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ON THE AERODYNAMICS OF WAKE VORTICES

Clinton E. Brown

Hydronautics, Incorporated

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13. ABSTRACT

The effect of wing span loading on the development of fully rolled up wing trailing vortices is discussed. It is shown that parabolic wing loadings produce potential flow maximum core rotary speeds which are finite and less than fifty percent of the downwash speeds at the plane of symmetry. The development of turbulent cores is analyzed and core growth is predicted to occur as the two thirds power of time whereas the peak velocities fall off as the inverse one third power. Axial flow effects of the wing profile drag and lifting system are shown to lead to axial jets on the vortex axis which may either follow the aircraft or exceed the free stream velocity depending on the ration of profile drag to induced drag.

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VORTEX WAKE						
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NOMENCLATURE

A	Wing aspect ratio
c	Wing chord, $\bar{c}$ = Wing mean chord
C_{D_p}	Section profile drag coefficient
C_L	Wing lift coefficient - $\frac{\pi\Gamma_o A}{4Us}$ for elliptic load
C_φ	$\frac{rdv}{vdr}$
C_z	$\frac{rdu}{vdr}$
E	Convected flow work per unit area
J	Vorticity moment of inertia
p	Pressure
q	Dynamic pressure = $\frac{\rho[(U+u)^2 + v^2]}{2}$
$\overline{q^2}$	Mean sum of the squares of the turbulent velocity fluctuations
r	Radial distance from vortex axis
r_c	Core radius
s	Wing half span
t	Time
u	Axial incremental velocity above free stream value
U	Free stream velocity (Flight Speed)

v	Rotational speed
v_p	Potential flow rotational speed
v_c	Rotational speed at outer core edge
x	Axial distance measured downstream
y	Lateral distance from the plane of symmetry

Greek Letters

β	Unknown constant
γ	Wing shed vorticity
Γ	Circulation
Γ_o	Wing root circulation
δC_{D_i}	Fraction of induced drag coefficient
ΔH	Total head loss
Δp	Pressure increment above ambient pressure
ρ	Fluid density

ABSTRACT

The effect of wing span loading on the development of fully rolled up wing trailing vortices is discussed. It is shown that parabolic wing loadings produce potential flow maximum core rotary speeds which are finite and less than fifty percent greater than the downwash speeds at the plane of symmetry. The development of turbulent cores is analyzed and core growth is predicted to occur as the two thirds power of time whereas the peak velocities fall off as the inverse one third power. Axial flow effects of the wing profile drag and lifting system are shown to lead to axial jets on the vortex axis which may either follow the aircraft or exceed the free stream velocity depending on the ratio of profile drag to induced drag.

INTRODUCTION

The twin tornado-like vortex systems created behind a lifting airplane wing have recently attracted much attention because of the potential hazard to aircraft that encounter them. The primary danger appears to be the development of high rolling rates for following aircraft that may fly directly into the center of the high velocity rotating air mass. Transverse flight through the whirling vortex system may also cause high gust loads on the encountering aircraft. Of particular concern are the upsetting effects on aircraft landing at busy air terminals where sufficient maneuvering altitude may not be available to recover from the rolling or pitching motions induced by the vortices. Presently several agencies of the government are studying the problem seeking understanding of the basic aerodynamics, means for detecting the presence of the vortices and also means for reducing their intensity. There is currently some debate on the longevity of the vortices once produced and various studies have been undertaken to measure and predict the vortex decay rates. It appears that atmospheric conditions, in particular the turbulence level, have much to do with the longevity.

The present paper is concerned with the basic aerodynamic patterns which govern the development and decay of the vortex system downstream of the generating aircraft. Calculations are carried out of the vortex downwash field for the case of parabolic wing loading and comparison is made with that of elliptically loaded wings. For elliptic load conditions the effect of the wing profile drag on the axial flows is considered and finally the conditions are derived under which self similar wake development can occur.

GENERAL DISCUSSION

Prandtl, Kaden and Betz, References 1, 2 and 3, respectively described the potential flow to be expected behind a lifting three dimensional wing. Betz, however, presented the first calculation method which described quantitatively the velocity and vorticity fields far behind an elliptically span loaded wing. The vortex sheet shed from the wing trailing edge was ultimately shown to roll up into a pair of spiral sheets with an infinite number of convolutions so that the vorticity and tangential velocity were essentially continuous functions of the distance from the center of the spiral. The vorticity was computed to be well spread out so that the singular tangential velocity at the vortex system center was found to vary as $r^{-\frac{1}{2}}$ rather than r^{-1} as would be the case for a concentrated line vortex.

The modification of the potential flow by viscosity or turbulent mixing was not discussed till 1954 when Squire Reference 4 attempted to calculate the decay rate of the vortex system as a function of time after generation. Squire's assumption of an initial concentrated line vortex and other simplifying assumptions such as the constancy of the turbulent eddy viscosity make his model somewhat questionable. Batchelor, Reference 5, has presented an excellent description of the rolling up process including effects of the wing viscous wake however, his very enlightening work on the development of the central core of the vortex assumes laminar flow and again he starts with the velocity field of a concentrated line vortex. The vortex wakes are generally found to grow at rates much slower than those of non-rotating wakes and it is now known that the rotation produces a certain stabilizing effect on the system and appears to cause a reduction in turbulent eddy diffusivity at least in the radial direction. Most authors dealing with the vortex wake problem have taken a concentrated line vortex as a starting point. Clearly the initial conditions of energy distribution and field stability must play important roles in the subsequent system decay and it is therefore important to establish these conditions and to investigate their consequences. We begin by looking into the effect of altered span loading on the final potential flow vortex velocity field.

Effect of Span Loading

The procedure of Betz has been applied to the case of elliptic loading by Betz himself, Reference 3 and to the case of a rectangular wing by Mason, Reference 6. In this method it is assumed that the development of the vortex spiral proceeds as a two-dimensional flow that is, the fore and aft influence at a downstream station is shown to be small and is neglected. The lateral position of the center of gravity of the vorticity for each side of the vortex system is shown to be constant as is the moment of inertia of vorticity about the lateral center of gravity. These considerations are shown to follow from Kelvin's original theorems. At this point Betz introduces the approximately correct assumption that the moment of inertia of each vorticity group about its own center of gravity is constant during the spiral roll up. Specifically, the vorticity shed from a region of the wing span from the tip to any inboard station, y, rolls up into a spiral having a radial moment of inertia of vorticity about the group center of gravity equal to the moment of inertia of the original vorticity distribution shed from the wing. Thus if the initial span loading is known the potential flow in the rolled up spiral downstream at infinity can be computed.

In most cases the shed vortex distribution is singular at the wing tip having an $(s-y)^{-\frac{1}{2}}$ variation at that point, here s is the half span distance and y is the distance to a point on the span measured from the center line or wing root. The calculations of the rolled up spiral velocity field show corresponding singular behavior in the core or center of the spiral. This effect will be discussed in more detail in the following sections. It is instructive to calculate the rolled up field for the case of parabolic loading where there is no singular behavior at the wing tip. We therefore take the span loading as

$$\Gamma(y) = \Gamma_o \left(1 - \frac{y^2}{s^2}\right) \qquad (1)$$

where Γ is the local circulation, Γ_o is the wing root circulation, y is the lateral distance from the root and s is the semi span. The vorticity, γ, shed in the wake close behind the wing is now

$$\gamma = -\frac{2\Gamma_o y}{s^2} \qquad (2)$$

Taking the subgroup of vorticity shed from y to s along the span we compute the location of the center of gravity, y_{cg} and the moment of inertia, J, as

$$y_{cg} = \frac{1}{\Gamma(y)}\int_y^s \gamma(y')y'dy' = \frac{2s}{3}\frac{[1 - (y/s)^3]}{[1 - (y/s)^2]} \qquad (3)$$

and

$$J(y)_{cg} = \int_y^s \gamma(y'-y_{cg})^2dy' = \frac{s^2\Gamma_o}{2}\left[1-(y/s)^4 - \frac{8[1-(y/s)^3]^2}{9[1-(y/s)^2]}\right] \qquad (4)$$

In the rolled up spiral, this group of vortices must be contained within a radius, r, and have the same moment of inertia about the core center, therefore

$$\Gamma(y) = \Gamma(r) \quad \text{and} \quad J(y) = J(r) \qquad (5)$$

From (5) and (1) we may write

$$\frac{d\Gamma}{dy}dy = \frac{d\Gamma}{dr}dr = -\frac{2\Gamma_o y}{s^2}\,dy \qquad (6)$$

and

$$\frac{dJ}{dy}dy = \frac{dJ(r)}{dr}\,dr \tag{7}$$

By definition

$$J(r) = \int_{o}^{r} \frac{d\Gamma}{dr}\,r^2 dr \tag{8}$$

and by combining (4), (6), (7) and (8) we obtain a relationship between r and y as

$$\frac{r^2}{s^2} = \frac{y^2}{s^2} + \frac{4}{9}\left[\frac{(1+\frac{y}{s}+\frac{y^2}{s^2})^2 - 3(\frac{y}{s})(1+\frac{y}{s})(1+\frac{y}{s}+\frac{y^2}{s^2})}{(1+\frac{y}{s})^2}\right] \tag{9}$$

This functional relationship is plotted in Figure 1 together with the corresponding functions for the elliptically loaded wing. Note that r/s approaches 2/3 for y/s = 0; this value corresponds to the center line of the double vortex system for the case of parabolic span loading. As pointed out by Betz, the interaction of the two systems which has been neglected would undoubtedly somewhat modify these results for large y/s.

The circulation distribution in the rolled up spiral vortex can be obtained as a function of r/s as

$$\Gamma(r) = \Gamma_o\left[1 - (\frac{y(r)}{s})^2\right] \tag{10}$$

Near the axis (9) can be approximated as

$$\left(\frac{y}{s}\right) = \left(1 - \frac{2r}{s}\right) \tag{11}$$

hence the circulation near the core is approximately

$$\Gamma\left(\frac{r}{s}\right) = \frac{4\Gamma_o r}{s} \tag{12}$$

The complete function is plotted in Figure 1.

As a result of the finite slope of the circulation distribution at r = 0 we infer that the velocities at the vortex center line would be finite. The rotary velocity field is given simply for axisymetrical flows as

$$v = \frac{\Gamma(r)}{2\pi r} \tag{13}$$

and at the axis the rotary speed is thus

$$v_{r\to 0} = \frac{2\Gamma_o}{\pi s} \tag{14}$$

The complete downwash pattern in the plane of the vortex pair has been calculated approximately from the above relations and is plotted in Figure 2; also shown is the downwash computed for the elliptic span load distribution. It should be noted that the center of gravity of the vortex system, (taken as the center of the vortex spiral) for the parabolic span loading is located at the two-thirds span position whereas that for the elliptical loading is at a point $\pi/4$ times the semi span.

Figure 2 brings out a major conclusion of this work, namely the maximum core velocity for parabolic wing loading is finite and less than 50% higher than the downwash velocity at the centerline. If one considers an aircraft encountering these velocity distributions, that is, flying with its fuselage along the axis of the vortex spiral it appears apparent, nonetheless, that the induced rolling moment would be greater for parabolic loading than for elliptic. The elimination of the high rotary speeds on the axis is consequently of little help in alleviating the flight hazard. It becomes clear that the reduction in rolling moments on encountering aircraft must come from reductions in the rotary speeds well out from the vortex axis. Far downstream from the aircraft the vortex system can be expected to decay under the influence of viscosity or turbulent diffusion of vorticity and it is important to determine whether the velocity field is substantially altered only near the axis in a central core or whether the angular momentum of more distant regions can be reduced. The answer to this question involves the stability of the system and some consideration of these questions will be given in the following sections.

Returning to the elliptical load distribution case, it is seen in Figure 2 that very high rotational speeds are predicted close to the axis. Confirmation of the velocity field calculations has been obtained in several places, References 6 and 7, and recently tests sponsored by the Langley Research Center of the NASA have provided the data shown in Figure 3. These hot film probe data were obtained for a model of the Boeing 747 aircraft at its cruise flight condition. The tests were carried out in the HYDRONAUTICS, Incorporated Ship Model Basin at a Reynolds number of 1.5×10^6 based on wing mean chord. Note that at a distance of 63 wing spans behind the aircraft corresponding to 2.3 miles full scale, a peak velocity of over six times the centerline downwash value was measured at a radius of just over one percent of the wing span. Although there is scatter in the data they yield a remarkably good fit to the theoretically derived curve. No substantial reductions in outerfield rotary speed is noted at the 110 span (4.1 miles) downstream position.

Considerations of Axial Flows

The type of radial circulation distribution associated with potential flow in a rolled up spiral sheet is generally a monotonic increasing function of radial distance from the spiral center (see Figure 1). Such distributions in the absence of axial flows are known to be stable with respect to radial disturbances (Rayleigh, Reference 8 and Synge, Reference 9). One may ask how then a turbulent core develops and grows with downstream distance. One answer can be found in the presence of the axial flows associated with the profile drag of the wing in combination with the axial flows produced by the rotary flow itself. The latter flow has been discussed by Batchelor, Reference 5, who derived the equations for the axial flow associated with rotary flow fields.

The pressure equations of pertinence of axisymmetric rotary flows are:

$$\Delta p = -\rho U u - \frac{\rho v^2}{2} - \Delta H \tag{20}$$

and

$$\Delta p = -\int_r^\infty \frac{\rho v^2}{r}\, dr + \text{constant} \tag{21}$$

in which Δp is the pressure increment above ambient pressure, ρ is the fluid density and ΔH is the total head loss. The constant in (21) is zero to first order. If we assume the head loss, ΔH, occurs as a result of the profile drag losses on the wing it is possible to relate ΔH and profile drag through the axial momentum equation and the assumption that the wing wake rolls up with the vortex system so that the axial force distributes itself exactly as does the vorticity. This assumption is a rather natural one because the trailing vortex lines are essentially embedded in the wing viscous wake as both leave the wing trailing edge. It therefore follows that the low energy wake air will follow the axial vortex lines during the roll up process. We therefore take the profile drag on an outboard portion of the wing and assume that this force appears in the rolled up spiral within a radius defined by the Betz procedure, that is, the circle within which the vorticity shed from the corresponding portion of the wing would appear. Following this argument we write

$$\int_y^s (C_{D_p} + \delta C_{D_i}) qc dy = -2\pi \int_0^r [\Delta p + \rho U u] r dr \qquad (22)$$

Here C_{D_p} is the wing local profile drag coefficient, q is the stream dynamic pressure, c is the local wing chord, and δC_{D_i} is some small fraction of the lost momentum associated with lift which appears in the rolled up spiral within the circle of radius r. Note that the momentum defect in the spiral for small r as a result of the drag due to lift is only a fraction of the total induced drag because the bulk of this drag is distributed as kinetic energy well outside the cores. δC_{D_i} should be independent of profile drag and can be evaluated by setting $\Delta H = 0$ and $C_{D_p} = 0$.

If we now differentiate (22) and make use of Betz's approximate relation connecting r and y for the case of elliptic span loading

$$r/s = \frac{2}{3}(1 - y/s) \tag{23}$$

we obtain making use of (20)

$$(C_{D_p} + \delta C_{D_i})qc = \frac{4\pi r}{3}\left[\frac{\rho v^2}{2} + \Delta H\right] \tag{24}$$

From the independence of profile drag and drag due to lift

$$\Delta H(r) = \frac{3C_{D_p} qc}{4\pi r} \tag{25}$$

Note that the equation indicates a singular behavior of the total head on the axis. In reality of course, laminar or turbulent mixing would preclude such a possibility, however, the rolling up process is clearly shown to result in a concentration of low energy air along the vortex axis. Using Betz's result for elliptic loading, the potential flow velocity field for small r is given by

$$v = \frac{\Gamma_o}{2\pi\left(\frac{2rs}{3}\right)^{\frac{1}{2}}} \tag{26}$$

hence (21) becomes

$$\Delta p = -\frac{3\Gamma_o^2}{8\pi^2 sr} \tag{27}$$

Making use of this with (25) and (20) enables the calculation of the axial velocity distribution as

$$\frac{u}{U} = \left[\frac{3\Gamma_o^2}{16\pi^2 U^2 s^2} - \frac{3C_{D_p} c}{8\pi s}\right]\frac{s}{r} \tag{28}$$

Expressed in terms of the usual parameters, the axial velocity distribution in the rolled up core is found to be

$$\frac{u}{U} = \left[\frac{3C_L^2}{4\pi^4 A^2} - \frac{3C_{D_p} c(r/s)}{8\pi A\bar{c}}\right]\frac{s}{r} \tag{29}$$

where A is the wing aspect ratio, C_L the wing lift coefficient, and $c/\bar{c}$ the ratio of local wing chord to mean chord, $\bar{c}$.

It can be seen from Equation (29) that the incremental axial velocity can be either forward or rearward depending on the lift and profile drag coefficients of the wing. Also apparent from (29) is the tendency for the axial flow to focus near the vortex axis as indicated by the r^{-1} dependence.

It should be noted that Equation 29 should be expected to hold only at relatively short distances behind the aircraft where the spiral system is well rolled-up but before the field has been greatly spread and reduced in intensity by turbulent diffusion. Much farther back the processes described by Batchelor, Reference 5, must come into play and the field would always present a wake following the aircraft. That is, both the profile drag and drag due to lift must ultimately show up downstream as a momentum defect, hence, a following wake must occur even if the system is broken up by three dimensional instabilities of the vortex pair.

Stability of the Rotating System

Having obtained an approximate description of the flow fields to be expected in the rolled-up trailing vortex system at distances of a few span lengths behind the aircraft, it is now possible to discuss the stability of the flow. It is of interest to determine whether the flow field or certain portions of the flow field have a natural tendency to produce turbulence by feeding energy from the mean flow motion into random motion or whether the system is one which tends to damp out turbulent fluctuations. Unfortunately, there is some disagreement among investigators as to general conditions for stability of rotating flows. Most pertinent to the present case is the work of Ludwieg, Reference 10, who has presented a general stability map for rotating flow which involves two parameters, namely, the dimensionless velocity gradients defined as

$$C_\varphi = \frac{r\,\frac{dv}{dr}}{v} \tag{30}$$

and

$$C_z = \left(\frac{r\,\frac{du}{dr}}{v}\right) \tag{31}$$

Ludwieg's condition for stability is given as

$$(1 - C_\varphi)(1 - C_\varphi^{2}) - \left(\frac{5}{3} - C_\varphi\right) C_z^{2} > 0 \tag{32}$$

This relation is plotted in Figure 4 with the stable and unstable regions noted. Also shown is the stability boundary found in an experimental study making use of a test arrangement having rotating cylindrical walls with provision for axial translation of the inner one. Introducing now the results derived for the rolled-up vortex sheet of an elliptically loaded wing, equations 26 and 29, there is obtained

$$C_\varphi = -1/2, \quad C_z = \left(\frac{3C_L^2}{4\pi^4 A^2} - \frac{3C_{D_p} c}{8\pi A\bar{c}}\right)\left(\frac{\pi^2 A}{2C_L}\right)\left(\frac{2s}{3r}\right)^{\frac{1}{2}} \qquad (33)$$

Considering Figure 4 and Equation 33 it can be seen that the flow field is predicted to be unstable on the vortex axis ($C_z = -\infty$) but to be stable in the outerfield as C_z falls below a value of 0.72. Substitution of any reasonable values in equation 33 shows that the region of instability is confined to an extremely small radius, but nevertheless a region does exist capable of initiating turbulent motions which could ultimately produce a turbulent core and modify the potential flow field.

It should be noted that Howard and Gupta, Reference 11, have criticized Ludwieg's result on the grounds that the analysis is not mathematically rigorous. In their own work they failed to find any general condition for stability to non-axisymmetric disturbances, however, as they pointed out, this does not imply instability either. Donaldson, Reference 12, derives a stability criterion which predicts instability for all values of C_φ less than -0.37 when no axial flow is present but his calculation also shows markedly reduced Reynold's stress for the case under consideration of $C_{\varphi} = -0.5$. Donaldson's method is not a rigorous proof as it contains certain arbitrary assumptions concerning the invariance of triple correlation coefficients of the fluctuating velocity components under changing conditions of the mean flow. It appears, therefore, that the question of whether the basic flow field having a $v \sim r^{-\frac{1}{2}}$ dependency is stable or not must await further analysis or controlled experiments. Ludwieg's experiments summarized in Figure 4 predict stability for the case in question and are quite persuasive. We will proceed, therefore, with some reservations on the assumption that Ludwieg's theory is correct.

The complete stability of the flow field would mean that all initial turbulence would decay and no additional turbulent energy would be extracted from the field; consequently only molecular diffusion of vorticity would occur. However, because of turbulence produced in the low stability region of the axis or more probably because of the initial turbulent energy from the wing profile drag which is concentrated on the axis, a turbulent core develops which modifies the potential flow field and in some way establishes conditions of instability necessary for the conversion of rotary motion into turbulence. In the turbulent core region eddy viscosity must certainly be much larger than molecular viscosity even though substantial suppression of radial eddy mixing occurs as a result of the stable rotary velocity field. In the region outside the core only molecular viscosity is available to transmit shear stress and as a result the outer field must remain substantially unaltered until it is engulfed by the expanding turbulent core. This situation is unlike the case of laminar decay (Reference 13) in which the outer field decelerates the inner field by viscous shearing stresses and in which the internal angular momentum is transported outward to infinity without local increases in rotary speed. Rather in the present case the original angular momentum within the core radius must be largely conserved so that some local increase in rotary speed must be anticipated as the turbulent shearing stresses modify the potential flow rotary field. If the turbulent shearing stresses were large or if sufficient time is allowed, the core region would tend toward the minimum kinetic energy state consistent with the conservation of angular momentum. This condition is the stress free state of uniform rotation and is of course only approachable as a limiting condition however it is instructive to bring out the consequences in the flow of these tendencies. The velocity and circulation distributions for this limiting case are shown as the solid lines in Figure 5. It should be noted that conservation of momentum requires core edge velocities to increase relative to the original potential flow values and therefore a circulation "overshoot" occurs as found in References 7 and 8 for the case of a line vortex. As the core grows, it engulfs a portion of the outer field having a lower angular speed than that of the core hence the angular core speed continually diminishes with time. The tendency toward the circulation discontinuity provides the necessary instability ($d\Gamma^2/dr < 0$) to generate turbulent motions at the advancing core boundary, however the calculation of the final velocity field is beyond the scope of the present report; it is possible however that the methods of References 7 or 8 can be applied. The dashed lines shown in Figure 5 represent the author's estimate of the form of the rotary velocity and circulation distributions. As pointed out by Govindaraju and Saffman (Reference 8) the experimental verification of the velocity field will be extremely difficult because of the usual unsteadiness of vortex flows which would tend to smear out any steep velocity gradients. Finally it must be remarked that the speed of advance of the turbulent front into the unperturbed fluid must be affected by the rotation induced stability of that region and one should expect the growth to be substantially less rapid than that of simple wakes such as those produced by bodies of revolution, disks, etc.

Vortex Core Growth

As indicated by Squire, Reference 4, it is probable that the core grows in a manner satisfying certain self similar flow relationships and it is of interest to examine the physics of the flow for their existence. As conditions for the self similar development we write

$$v = v_c \, f(r/r_c) \tag{34}$$

$$\Delta H = \rho v_c^2 \, g(C_{D_p}/C_L^2, \; r/r_c) \tag{35}$$

$$u = \frac{v_c^2}{U} \, h(C_{D_p}/C_L^2, \; r/r_c) \tag{36}$$

$$\Delta p = \rho v_c^2 \, \ell(C_{D_p}/C_L^2, \; r/r_c) \tag{37}$$

Equations (36) and (37) follow from (34), (35), (20) and (21). The rate of turbulent diffusion is known to be proportional to the root mean square of the turbulent velocity fluctuations and we therefore write

$$\frac{dr_c}{dt} = \beta \sqrt{\overline{q^2}} \tag{37}$$

where β is a constant which should depend in an inverse way on the stability of the outer velocity field and $\overline{q^2}$ is the mean intensity of turbulence in the core. The turbulent energy arises from the change in rotary kinetic energy from the initial potential flow state to the final state but may be reduced by dissipation or altered by energy flow convected by the axial currents. Ignoring both latter losses for the moment, we calculate the turbulent intensity within the circle of radius r_c as

$$\overline{q^2} = \frac{2}{r_c^2} \int_0^{r_c} (v_p^2 - v^2) r dr \sim \frac{\Gamma_o^2}{s r_c} \tag{38}$$

in which v_p is the initial potential flow rotary speed and we have made use of Equations (26) and (34). It is seen that the turbulent intensity falls off as the inverse of the core radius whereas for a line vortex the intensity would vary as r_c^{-2}. Substitution of (38) i (37) yields

$$\frac{dr_c}{dt} \sim \frac{\Gamma_o}{\sqrt{s r_c}} \tag{39}$$

and upon integration we obtain

$$r_c \sim \Gamma_o^{2/3}\, s^{-1/3}\, t^{2/3} \tag{40}$$

We conclude from (40) that the conditions for self similar growth are satisfied if the core radius grows like $t^{2/3}$ or, because $t = x/U$, like $x^{2/3}$. The maximum rotary speed in the core is proportional to v_c and we may therefore write

$$v_{max} \sim \frac{\Gamma_o}{\sqrt{sr_c}} \sim \frac{\Gamma_o^{2/3}}{s^{1/3}\, t^{1/3}} \tag{41}$$

and hence the maximum velocity also varies as $x^{-1/3}$. These results are in contrast to the findings for line vortices, $r_c \sim x^{1/2}$ and $v_{max} \sim x^{-1/2}$, solely because of the initial vorticity distribution differences between the two cases. It is important to note that the result given by Equation (40) should be generally valid for real flows because most vortex motions are generated by rolling up of spiral vortex sheets in which the initial vorticity should vary like $r^{-3/2}$ as in the present calculation. It should also be mentioned that many correlations of flight test data have been attempted using the $x^{1/2}$ relationship which now appears to be in error. Other calculations such as the speed of vortex rings having turbulent cores should also be reexamined in light of the above results.

With dissipation and axial flow present the similarity is not destroyed because (1) the dissipation proceeds at a rate proportional the basic energy release and (2) the axial convection of energy is constant to first order. The dissipation in a turbulent flow can be generally written (Reference 13)

$$\frac{\partial \overline{q^2}}{\partial t} \sim - \frac{(\overline{q^2})^{3/2}}{r_c} \tag{42}$$

and upon integration introducing Equation (40) we compute the energy dissipated by viscosity

$$\Delta \overline{q^2}/2 \sim - \frac{\Gamma_o^2}{s r_c} \tag{43}$$

Comparison of (43) with (38) shows that the dissipation is directly proportional to the kinetic energy released and therefore the self similar development is preserved.

The convection of turbulence and flow work by the axial currents can be written

$$\left(\frac{\partial \overline{q^2}}{\partial t}\right) = \frac{\partial}{\partial x}\left\{\frac{2}{r_c^2}\int_o^{r_c} u(\overline{q^2})\,r\,dr\right\} \tag{44}$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{2}{r_c^2}\int_o^{r_c} u\Delta p\,r\,dr\right\} \tag{45}$$

Introducing Equations (36), (37) and (38) into the above we find that the terms within the braces become independent of r_c or in physical terms the lateral convection of energy in the core is constant to first order. As a consequence the lateral energy flow does not affect the energy balance and the conditions for self-similar development are not violated.

CONCLUDING REMARKS

The development of turbulent vortex cores in a twin spiral rolled up vortex sheet behind a lifting wing has been discussed. The effect of wing span loading is shown to be important in regard to the peak rotary speeds developed in the vortex system. Calculations indicate that parabolic span loading on the wing produces core maximum rotary speeds less than fifty percent higher than the centerline downwash velocity.

Under the plausible assumption that the viscous wing wake rolls up in the vortex spiral with the trailing vortices, it is found that in the early stage of vortex development axial currents are formed at the vortex center which may be either greater or less than the flight speed depending on the ratio of profile drag to induced drag.

For aircraft having elliptic span loadings or indeed for all vortical flows where the shed vorticity has an inverse square root type singularity at the generating surface, the turbulent core of the rolled-up vortex system is predicted to grow as the two-thirds power of time or the downstream distance.

The peak rotary speeds fall off as the inverse one third power. These results are believed to be new and in contradiction to earlier analyses made assuming the vortex system originated as discrete concentrated vortices.

Finally, it is concluded that the rolling moment induced on aircraft entering the vortex wake should be little affected by the core decay until the core diameter reaches a size comparable with the encountering aircraft's wing span.

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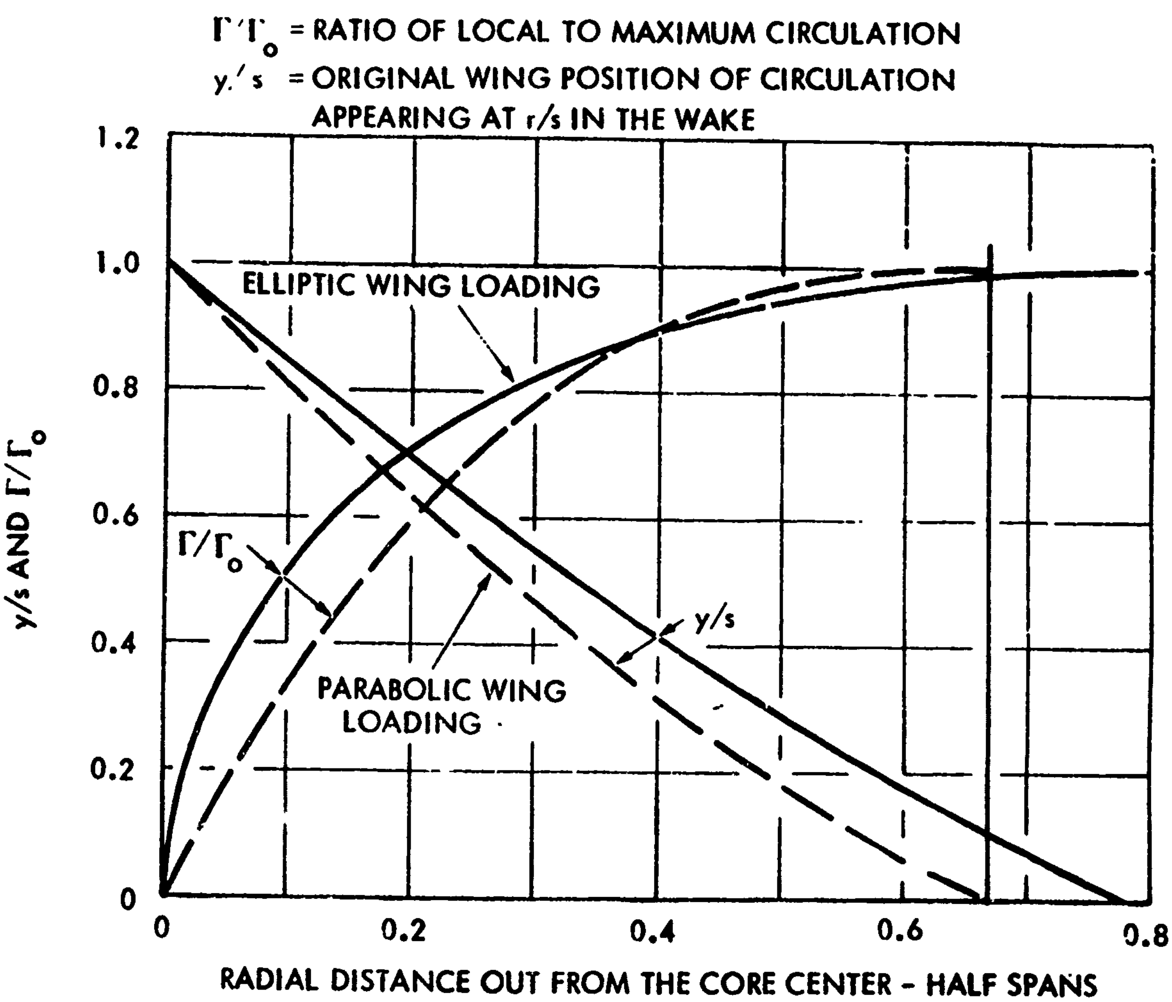

FIGURE 1 - CIRCULATION DISTRIBUTION FOR ELLIPTIC AND PARABOLIC LOADINGS

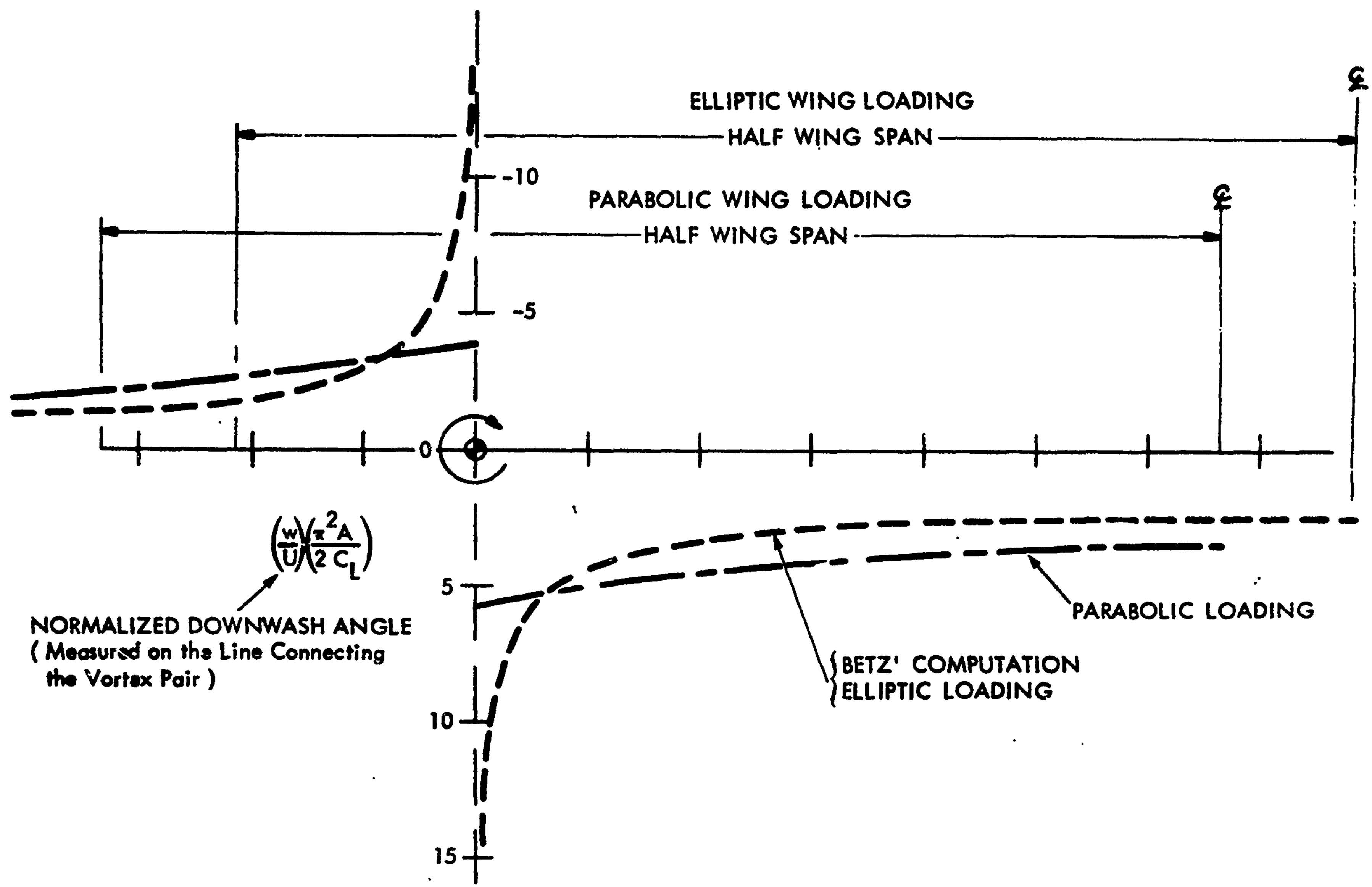

FIGURE 2 - DOWNWASH ANGLE IN THE TREFTZ PLANE

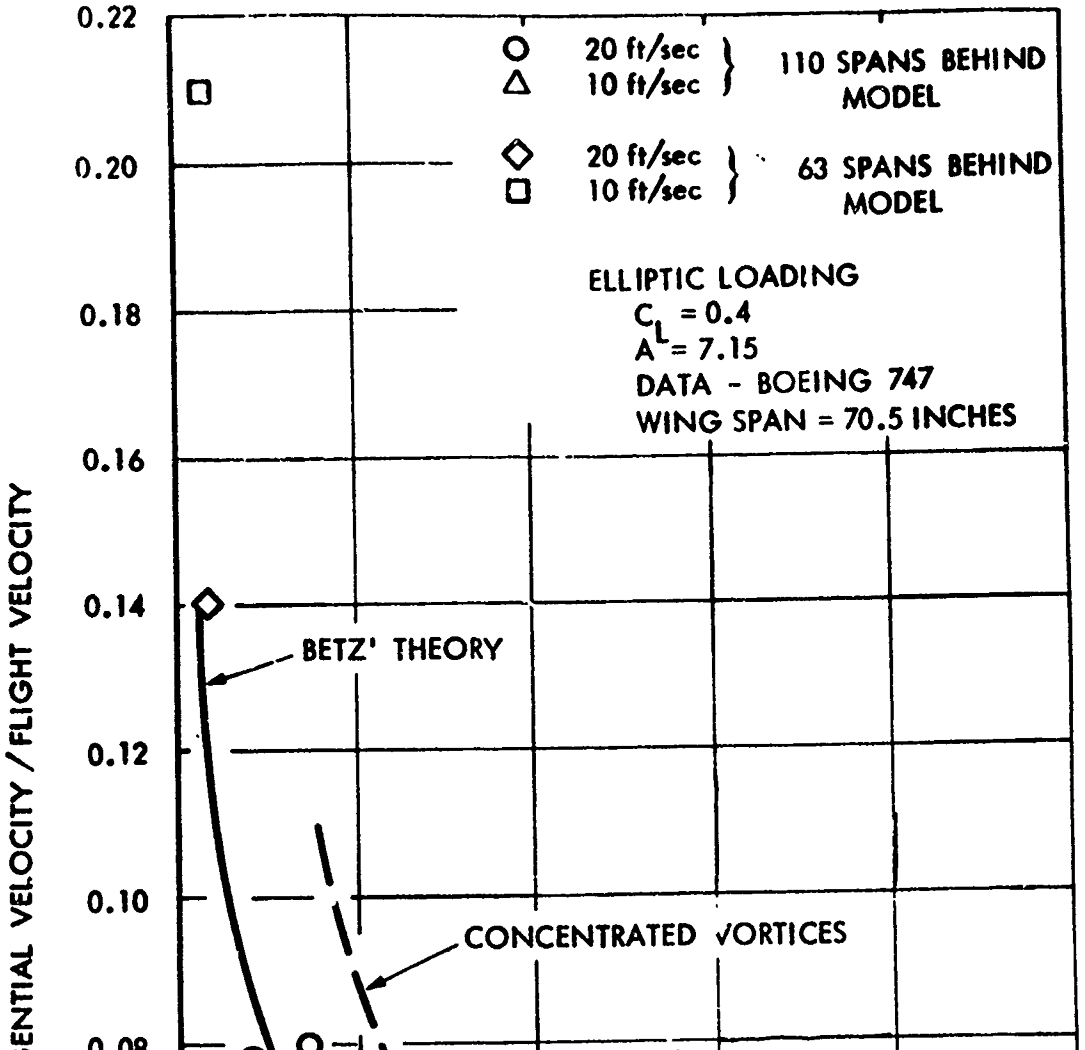

FIGURE 3 · COMPARISON OF THEORY AND EXPERIMENT FOR WAKE DOWNWASH

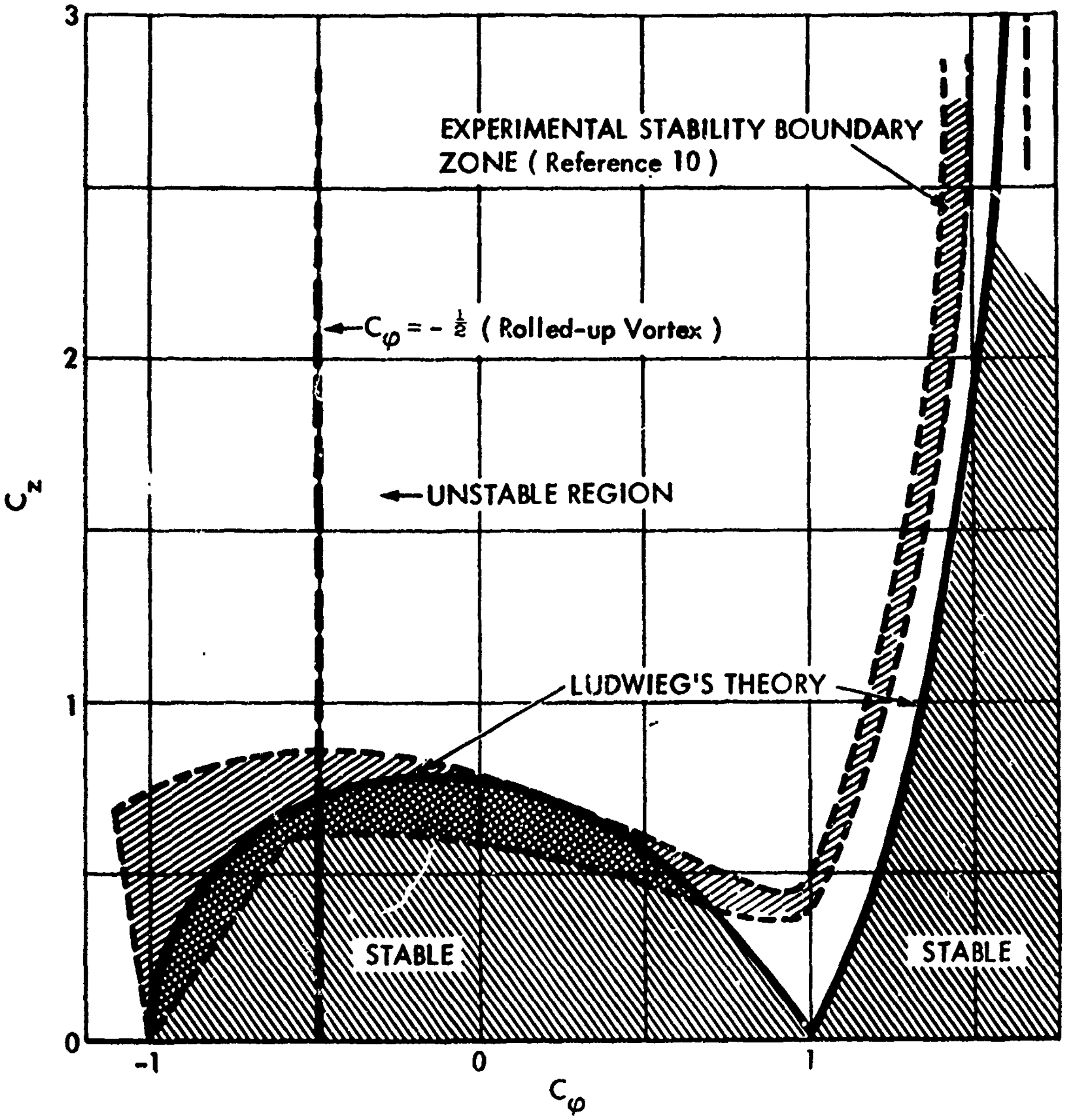

FIGURE 4 - LUDWIEG'S STABILITY MAP

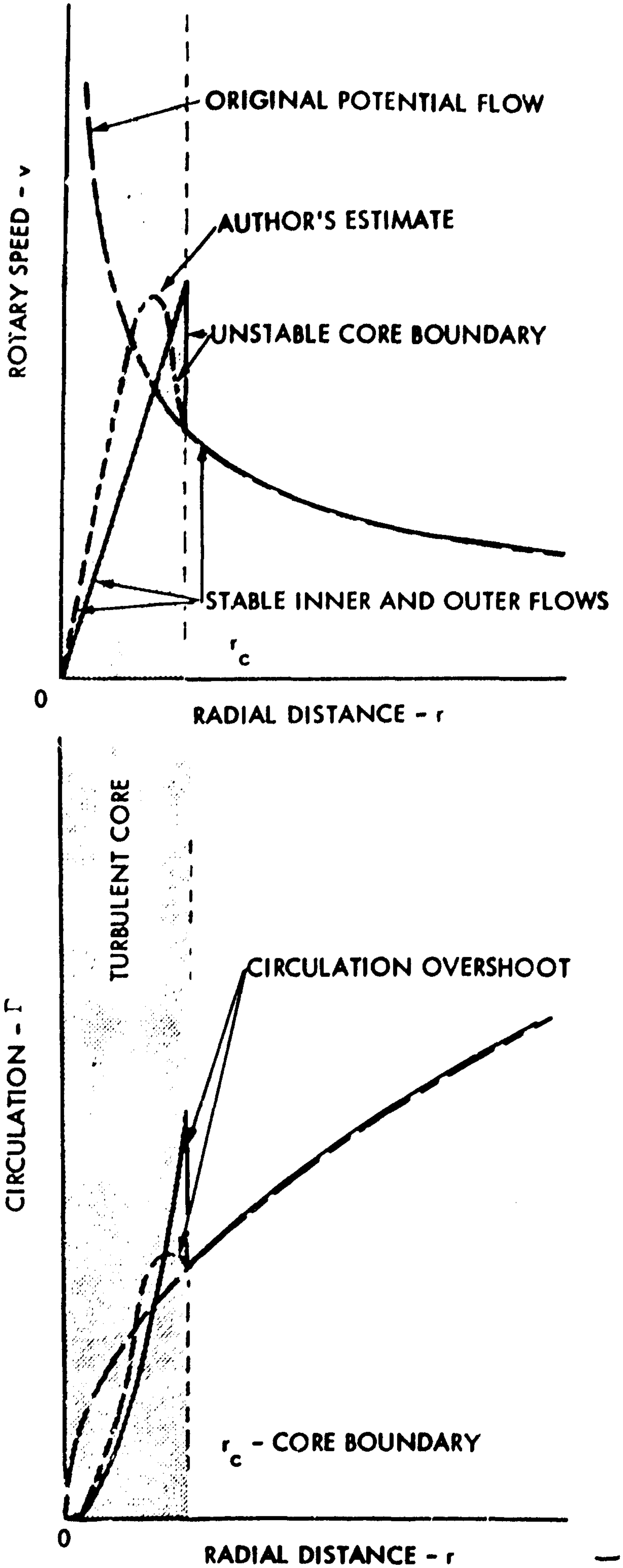

FIGURE 5 - LIMITING VELOCITY AND CIRCULATION PROFILES