3 times smaller than $\partial\bar{E}(E_0'')/\partial J$. For both planets, then, final partials are good to about three significant figures. Not as good as we would like, but probably adequate for the purpose.

Back to the derivation. Except for the factor $t - t_0$, which appears in both terms on the right of (59), all symbols can be replaced by their numerical values. Since the semimajor axis has only short-periodic perturbations, $\partial_{a_0}/\partial J$ is obtained from that very expression in Brouwer's theory. We employed the following values in these calculations:

	Mercury	Venus	Solar Equator
Inclination to ecliptic	7°0′	3°24′	7°15′
Node on ecliptic	47°44′	76°14′	75°4′
Inclination to solar equator (derived)	3°22′	3°51′	
Semimajor axis	.38710	.72333	
Eccentricity	.20561	.00682	
Mean motion	.071425	.027962	
Radius			.0046524
Mean anomaly	324°9′	272°1′	
Argument of perigee	28°49′	54°27′	

Now everything in (59) can be computed. We find for

Mercury:

$$\frac{\partial\bar{E}}{\partial J} = 10^{-5}(t - t_0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2.4961 \\ 3.3598 \\ -1.6843 \end{pmatrix}$$

Venus:

$$\frac{\partial \bar{E}}{\partial J} = 10^{-5}(t - t_0) \begin{pmatrix} 0 \\ 0 \\ 0 \\ .17287 \\ .34511 \\ -.17314 \end{pmatrix} \qquad (65)$$

These represent the last factor on the right of (56).

The next step is the premultiplication of $\partial E / \partial J$ by

$$\frac{\partial \bar{r}_s}{\partial \bar{E}_s^T} = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial e} & \cdots & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial a} & & \cdots & \\ \frac{\partial z}{\partial a} & & \cdots & \frac{\partial z}{\partial h} \end{pmatrix} \qquad (66)$$

Since the first three rows of $\partial \bar{E} / \partial J$ are zero, the first three columns of (66) are not needed. With the aid of (36) the other three columns become

$$\frac{\partial \bar{r}}{\partial l} = \frac{1}{n} \dot{\bar{r}}$$

$$\frac{\partial \bar{r}}{\partial g} = F\bar{r} + G\dot{\bar{r}} \qquad \text{where} \quad F = \frac{-e \sin E}{\sqrt{1-e^2}}$$

$$\frac{\partial \bar{r}}{\partial h} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad G = \frac{r^2}{a^2 \cdot \sqrt{1-e^2}} \frac{1}{n}$$

If we temporarily let the elements in the vectors (64) and (65) be α, β, and γ, then the product we are after can be put in the form

$$\frac{\partial \bar{r}_s}{\partial \bar{E}_s^T}\frac{\partial \bar{E}_s}{\partial J} = 10^{-5}(t - t_0)\left[\beta F\bar{r}_s + \left(\frac{\alpha}{n} + \beta G\right)\dot{\bar{r}}_s + \gamma\begin{pmatrix} -y_s \\ x_s \\ 0 \end{pmatrix}\right] \tag{67}$$

The subscript S was added to $\partial \bar{E} / \partial J$ at this stage since it will be needed at once.

According to (56), we now premultiply (67) by $\partial \bar{r}/\partial \bar{r}_s^T$. If the coordinates referred to the solar and terrestrial equators are related through

$$\bar{r}_s = N\bar{r}$$

then the matrix

$$\frac{\partial \bar{r}}{\partial \bar{r}_s^T} = N^T$$

Combining the last three equations, we obtain

$$\frac{\partial \bar{r}}{\partial \bar{r}_s^T}\frac{\partial \bar{r}_s}{\partial \bar{E}_s^T}\frac{\partial \bar{E}_s}{\partial J} = N^T \; 10^{-5}(t - t_0)\left[\beta FN\bar{r} + \left(\frac{\alpha}{n} + \beta G\right)N\dot{\bar{r}} + \gamma\begin{pmatrix} -y_s \\ x_s \\ 0 \end{pmatrix}\right] \tag{68}$$

Since it can be shown that

$$N^T\begin{pmatrix} -y_s \\ x_s \\ 0 \end{pmatrix} = \bar{N}_3 \times \bar{r}$$

where $\bar{N}_3$ is the last row of N, (68) becomes

$$\frac{\partial \bar{r}}{\partial \bar{r}_s^T}\frac{\partial \bar{r}_s}{\partial \bar{E}_s^T}\frac{\partial \bar{E}_s}{\partial J} = 10^{-5}(t - t_0)\left[\beta F\bar{r} + \left(\frac{\alpha}{h} + \beta G\right)\dot{\bar{r}} + \gamma\bar{N}_3 \times \bar{r}\right]$$

The remaining steps are trivial. We recall that α, β, and γ were numbers. The same is true for $\overline{N}_3$, used in a previous section. Also, various parts of F and G are constant. Hence, we substitute numbers wherever possible. It becomes advantageous to introduce a few new symbols, but there is no need to record any relations since everything is easily verified. Finally all is put into (56) and we obtain

$$\frac{\partial \overline{D}_i}{\partial J_\odot} = \frac{\partial \overline{D}_i}{\partial \overline{r}_i^T} 10^{-5}(t - t_0)\left[B_i \overline{r}_i + \left(C_i + D_i\right)\dot{\overline{r}}_i + \overline{H}_i \times \overline{r}_i\right] \tag{69}$$

where

$b_1 = -.70590$

$C_1 = 34.9470$

$d_1 = 320.776$

$$\overline{H}_1 = \begin{pmatrix} -.20538 \\ .71504 \\ -1.51107 \end{pmatrix}$$

$b_2 = -.00235$

$C_2 = 6.1823$

$d_2 = 23.5894$

$$\overline{H}_2 = \begin{pmatrix} -.02111 \\ .07351 \\ -.15534 \end{pmatrix}$$

$$r_i = |\overline{r}_i|$$

$$B_i = b_i \sin E_i$$

$$D_i = d_i r_i^2$$

The subscripts 1 and 2 refer to Mercury and Venus. Recall that all but the first three significant figures are illusory.

Partials for Clock Corrections

We decided to solve for corrections to our observation clock at a number of equally spaced discrete points. Since the program still had room for another 15 to 20 unknowns, we arbitrarily picked the 15 dates 1912.5, 1916.5, ··· 1968.5 to be thus improved. As long as ET was used, we had to assume at least two points known. Hence, the values of ET at 1912.5 and 1968.5 were held fixed. After switching at AT, we solve for corrections from 1912.5 to 1952.5. Since AT is well known from laboratory determination from 1955 on, we do not solve for corrections to the points 1956.5 to 1968.5.

The principle of finding our corrections is a simple one. Let

$$T = T^0 + \epsilon$$

$$T = T^0 + \epsilon$$

where T is the corrected time, T^0 the initial estimate, and ϵ the increment to be solved for. Since ϵ is quite small, the right ascension of any one of our objects can be written as

$$\alpha(T) = \alpha(T^0) + \dot{\alpha}\epsilon \tag{70}$$

The same holds for δ. Assume further that the ϵ for any observation time t is given by the four-point interpolation

$$\epsilon = \epsilon_{-1} + p(\epsilon_1 - \epsilon_{-1}) + \frac{1}{4} p(p - 1)(\epsilon_{-2} - \epsilon_{-1} - \epsilon_1 + \epsilon_2) \tag{71}$$

where

$$p = \frac{t - t_{-1}}{1461}$$

and 1461 is the number of days in four years. The subscripts do not follow adopted notation, but they provide a more symmetric form. It is almost obvious that $t_{-2} < t_{-1} < t < t_1 < t_2$. The t and ϵ with subscripts belong to grid points for which corrections are computed.

The partial required is $\partial\alpha/\partial\epsilon_i$. This can be written as

$$\frac{\partial\alpha}{\partial\epsilon_i} = \frac{\partial\alpha}{\partial\epsilon}\frac{\partial\epsilon}{\partial\epsilon_i} \tag{72}$$

The first factor on the right is found with the aid of formula (70), that is

$$\frac{\partial\alpha}{\partial\epsilon} = \dot{\alpha}$$

From the relations between coordinates and α and δ we obtain

$$\dot{\alpha} = \frac{\Delta x \Delta\dot{y} - \Delta y \Delta\dot{x}}{(\Delta x)^2 + (\Delta y)^2}$$

$$\dot{\delta} = \frac{\left[(\Delta x)^2 + (\Delta y)^2\right] \Delta\dot{z} - \Delta z(\Delta x \Delta\dot{x} + \Delta y \Delta\dot{y})}{\sqrt{(\Delta x)^2 + (\Delta y)^2}\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]} \qquad (73)$$

where again

$$\Delta\bar{r} = \bar{r}_i - \bar{r}_3$$

and

$$\Delta\dot{\bar{r}} = \dot{\bar{r}}_i - \dot{\bar{r}}_3$$

Above relations are good for all observations remembering that

$$\bar{r}_{10} = \dot{\bar{r}}_{10} = \bar{0}$$

The second factor on the right of (72) follows from equation (71). We find

$$\frac{\partial\epsilon}{\partial\epsilon_{-2}} = \frac{1}{4}\, p(p-1) \qquad \frac{\partial\epsilon}{\partial\epsilon_1} = p - \frac{1}{4}\, p(p-1)$$

$$\frac{\partial\epsilon}{\partial\epsilon_{-1}} = 1 - p - \frac{1}{4}\, p(p-1) \qquad \frac{\partial\epsilon}{\partial\epsilon_2} = \frac{1}{4}\, p(p-1) \qquad (74)$$

This scheme is used for all observations after 1916.5. For those before 1916.5, there is only one grid point to the left, namely 1912.5. Hence, we will use linear interpolation in the interval 1912.5 to 1916.5. This is accomplished by changing (74) to

$$\frac{\partial \epsilon}{\partial \epsilon_{-2}} = 0 \qquad \frac{\partial \epsilon}{\partial \epsilon_1} = p$$

$$\frac{\partial \epsilon}{\partial \epsilon_{-1}} = 1 - p \qquad \frac{\partial \epsilon}{\partial \epsilon_2} = 0 \tag{75}$$

To summarize, the required partials are

$$\frac{\partial \overline{D}}{\partial \epsilon_i} = \dot{\overline{D}} \frac{\partial \epsilon}{\partial \epsilon_i}$$

with $\dot{\overline{D}}$ given by (73) and $\partial\epsilon/\partial\epsilon_i$ by (74) or (75).

After a solution for the ϵ's if made, values are computed for the years between grid points. For the years 1913.5, 1914.5, and 1915.5, this is again done by linear interpolation. Four-point interpolation is used from 1917.5 on. The complete list of ϵ's then replaces the ΔT - table in our Data Conversion Program, discussed in thė section on Data Preparation.

Partials for Lunar Declination Bias

Once lunar residuals came down to a reasonable level, it became apparent that there existed a severe bias in declination. The same has been seen by other investigators of lunar observations, and the cause is still under investigation. We elected to "remove" the problem by solving for a bias parameter which, in turn, is applied to the declination residuals of the next iteration. If we let the observed declination be

$$\delta = \delta^0 + \Delta\delta$$

where $\Delta\delta$ is the parameter we want, then

$$\frac{\partial \delta}{\partial \Delta\delta} = 1$$

With this definition, $\Delta\delta$ is added to the . ɔmputed declination.

7. NORMAL EQUATIONS AND THEIR SOLUTIONS

Let us start with the equations of condition

$$\frac{\partial \overline{D}}{\partial \overline{E}^T} \Delta \overline{E} = \Delta \overline{D} \tag{76}$$

As before, $\overline{D}$ contains the observables α and δ. E is any parameter we solve for, not just an orbital element. The normal equations are then

$$\frac{\partial \overline{D}^T}{\partial \overline{E}} W \frac{\partial \overline{D}}{\partial \overline{E}^T} \Delta \overline{E} = \frac{\partial \overline{D}^T}{\partial \overline{E}} W \Delta \overline{D}$$

which we frequently abbreviate

$$B \Delta \overline{E} = \Delta \overline{N} \tag{77}$$

The weight matrix W contains only 22 distinctly different elements, one each for the α and δ of the ten bodies observed, with an extra set for the moon. Hence, only 22 numbers need be stored. The weights themselves are obtained as

$$W_a = \frac{1}{\sigma_\alpha^2} \quad \text{and} \quad W_d = \frac{1}{\sigma_\delta^2}$$

where σ_α and σ_δ are the formal standard deviations of each planet's residuals. As such they would be numbers obtained in the previous interation. Their derivation will be discussed later.

Since in most problems of this nature the number of observations is large, the dimensions of (76) are enormous. Hence, it is impractical to evaluate (76) and do (77) numerically. We therefore evaluate (77) at once. We also found it advantageous to take one observation at a time and complete various calculations before going to the next one. Let's look at this now.

The partials for the various parameters are available. They are of the form $\partial\alpha / \partial E$ and $\partial\delta / \partial E$. Since the residuals in α are scaled by $\cos \delta$, we will have to do the same for the corresponding partial derivatives. For the sake of brevity, let

$$a = \frac{\partial \alpha}{\partial E} \cos \delta \qquad\qquad d = \frac{\partial \delta}{\partial E}$$

Then the normal matrix becomes

$$B = \begin{pmatrix} a_1 W_a a_1 + d_1 W_d d_1 & a_1 W_a a_2 + d_1 W_d d_2 & \cdots & a_1 W_a a_p + d_1 W_d d_p \\ 0 & a_2 W_a a_2 + d_2 W_d d_2 & \cdots & a_2 W_a a_p + d_2 W_d d_p \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_p W_a a_p + d_p W_d d_p \end{pmatrix}$$

and the right hand side is

$$\Delta \overline{N} = \begin{pmatrix} a_1 W_a \Delta\alpha + d_1 W_d \Delta\delta \\ a_2 W_a \Delta\alpha + d_2 W_d \Delta\delta \\ \vdots \\ a_p W_a \Delta\alpha + d_p W_d \Delta\delta \end{pmatrix}$$

Since all this is done for one observation at a time, no subscript is needed to identify the observation. Hence, the subscripts attached to a and d refer to the parameter involved, and p is the total number of parameters. Since B is symmetric, only $(p^2 + p)/2$ distinct elements need be computed while the others are supplied by $b_{ji} = b_{ij}$ upon completion. It is important to recall that the residual $\Delta\alpha$ already contains $\cos\delta$, as discussed in an earlier section.

While the above numbers are available, it is advantageous to perform a few other calculations at this point. Hence, for each observation we add to the sums

$$\Sigma W_a (\Delta\alpha)^2 \qquad \text{and} \qquad \Sigma W_a$$

After processing the last observation, we calculate

$$\sigma_\alpha = \sqrt{\frac{\Sigma\, W_a (\Delta\alpha)^2}{\Sigma\, W_a}}$$

$$S/N_\alpha = \sqrt{\frac{\Sigma\, W_a (\Delta\alpha)^2}{n_\alpha}}$$

and similar expressions for the declination. Here n_α is the total number of data points in α. The first gives the standard deviation of a single α, the other its signal-to-noise ratio. We did not find the latter to be useful. Both formulae were evaluated for individual planets as well as the solar system as a whole. Again note that σ contains the scale factor $\cos\delta$ which, via the weights, enters B and $\Delta\overline{N}$.

The solution of the normal equations (77) is

$$\Delta\overline{E} = B^{-1}\ \Delta\overline{N} \tag{78}$$

If we let the elements of the inverse matrix B^{-1} be b_{ij}^{-1}, the formal standard deviation of the parameters ΔE in (78) are given by

$$\sigma_i = \sqrt{b_{ii}^{-1}} \tag{79}$$

and the correlation coefficients between parameters by

$$\rho_{ij} = \frac{b_{ij}^{-1}}{\sigma_i \sigma_j} \tag{80}$$

It would seem that $\Delta\overline{E}$ is now to be added to the initial estimates of the parameters. We found quite some time ago, and so did others, that this procedure often does not converge. It is immaterial whether we talk about our nonsingular elements, four of which contain the semimajor axis, or the rectangular coordinates. The problem arises whenever the semimajor axis is not decoupled from the other five orbital parameters.

It is not difficult to see the reason for this behavior. Let us assume that we make two least-squares fits to the same data set, solving for corrections first to the classical elements $\overline{A} = (a,e,I,l,g,h)^T$ and then the coordinates $\overline{X} = (x,y,z,\dot{x},\dot{y},\dot{z},)^T$. The notation was chosen to set this pheripheral discussion off from the rest of the chapter. Let the solution of the former be $\Delta\overline{A}$, the other $\Delta\overline{X}$. Now it is clear that the crucial part of the first solution is a good Δa. Any error in $\underline{a}$ will produce a secular effect in the computed places. Since this is not so for the other five elements, their adjustment is less critical. We have to make the assumption that we know how to formulate a least-squares algorithm that will yield a "good" Δa which is linear in the residuals. If we now take our coordinate solution and determine the increment in $\underline{a}$ associated with it, we find

$$\Delta\hat{a} = a(\overline{X}+\Delta\overline{X}) - a(\overline{X})$$

where, of course, $\overline{X}$ was the initial estimate. We will now proceed to show that $\Delta\hat{a}$ can be quite different from Δa, and, hence, cannot also be a good adjustment.

A Taylor expansion about $\overline{X}$ gives

$$\Delta\hat{a} = \frac{\partial a}{\partial\overline{X}^T}\Delta\overline{X} + \frac{1}{2}\Delta\overline{X}^T\frac{\partial_a^2}{\partial\overline{X}\partial\overline{X}^T}\Delta\overline{X} + \cdots$$

Next we will show that the first term on the right of this series is nothing but the good Δa discussed above. It is linear in the residuals because $\Delta\overline{X}$ is. We can start with the normal equations in the coordinates, namely

$$\frac{\partial\overline{D}^T}{\partial\overline{X}} W \frac{\partial\overline{D}}{\partial\overline{X}^T}\Delta\overline{X} = \frac{\partial\overline{D}^T}{\partial\overline{X}} W \Delta\overline{D}$$

Since

$$\frac{\partial\overline{D}}{\partial\overline{X}^T} = \frac{\partial\overline{D}}{\partial\overline{A}^T}\frac{\partial\overline{A}}{\partial\overline{X}^T}$$

this becomes

$$\frac{\partial \overline{A}^T}{\partial \overline{X}} \frac{\partial \overline{D}^T}{\partial \overline{A}} W \frac{\partial \overline{D}}{\partial \overline{A}^T} \frac{\partial \overline{A}}{\partial \overline{X}^T} \Delta \overline{X} = \frac{\partial \overline{A}^T}{\partial \overline{X}} \frac{\partial \overline{D}^T}{\partial \overline{A}} W \Delta \overline{D}$$

Eliminating the leftmost matrix on both sides, there remain the normal equations in the elements. Hence, we must have

$$\frac{\partial \overline{A}}{\partial \overline{X}^T} \Delta \overline{X} = \Delta \overline{A}$$

or

$$\Delta a = \frac{\partial a}{\partial \overline{X}^T} \Delta \overline{X}$$

Q.E.D. If we can also show that the second and higher order terms in above Taylor series can be significant, then $\Delta \hat{a}$ will be nonlinear in the residuals and poor whenever this occurs.

As an example, we picked an intermediate solution for Mercury. The element solution calls for

$$\Delta a = -.700 \times 10^{-8} \text{ a.u.}$$

Solving the same normal equations for coordinates and substituting these into above Taylor expansion, we obtain

$$\Delta \hat{a} = (-.700 + .175) \times 10^{-8} \text{ a.u.}$$

The second-order term, namely 0.175, is relatively large, but not large enough to prevent convergence. However, in our example Mercury's orbital parameters are quite close to their final values. In fact, the Δa of $.7 \times 10^{-8}$ a.u. corresponds to only 8″

in the orbital longitude after 20,000 days, some 227 revolutions of Mercury. Since the ratio of the second to the first-order term in above Taylor series is proportional to ΔX, it is clear that this ratio can exceed unity if ΔX is large enough. In our example, if ΔX had been greater than .700/.175 = 4 times the actual ΔX, $\Delta \hat{a}$ would have the wrong sign. In other words, if we had an initial estimate of Mercury's orbit, at our particular epoch, with an error of about 3×10^{-8} a.u. in the semimajor axis, any attempt to differentially correct the rectangular coordinates would lead to divergence.

It can be argued that choosing the mean motion as one of the solution parameters would be superior to improving the semimajor axis. Indeed, the right ascensions of a planet are linear in n except for periodic terms. But it can be shown quickly that there is very little difference between solving for a and n. Suppose a solution in a has been made. Then the corresponding correction for n is obtained from

$$\Delta n = n(a + \Delta a) - n(a)$$

Again, by Taylor's theorem,

$$\Delta n = \frac{\partial n}{\partial a} \Delta a + \frac{1}{2} \frac{\partial^2 n}{\partial a^2} (\Delta a)^2 + \cdots$$

For our example, we find

$$\Delta n = (.194 + .439 \times 10^{-8}) \times 10^{-8}$$

Hence the second-order term is down by eight orders of magnitude, a very comfortable ratio.

Let us now return to the discussion of the solution process. We have seen that it will not do to add $\Delta \overline{E}$ to the initial estimates of $\overline{E}$. Hence we will convert $\Delta \overline{E}$ to increments in the classical elements, and these will be added to the old values to get the new ones. We will go to the classical elements via rectangular coordinates, but this intermediate step is taken only for convenience, as set forth earlier. We now look at this sequence of operations.

Let $\overline{C}$ stand for position and velocity, and $\overline{E}$ for the nonsingular elements. Then there exists a relation

$$\overline{C} = \overline{C}(\overline{E})$$

from which we obtain

$$\Delta\overline{C} = \frac{\partial\overline{C}}{\partial\overline{E}^{T}} \Delta\overline{E}$$

The matrix of partials will be labeled M. Hence

$$\Delta\overline{C} = M \ \Delta\overline{E} \tag{81}$$

permits us to express the solution in the nonsigular elements in terms of equatorial coordinates. Cohen and Hubbard (1962) furnish the elements of M, as explained under Partial Derivatives.

In order to get the variance-covariance matrix, rewrite (78) in the form

$$\Delta\overline{E} = B^{-1}\frac{\partial\overline{D}^{T}}{\partial\overline{E}} W \Delta\overline{D}$$

A similar equation exists for the coordinates, namely

$$\Delta\overline{C} = B_c^{-1} \frac{\partial\overline{D}^{T}}{\partial\overline{C}} W \Delta\overline{D}$$

These expressions are related through (81), and we find

$$B_c^{-1} \frac{\partial\overline{D}^{T}}{\partial\overline{C}} = MB^{-1} \frac{\partial\overline{D}^{T}}{\partial\overline{E}} \tag{82}$$

Since also

$$\overline{D} = \overline{D}\,[\overline{C}(\overline{E})]$$

there follows, after differentiating w.r.t. $\overline{E}$,

$$\frac{\partial \overline{D}^T}{\partial \overline{E}} = \frac{\partial \overline{C}^T}{\partial \overline{E}} \frac{\partial \overline{D}^T}{\partial \overline{C}}$$

Substitution of the last relation into (82) gives the desired result

$$B_c^{-1} = MB^{-1}\,M^T \tag{83}$$

This equation is used to get the B^{-1} - matrix for the coordinates. Standard deviations and correlation coefficients are computed similar to (79) and (80).

The transformation into ecliptic elements can almost be written down from inspection. We appeal to the relation

$$\overline{E}_E = \overline{E}_E\,\left[\overline{C}_E\,(\overline{C})\right]$$

where $\overline{E}_E$ are the classical Keplerian elements in the ecliptic, and $\overline{C}_E$ the corresponding rectangular coordinates. Then

$$\Delta \overline{E}_E = \frac{\partial \overline{E}_E}{\partial \overline{C}_E^T} \frac{\partial \overline{C}_E}{\partial \overline{C}^T}\,\Delta \overline{C}$$

We let the first matrix be N, the second ϵ^T. Then

$$\Delta \overline{E}_E = N\epsilon^T\,\Delta \overline{C} \tag{84}$$

The elements of the inverse of N are given by (36) and (37). Hence, we make N^{-1} by using these relations and N by numerical inversion. Note again that those calculations are made at $t = t_0$. The matrix ϵ^T is also available and defined by (12). It is now easy to see that

$$B_E^{-1} = N\epsilon^T B_C^{-1} \epsilon N^T \tag{86}$$

Again, standard deviations and correlation coefficients for the ecliptic elements are calculated.

The $\Delta\overline{E}_E$ obtained by (84) are added to the initial estimates of these parameters. This step completes the logical loop of the orbit improvement program. The improved $\overline{E}_E$ are now converted to equatorial coordinates and nonsingular elements. This is the procedure for the elements of all planets, including the earth.

Treatment of the other parameters differs somewhat. Their contributions to the matrices M and $N\epsilon^T$ are identity matrices of the appropriate dimensions. From this point on, all but the lunar elements are handled almost like the planetary orbital parameters.

In case of the moon, we obtain $\Delta\overline{r}_G$ and $\Delta\dot{\overline{r}}_G$, the increments in equatorial geocentric coordinates. We then calculate

$$\overline{r}_{11,\,\mathrm{FINAL}} = \overline{r}_{3,\,\mathrm{NEW}} + \overline{r}_{G,\,\mathrm{OLD}} + \frac{m_3}{m_3 + m_{11}} \Delta\overline{r}_G \tag{86}$$

$$\overline{r}_{3,\,\mathrm{FINAL}} = \overline{r}_{3,\,\mathrm{NEW}} - \frac{m_{11}}{m_3 + m_{11}} \Delta\overline{r}_G \tag{87}$$

and similar equations for the velocities. This requires some explanations.

The coordinates labelled "OLD" are initial estimates. Those with the subscript "NEW" are the results of planetary orbit adjustments discussed just above. One of the functions of (86) is to apply the same corrections to the heliocentric coordinates of the moon that were determined for the earth from observations on sun and planets. In other words, any such adjustments are made to the earth-moon barycenter. This step improves the speed of convergence considerably. Moreover, equations (86) and (87) also show that the corrections found for the geocentric orbit of the moon are distributed to the heliocentric coordinates of earth and moon such that their barycenter remains unchanged.

It is not necessary to solve for all parameters that enter the normal equations. The program permits suppression of any number of unwanted parameters in the solution. Suppose the k^{th} unknown is not needed. The program sets the k^{th} row and column of B, see equatioion (77), equal to zero except for a 1 in the main diagonal position b_{kk}. Also the k^{th} element of ΔN is put to zero. The solution then proceeds in a normal fashion. A secondary solution program allows us to try as many combinations of suppressed parameters as we wish to test with a given set of normal equations.

VII. CHECKOUT AND EXPERIMENTS

The original formulation of our computer program was checked with care, and practically every phase was verified by independent hand calculations. Later additions and modifications were perhaps not all examined with identical thoroughness, but none was employed without some kind of checkout. Only the important or interesting results of checks and experiments will be reported in the following pages.

1. INTEGRATION ROUTINE

Although known to be of limited value, we numerically integrated a Keplerian orbit approximating that of the earth. The stepsize was determined such that the orbit would close after 460 steps, simply for the sake of convenience. Maximum error in any coordinate was 5 units in the last place of the 14 digit word. Velocity components did even better. All supporting hand calculations were done to 20 significant figures.

2. TRUNCATION ERRORS

Initial tests were made when our plans did not include integration of the lunar orbit. Most test runs were done for the pair Mercury and Jupiter. Mercury, of course, has the shortest period and Jupiter the largest mass. But there are additional considerations. We have seen in several other applications of numerical integration that the highest frequency present in a system will determine the truncation error, even if this frequency is only a perturbation of small amplitude. In other words, planets such as Venus and Earth would probably require the same step size as Mercury simply because the latter's frequency appears in the higher differences. Also, Mercury's eccentricity of 0.21 is likely to produce a noticable effect.

We made very extensive tests, varying all possible parameters. It would go beyond the scope of this paper to describe the results. Be that as it may, all this is of academic interest anyway since we introduced the moon. It is clear that the lunar period will now control truncation error, and, consequently, step size.

Before we turn to the moon, a few observations of general nature. from the Mercury-Jupiter experiment, should be recorded. The position of Mercury in its orbit at the epoch of integration is quite important. This, obviously, is a consequence of the relatively large eccentricity. Any Truncation errors thus produced can be minimized by the proper selection of the starting routine controls a and b which are described above in NUMERICAL INTEGRATION ROUTINE. Unfortunately, a and b would then at least be a function of initial phase of a planet and its eccentricity. This is a complication we do not wish to pay for. Further tests showed that $a = b$ and $a + b = c$ is the optimum choice for the general case.

Also seen in earlier studies was the effect of the initial relative position of the principal perturbing body. This remark as well as some made above applies only, of course, when the step size approaches its allowable upper limit. With an integration interval several times as small any such effects disappear.

A number of considerations led us to the decision to integrate all bodies with the same step size. It does not seem that we pay much extra for computer time, but we do have a simpler program.

Let us now look at the truncation errors in the presence of the moon. It is clear that the maximum errors occur in the earth-moon system. We soon found that the heliocentric coordinates of the moon show larger errors than those of the earth. For convenience, we put results into the geocentric frame and convert truncation errors in the coordinates into angular measure. For an integration over 57 years (20,800 days) with different step sizes, the errors found are given in Table II.

Since we are using a step size of $0^{d}.4$, the error of $0''.08$ seems to be considerably in excess of our design accuracy. Fortunately, the least-squares fit will remove a good portion of this signal. As long as the same program with the same order and step size is used in subsequent runs, the effect on the initial conditions is of no consequence. Hence, the errors actually encountered are the quadratic and higher degree parts that cannot be absorbed. Our tests show that the maximum truncation error left is only $0''.013$. Figure 5 shows this to occur about 8,000 days after epoch. The plot depicts angular truncation errors after fitting a lunar orbit integrated with $h = 0^{d}.4$ to synthetic observations which were generated with a much smaller step size.

However, when the program is exercised with the lunar tidal coefficient as a free parameter, the truncation error signal is practically absorbed altogether. This may change the tidal coefficient by about 1%. Since its standard deviation is more like 20%, the error is insignificant.

TABLE II

Maximum Truncation Errors of Moon After 57 Years

Step size	in $\alpha \cos \delta$	in δ
$0^{d}_{.}8$	550″	180″
0.5	1.2	0.9
0.4	0.08	0.06

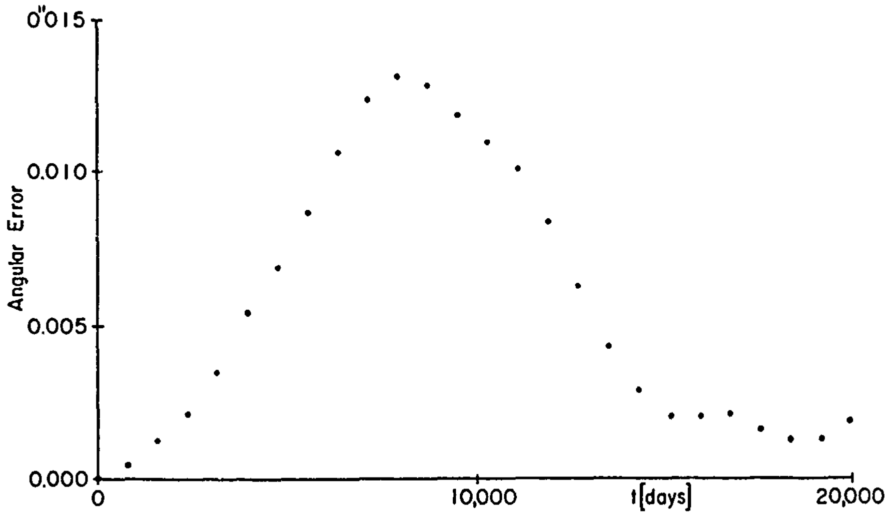

FIGURE 5

Effective Truncation Error of Moon Over 55 Years

3. COMPARISON WITH OTHER INTEGRATIONS

Since our integration routine evolved from other in-house Cowell algorithms, it was carefully compared with the "best" of these programs. Naturally, any such checkout is concluded only after all discrepancies are removed or well below the threshold of significance. If the same computer is used, such differences can often be pushed down to the last significant figure or two because of many similarities in the algorithms.

Comparison with programs developed elsewhere is usually a more convincing exercise. However, there are always differences in computers and the mathematical models that have to be considered or circumvented. We made arrangements with O'Handley and Mulholland at JPL for such tests. At that time, their programs ran on the IBM 7094 and Univac 1108, with all or most calculations in double precision. Also, JPL uses two separate programs to handle the major planets and the moon. In the former, the earth-moon barycenter is integrated numerically and the motion of the two bodies about this point is supplied through additional input, not generated internally. The lunar program contains the planets, of course, as perturbing bodies, but their orbits are not subject to correction. There were differences in the relativistic terms, oblateness perturbations, and other small effects. It was relatively easy, however, to adjust our program to simulate the two JPL routines.

Results of the planetary test portion are quite satisfactory. Experiments were run over 12 years with step sizes from $0^{d}.25$ to $1^{d}.00$. However, the larger step sizes are likely to reflect the differences of the algorithms. Since this facet is of no interest here, we will restrict our comparison to $h = 0^{d}.25$. Table III contains the principal results. The differences in coordinates have been converted to angles in seconds of arc, third column, as seen from the earth. We did not investigate the relatively large coordinate difference of the earth. This number may be due to some modeling difference we failed to catch. Even if correct, it would increase some of the angular discrepancies by a factor of ten and would still leave excellent agreement for the purpose at hand.

Comparison of the lunar programs did not fare as well. In one run for 4,400 days (12 years) at $h = 0^{d}.25$ the final position difference between NWL and JPL was 2.2×10^{-8} a.u. This corresponds to an unacceptable angular error of $1''.8$. Material for additional tests did not become available. However, we feel confident about our routine. Since we have only one program in which the moon is but one of n planets, the various tests discussed above check planetary and lunar orbits.

TABLE III

NWL vs. JPL Coordinates After 4,400 Days

Planet	$\lvert\Delta \bar{r}\rvert$	Angular measure
Mercury	6.9×10^{-2} a.u.	$''.2 \times 10^{-5}$
Venus	8.2	$.6 \times 10^{-5}$
E-M Barycenter	24.9	–
Mars	2.4	$.9 \times 10^{-6}$
Jupiter	2.8	$.1 \times 10^{-6}$
Saturn	2.9	$.7 \times 10^{-7}$
Uranus	6.3	$.7 \times 10^{-7}$
Neptune	10.8	$.8 \times 10^{-7}$
Pluto	4.2	$.2 \times 10^{-7}$

4. PHASE EFFECTS FOR VENUS, MERCURY, AND MOON

Even our earliest solutions indicated that there was a problem with the Venus residuals. The root mean squares of residuals in both right ascension and declination were larger than those of other planetary transit circle observations. Closer examination quickly revealed a signal with a 585 day period or so, clearly the synodic period of Venus. This suggested that the phase corrections applied by the observers did not adequately reduce observations to the center of the planet. Figure 6 shows α residuals over a bit more than a synodic cycle and also a quadratic fitted to the combined material of several cycles spread over the entire 20,000 days. The polynomial is given in the section on residuals. It is no surprise that correction of residuals in all subsequent runs considerably improved the r.m.s. The discontinuity of about $3''$ occurs at inferior conjunction, as one would expect. Residuals in α were also examined for their dependence on δ, but no relation was found.

There is also a distinct phase correction in declination, although it is not as spectacular. Nevertheless, a polynomial of third degree was needed to describe this effect within comparable tolerances.

The residuals of Mercury do not show problems of this magnitude so that we did not make a search for phase dependence. However, it is likely that such an additional phase correction can be found.

For the moon, all α residuals for 16 consecutive synodic cycles were examined, and nothing of any consequence to this study was found. There is some indication of a possible discontinuity near new moon in the same sense as observed for Venus. If real, the effect would be small. We applied no corrections.

5. OBLATENESS PERTURBATIONS

The effects of the leading zonal harmonics of the earth on the motion of the moon can be assessed easily using one of the existing general theories. One finds that the largest secular perturbation, that in the argument of perigee, amounts to about $700''$ after 20,000 days. The other two secular terms are about half that size. Coefficients of periodic terms can be found, too, but they are very small.

It takes a more powerful theory to obtain coupling between oblateness and solar terms. We found one such relatively large and important term by coincidence, investigating a different problem. It is shown in Figure 7 and represents

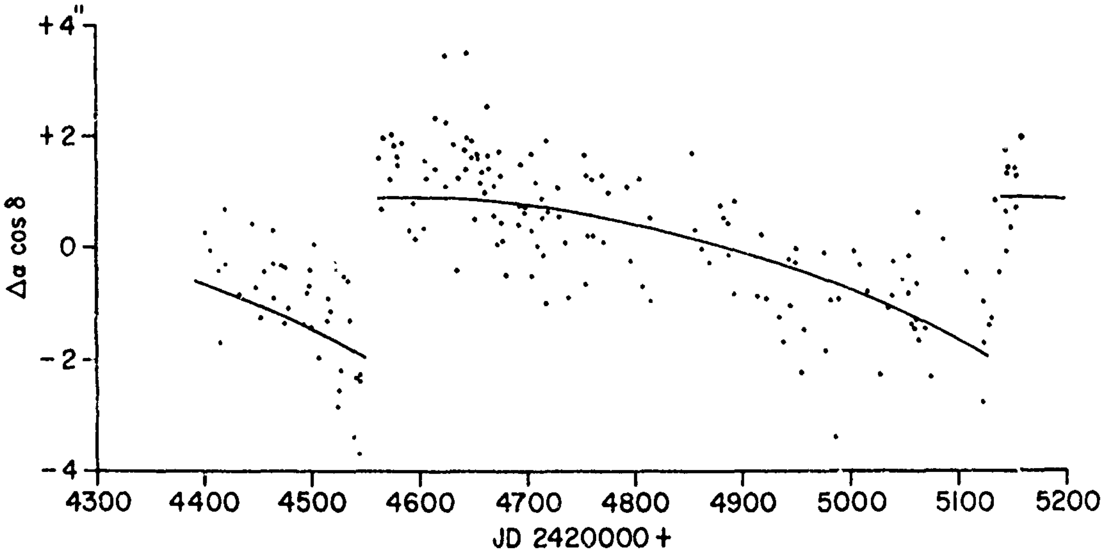

FIGURE 6

Venus Residuals in Right Ascension Uncorrected for Phase Effects

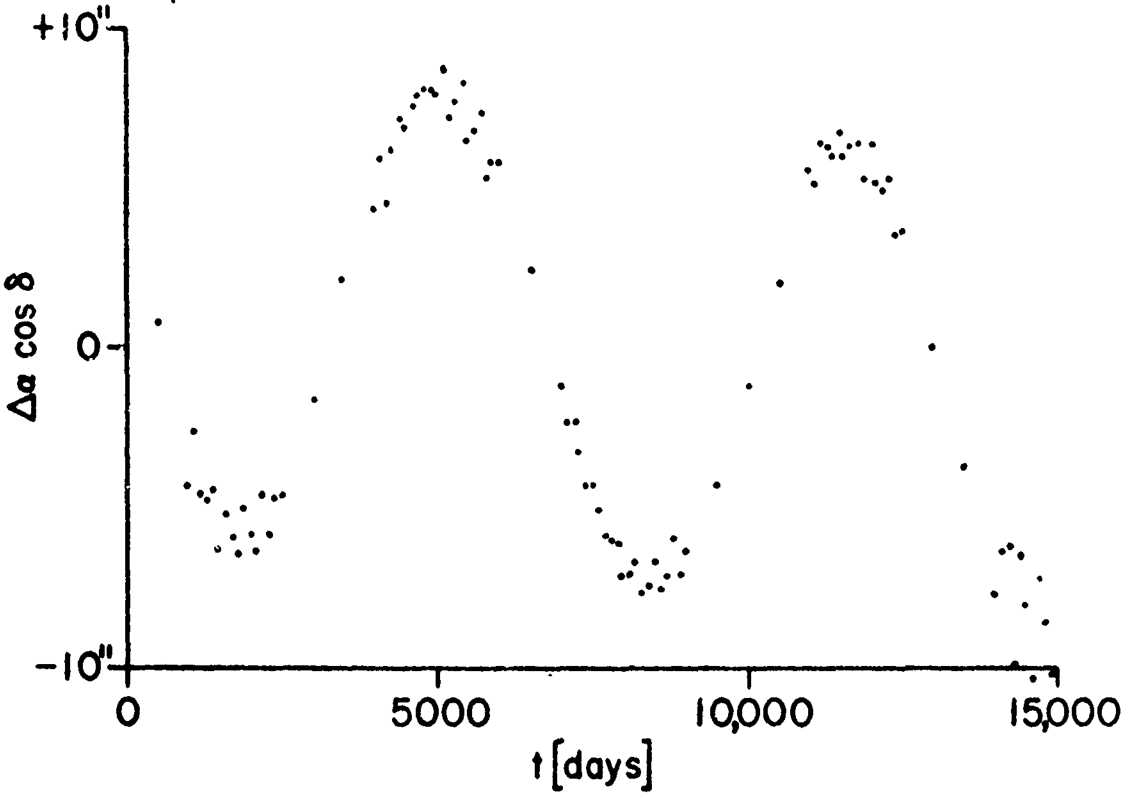

FIGURE 7

Effects on Lunar Residuals Due to J_2 of Earth and Precession of Lunar Orbit the effect of suppressing J_2 of the earth in the presence of solar perturbations. Its amplitude is about 7″ and it has the nodal period of about 18 years.

The leading zonal harmonic of the moon produces effects which are considerably smaller. The secular perturbation in g is only about 11″ after 20,000 days. The new feature is that here the precession of the lunar equator must be taken into account. According to Cassini, it has the same rate as the node of the orbit. Hence, the lunar equatorial plane completes one precession cycle in 18.6 years. The effect on the lunar α residuals is mostly secular, something like 6″ after 20,000 days.

Partial derivatives of right ascension and declination with respect to the solar oblateness coefficient J_2 were tested for a series of observations on Mercury and Venus against finite difference ratios. It seems that the partials for Mercury usually agree to about two significant figures. This is by no means spectacular, but we still hope that they serve in the very limited role they have. Venus partials are sometimes 10 - 15% off, but they enter with considerably less weight.

6. LUNAR TESSERAL HARMONICS

The second degree terms with coefficients c_{22} and s_{22} were introduced into our program at a rather late stage. Admittedly, these additions were made without careful analysis of the size of the resulting perturbations but rather prompted by the thought that the earth always remains near the first meridian of the moon.

The first test run immediately indicated a problem since the right ascension residuals increased rapidly with time. It looked like a growth cubic in time which, extrapolated to 20,000 days, would amount to several degrees. However, any growth in α other than linear must be erroneous since the total work done by such an acceleration must be zero, taken over a long enough time. There could be such an effect, of course, if the earth were to oscillate about the first meridian with a bias. Hence, differential corrections were made, with the aid of finite difference ratios, to the initial orientation of the moon and to its orbital rate. The first is given by the argument of latitude of the first meridian at epoch, the second must be the draconic month. We found that our residuals called for an increase in said argument of 0°.29 and a decrease in the period by 0°.40 X 10^{-4}/day. Since the latter amounts to only 0°.80 after 20,000 days, both numbers sound reasonable. However, all that was accomplished was a change from a signal cubic in time to a quadratic one, still leaving a residual of $-150''$ in $\Delta\alpha \cos\delta$ after 20,000 days. It should be noted that the real problem is larger than the $150''$ indicate since much of the signal has been removed by the simulataneous corrections to the lunar elements.

Careful checks did not reveal any errors in either formulation or computer program; they appear to be well hidden. We finally decided to bypass the tesseral harmonics altogether. It can be shown, fortunately, that the effects of this step are inconsequential. The neglected accelerations are about 1.5×10^{-9} times the central term. Since we are dealing with an almost constant force, the missing accelerations can be simulated by changing the combined masses of earth and moon by a like percentage. Note that this sum of masses is known to only six or seven figures. However, since we do not adjust the masses, the effects of the missing terms must be absorbed by some other parameter. It cannot be the mean motion because of the enforced fit to observations. That leaves the semimajor axis. The relative error da/a is then about 4.5×10^{-9} which is much smaller than the computed standard deviation.

7. PARTIALS FOR TIDAL EFFECTS

These partials were tested in two ways. Both lunar and solar tide partial derivatives were compared with results from finite difference ratios. Since those for the moon are obtained by numerical integration, they are expected to be quite accurate. Agreement is usually found to three or four places. In case of the sun, partials are analytic approximations. Here a number of discrepancies up to 7% were noted. In view of the marginal value of this particular parameter, no other tests were made.

The lunar partials were subjected to an additional and more comprehensive test. Synthetic observations were generated without tidal terms. Next an orbit was fitted to these data by improving only the tidal coefficient. Its initial input value was -0.80×10^{-13}, and this number was driven down to -0.76×10^{-17} in three iterations. In the first iteration, residuals were about $9''$ at 1,000 days from epoch, which came down to $0\overset{''}{.}007$ in the third iteration. This rate of convergence instills some confidence in the procedure.

VIII. CONSTANTS AND NUMERICAL RESULTS

In order to keep all numerical data together in one place, we are listing the input constants as well as our results in this chapter.

The planetary masses were changed several times during the course of this study. The set finally adopted and given in Table IV is identical to that used by O'Handley in his Development Ephermis 69, but the decision to go with these particular numbers was somewhat arbitrary. O'Handley's figures for the terrestrial planets benefit from electronic observations.

More recent mass solutions exist, based on more electronic data, but they are in a constant state of flux and agreement between the various results is not at all satisfactory. Also, several of these determinations employ procedures that we have investigated and found to be inadequate.

Other constants used are as follows:

$$k = 0.017\ 2020\ 9895$$

$$c = 299\ 792.5 \text{ km/sec}$$

$$1 \text{ a.u.} = 149\ 597\ 900 \text{ km}$$

$$C_{20} = -.001\ 082\ 610$$

$$C_{30} = .000\ 002\ 539$$

$$c_{20} = -.000\ 207\ 5$$

The speed of light and the a.u. are used only in the light time correction.

In an earlier part of this paper we explained why the program fails when attempting to solve for all parameters simultaneously. What happens is that errors creep into the lunar element corrections large enough to produce residuals of 50″ or so after 55 years. However, it is possible to iterate successfully to a least-squares solution by alternating between two carefully chosen blocks of parameters. Even then one must proceed with care. The lunar initial conditions are exceedingly

TABLE IV

Reciprocal Masses

Body	Reciprocal Mass
Mercury	5 983 000
Venus	408 522
Earth	332 945.561 925 44[1]
Mars	3 098 700
Jupiter	1 047.390 8
Saturn	3 499.2
Uranus	22 930
Neptune	19 260
Pluto	1 812 000
Moon	27 068 807.130 100[1]

1) The numbers for earth and moon are a consequence of a reciprocal mass of 328 900.1 for the earth-moon system and a mass ratio of 81.301. Fourteen significant figures were retained in the individual masses because of the wordlength of our computer.

sensitive to minute changes in other parameters. Small adjustments of planetary elements, very close to final convergence, can easily cause signals of 20″ in the residuals of the moon. Hence, it should be no surprise that the last three iterations were devoted to correcting the lunar elements, the clock corrections, the declination bias, and the lunar tidal coefficient. The normal equations at each iteration were subjected to a variety of test solutions. Hence, it was easy to ascertain that any additional planetary corrections called for were always well within their own standard deviations.

Table XI lists the number of observations used, weights, and the standard deviations (root mean square) of their final residuals. Observations in right ascension and declination are now counted separately. This procedure has recently been adopted by other researchers in combining optical data with other types of observations, such as radar ranges. The weights are those actually employed in the final run. They are $1/\sigma^2$, where σ is the standard deviation of the residuals in the previous iteration, converted to radian measure. It is easily verified that the sigmas in this table would lead to very similar weights. The derivation of the sigmas was discussed previously. It is important to note, however, that the weight of all residuals larger than 4σ was put to zero. In the case of Pluto, this gate is probably too narrow. In a test with a 20σ cut off, the sigmas for Pluto went up to 1″.43 and 1″.66, in α and δ! In the same test, the σ for all other planets were ″.01 or ″.02 larger than those given in Table XI. This indicates that, except for Pluto, very few residuals fall outside the 4σ-band. Hence, 4σ seems to be a comfortably conservative cutoff.

Among the principal results of this study are the improved orbital parameters. All adjustments were made at the epoch of integration, namely JD 242 0000.5. We will give coordinates, Table XII, at this date as well as the three following future dates:

JD 242 0000.5 = 1913 August 21.0

244 1200.5 = 1971 September 6.0

244 1600.5 = 1972 October 10.0

244 1000.5 = 1973 November 14.0

Osculating elements will also be listed, but only for the first and last of above times. Formal standard deviations are given for coordinates and elements at epoch. We wish to stress the fact that these as well as all other standard deviations quoted for the final results are obtained from a total solution for all parameters. Such numbers are larger than the sigmas associated with a partial solution and, we hope, more realistic. All coordinates are tabulated to the full 14 place capacity of our computer in order to facilitate using our numbers as initial conditions for other runs. The elements, on the other hand, are truncated in keeping with their formal standard deviations. Since the improvement of the lunar elements was done in the somewhat unusual equatorial frame, we will list, at epoch, both the equatorial and the derived ecliptic elements. See Tables XIII and XIV.

In using any of our figures, the following must be kept in mind. Data given for JD 242 0000.5 incorporate the adjustments procuded by the last iteration. All others are consistent with the initial conditions to the last run and, hence, reflect the next to last improvement cycle. The differences are of academic interest only since they are always much smaller than the respective sigmas.

As backup to the published tables we intend to save our tapes which contain coordinates at intervals of 400 days.

A few comments should be made about the standard deviations given with our coordinates and elements. Table XII shows that both y and $\dot{y}$ for Pluto are considerably weaker than the other coordinates. It is interesting to speculate by considering the geometry. Since Pluto lies approximately along the y-axis around epoch, it would seem likely that angular observations yield less resolution in this direction.

It is not difficult to see why some sigmas are given to two significant figures. We aimed at quoting the same number of decimals for all three components of any given position or velocity vector. In each case, the smallest sigma determined the number of decimals retained.

Up to four significant figures are given in certain standard deviations in Table XIV. In each such case we are dealing with a small eccentricity or inclination. We would guess that the nonsingular elements $l + g$ and $g + h$, respectively, would have sigmas in the last decimal quoted. Although the latter values can be calculated, we truncated our table simply by inspection.

Table XIV also shows that the accuracy of the semimajor axes deteriorates quickly as we go to the outer planets. From Saturn to Pluto, the corresponding sigmas increase by three orders of magnitude. This is partly explained by comparison of their periods to our observation span of 55 years.

In the case of Pluto, it made no sense to quote l, g, and h to the same number of decimals. The in-plane elements l and g also suffer from the relatively short data span. The sigma for h, on the other hand, shows that the node is sharply defined, thanks to the substantial inclination.

The ecliptic elements for the moon are derived quantities. The principal least-squares solution was made in terms of equatorial elements, given in Table XIII. Again, it is seen that the large equatorial inclination leads to a much better defined node than the ecliptic counterpart.

We did not think it necessary to list all correlation coefficients of our 73 X 73 matrix. In fact, since the computer prints coefficients for the nonsingular elements, equatorial coordinates, and ecliptic elements, there is a total of about 8,000 numbers. We decided to tabulate only the correlation coefficients between the six elements of any given planet and all others exceeding about 0.10. Table XV gives the former. It is arranged to conserve space, but it must not be misread to contain correlation coefficients across planets.

Table XVI gives all planetary elements which show correlation coefficients larger than 0.10 with the lunar elements and the clock corrections. They are seen to be the elements a and l of the terrestrial planets and a for Jupiter. C is the lunar tidal coefficient. Note that the lunar elements I and h are missing because of very small coefficients. The l for Venus was left in order to complete the upper portion of the table.

Practically all remaining planetary coefficients greater than about 0.10 are listed in Table XVII. About half a dozen were ignored so that we would not have to add another line for just one number. The largest of these was a 0.13 between the a's of Mars and Jupiter.

Finally, Table XVIII gives the complete set of correlation coefficients between the tidal term C, the lunar elements, and the clock corrections. Some of the large numbers are not unexpected.

The only parameter not mentioned at all so far is the declination bias $\Delta\delta$. We found it only very weakly correlated with anything else.

One of the principal results of this study is the extrapolation of the atomic time scale back to 1912.5. The final figures are given in Table XIX. The column labelled ΔT_0 is taken from Brouwer's work (1952) through 1948.5. From 1949.5 on, these numbers are obtained from the American Ephemeris. The next column contains our solution "atomic time minus universal time". The symbols AT and UT were chosen purposely different from the abbreviations A.1, U.T., and U.T.1, which have a very specific meaning. As stated elsewhere, AT-UT was solved for at intervals of four years, and intermediate values were supplied by four point interpolation. The standard deviations listed are again obtained from a global solution. In a partial solution for clock corrections and lunar elements only, some of the sigmas were only 30% of the tabulated ones.

The final column in Table XVII, AT-ET, is simply (AT-UT) - ΔT_0. It is a measure of the disparity between the atomic and ephemeris time scales. Figure 8 is a plot of these numbers. While there is a total excursion of six seconds or so, the fine structure on the left is well within the noise. However, some of the fluctuations in the right hand half of the figure seem reasonably well established.

The last two parameters of our principal solution are the lunar tidal coefficient and the declination bias. For better visibility, they are put in Table V. The product KT^2 would be added to the computed mean longitude of the moon, with T measured in centuries. Over our data span of 55 years, this effect amounts to a sizeable 6″.

The bias term $\Delta\delta$ of $-0\rlap{.}''33$ is to be added to the computed declinations of the moon in order to match observations. These computed places are, of course, dynamically consistent with all other ephemerides of our solar system. Most of the observed declinations are those recently uniformly reduced (Adams, Klock, and Scott, 1969) and, in particular, adjusted using Watts' limb corrections. The data before 1925, not treated by above authors, are effectively in the same declination system since we had added a constant bias correction of $0\rlap{.}''57$ to these early residuals. Perhaps the center of Watts' reference sphere is $0\rlap{.}''33$ south of the center of mass.

Figures 9 to 28 are plots of the residuals in right ascension and declination. In viewing these it should be recalled that all points greater than 4σ have been deweighted and do not contribute to the solution. The Venus residuals still seem too noisy, but no additional systematic signal was found. From Mars to Neptune, some plots exhibit signals which appear to be above the noise level. Clearly, these trends were not removable by improving the initial conditions. Whether mass corrections can be extracted remains to be seen.

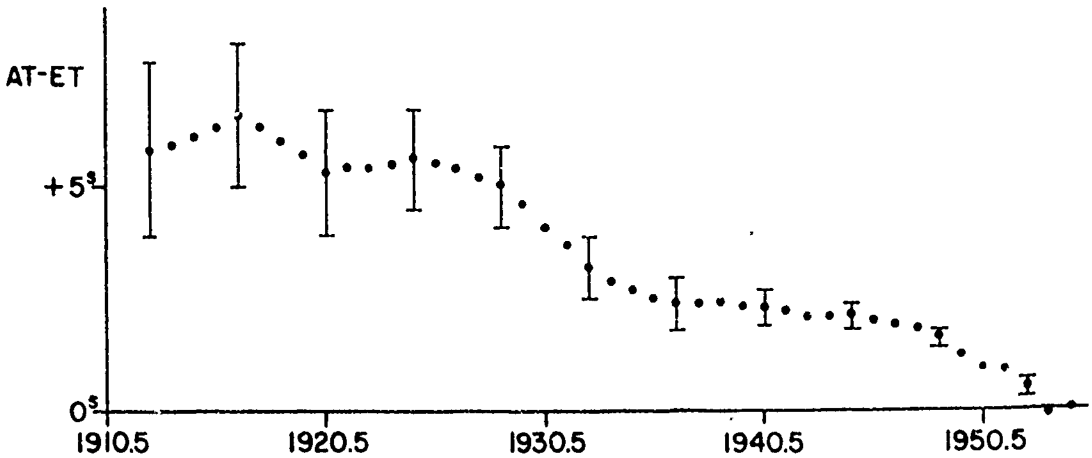

FIGURE 8

Atomic Minus Ephemeris Time, Extrapolated Back to 1912.5

TABLE V

Tidal Coefficient and Declination Bias

K	$(-19'' \pm 4'')$ / century^2
$\Delta\delta$	$-''\!.33 \pm ''\!.01$

Note: $K = -\frac{3}{2}\frac{C}{a}$

Table VI contains two parameters which were not part of our principal solution. The numbers shown here stem from various experimental solutions, but they are still interesting.

$K_\odot$ results from modeling a tidal term in the apparent motion of the sun or, what amounts to the same, in the equations of motion of the earth. Van Flandern correctly predicted that we should not find any term of significance, and the result bears him out. $K_\odot$ is seen to be a positive quantity. Since we are not aware of any forces which would increase the earth's mean motion, we omitted this term from our main solution. Since the numerical value is 3σ, it has some statistical significance. However, the total effect of this term would be only $0\overset{''}{.}2$ over our data span. It is more than likely that there are a number of unmodeled forces which could give rise to perturbations of this size.

The other quantity in this table, $J_\odot$, is a bit of a disappointment. We had hoped to get the leading zonal harmonic of the sun's oblate field. However, since it is only of the order of its own sigma, it has really no significance. Hence, the wrong sign means little. If $J_\odot$ were actually of this size, it would produce a secular perturbation in Mercury's perihelion of about 2″ in 55 years, and about 1″ in l and h. All that can be said from our result is that the sun's J_2 is not likely to be much larger than 1.4×10^{-5}.

TABLE VI

Suppressed Parameters

$K_{\odot}$	$(+.''6 \pm .''2)$ / century^2
$J_{\odot}$	$(-.17 \pm .14) \times 10^{-4}$

IX. SUMMARY AND RECOMMENDATIONS

We have derived and presented very accurate orbital elements for the moon and the major planets. It is our belief that ephemerides based on our initial conditions are a measurable improvement over numbers presently published by the international astronomical community. In fact, our solution appears to be almost as refined as the present state of the art and the pocketbook permit.

Our investigations also yielded some by-products, and two of these have considerable scientific significance. The backward extrapolation of the atomic time scale extends from 15 to almost 60 years the span over which a clock with constant rate is now available. The secular deceleration of the moon has been determined over the same interval. Since this result is also believed to be a refinement over previous numbers, it may well have a bearing on studies of the past history of lunar motion.

In the introduction of this report we have explained that certain deficiencies in the available ephemerides prompted us to undertake this study. Now that good results are in hand, our initial conditions, or those stemming from another modern global solution, should be made the basis for future publications of the positions of sun, moon, and planets. We believe that this suggestion merits careful consideration particularly in view of the fact that the International Astronomical Union plans to extend the present set of tables for another decade.

At the moment, we feel most comfortable with our own results and procedures. As explained earlier, the simultaneity of solving for the orbits of moon and planets is a crucial feature expecially in deriving highly accurate lunar elements. Also, we have subjected our model and program to a variety of tests, and they now appear to be free of detectable errors. Nevertheless, we wish to recognize the somewhat similar efforts of our friends at MIT and JPL. Theirs are also modern solutions having their own strong points.

The introduction of ephemerides computed by numerical integration and based on global solutions is seen as a first step only. In order to maintain a product that is to be useful in these days of space age technology, improvements must be made continuously. Different and more precise observational data will become available so that both the mathematical model and the computational tools require updating. At the same time the needs of interplanetary navigation will become more stringent. Researchers will continue to find that, in filling the practical requirements, exciting scientific discoveries are made.

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APPENDIX A

AUXILIARY TABLES

TABLE VII

USNO Transit Circle Data Punched at Dahlgren

Volume	Instrument	Time Span	Source Codes
XI	6″	1911 - 1918	D 6
XIII	9″	1913 - 1925	D 7
XV Part V	9″	1935 - 1944	D 8
XVI Part I	6″	1925 - 1941	D 9
XVI Part III	6″	1941 - 1948	D 0

TABLE VIII

Observations on Dahlgren Master Tape

Planet Source Code	Mercury	Venus	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto	Sun	Moon	Time Span of Observation Series: Date	JD		Date	JD	Cards Supplied by	Source
D 6	X	X							X		1911 5 17	(241 9174.21)	-	1918 8 16	(242 1822.21)	NWL	V XI
D 7	X	X	X	X	X	X	X		X	X[1]	1913 8 7.2	(241 9987.40)	-	1925 6 22.1	(242 4323.77)	NWL	V XIII
D XX								X			1914 1 23.79	(242 0156.29)	-	1965 5 2.83	(243 8883.33)	NWL	Astron J 72,973
UTVF										X	1925 1 2.99	(242 4153.49)	-	1968 6 16.42	(244 0023.92)	USNO	V XIX Pt III
D 9	X	X	X	X	X	X	X		X		1925 5 16.5	(242 4287.22)	-	1941 5 24.5	(243 0139.21)	NWL	V XVI Pt I
D 8	X	X	X	X	X	X	X		X		1935 6 4.5	(242 7958.21)	-	1944 12 19.5	(243 1444.21)	NWL	V XV Pt V
D 0	X	X	X	X	X	X	X		X		1941 6 6.5	(243 0152.21)	-	1948 12 28.5	(243 2914.21)	NWL	V XVI Pt III
D 1	X	X	X	X	X	X	X		X		1949 1 7.16	(243 2923.66)	-	1955 12 28.80	(243 5470.30)	JPL	V XIX Pt I
U 004	X	X	X	X	X	X	X		X		1956 1 19.22	(243 5491.72)	-	1962 9 1.21	(243 7908.71)	USNO	V XIX Pt II
D 3	X	X	X	X	X	X	X		X		1963 3 8.11	(243 8096.61)	-	1964 7 2.73	(243 8579.23)	JPL	C 105
D 4	X	X	X	X	X	X	X		X		1964 7 7.64	(243 8584.14)	-	1965 6 27.07	(243 8938.57)	JPL	C 108, C 115[3]
U 002	X	X	X	X	X	X	X		X		1966 6 1.60	(243 9278.10)	-	1967 7 7.84	(243 9679.34)	USNO	C 118
U 005	X	X	X	X	X	X	X		X	X[2]	1967 7 12.71	(243 9684.21)	-	1969 1 1.71	(244 0223.21)	USNO	C 124, C 127[4]

1) to 1924 12 29.2 (242 4149.36)
2) from 1968 7 6 04 (244 0043.54)
3) part of C 115 was lost
4) contains only part of C 127

TABLE IX

Identification of Pluto Observations

Source Code	Meaning	Source
D PR	Prediscovery observations	A.J. 72,973 Table I
D SB	Sharaf-Budnikova normal places	A.J. 72,973 Table II
D YM	Yerkes-McDonald observations	A.J. 72,973 Table III
D LO	Lowell observations	A.J. 72,973 Table IV
D YE	Yerkes observations	A.J. 72,973 Table V
D HA	Halliday observations	A.J. 72,973 Table VI
DL 1	Dahlgren List 1	Chernykh and Chernykh

TABLE X

Adams and Cowell Coefficients

j	a_j	c_j
0	+32,011,868,528,640,000	+32,011,868,528,640,000
1	-16,005,934,264,320,000	-32,011,868,528,640,000
2	-2,667,655,710,720,000	+2,667,655,710,720,000
3	-1,333,827,855,360,000	0
4	-844,757,641,728,000	-133,382,785,536,000
5	-600,222,534,912,000	-133,382,785,536,000
6	-456,783,110,784,000	-116,974,585,728,000
7	-363,891,528,000,000	-100,566,385,920,000
8	-299,520,219,398,400	-86,707,632,211,200
9	-252,655,401,398,400	-75,398,324,601,600
10	-217,227,737,563,200	-66,193,573,118,400
11	-189,640,115,028,000	-58,648,487,788,800
12	-167,636,336,098,320	-52,401,453,198,480
13	-149,735,464,049,160	-47,174,128,491,600
14	-134,928,496,929,540	-42,755,108,505,900
15	-122,506,205,369,730	-38,983,584,907,800
16	-111,956,703,448,001	-35,736,323,456,205

D = 32,011,868,528,640,000

All numbers are to be divided by the common denominator D. Table applicable to all orders up to 16.

TABLE XI

Number and Quality of Observations

Body	Number of Observations	[1]Weight × 10^{-11} in $\Delta\alpha \cos\delta$	$\Delta\delta$	Final Standard Deviation σ in $\Delta\alpha \cos\delta$	$\Delta\delta$
System	40 972	..	..	.65	.72
Mercury	4 380	.52995	.59585	.90	.84
Venus	6 058	.53712	.50817	.89	.92
Mars	1 284	1.42713	1.39628	.55	.55
Jupiter	1 748	2.06414	1.56738	.45	.52
Saturn	1 750	1.66827	1.49200	.51	.53
Uranus	1 822	3.21105	1.76477	.36	.49
Neptune	1 796	2.50643	1.92599	.41	.47
Pluto	1 110	.54076	.62207	.89[2]	.83[2]
Sun	14 148	.74048	.63428	.76	.82
Moon <1925 Moon ≥1925	6 876	.52525 .68170	.49191 .68170	.93 .70	.93 .79

[1] Many significant figures in weights are meaningless but may be required for duplication of final results.

[2] Unrealistic. See text.

TABLE XII

Heliocentric Equatorial Coordinates. Mean Equator and Equinox of 1950.0

	x[a.u.] z	y	z	ẋ[a.u./100 d]	ẏ	ż
Moon	.8588 3367 5247 52	-.4935 3999 1945 23	-.2139 9780 2491 52	.8744 2800 8809 1	1.3777 6130 6977 4	.6028 7047 0591 2
	±.0000 002	±.0000 002	±.0000 001	±.0000 007	±.0000 002	±.0000 001
	.9635 6034 3737 03	-.2790 7250 2798 48	-.1208 6492 9786 22	.4962 5753 8514 9	1.5547 4780 8923 0	.6788 9607 3746 7
	.9956 6103 2436 13	.2579 6556 8210 20	.1116 2523 7890 37	-.4765 8242 0818 4	1.4714 6613 3205 6	.6401 9102 3520 3
	.6203 9694 3382 46	.7089 7782 4823 32	.3073 9017 2555 56	-1.4287 2961 5261 0	.9804 5133 8019 7	.4191 2645 6254 7
Mercury	.3014 1266 8253 89	.1285 9494 7178 22	.0378 4152 5646 82	-1.6816 6807 8098 1	2.1334 6003 7844 4	1.4224 5923 7170 9
	±.0000 04	±.0000 005	±.0000 003	±.0000 05	±.0000 03	±.0000 01
	.3110 8140 3238 44	.1144 9436 1824 89	.0293 5912 9522 66	-1.5331 7648 8888 3	2.3917 3418 7714 7	1.4383 5667 4930 7
	-.2164 5745 6053 08	-.3695 2221 2530 23	-.1756 5723 0842 97	1.9216 4576 3167 2	-.9677 3959 3182 4	-.7156 5667 7999 6
	.1071 8782 8756 78	.2588 7850 9684 44	.1276 2912 8612 58	-3.1987 7605 9099 9	.8533 7022 2128 5	.7855 8542 5269 0
Venus	5359 7737 9674 19	.4544 7200 9303 60	.1708 3667 1726 30	-1.3634 1422 7994 0	1.3285 2669 2131 6	.6845 9799 0991 7
	± 0000 003	±.0000 003	±.0000 002	±.0000 008	±.0000 007	±.0000 003
	-.7038 2500 9735 15	.1142 2571 1335 84	.0959 2622 7409 56	-.4132 5160 6386 7	-1.8248 8513 1791 1	-.7962 5456 6676 7
	.0207 6300 4552 03	.6563 6217 8221 29	.2944 6861 7428 56	-2.0288 5517 0096 8	-1.0029 4566 6254 8	.1268 0824 3086 3
	.7156 6047 7259 35	.1211 9671 1864 25	.0094 1807 2364 10	-.3284 7842 0152 7	1.8050 3378 1333 6	.8341 5954 4894 0
Earth	.8561 5698 3323 07	-.4940 6304 5723 79	-.2143 0905 0762 53	.8879 8343 9180 7	1.3300 0253 7802 8	.5768 7810 9896 4
	±.0000 002	±.0000 002	±.0000 001	±.0000 004	±.0000 002	±.0000 001
	.9611 7879 0827 28	.2787 9403 8580 68	-.1208 9770 9468 64	.4903 5739 1990 8	1.4987 4121 8202 8	.6498 3470 8238 5
	.9575 4044 5799 32	.2596 5982 6741 67	.1125 9433 8915 74	-.5162 9383 7061 5	1.5085 9391 7224 1	.6541 4091 0829 9
	.6207 2446 9576 41	.7067 5363 0648 05	.3064 6351 0847 34	-1.3670 5375 6473 4	.9848 2164 1629 5	.4271 2312 8724 3
Mars	1.2054 0598 2723 6	.7272 0270 1740 7	.3012 6927 7695 5	-.7092 8352 6874 9	1.1656 0437 6444 3	.5541 3496 6121 4
	±.0000 002	±.0000 003	±.0000 001	±.0000 003	±.0000 002	±.0000 001
	1.2369 8921 9494 6	-.5459 4727 7825 0	-.2837 9252 4838 4	.6771 0813 9361 0	1.2539 4239 8580 5	0.5574 0968 9044 1
	-1.6509 1949 7293 2	.0120 0050 9271 8	.0498 0540 0271 3	.0278 4376 5436 8	-1.1636 4188 1675 5	-.5348 7825 2959 8
	1.0712 4137 4575 1	.9098 9426 0971 4	.3889 2016 2376 7	-.8943 3779 4008 6	1.0333 9454 9838 1	.4983 4157 4832 6
Jupiter	1.5413 7062 0191 5	-4.5249 9586 3662 5	-1.9791 7225 7169 8	.7124 6623 2960 00	.2456 5410 7252 18	.0879 6486 3140 30
	±.0000 008	±.0000 004	±.0000 005	±.0000 0005	±.0000 0010	±.0000 0008
	-1.8610 0233 0967 8	-4.6105 7708 0498 1	-1.9324 4709 4718 0	.6992 3349 8539 60	-.2035 6349 7166 35	-.1044 1878 3033 68
	1.0770 8512 1618 3	-4.6635 1617 5876 2	-2.0269 6797 9931 6	.7302 8771 0993 40	.1828 2558 3103 16	.0605 8337 0215 14
	3.6484 4242 3167 1	-3.1885 6285 6184 3	-1.4570 5941 3855 1	.5145 8739 1602 44	.5377 0578 5677 46	.2180 9890 3370 32
Saturn	2.9214 7119 6571 6	7.9536 0576 5225 5	3.1615 8784 8402 5	-.5578 1909 5390 19	.1571 3268 4735 8	.0890 4565 7495 96
	±.0000 02	±.0000 009	±.0000 01	±.0000 0004	±.0000 0010	±.0000 0007
	4.5772 2823 0469 5	7.3220 8096 4855 8	2.8296 3119 9724 2	-.5114 6110 5402 91	.2509 4791 2980 46	.1258 8742 4372 26
	2.4085 2087 1038 1	8.0845 3365 0003 4	3.2387 1645 5799 7	-.5673 2921 3262 52	.1279 1469 6090 14	.0774 3670 1564 28
	.0860 8120 0826 8	8.3323 9150 3350 2	3.4416 8524 6807 7	-.5877 1844 0129 93	-.0048 0779 5380 90	.0234 4003 6257 52

TABLE XII (Continued)

	x [a.u.]	y	z	$\dot{x}$ [a.u./100 d]	$\dot{y}$	$\dot{z}$
Uranus	11.7455 7044 8084	−14.5716 6410 0264	−6.5511 7579 4430	.3142 1052 2768 08	.1985 9135 6592 66	.0825 6797 6774 39
	±.0000 07	±.0000 02	±.0000 02	±.0000 0014	±.0000 0007	±.0000 0005
	−17.8627 5625 7709	−3.9429 9700 0301	−1.4752 3941 2803	.0874 3863 1447 54	−.3671 7358 5979 09	−.1621 2038 1396 26
	−17.4449 9221 4571	−5.3940 9389 5945	−2.1169 7689 2333	.1212 3866 1881 29	−.3579 1122 3237 65	−.1585 4177 8402 38
	−16.8945 8032 1104	−6.8027 9027 4465	−2.7420 1520 2544	.1537 1000 2472 55	−.3460 0609 5920 10	−.1537 8414 5913 29
Neptune	−13.3527 4430 2403	24.7256 3582 3790	10.4645 8160 7667	−.2823 2710 4702 41	−.1298 3396 3363 39	−.0460 6982 2601 27
	±.0000 2	±.0000 2	±.0000 1	±.0000 002	±.0000 003	±.0000 001
	−14.2179 7116 6151	−24.9047 7051 1434	−9.8441 7845 2524	.2758 0291 3039 22	−.1322 1274 5403 71	−.0611 0233 3417 57
	−13.1028 7494 6495	−25.4116 2818 7726	−10.0798 6345 5523	.2816 2437 2295 40	−.1211 7150 7879 05	−.0567 1854 4702 80
	−11.9658 6398 0240	−25.8739 3439 5511	−10.2978 2202 8119	.2867 5086 9894 66	−.1099 5281 7836 36	−.0522 4555 2231 66
Pluto	−.1968 0838 0477	42.6773 9127 0856	13.5391 6517 0838	−.2240 1847 3997 99	−.0686 0971 5722 25	.0464 0707 7004 21
	±.0000 7	±.0003 2	±.0001 2	±.0000 004	±.0000 030	±.0000 009
	−30.1337 0797 5330	−3.0490 3217 5568	8.1684 3194 8738	.0515 5204 5815 39	−.3134 8842 5897 75	−.1145 1069 3906 85
	−29.9041 7074 1965	−4.2996 5208 5851	7.7044 7483 2341	.0631 9480 0767 90	−.3117 2817 1671 96	−.1174 4240 5588 56
	−29.6282 5533 6199	−5.5422 3795 0558	7.2290 4099 3881	.0747 2897 6766 80	−.3094 8199 5153 22	−.1202 5228 5627 57

Line 1: Coordinates for JD 242 0000.5
Line 2: Formal Standard Deviations for JD 242 0000.5
Line 3: Coordinates for JD 244 1200.5
Line 4: Coordinates for JD 244 1600.5
Line 5: Coordinates for JD 244 2000.5

TABLE XIII

Lunar Osculating Geocentric Equatorial Elements
Mean Equator and Equinox of 1950.0

Element	Value and Standard Deviation at JD 242 0000.5	
a	.002 563 725 2	± .000 000 000 2 a.u.
e	.058 227 02	± .000 000 09
I	28°.680 189	±° .000 005
l	198.587 90	± .000 25
g	177.267 78	± .000 06
h	359.000 37	± .000 01

TABLE XIV

Osculating Ecliptic Elements. Mean Ecliptic and Equinox of 1950.0.

	a [a.u.]	e	I	l	g	h
Moon	.002 563 725 2	.058 227 02	5°252 621	198°587 90	181°616 29	354°752 57
	±.000 000 000 2	±.000 000 09	±.000 005	±.000 25	±.000 08	±.000 06
	.002 561 126 2	.054 442 37	5.200 962	20.647 11	166.091 05	268.691 91
Mercury	.387 097 842 2	.205 624 5	7.006 20	324.152 8	28.836 5	47.783 8
	±.000 000 000 4	±.000 000 3	±.000 02	±.000 1	±.000 2	±.000 2
	.387 098 432 3	.205 620 7	7.002 53	355.505 5	29.009 8	47.707 7
Venus	.723 325 568 1	.006 855 96	3.394 451	271.975 55	54.568 24	76.331 82
	±.000 000 000 8	±.000 000 08	±.000 007	±.000 65[1]	±.000 66[1]	±.000 11
	.723 326 653 7	.006 796 79	3.393 910	238.613 45	54.968 21	76.168 18
Earth	.999 416 569 5	.016 945 64	.003 033	225.239 00	92.912 02	11.037 81
	±.000 000 000 8	±.000 000 03	±.000 003	±.000 11	±.055 80[2]	±.055 80[2]
	.999 330 544 3	.016 095 09	.002 544	309.290 78	333.814 72	129.478 95
Mars	1.523 662 703	.093 219 38	1.852 889	49.430 12	285.680 77	49.279 46
	±.000 000 002	±.000 000 04	±.000 004	±.000 02	±.000 12	±.000 12
	1.523 634 008	.093 331 90	1.848 036	57.912 07	286.102 63	49.098 38
Jupiter	5.202 965 95	.048 091 38	1.307 499	279.423 14	273.532 31	99.861 20
	±.000 000 03	±.000 000 04	±.000 006	±.000 05	±.000 25	±.000 25
	5.202 854 11	.048 083 11	1.306 451	307.147 26	273.579 39	99.981 55
Saturn	9.523 632 9	.053 680 17	2.489 648	340.873 77	339.048 64	113.346 71
	±.000 000 2	±.000 000 06	±.000 007	±.000 06	±.000 16	±.000 14
	9.528 437 1	.053 852 72	2.490 769	358.108 15	338.364 29	113.178 83
Uranus	19.280 385	.044 255 51	.773 712	128.117 5	100.550 9	73.803 8
	±.000 003	±.000 000 09	±.000 006	±.000 2	±.000 5	±.000 5
	19.172 780	.045 956 81	.772 571	29.705 3	97.248 0	73.759 3
Neptune	29.985 79	.008 229	1.775 952	89.577 3	254.687 1	131.229 2
	±.000 05	±.000 001	±.000 006	±.004 6 [1]	±.004 8 [1]	±.000 2
	30.129 69	.007 869	1.773 438	220.834 3	255.177 9	131.300 2

TABLE XIV (Continued)

	a [a.u.]	e	i	l	g	h
Pluto	39.383 9	.249 779	17°182 67	248°747	114°399	109°584 06
	±.000 7	±.000 003	±.000 04	±.004	±.003	±.000 05
	39.399 3	.248 065	17.149 76	336.543	114.186	109.706 18

Line 1: Elements for JD 242 0000.5
Line 2: Formal standard deviations for JD 242 0000.5
Line 3: Elements for JD 244 2000.5

1) Small eccentricity. Elements l and g poorly defined.
2) Small inclination. Elements g and h poorly defined.

TABLE XV

Correlation Coefficients

		a	e	I	l	g	h	
	a		-.04	.01	.03	.02	-.03	
	e	-.01		-.01	-.03	.04	-.01	
Mercury	I	.01	.00		.01	-.01	-.01	Venus
	l	.70	.07	.02		-.98	.01	
	g	.00	-.06	-.06	-.28		-.18	
	h	.00	.01	.07	.00	-.90		
	a		.05	.02	.39	.03	-.02	
	e	.02		-.01	-.01	.01	.00	
Earth	I	.06	.01		.01	.01	-.01	Mars
	l	.17	.04	.01		-.16	-.01	
	g	.01	.00	-.02	.00		-.98	
	h	-.01	.00	.02	-.01	-1.00		
	a		-.27	.00	-.20	.13	.00	
	e	.11		.00	.01	-.02	.00	
Jupiter	I	.00	.00		.00	-.03	.03	Saturn
	l	.07	.03	.00		-.38	.00	
	g	.01	.00	-.03	-.21		-.92	
	h	.01	.00	.03	.00	-.98		
	a		-.99	-.01	-.96	.96	-.01	
	e	-.67		.01	.93	-.93	.01	
Uranus	I	.04	-.09		.01	-.03	.46	Neptune
	l	.98	-.62	.04		-1.00	.01	
	g	-.45	.25	.03	-.48		-.05	
	h	.08	.00	-.05	.09	-.92		
	a		-.85	-.03	-.98	-.55	-.02	
	e	.79		.00	.85	.39	.09	
Pluto	I	.02	.02		-.03	.44	-.11	Moon
	l	1.00	.80	.02		.37	.04	(geocentric)
	g	-1.00	-.76	-.03	-1.00		-.24	equatorial)
	h	-.04	.03	.69	- .04	.02		

Correlations between the six elements of any one planet.
Do not read across planets.

TABLE XVI

Correlation Coefficients

		Mercury		Venus		Earth		Mars		Jupiter
		a	l	a	l	a	l	a	l	a
Venus	a	.51	.48	1.00		.82	.14	.63	.31	.13
Earth	a	.58	.55	.82		1.00		.77	.37	.17
	e		-.10		.28					
	l	.10	.29	.14	.53		1.00	.12	-.23	
Mars	a	.43	.41	.63		.77	.12	1.00		
	e		.11		.13		.40			
	l	.24	.20	.31	-.23	.37	-.23		1.00	
Jupiter	a			.13		.17				1.00
C		-.64	-.57	-.73		-.86	-.14	-.65	-.33	-.13
Moon	a	-.68	-.62	-.79		-.93	-.16	-.70	-.36	-.14
	e	.58	.53	.67		.79	.13	.60	.31	.12
	l	.67	.61	.77		.91	.15	.69	.36	.14
	g	.36	.33	.42		.50		.37	.20	
ϵ_1		-.67	-.61	-.78		-.91	-.15	-.69	-.36	-.14
ϵ_2		-.68	-.62	-.79		-.92	-.15	-.70	-.36	-.14
ϵ_3		-.68	-.62	-.79		-.92	-.16	-.70	-.36	-.14
ϵ_4		-.68	-.62	-.79		-.92	-.15	-.70	-.36	-.14
ϵ_5		-.67	-.62	-.79		-.92	-.16	-.69	-.36	-.14
ϵ_6		-.67	-.61	-.78		-.91	-.15	-.69	-.36	-.13
ϵ_7		-.65	-.60	-.76		-.89	-.15	-.67	-.36	-.13
ϵ_8		-.63	-.58	-.74		-.86	-.15	-.65	-.34	-.13
ϵ_9		-.58	-.55	-.69		-.80	-.14	-.61	-.32	-.12
ϵ_{10}		-.50	-.47	-.60		-.60	-.12	-.52	-.28	-.10
ϵ_{11}		-.37	-.37	-.45		-.52	-.09	-.40	-.22	-.08

TABLE XVII

Correlation Coefficients

		Mercury	Venus				Earth			
		e	e	I	g	h	e	I	g	h
Venus	l	.17								
	g	-.16			1.00					
Earth	e	.31	.52		-.28		1.00			
	I			.24		-.41		1.00		
	l	.17	-.25		.52					
	g			-.45		-.21			1.00	
	h			.45		.21				1.00
Mars	e		-.27		-.13		-.30			
	I			.23				.36	-.34	
	l	-.15	-.10		.24		-.36			
	g					-.22		.35	.34	-.34
	h					.22		-.35	-.34	.34

TABLE XVIII

Correlation Coefficients

		C	Moon a	Moon e	Moon I	Moon l	Moon g	Moon h	Clock Corrections ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7	ϵ_8	ϵ_9	ϵ_{10}	ϵ_{11}
C		1.00	.95	-.80		-.93	-.50		.94	.93	.92	.91	.89	.86	.82	.76	.66	.53	.32
Moon	a		1.00	-.85		-.98	-.55		.99	.99	.99	.99	.98	.97	.95	.91	.84	.72	.54
	e			1.00		.85	.39	.09	-.84	-.85	-.84	-.84	-.84	-.83	-.81	-.78	-.72	-.61	-.47
	I				1.00	-.08	.44	-.11											
	l					1.00	.37		-.97	-.98	-.98	-.97	-.97	-.95	-.93	-.89	-.83	-.70	-.53
	g						1.00	-.24	-.53	-.53	-.53	-.53	-.53	-.52	-.51	-.50	-.45	-.41	-.28
	h							1.00											
	ϵ_1								1.00	.97	.98	.97	.97	.96	.93	.90	.83	.71	.53
	ϵ_2									1.00	.99	.99	.98	.97	.95	.91	.85	.73	.55
	ϵ_3										1.00	.99	.99	.97	.96	.92	.86	.74	.57
	ϵ_4											1.00	.98	.98	.96	.93	.87	.76	.59
	ϵ_5												1.00	.98	.97	.94	.89	.78	.61
	ϵ_6													1.00	.96	.95	.90	.80	.64
	ϵ_7														1.00	.94	.92	.81	.67
	ϵ_8															1.00	.89	.84	.69
	ϵ_9																1.00	.79	.75
	ϵ_{10}																	1.00	.65
	ϵ_{11}																		1.00

TABLE XIX

Atomic Time 1912.5 to 1954.5

t	ΔT_0 = E.T.0.-U.T.2	AT-UT	Our Solution σ	AT-ET
1912.5	13^{s}0	18^{s}8	1^{s}9	5^{s}8
1913.5	14.2	20.1		5.9
1914.5	15.3	21.4		6.1
1915.5	16.4	22.7		6.3
1916.5	17.4	24.0	1.6	6.6
1917.5	18.3	24.6		6.3
1918.5	19.1	25.1		6.0
1919.5	19.8	25.5		5.7
1920.5	20.5	25.8	1.4	5.3
1921.5	21.1	26.5		5.4
1922.5	21.6	27.0		5.4
1923.5	22.0	27.5		5.5
1924.5	22.3	27.9	1.1	5.6
1925.5	22.6	28.1		5.5
1926.5	22.7	28.1		5.4
1927.5	22.8	28.0		5.2
1928.5	22.9	27.9	.9	5.0
1929.5	23.0	27.6		4.6
1930.5	23.2	27.3		4.1
1931.5	23.3	27.0		3.7
1932.5	23.5	26.7	.7	3.2
1933.5	23.6	26.5		2.9
1934.5	23.6	26.3		2.7
1935.5	23.6	26.1		2.5
1936.5	23.6	26.0	.6	2.4
1937.5	23.6	26.0		2.4
1938.5	23.8	26.2		2.4
1939.5	24.0	26.3		2.3
1940.5	24.3	26.6	.4	2.3
1941.5	24.7	26.9		2.2
1942.5	25.2	27.3		2.1
1943.5	25.6	27.7		2.1
1944.5	26.1	28.2	.3	2.1
1945.5	26.6	28.6		2.0

TABLE XIX (Continued)

t	ΔT_0 = E.T.0.-U.T.2	AT-UT	Our Solution σ	AT-ET
1946.5	27.1	29.0		1.9
1947.5	27.6	29.4		1.8
1948.5	28.2	29.8	.2	1.6
1949.5	28.9	30.1		1.2
1950.5	29.4	30.3		.9
1951.5	29.7	30.6		.9
1952.5	30.3	30.8	.2	.5
1953.5	31.0	30.9		-.1
1954.5	31.1	31.1		.0

APPENDIX B

AUXILIARY FIGURES

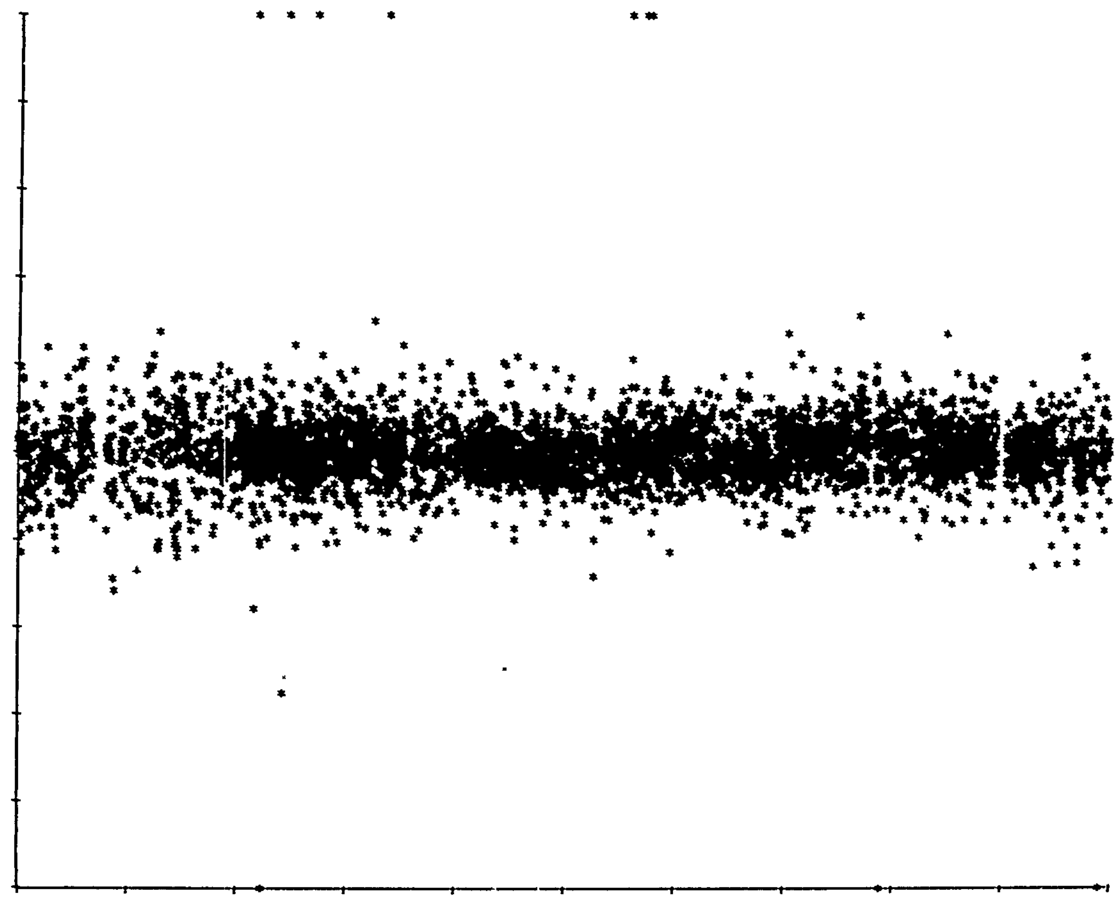

FIGURE 9

Moon. Residuals in Right Ascension Times cos δ

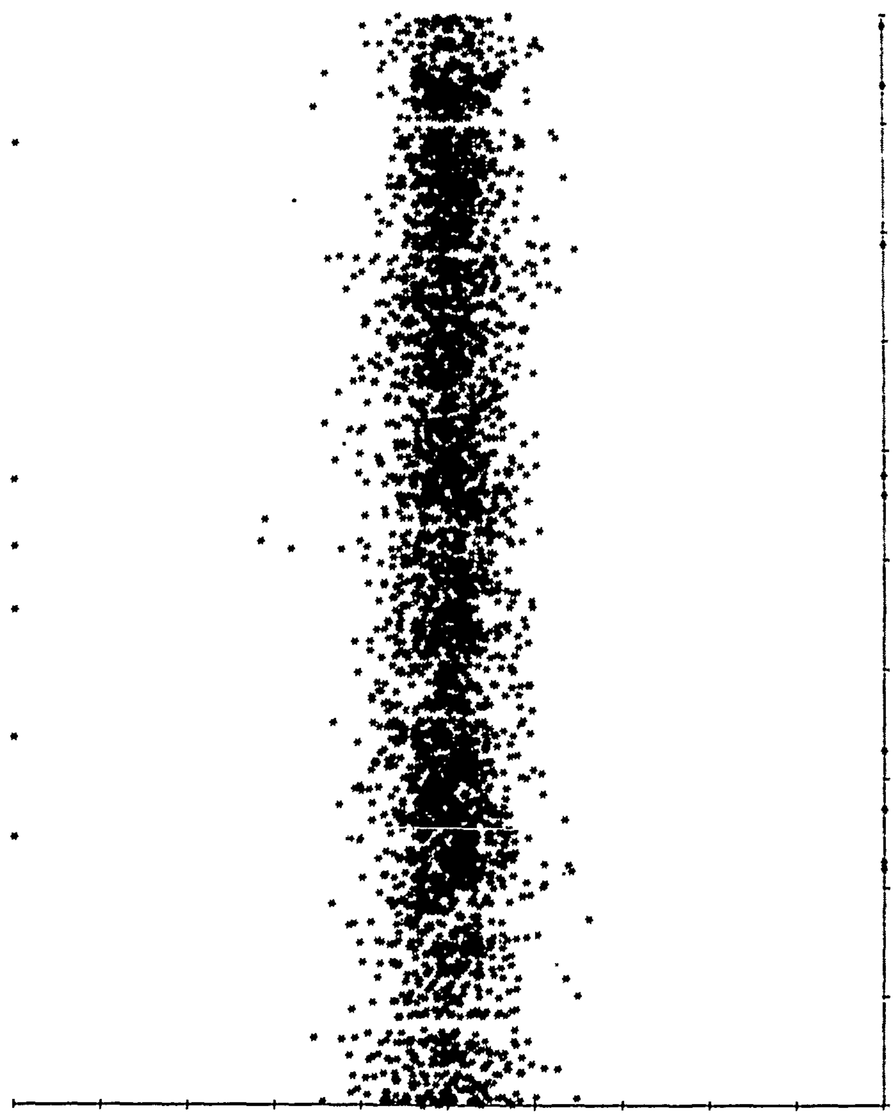

FIGURE 10

Moon. Residuals in Declination

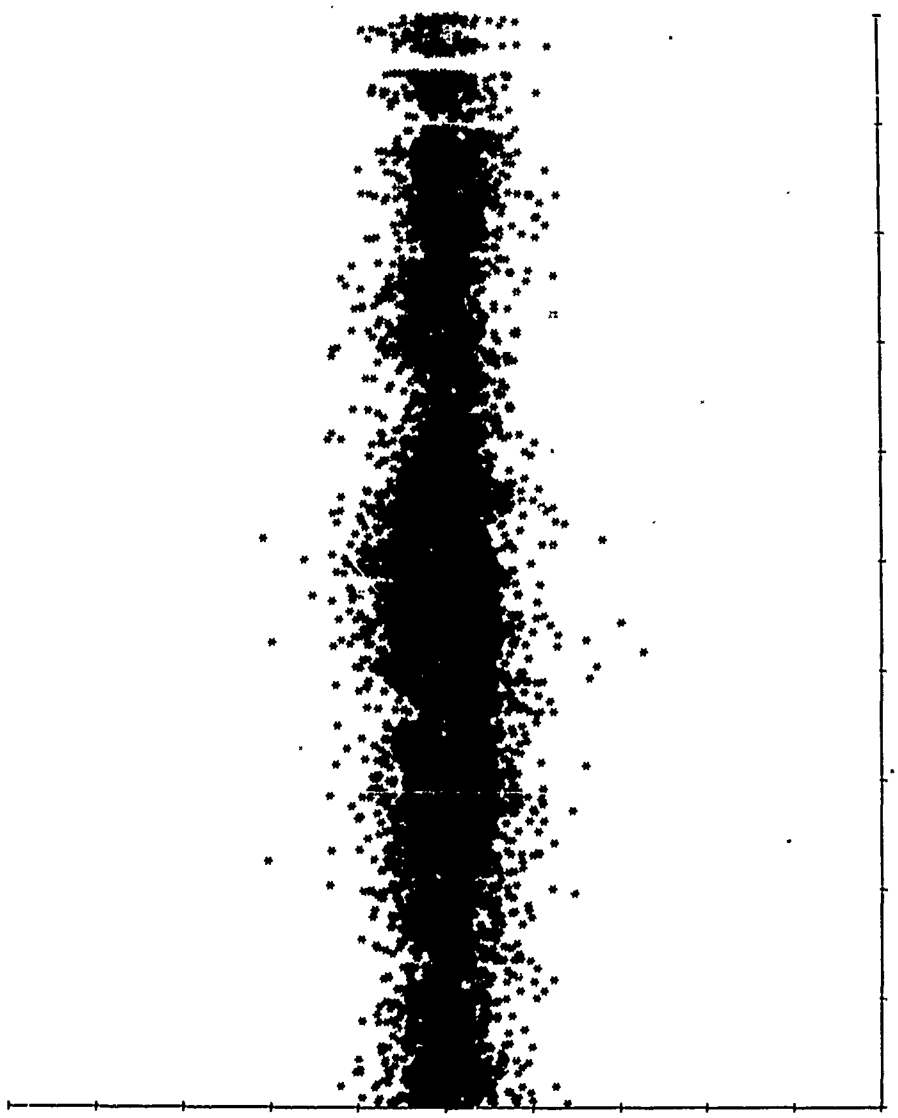

FIGURE 11

Sun. Residuals in Right Ascension Times cos δ

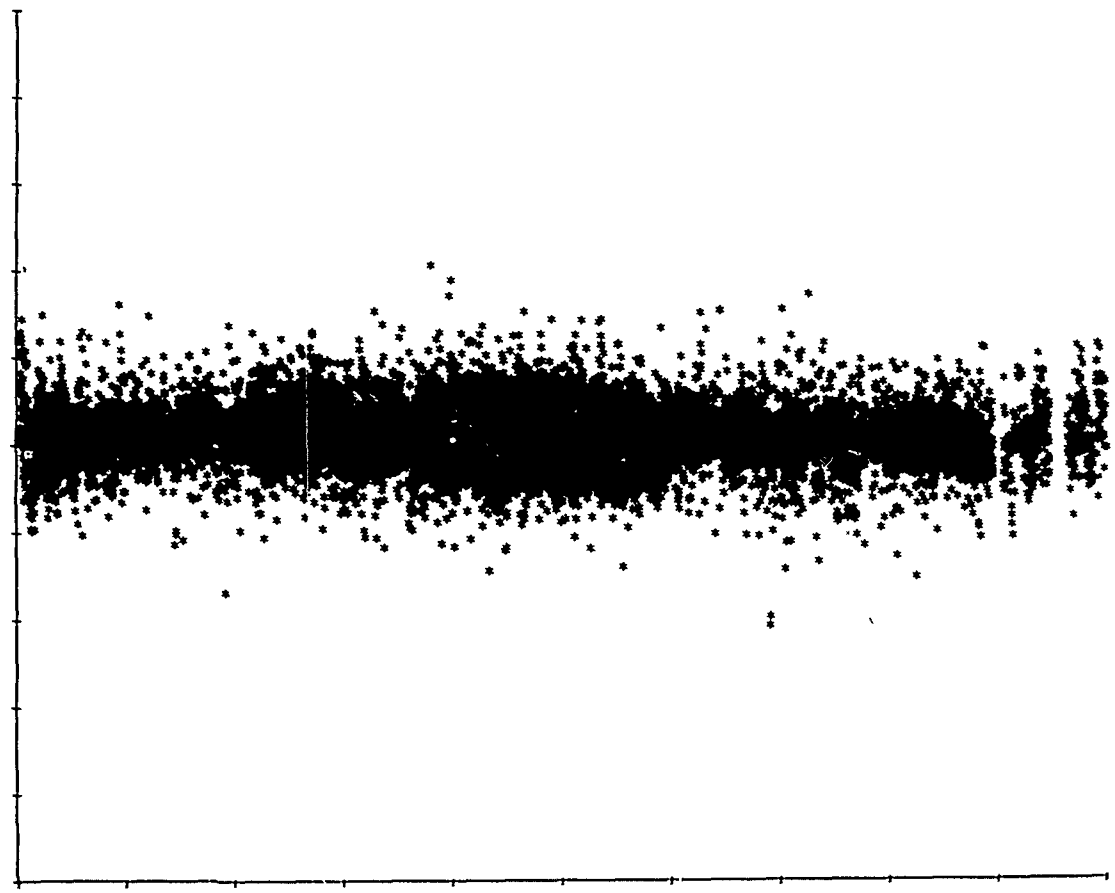

FIGURE 12

Sun. Residuals in Declination

FIGURE 13

Mercury. Residuals in Right Ascension Times cos δ

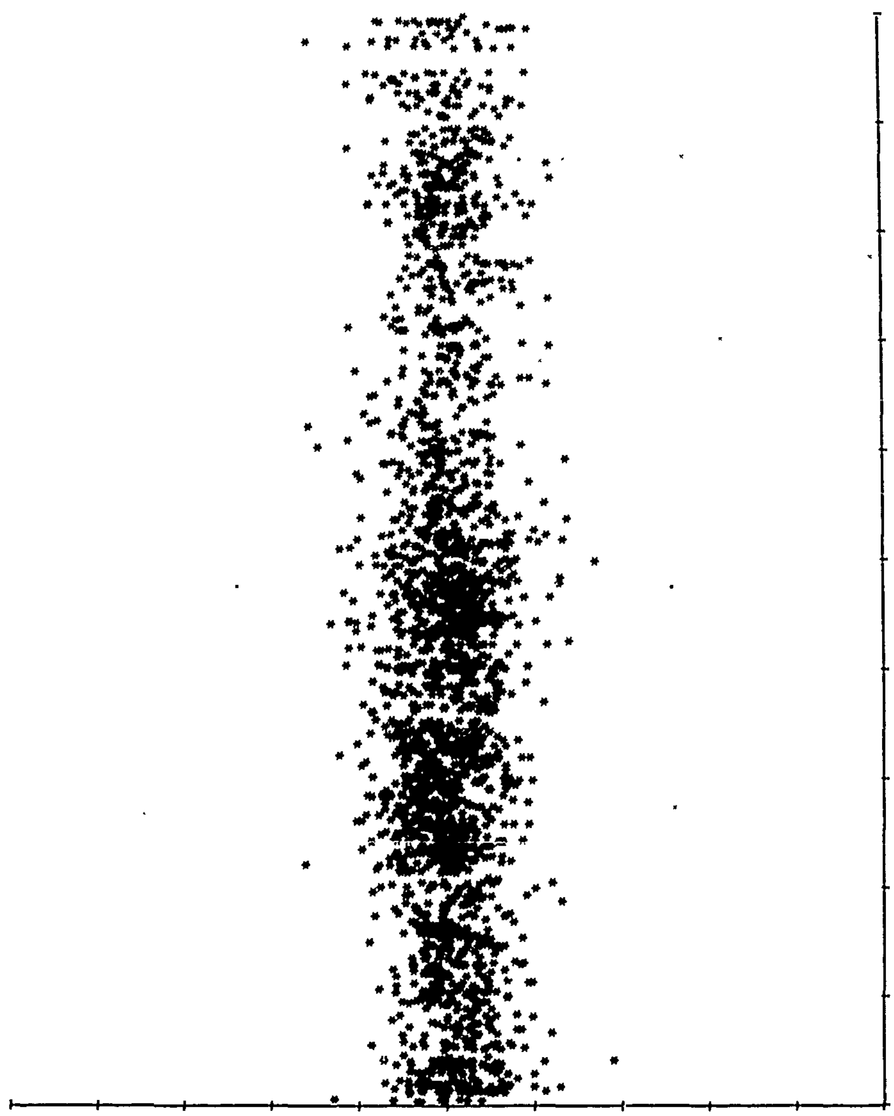

FIGURE 14

Mercury. Residuals in Declination

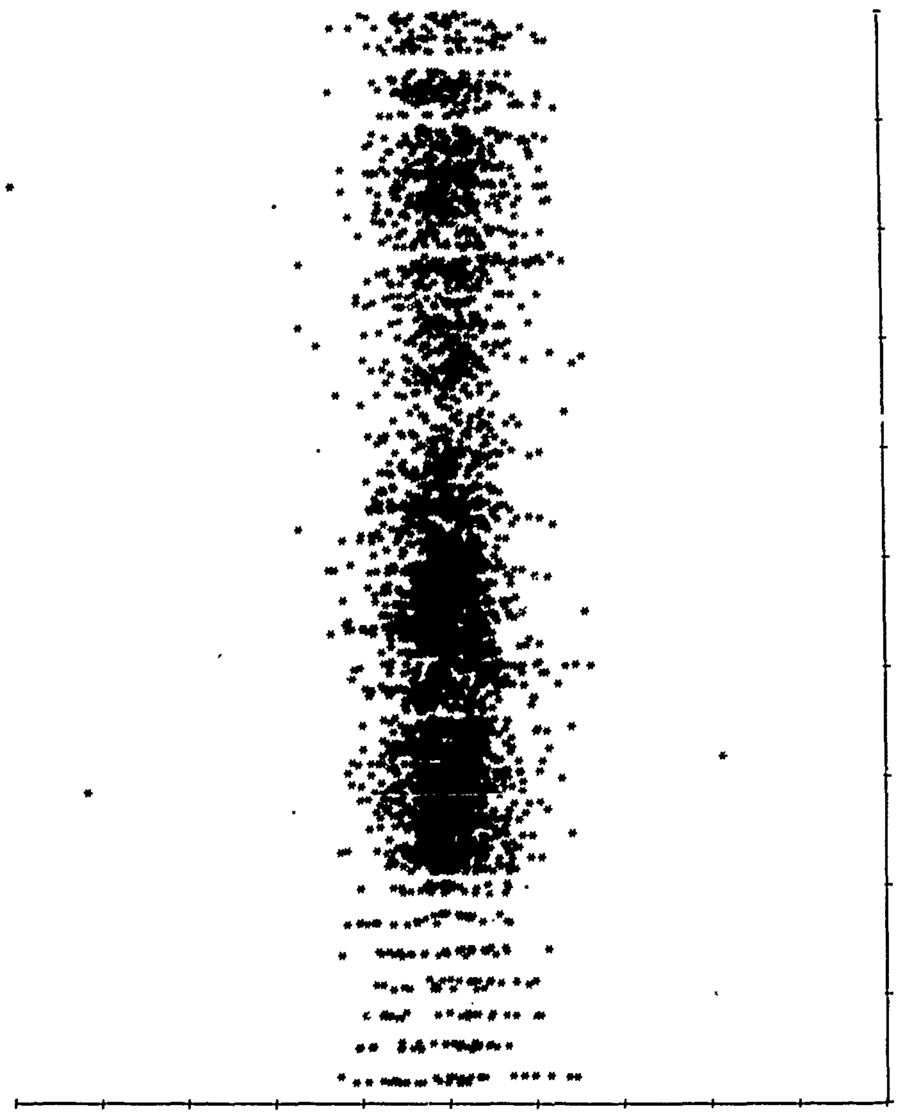

FIGURE 15

Venus. Residuals in Right Ascension Times cos δ

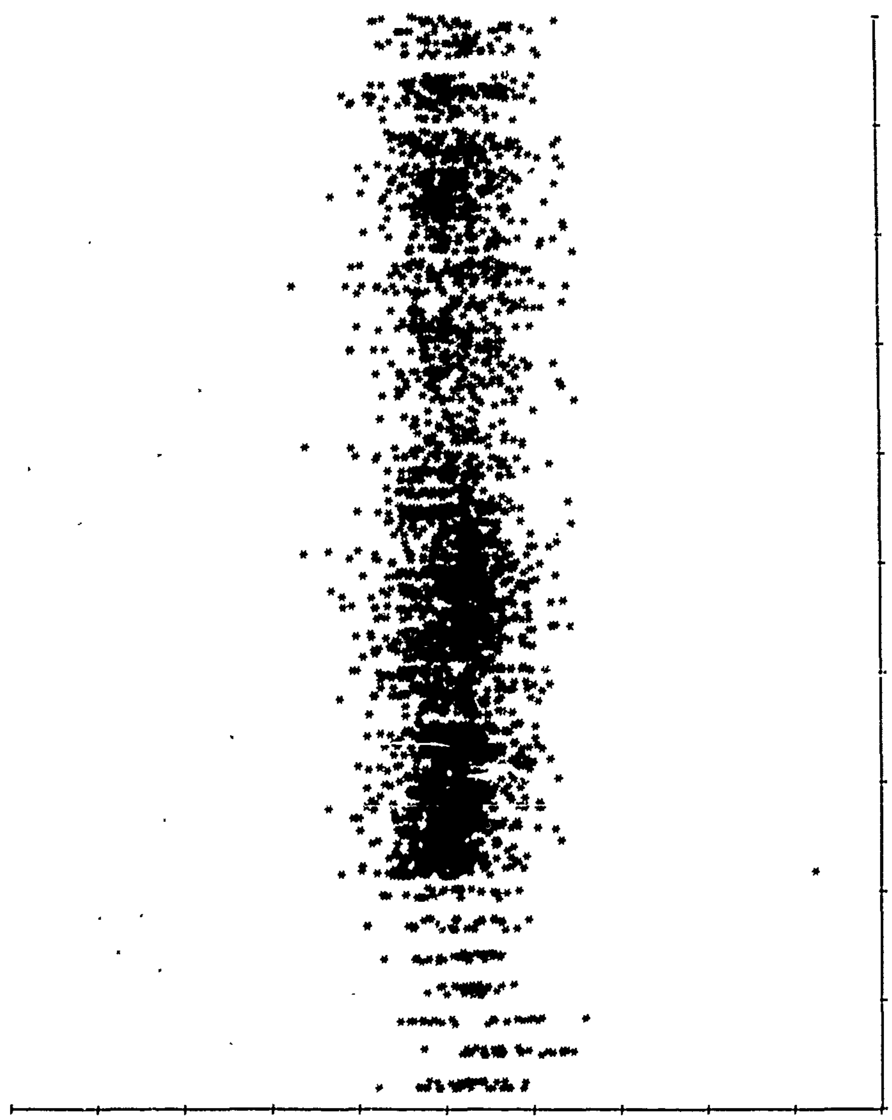

FIGURE 16

Venus. Residuals in Declination

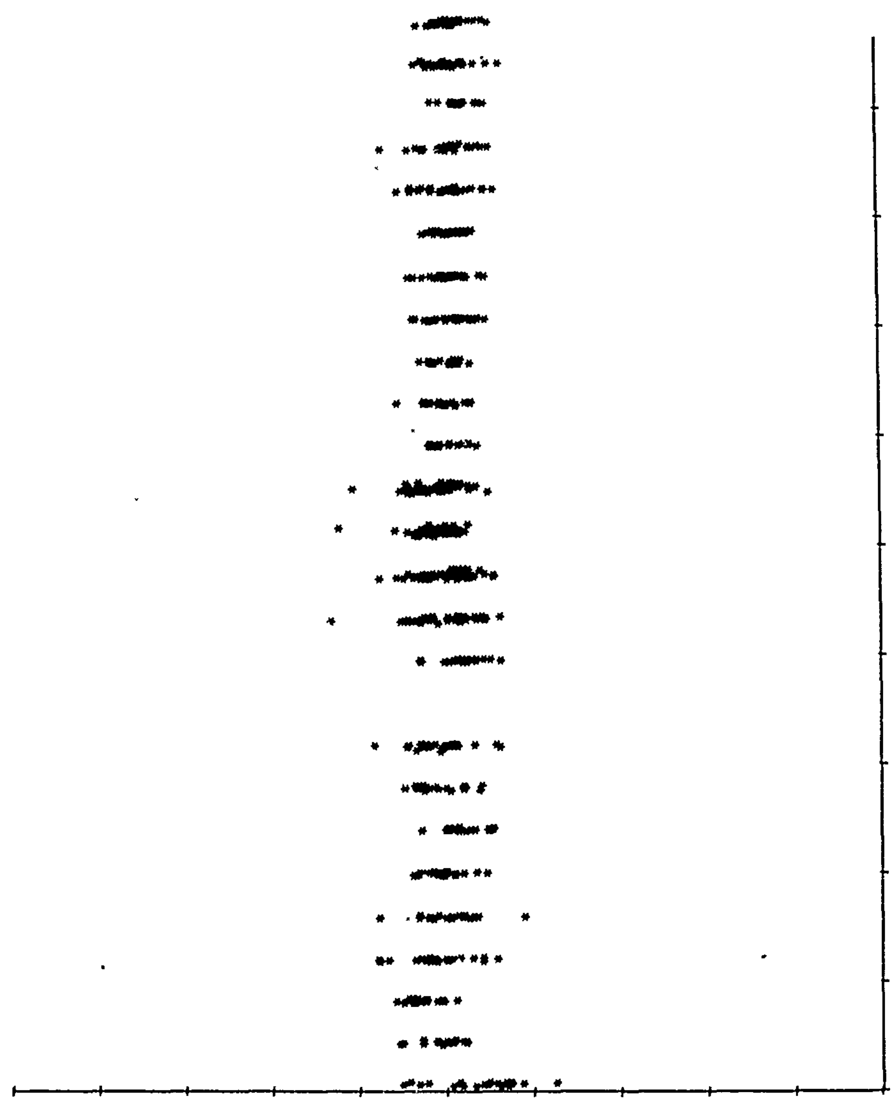

FIGURE 17

Mars. Residuals in Right Ascension Times cos δ

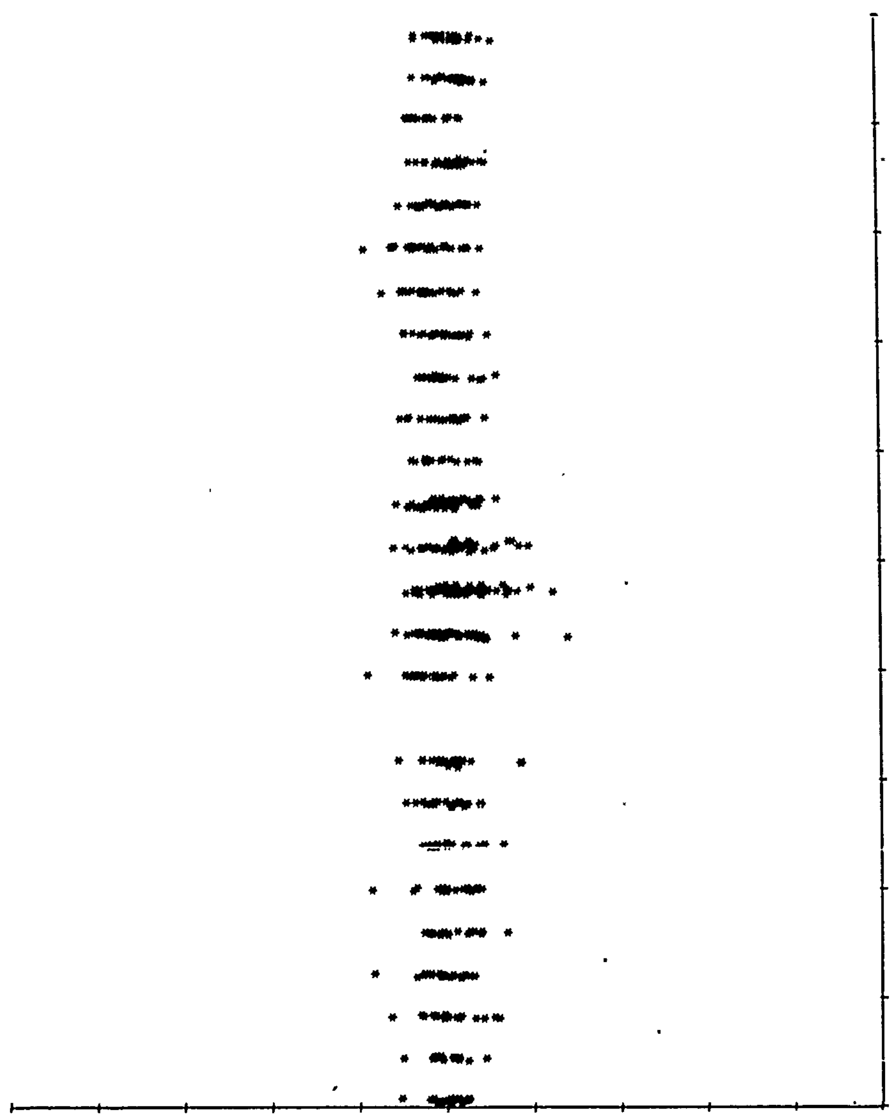

FIGURE 18

Mars. Residuals in Declination

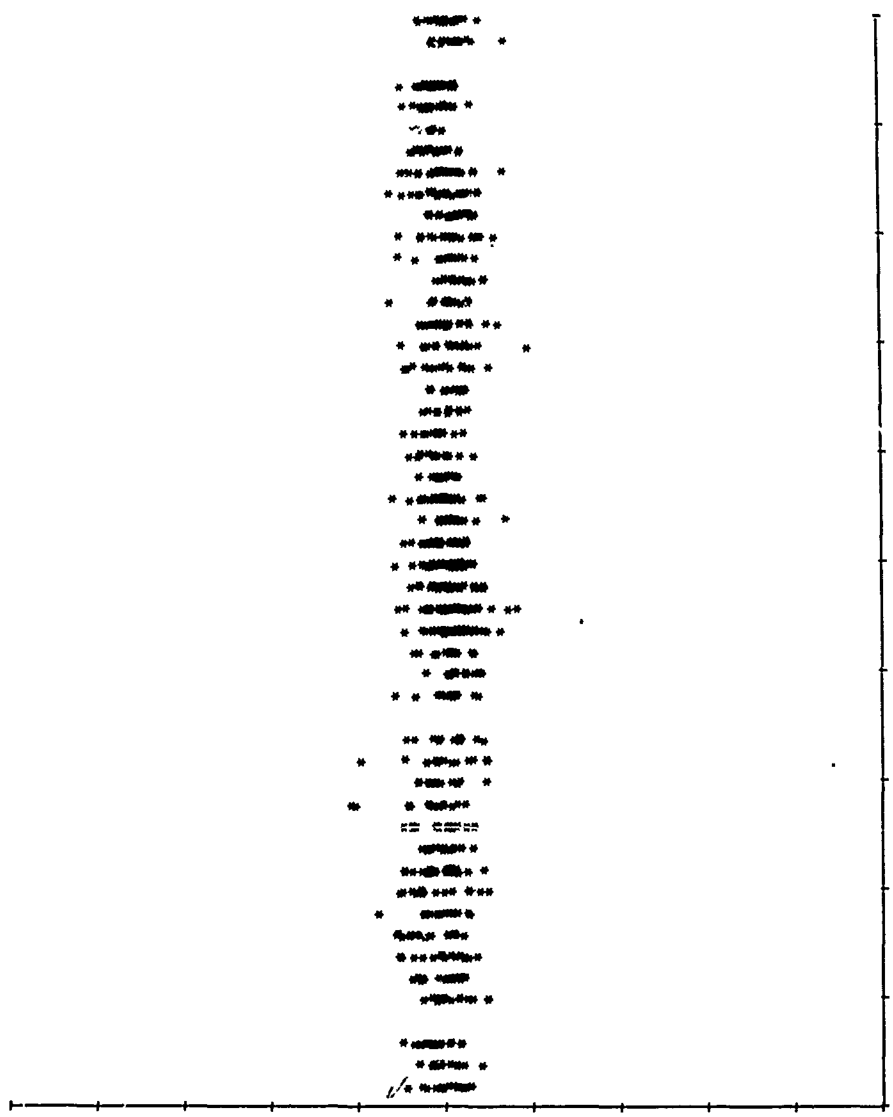

FIGURE 19

Jupiter. Residuals in Right Ascension Times cos δ

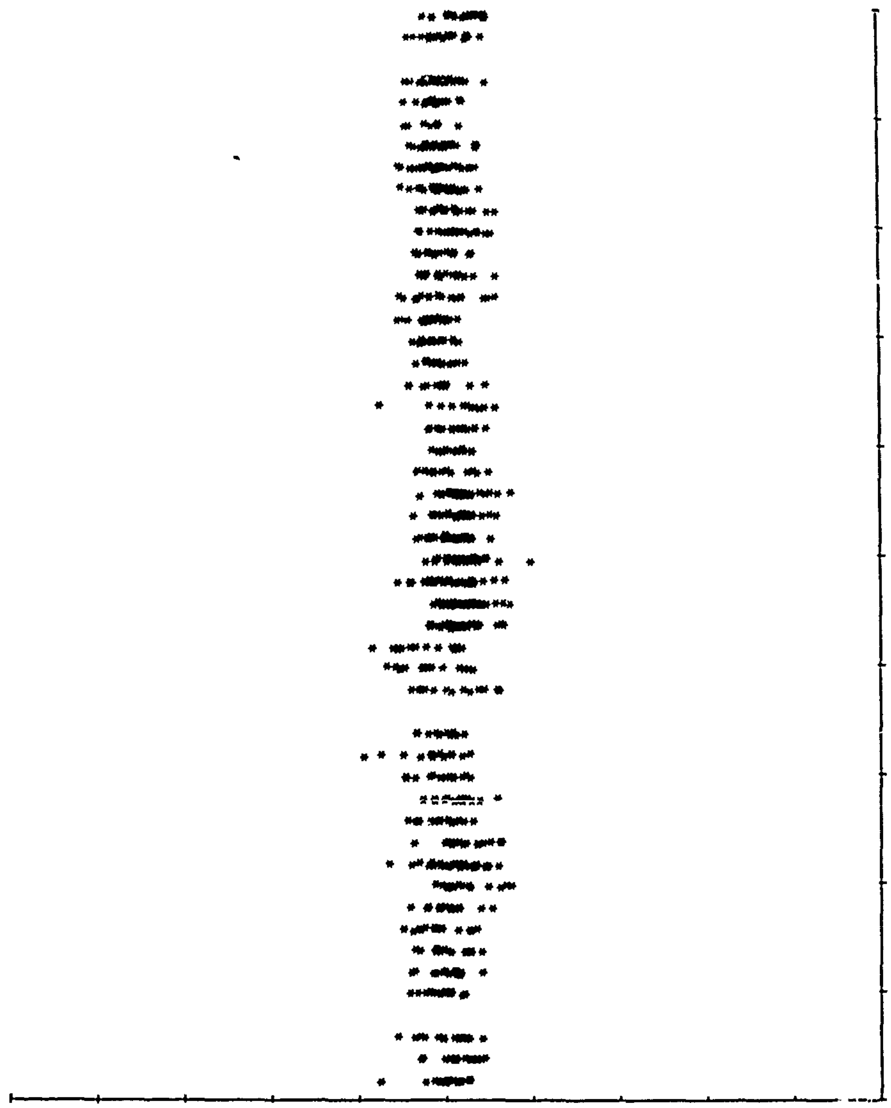

FIGURE 20

Jupiter. Residuals in Declination

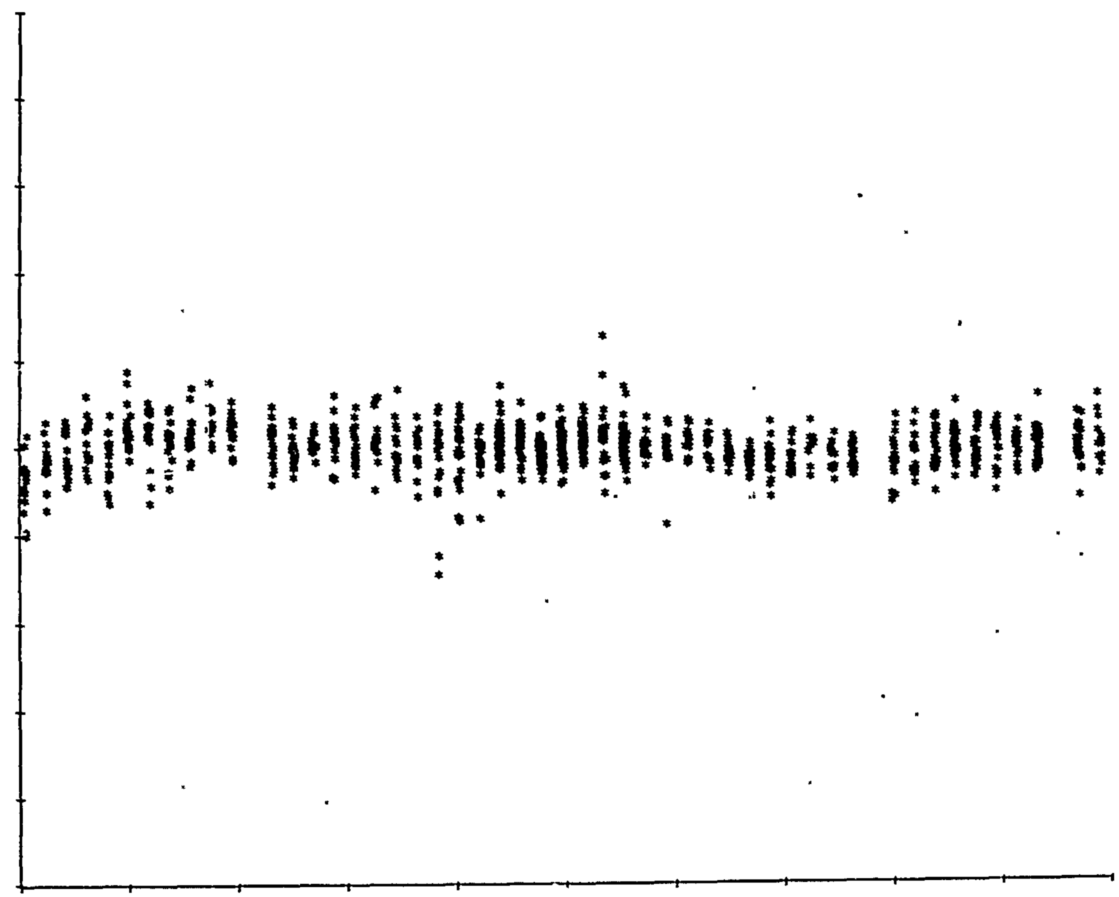

FIGURE 21

Saturn. Residuals in Right Ascension Times cos δ

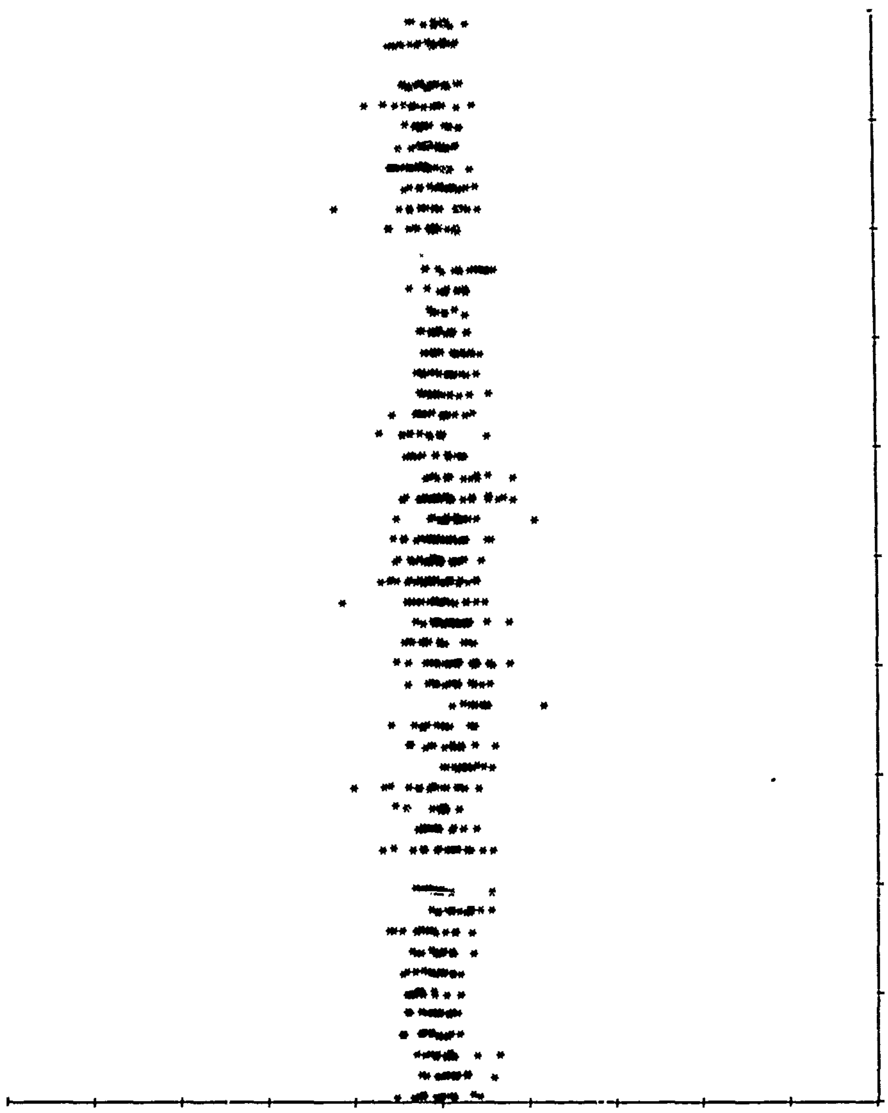

FIGURE 22

Saturn. Residuals in Declination

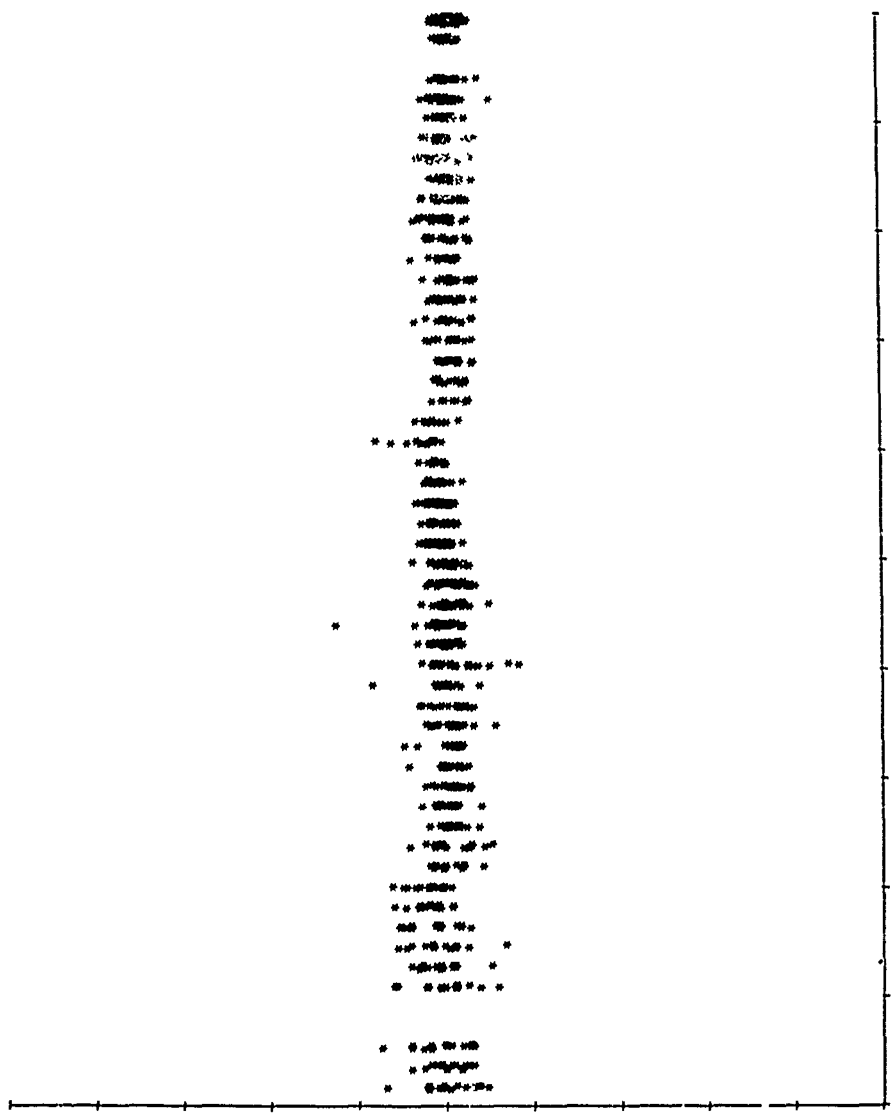

FIGURE 23

Uranus. Residuals in Right Ascension Times cos δ

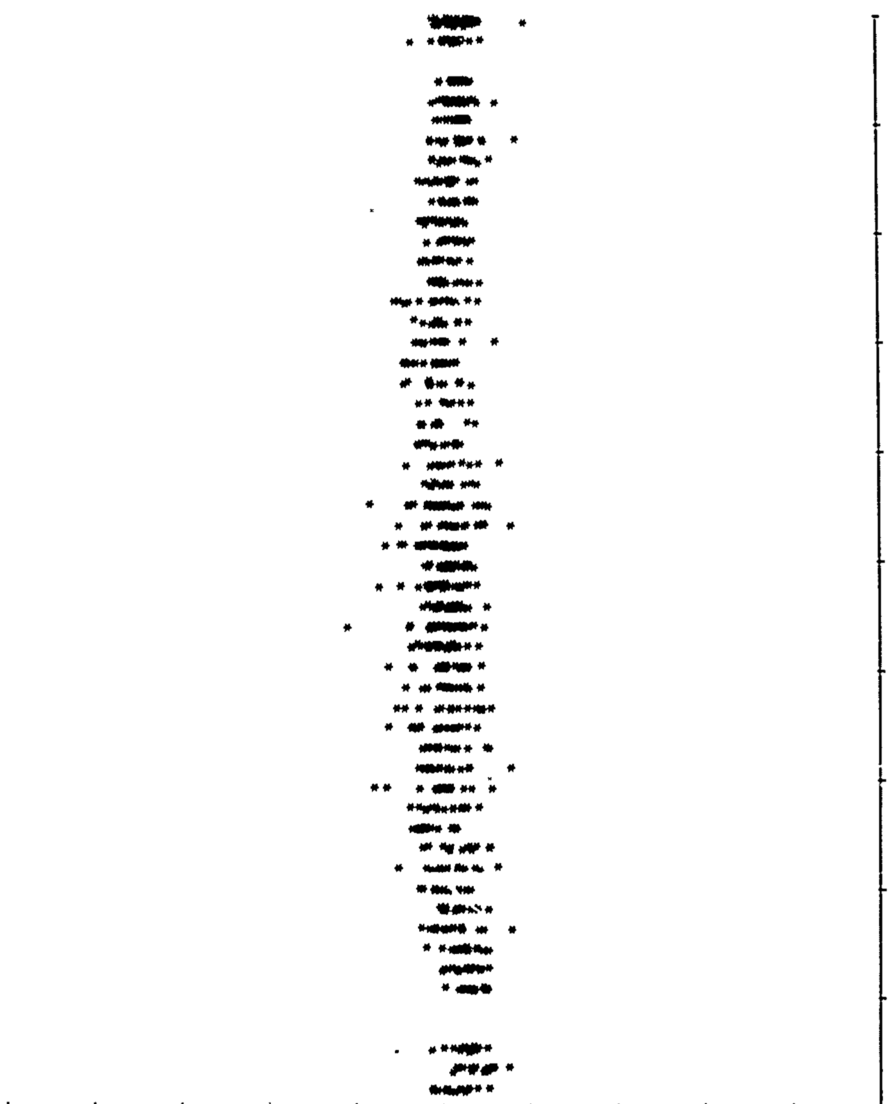

FIGURE 24

Uranus. Residuals in Declination

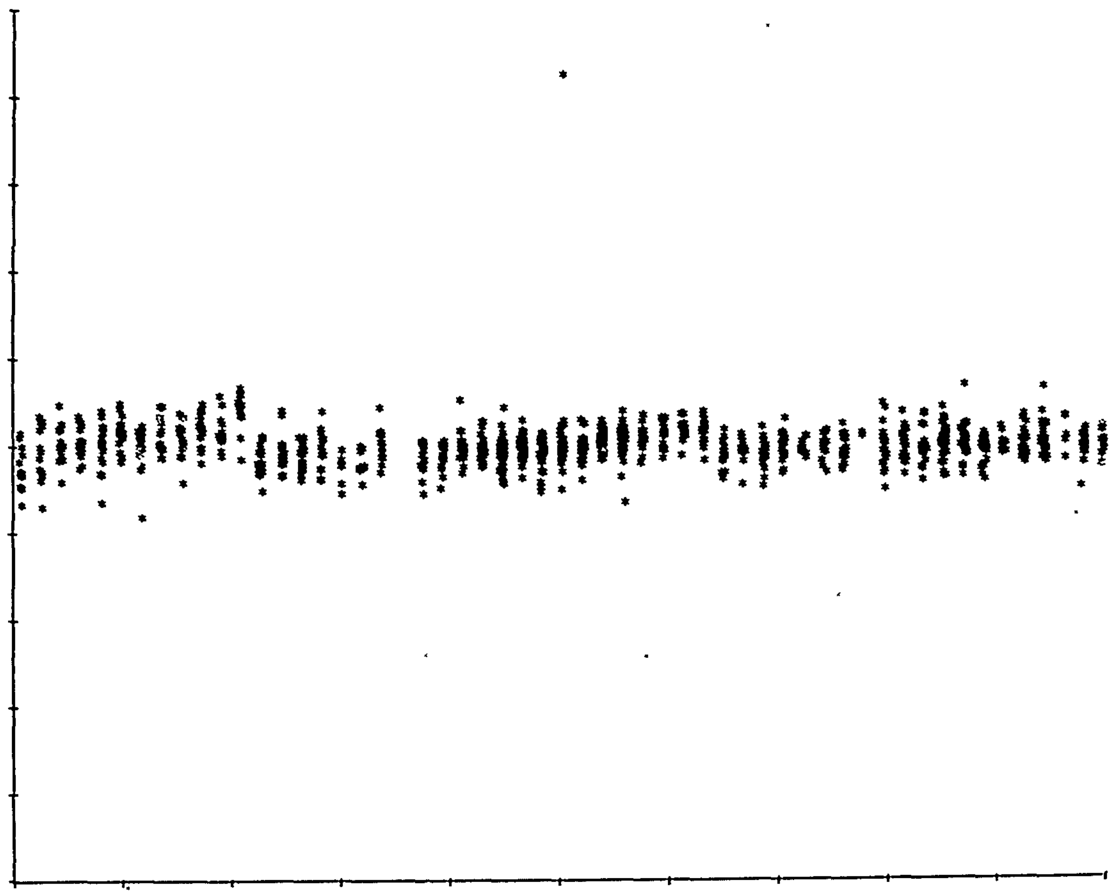

FIGURE 25

Neptune. Residuals in Right Ascension Times cos δ

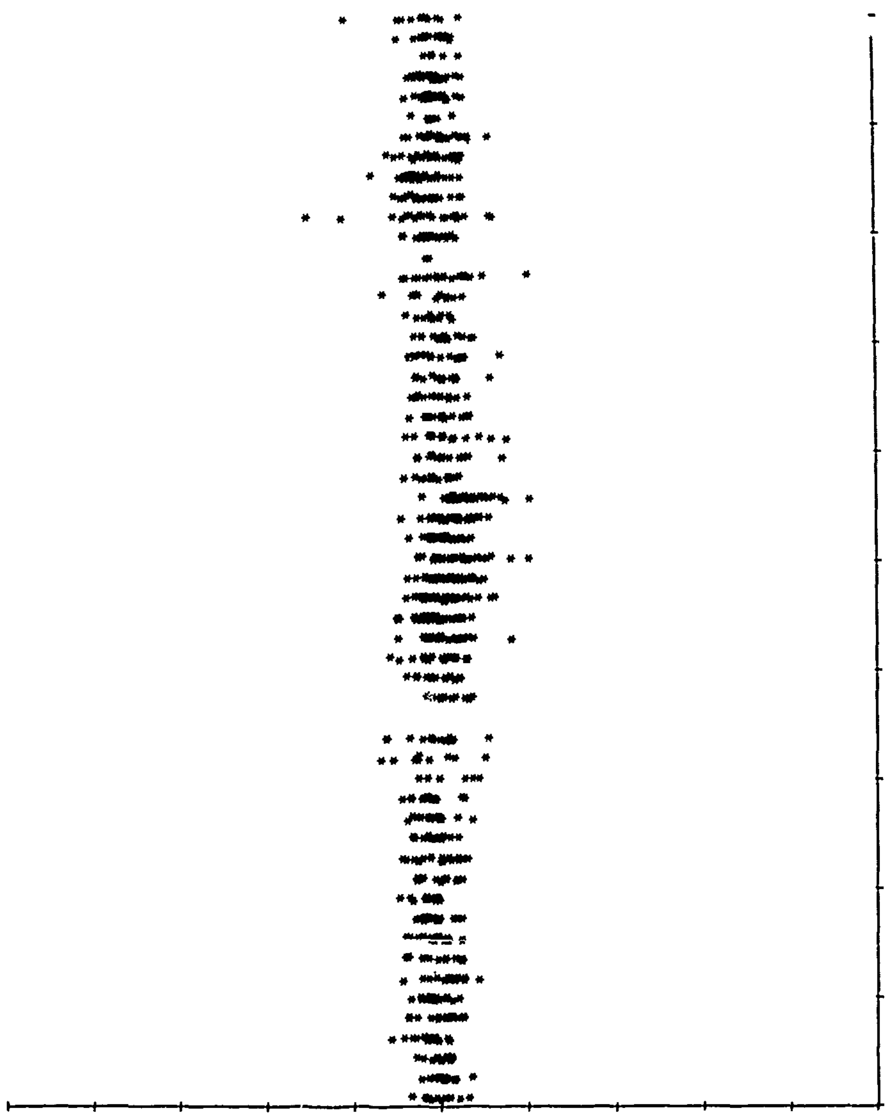

FIGURE 26

Neptune. Residuals in Declination

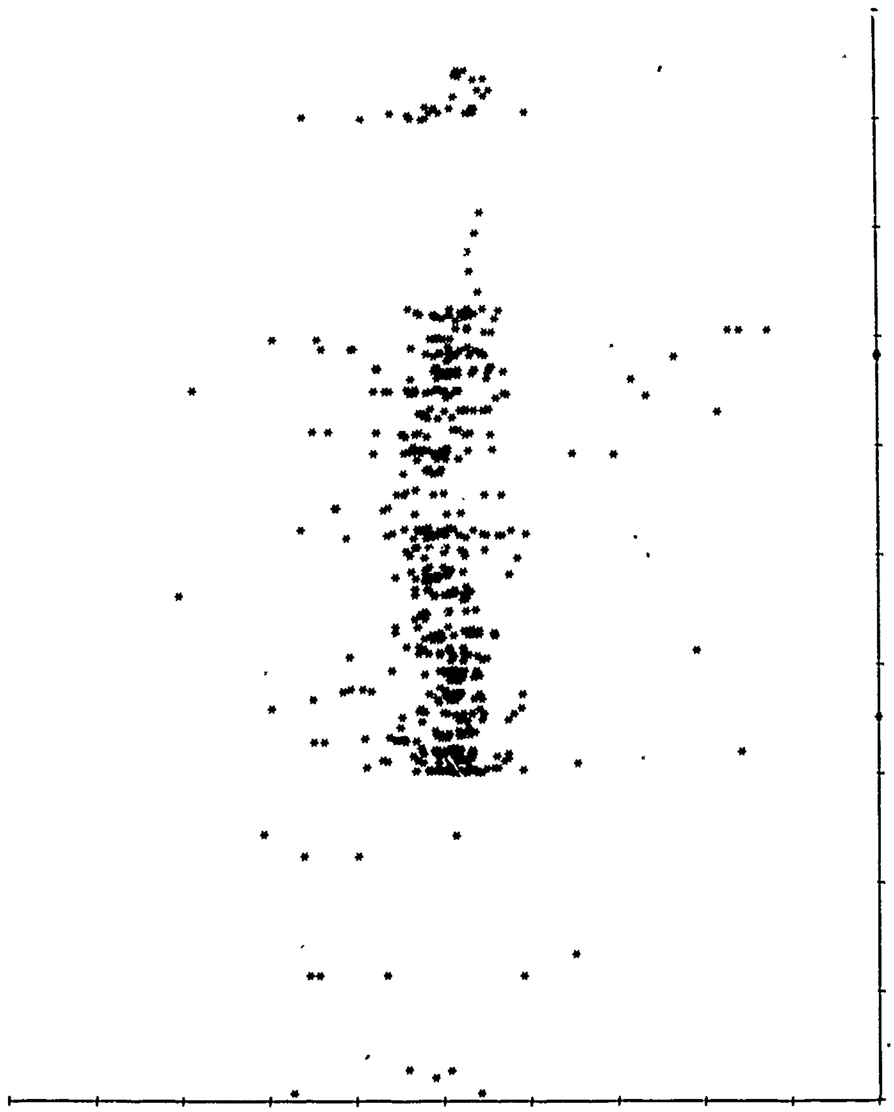

FIGURE 27

Pluto. Residuals in Right Ascension Times cos δ

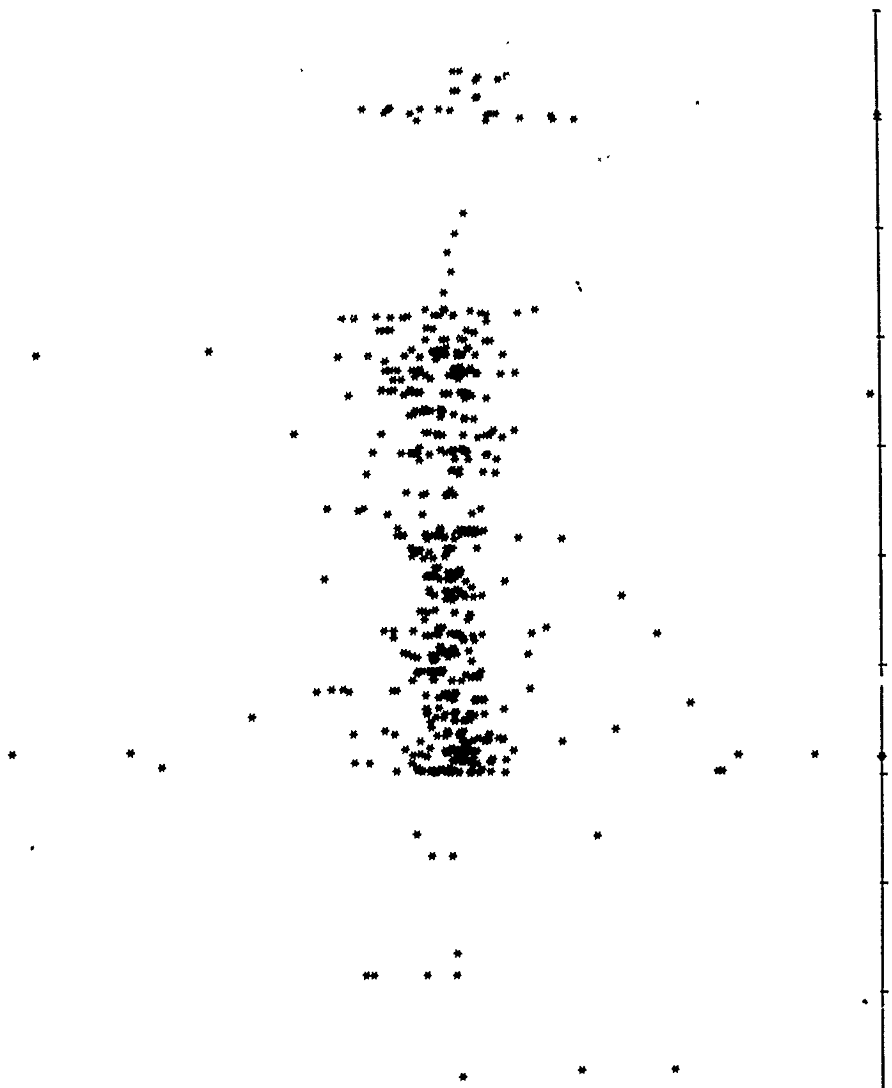

FIGURE 28

Pluto. Residuals in Declination