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MULTIWAY CONTINGENCY TABLE ANALYSIS APPLIED TO THE CLASSIFICATION OF MULTIVARIATE DICHOTOMOUS POPULATIONS

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by

S. KULLBACK

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Multiway contingency tables, or cross-classifications of vectors of discrete random variables, provide a useful approach to the analysis of multivariate discrete data. In the particular application we shall consider herein, the individual variates are dichotomous or binary. We shall use techniques and concepts presented and discussed by the author in previous papers. We note that the procedures and analysis are not restricted to dichotomous or binary data but are also applicable to polychotomous variates. The procedure we shall use is based on the principle of minimum discrimination information estimation applied to the analysis of multiway contingency tables. It yields results practically equivalent to procedures proposed by other investigators. When the minimum discrimination information estimates provide a satisfactory fit to a set of data, a complete analysis, including significance tests and estimates describing the pattern of observations is provided.

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II

Multiway Contingency Table Analysis Applied to the Classification of Multivariate Dichotomous Populations

by

S. Kullback

Introduction

Multiway contingency tables, or cross-classifications of vectors of discrete random variables, provide a useful approach to the analysis of multivariate discrete data. In the particular application we shall consider herein, the individual variates are dichotomous or binary. We shall use techniques and concepts presented and discussed in [4] and [6]. We note that the procedures and analysis are not restricted to dichotomous or binary data but are also applicable to polychotomous variates.

For background on the study and problem which gave rise to the data we shall analyze see [8]. In [3], procedures further developed in [4] and [6], were applied to problems of multivariate binary data in information systems, such as communication, pattern recognition, and learning systems. In [1] there is a review of methods and models for the analysis of multivariate binary data. Solomon's data, which we shall analyze herein, is given as a typical example. In [7] there is developed a model based on a set of orthogonal polynomials and applied to Solomon's data. We remark that the procedure we shall use, based on the principle of minimum discrimination information estimation applied to the analysis of multiway contingency tables yields a result practically equivalent to that in [7].

"Multivariate data analysis needs a large and flexible class of hypothetical distributions of free variables indexed by the values of fixed variables. From this class, appropriate subfamilies would be chosen for fitting to specific data sets" [2]. The principle of minimum discrimination information estimation, and its basis, the minimum discrimination information theorem which is quite general in its formulation, lead to exponential families of distributions [4], [5], [6], The exponential families have very useful and desirable statistical properties and contain many subfamilies in common use [2]. "The data analytic attitude to models is empirical rather than theoretical... when detailed theoretical understanding is unavailable, a more empirical attitude is natural, so that estimation of parameters in models should be seen less as attempts to discover underlying truth and more as data calibrating devices which make it easier to conceive of noisy data in terms of smooth distributions and relations. Exponential families are viewed here as intended for use in the empirical mode. With a given data set, a variety of models may be tried on, and one selected on the ground of looks and fit" [2]. When the minimum discrimination information estimates provide a satisfactory fit to a set of data, a complete analysis, including significance tests and estimates describing the pattern of observations is provided.

Solomon's Data

A total of 2982 high-school seniors were given an attitude questionnaire to assess their attitude towards science. The students were also classified on the basis of an IQ test into high IQ, the upper half, and low IQ, the lower half. The sixteen possible response vectors to each of four agree-disagree responses were tabulated. The data is given in table 1, where x_1, x_2, x_3, x_4 indicate the statements ([8,p.416]), agree and disagree were coded as 1 and 0 respectively, and listed as low IQ and high IQ. The problem of interest was to determine whether the response vectors could be used as a basis for classifying the students into one of two classes and evaluate possible classification procedures.

Contingency Table Analysis

We shall treat the data as a five-way $2\times2\times2\times2\times2$ contingency table, denoting the original observations by $x(hijk\ell)$, where

$h=1$, low IQ, $h=2$, high IQ ;

$i=1$, response to x_1 coded 0, $i=2$, response to x_1 coded 1;

$j=1$, response to x_2 coded 0, $j=2$, response to x_2 coded 1;

$k=1$, response to x_2 coded 0, $k=2$, response to x_3 coded 1;

$\ell=1$, response to x_4 coded 0, $\ell=2$, response to x_4 coded 1.

As a first overview of the data to determine the marginals and their related interaction parameters which may be considered to furnish significant values in the log-linear representation of the exponential family of the estimates [6], we list in table 2a, Analysis of Information, a sequential study of interaction and effect type measures [4], [6].

We remark that the first estimate is

$$x_a^*(hijk\ell) = x(h\cdot\cdot\cdot\cdot)x(\cdot ijk\ell)/n$$

and the minimum discrimination information statistic (interaction type measure)

$$2I(x:x_a^*) = 2\Sigma\Sigma\Sigma\Sigma\Sigma\ x(hijk\ell)\ell n \frac{x(hijk\ell)n}{x(h\cdots\cdot)x(\cdot ijk\ell)}$$

tests a null hypothesis that the IQ groupings are homogeneous over the sixteen response vectors [5,Chap.8], [4]. This null hypothesis is rejected and the subsequent study of effect and interaction type measures is an attempt to get a good fit to the data and account for the variation. Although the association between IQ and the response to the first statement is not significant, $2I(x_b^*:x_a^*) = 2.376$, 1 D.F., it was decided to examine in detail the estimate $x_e^*(hijk\ell)$ whose numerical values are given in table 1. (We remark that it may be shown that

$$2I(x_b^*:x_a^*) = 2\Sigma\Sigma\ x(hi\cdots)\ell n \frac{x(hi\cdots)n}{x(h\cdots\cdot)x(\cdot i\cdots)}$$

and tests a null hypothesis that IQ is homogeneous over the response to the first question). The estimate $x_e^*(hijk\ell)$ was selected because it does not differ significantly from the observed values, $2I(x:x_e^*) = 16.307$, 11D.F. (represents an acceptable fit), is symmetric with respect to the four statements, and is comparable to the first-order model estimate of [7], whose values are also listed in table 1.

From the log-linear representation in figure 1 [6], we obtain the parametric representation for the log-odds (low IQ/high IQ)

$$\ell n(x_e^*(1ijk\ell)/x_e^*(2ijk\ell))$$

over the sixteen response vectors as given in table 3a. Thus, for example

$$\ell n \frac{x_e^*(11111)}{x_e^*(21111)} = \tau_1^h + \tau_{11}^{hi} + \tau_{11}^{hj} + \tau_{11}^{hk} + \tau_{11}^{h\ell} ,$$

that is, a linear regression of the log-odds in terms of a constant τ_1^h and the main effects of each component of the response vector, namely, τ_{11}^{hi}, τ_{11}^{hj}, τ_{11}^{hk}, $\tau_{11}^{h\ell}$. The numerical values of the log-odds and the parameters are easily obtained from the entries in the computer output and are also given in table 3a [6].

We note from table 3a that

$$\ell n \frac{x_e^*(1ijk1)}{x_e^*(2ijk1)} - \ell n \frac{x_e^*(1ijk2)}{x_e^*(2ijk2)} = \tau_{11}^{h\ell} = 0.3338 ,$$

that is, a change from disagree to agree on the fourth statement is associated with an increase of 0.3338 in the log-odds (low IQ/high IQ). Note also that $\tau_{11}^{h\ell}$ represents the association between IQ and response to the fourth statement as measured by the log-cross-product - ratio

$$\tau_{11}^{h\ell} = \ell n \frac{x_e^*(1ijk1)x_e^*(2ijk2)}{x_e^*(2ijk1)x_e^*(1ijk2)}$$

and is the same for all eight levels of the responses to statements one, two and three.

Similarly, it is found that

$$\ell n \frac{x_e^*(1ij1\ell)}{x_e^*(2ij1\ell)} - \ell n \frac{x_e^*(1ij2\ell)}{x_e^*(2ij2\ell)} = \tau_{11}^{hk} = 0.3411 ,$$

$$\ell n \frac{x_e^*(1i1k\ell)}{x_e^*(2i1k\ell)} - \ell n \frac{x_e^*(1i2k\ell)}{x_e^*(2i2k\ell)} = \tau_{11}^{hj} = 0.1240 ,$$

$$\ell n \frac{x_e^*(11jk\ell)}{x_e^*(21jk\ell)} - \ell n \frac{x_e^*(12jk\ell)}{x_e^*(22jk\ell)} = \tau_{11}^{hi} = -0.2030 .$$

Classification

Since $x(1\cdots\cdot) = x_e^*(1\cdots\cdot) = 1491$, and $x(2\cdots\cdot) = x_e^*(2\cdots\cdot) = 1491$, we assign a response vector $(ijk\ell)$ to the region

E_1: classify as population $h=1$ (low IQ), when

$$\ell n \frac{x_e^*(1ijk\ell)}{x_e^*(2ijk\ell)} \geqq 0$$

and to the complementary region

E_2: classify as population $h=2$ (high IQ), when

$$\ell n \frac{x_e^*(1ijk\ell)}{x_e^*(2ijk\ell)} < 0 \; .$$

If we set

$$\mu_1(E_1) = \sum_{(ijk\ell)\epsilon E_1} \frac{x_e^*(1ijk\ell)}{1491} \, , \quad \mu_2(E_1) = \sum_{(ijk\ell)\epsilon E_1} \frac{x_e^*(2ijk\ell)}{1491} \, ,$$

then the probability of error of the classification procedure is [5,pp.4,69,80]

$$\text{Prob Error} = p\mu_2(E_1)+q\mu_1(E_2) = (\mu_2(E_1)+\mu_1(E_2))/2$$

since here $p = x(2\cdots\cdot)/2982 = \frac{1}{2}$, $q = x(1\cdots\cdot)/2982 = \frac{1}{2}$.

The relevant computations with $x_e^*(hijk\ell)$ are given in table 4(b) and show that the Prob. Error = 0.444. The corresponding computations with the original data $x(hikj\ell)$ are given in table 4(a) and yield Prob. Error = 0.441.

Other Estimates

In view of the measure of the effect of the marginal $x(hi\cdot\cdot\ell)$ (and the associated interaction parameters) in table 2a, $2I(x_m^*:x_g^*) = 4.316$, 1D.F.

and the marginal $x(h\cdot j\cdot \ell)$, $2I(x^*_p : x^*_n) = 3.181$, 1 D.F., the estimate $x^*_v(hijk\ell)$ fitting the marginals $x(\cdot ijk\ell)$, $x(h\cdot j\cdot\cdot)$, $x(h\cdot\cdot k\cdot)$, $x(hi\cdot\cdot\ell)$ and the estimate $x^*_w(hijk\ell)$ fitting the marginals $x(\cdot ijk\ell)$, $x(h\cdot\cdot k\cdot)$, $x(hi\cdot\cdot\ell)$, $x(h\cdot j\cdot\ell)$ were computed. The estimates are given in table 1 and the relevant analysis of information given in table 2b.

The values of the log-odds, parametric representation, and the values of the associated interaction parameters are given in table 3b for $x^*_v(hijk\ell)$ and in table 3c for $x^*_w(hijk\ell)$. Note from table 3b that

$$\ell n \frac{x^*_v(11jk1)}{x^*_v(21jk1)} - \ell n \frac{x^*_v(11jk2)}{x^*_v(21jk2)} = \tau^{h\ell}_{11} + \tau^{hi\ell}_{111} = 0.6469 ,$$

$$\ell n \frac{x^*_v(12jk1)}{x^*_v(22jk1)} - \ell n \frac{x^*_v(12jk2)}{x^*_v(22jk2)} = \tau^{h\ell}_{11} = 0.2680 ,$$

$$\ell n \frac{x^*_v(11jk1)}{x^*_v(21jk1)} - \ell n \frac{x^*_v(12jk1)}{x^*_v(22jk1)} = \tau^{hi}_{11} + \tau^{hi\ell}_{111} = -0.0276$$

$$\ell n \frac{x^*_v(11jk2)}{x^*_v(21jk2)} - \ell n \frac{x^*_v(12jk2)}{x^*_v(22jk2)} = \tau^{hi}_{11} = -0.4065$$

reflecting the interaction of the responses to the first and fourth statements.

From table 3c, it is found for example, that

$$\ell n \frac{x^*_w(111k1)}{x^*_w(211k1)} - \ell n \frac{x^*_w(111k2)}{x^*_w(211k2)} = \tau^{h\ell}_{11} + \tau^{hi\ell}_{111} + \tau^{hj\ell}_{111} = 0.5806$$

$$\ell n \frac{x^*_w(121k1)}{x^*_w(221k1)} - \ell n \frac{x^*_w(121k2)}{x^*_w(221k2)} = \tau^{h\ell}_{11} + \tau^{hj\ell}_{111} = 0.2030$$

$$\ell n \frac{x^*_w(112k1)}{x^*_w(212k1)} - \ell n \frac{x^*_w(112k2)}{x^*_w(212k2)} = \tau^{h\ell}_{11} + \tau^{hi\ell}_{111} = 0.9371$$

$$\ell n \frac{x_w^*(122k1)}{x_w^*(222k1)} - \ell n \frac{x_w^*(122k2)}{x_w^*(222k2)} = \tau_{11}^{h\ell} = 0.5595$$

reflecting the interactions of the responses to the first, second and fourth statements.

The computation of the probability of error using the estimates $x_v^*(hijk\ell)$ and $x_w^*(hijk\ell)$ is shown in table 4(c) and 4(d) respectively, and yields probabilities of error 0.444 and 0.446.

Measure of Divergence

As a measure of the divergence between the low IQ and high IQ observed and estimated values, we computed the values of

$$J(1,2) = \frac{1}{2} \Sigma\Sigma\Sigma\Sigma(x(1ijk\ell)-x(2ijk\ell))\ell n \frac{x(1ijk\ell)}{x(2ijk\ell)}$$

for $x(hijk\ell)$, $x_e^*(hijk\ell)$, $x_v^*(hijk\ell)$, $x_w^*(hijk\ell)$ [5,p.130]. The resulting values and their ratios to the respective degrees of freedom are given in table 5. As is to be expected from the properties of the discrimination information we note that

$$J(1,2;x_e^*) < J(1,2;x_v^*) < J(1,2;x_w^*) < J(1,2;x) .$$

However the ratio to the respective degrees of freedom leads to the inequalities

$$J(1,2;x)/\text{D.F.} < J(1,2;x_e^*)/\text{D.F.} < J(1,2;x_v^*)/\text{D.F.} < J(1,2;x_w^*)/\text{D.F.}$$

Remark

Martin and Bradley [7] examined Solomon's data in terms of an estimate they called a first-order or linear model. These estimated values are given in table 1. It turns out that although the underlying approaches are different, the Martin and Bradley parameters and estimates are practically the same as those for $x_e^*(hijk\ell)$. From [7,pp.216-217] we note that

$$\ell n \frac{x_e^*(12222)}{x_e^*(22222)} = \tau_1^h = \ell n \frac{1+a_o+a_1+a_2+a_3+a_4}{1-a_o-a_1-a_2-a_3-a_4}$$

$$\ell n \frac{x_e^*(12221)}{x_e^*(22221)} = \tau_1^h + \tau_{11}^{h\ell} = \ell n \frac{1+a_o+a_1+a_2+a_3-a_4}{1-a_o-a_1-a_2-a_3+a_4}$$

$$\ell n \frac{x_e^*(12212)}{x_e^*(22212)} = \tau_1^h + \tau_{11}^{hk} = \ell n \frac{1+a_o+a_1+a_2-a_3+a_4}{1-a_o-a_1-a_2+a_3-a_4}$$

$$\ell n \frac{x_e^*(12122)}{x_e^*(22122)} = \tau_1^h + \tau_{11}^{hj} = \ell n \frac{1+a_o+a_1-a_2+a_3+a_4}{1-a_o-a_1+a_2-a_3-a_4}$$

$$\ell n \frac{x_e^*(11222)}{x_e^*(21222)} = \tau_1^h + \tau_{11}^{hi} = \ell n \frac{1+a_o-a_1+a_2+a_3+a_4}{1-a_o+a_1-a_2-a_3-a_4}$$

or to a first approximation

$$\tau_1^h = 2a_o+2a_1+2a_2+2a_3+2a_4$$

$$\tau_1^h+\tau_{11}^{h\ell} = 2a_o+2a_1+2a_2+2a_3-2a_4$$

$$\tau_1^h+\tau_{11}^{hk} = 2a_o+2a_1+2a_2-2a_3+2a_4$$

$$\tau_1^h+\tau_{11}^{hj} \doteq 2a_o+2a_1-2a_2+2a_3+2a_4$$

$$\tau_1^h+\tau_{11}^{hi} = 2a_o-2a_1+2a_2+2a_3+2a_4 \ .$$

It is found that

$$\tau_{11}^{h\ell} = -4a_4$$

$$\tau_{11}^{hk} = -4a_3$$

$$\tau_{11}^{hj} = -4a_2$$

$$\tau_{11}^{hi} = -4a_1 \ .$$

The values of the parameters given in [7,table 3,p. 217] are

$$a_0 = -0.042, \quad a_1 = 0.049, \quad a_2 = -0.031, \quad a_3 = -0.084, \quad a_4 = -0.082$$

so that

$$\tau_{11}^{h\ell} = 0.3338 = 0.334, \ -4a_4 = 0.328$$

$$\tau_{11}^{hk} = 0.3411 = 0.341, \ -4a_3 = 0.336$$

$$\tau_{11}^{hj} = 0.1240 = 0.124, \ -4a_2 = 0.124$$

$$\tau_{11}^{hi} = -0.2030 = -0.203, \ -4a_1 = -0.196$$

The computation for the probability of error using the estimates in [7] are shown in table 4(e) and yields a probability of error 0.445. (Martin and Bradley give a value of the risk as 0.455).

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x^*_e		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓							✓	✓	✓	✓					✓	
x^*_v		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓			✓				✓	✓	✓	✓					✓	
x^*_w		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓			✓		✓		✓	✓	✓	✓					✓	

Figure 1

Solomon's Data-Classification Procedures

		Observed		Estimates			Observed		Estimates		
$x_1x_2x_3x_4$	ij kl	Low IQ $x(1ijk\ell)$	Martin & Bradley	$x_e^*(1ijk\ell)$	$x_v^*(1ijk\ell)$	$x_w^*(1ijk\ell)$	High IQ $x(2ijk\ell)$	Martin & Bradley	$x_e^*(2ijk\ell)$	$x_v^*(2ijk\ell)$	$x_w^*(2ijk\ell)$
11 11	22 22	62	74.56	74.589	76.097	70.156	122	109.45	109.414	107.904	113.844
11 10	22 21	70	67.30	67.296	66.198	71.600	68	70.71	70.703	71.802	66.400
11 01	22 12	31	31.32	31.329	31.943	29.827	33	32.68	32.671	32.057	34.173
11 00	22 11	41	37.74	37.780	37.337	39.884	25	28.26	28.219	28.662	26.115
10 11	21 22	283	266.76	266.570	271.120	275.979	329	345.24	345.429	340.879	336.020
10 10	21 21	253	259.17	259.322	254.876	250.769	247	240.83	240.679	245.125	249.232
10 01	21 12	200	193.45	193.625	196.841	200.037	172	178.55	178.376	175.160	171.963
10 00	21 11	305	314.50	314.491	310.589	306.748	217	207.50	207.508	211.411	215.252
01 11	12 22	14	12.10	12.156	10.866	9.914	20	21.90	21.844	23.135	24.085
01 10	12 21	11	9.20	9.182	9.929	10.760	10	11.80	11.818	11.071	10.240
01 01	12 12	11	9.68	9.659	8.776	8.102	11	12.32	12.341	13.224	13.898
01 00	12 11	14	12.02	12.010	12.855	12.756	9	10.98	10.990	10.144	9.244
00 11	11 22	31	33.63	33.623	30.125	30.820	56	53.37	53.375	56.874	56.179
00 10	11 21	46	47.37	47.263	50.789	50.001	55	53.63	53.737	50.211	50.999
00 01	11 12	57	47.54	47.450	43.233	44.163	64	53.46	53.550	57.767	56.837
00 00	11 11	82	74.67	74.656	79.426	78.482	53	60.33	60.346	55.574	56.517
		1491					1491				

Table 1

Table 2a

Analysis of Information

Marginals Fitted	Information	D.F.
a) x(.ijkℓ),x(h....)	$2I(x:x_a^*) = 68.369$	15
b) x(.ijkℓ),x(hi...)	$2I(x_b^*:x_a^*) = 2.376$	1
	$2I(x:x_b^*) = 65.993$	14
c) x(.ijkℓ),x(hi...),x(h.j..)	$2I(x_c^*:x_b^*) = 4.265$	1
	$2I(x:x_c^*) = 61.728$	13
d) x(.ijkℓ),x(hi...),x(h.j..),x(h..k.)	$2I(x_d^*:x_c^*) = 25.230$	1
	$2I(x:x_d^*) = 36.498$	12
e) x(.ijkℓ),x(hi...),x(h.j..),x(h..k.),x(h...ℓ)	$2I(x_e^*:x_d^*) = 20.191$	1
	$2I(x:x_e^*) = 16.307$	11
f) x(.ijkℓ),x(h..k.),x(h...ℓ),x(hij..)	$2I(x_f^*:x_e^*) = 3.016$	1
	$2I(x:x_f^*) = 13.291$	10
g) x(.ijkℓ),x(h...ℓ),x(hij..),x(hi.k.)	$2I(x_g^*:x_f^*) = 0.042$	1
	$2I(x:x_g^*) = 13.249$	9
m) x(.ijkℓ),x(hij..),x(hi.k.),x(hi..ℓ)	$2I(x_m^*:x_g^*) = 4.316$	1
	$2I(x:x_m^*) = 8.933$	8
n) x(.ijkℓ),x(hij..),x(hi.k.),x(hi..ℓ),x(h.jk.)	$2I(x_n^*:x_m^*) = 0.983$	1
	$2I(x:x_n^*) = 7.950$	7
p) x(.ijkℓ),x(hij..),x(hi.k.),x(hi..ℓ),x(h.jk.),x(h.j.ℓ)	$2I(x_p^*:x_n^*) = 3.181$	1
	$2I(x:x_p^*) = 4.769$	6
q) x(.ijklℓ),x(hij..),x(hi.k.),x(hi.ℓ),x(h.jk.),x(h.j.ℓ), x(h..kℓ)	$2I(x_q^*:x_p^*) = 0.219$	1
	$2I(x:x_q^*) = 4.550$	5
r) x(.ijkℓ),x(hi..ℓ),x(h.j.ℓ),x(h..kℓ),x(hijk.)	$2I(x_r^*:x_q^*) = 0.346$	1
	$2I(x:x_r^*) = 4.204$	4

Analysis of Information (continued)

Marginals Fitted	Information	D.F.
	$2I(x:x^*_r) = 4.204$	4
s) $x(.ijk\ell),x(h..k\ell),x(hijk.),x(hij.\ell)$	$2I(x^*_s:x^*_r) = 2.303$	1
	$2I(x:x^*_s) = 1.901$	3
t) $x(.ijk\ell),x(hijk.),x(hij.\ell),x(hi.k\ell)$	$2I(x^*_t:x^*_s) = 1.375$	1
	$2I(x:x^*_t) = 0.526$	2
u) $x(.ijk\ell),x(hijk.),x(hij.\ell),x(hi.k\ell),x(h.jk\ell)$	$2I(x^*_u:x^*_t) = 0.361$	1
	$2I(x:x^*_u) = 0.165$	1

Table 2b

Analysis of Information

Marginals Fitted	Information	D.F.
e) $x(.ijk\ell),x(hi...),x(h.j..),x(h..k.),x(h...\ell)$	$2I(x:x^*_e) = 16.307$	11
v) $x(.ijk\ell),x(h.j..),x(h..k.),x(hi..\ell)$	$2I(x^*_v:x^*_e) = 3.735$	1
	$2I(x:x^*_v) = 12.572$	10
w) $x(.ijk\ell),x(h..k.),x(hi..\ell),x(h.j.\ell)$	$2I(x^*_w:x^*_v) = 3.443$	1
	$2I(x:x^*_w) = 9.129$	9

Log-odds $\ell n \dfrac{x_e^*(1ijk\ell)}{x_e^*(2ijk\ell)}$

$ijk\ell$	Parametric representation					log-odds
1111	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	0.2128
1112	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$		-0.1210
1121	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$	-0.1284
1122	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$			-0.4621
1211	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	0.0888
1212	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$		-0.2450
1221	τ_1^h	$+\tau_{11}^{hi}$			$+\tau_{11}^{h\ell}$	-0.2524
1222	τ_1^h	$+\tau_{11}^{hi}$				-0.5861
2111	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	0.4158
2112	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$		0.0820
2121	τ_1^h		$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$	0.0746
2122	τ_1^h		$+\tau_{11}^{hj}$			-0.2592
2211	τ_1^h			$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	0.2918
2212	τ_1^h			$+\tau_{11}^{hk}$		-0.0420
2221	τ_1^h				$+\tau_{11}^{h\ell}$	-0.0494
2222	τ_1^h					-0.3831

$\tau_1^h = -0.3831$, $\tau_{11}^{hi} = -0.2030$, $\tau_{11}^{hj} = 0.1240$

$\tau_{11}^{hk} = 0.3411$, $\tau_{11}^{h\ell} = 0.3338$

Table 3a

$$\text{Log-odds} \quad \ell n \frac{x_v^*(1ijk\ell)}{x_v^*(2ijk\ell)}$$

$ijk\ell$	Parametric representation						log-odds
1111	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	0.3571
1112	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$			-0.2898
1121	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	0.0115
1122	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$				-0.6355
1211	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	0.2366
1212	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$			-0.4101
1221	τ_1^h	$+\tau_{11}^{hi}$			$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	-0.1088
1222	τ_1^h	$+\tau_{11}^{hi}$					-0.7557
2111	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$		0.3847
2112	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$			0.1167
2121	τ_1^h		$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$		0.0390
2122	τ_1^h		$+\tau_{11}^{hj}$				-0.2290
2211	τ_1^h			$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$		0.2644
2212	τ_1^h			$+\tau_{11}^{hk}$			-0.0036
2221	τ_1^h				$+\tau_{11}^{h\ell}$		-0.0813
2222	τ_1^h						-0.3492

$\tau_1^h = -0.3492$, $\tau_{11}^{hi} = -0.4065$, $\tau_{11}^{hj} = 0.1203$

$\tau_{11}^{hk} = 0.3457$, $\tau_{11}^{h\ell} = 0.2680$, $\tau_{111}^{hi\ell} = 0.3789$

Table 3b

Log-odds $\ell n \dfrac{x_w^*(1ijk\ell)}{x_w^*(2ijk\ell)}$

$ijk\ell$	Parametric representation							log-odds
1111	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	$+\tau_{111}^{hj\ell}$	0.3283
1112	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$				-0.2523
1121	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$	$+\tau_{111}^{hj\ell}$	-0.0197
1122	τ_1^h	$+\tau_{11}^{hi}$	$+\tau_{11}^{hj}$					-0.6004
1211	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$		0.3976
1212	τ_1^h	$+\tau_{11}^{hi}$		$+\tau_{11}^{hk}$				0.5396
1221	τ_1^h	$+\tau_{11}^{hi}$			$+\tau_{11}^{h\ell}$	$+\tau_{111}^{hi\ell}$		0.0495
1222	τ_1^h	$+\tau_{11}^{hi}$						-0.8876
2111	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$		$+\tau_{111}^{hj\ell}$	0.3542
2112	τ_1^h		$+\tau_{11}^{hj}$	$+\tau_{11}^{hk}$				0.1512
2121	τ_1^h		$+\tau_{11}^{hj}$		$+\tau_{11}^{h\ell}$		$+\tau_{111}^{hj\ell}$	0.0061
2122	τ_1^h		$+\tau_{11}^{hj}$					-0.1968
2211	τ_1^h			$+\tau_{11}^{hk}$	$+\tau_{11}^{h\ell}$			0.4235
2212	τ_1^h			$+\tau_{11}^{hk}$				-0.1360
2221	τ_1^h				$+\tau_{11}^{h\ell}$			0.0754
2222	τ_1^h							-0.4841

$\tau_1^h = -0.4841,\ \tau_{11}^{hi} = -0.4035,\ \tau_{11}^{hj} = 0.2873$

$\tau_{11}^{hk} = 0.3481,\ \tau_{11}^{h\ell} = 0.5595,\ \tau_{111}^{hi\ell} = 0.3776$

$\tau_{111}^{hj\ell} = -0.3565$

Table 3c

E_1: {ijkℓ: ℓn odds $\geq$ 0}

E_1: Observations

ijkℓ	x(1ijkℓ)	x(2ijkℓ)
1111	82	53
1211	14	9
1221	11	10
2111	305	217
2112	200	172
2121	253	247
2211	41	25
2221	70	68
	976	801

$$\mu_2(E_1) = \frac{801}{1491}, \quad \mu_1(E_2) = \frac{1491-976}{1491}$$

$$\text{Prob. Error} = \frac{1}{2}\,\frac{801+515}{1491}$$

$$= \frac{1316}{2 \times 1491} = 0.441$$

(a)

E_1: x_e^*

ijkℓ	x_e^*(1ijkℓ)	x_e^*(2ijkℓ)
1111	74.656	60.346
1211	12.010	10.990
2111	314.491	207.508
2112	193.625	178.376
2121	259.322	240.679
2211	37.780	28.219
	891.884	726.118

$$\mu_2(E_1) = \frac{726.118}{1491}, \quad \mu_1(E_2) = \frac{1491-891.884}{1491}$$

$$\text{Prob. Error} = \frac{1}{2}\,\frac{726.118+599.116}{1491}$$

$$= \frac{1325.234}{2982}$$

$$= 0.444$$

(b)

Table 4

$E_1: x^*_v$

$ijk\ell$	$x^*_v(11jk\ell)$	$x^*_v(21jk\ell)$
1111	79.426	55.574
1121	50.789	50.211
1211	12.855	10.144
2111	310.589	211.411
2112	196.841	175.160
2121	254.876	245.125
2211	37.337	28.662
	942.713	776.287

$$\mu_2(E_1) = \frac{776.287}{1491}$$

$$\mu_1(E_2) = \frac{1491-942.713}{1491}$$

$$\text{Prob. Error} = \frac{1}{2}\ \frac{776.287+548.287}{1491}$$

$$= \frac{1324.574}{2982}$$

$$= 0.444$$

(c)

$E_1: x^*_v$

$ijk\ell$	$x^*_v(11jk\ell)$	$x^*_v(21jk\ell)$
1111	78.482	56.517
1211	13.756	9.244
1212	8.102	13.898
1221	10.760	10.240
2111	306.748	215.252
2112	200.037	171.963
2121	250.769	249.232
2211	39.884	26.115
2221	71.600	66.401
	980.138	818.862

$$\mu_2(E_1) = \frac{818.862}{1491}$$

$$\mu_1(E_2) = \frac{1491-980.138}{1491}$$

$$\text{Prob. Error} = \frac{1}{2}\ \frac{818.862+510.862}{1491}$$

$$= \frac{1329.724}{2982}$$

$$= 0.446$$

(d)

Table 4

E_1	$\hat{x}(11jk\ell)$	$\hat{x}(21jk\ell)$
1111	74.67	60.33
1211	12.02	10.98
2111	314.50	207.50
2112	193.45	178.55
2121	259.17	240.83
2211	37.74	28.26
	891.55	726.45

$$\mu_2(E_1) = \frac{726.45}{1491}, \quad \mu_1(E_2) = \frac{1491-891.55}{1491}$$

$$\text{Prob. Error} = \frac{1}{2}\,\frac{726.45+599.45}{1491}$$

$$= \frac{1325.90}{2982}$$

$$= 0.445$$

Table 4(e)

Divergence Between Low IQ and High IQ
Observations and Estimates

$$\frac{1}{2}\Sigma\Sigma\Sigma\Sigma(x(1ijk\ell)-x(2ijk\ell))\ell n\,\frac{x(1ijk\ell)}{x(2ijk\ell)} = 69.132$$

$$69.132/15 = 4.61/\text{D.F.}$$

$$\frac{1}{2}\Sigma\Sigma\Sigma\Sigma(x_e^*(1ijk\ell)-x_e^*(2ijkl))\ell n\,\frac{x_e^*(1ijk\ell)}{x_e^*(2ijk\ell)} = 52.374$$

$$52.374/11 = 4.76/\text{D.F.}$$

$$\frac{1}{2}\Sigma\Sigma\Sigma\Sigma(x_v^*(1ijk\ell)-x_v^*(2ijk\ell))\ell n\,\frac{x_v^*(1ijk\ell)}{x_v^*(2ijk\ell)} = 56.249$$

$$56.249/10 = 5.62/\text{D.F.}$$

$$\frac{1}{2}\Sigma\Sigma\Sigma\Sigma(x_w^*(1ijk\ell)-x_w^*(2ijk\ell))\ell n\,\frac{x_w^*(1ijk\ell)}{x_w^*(2ijk\ell)} = 59.815$$

$$59.815/9 = 6.65/\text{D.F.}$$

Table 5

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