INTENSITY INTERFEROMETRY IN THE SPATIAL DOMAIN

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by

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BALLISTIC RESEARCH LABORATORIES
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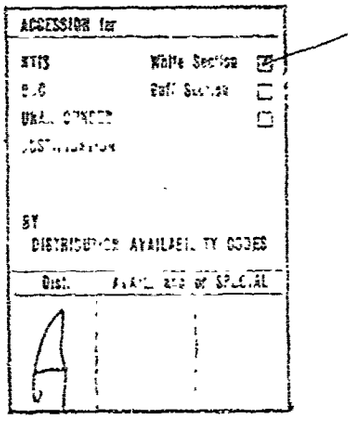

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Intensity interferometry, as developed by Hanbury Brown and Twiss for stellar observation, has shown relative insensitivity to atmospheric scintillation. However, with classical sources, the limitations placed on this technique by quantum noise and detector efficiency are somewhat severe. This situation is vastly improved when laser illumination is employed. Generalizing a methodology of Marchand and Wolf, the far-zone behavior of the mutual intensity function is derived for an intermediate time average. This result is used to reconstruct the irradiance distribution of a spatially-incoherent source. The far-field intensity distribution is recorded spatially for one time-resolution unit of the detector. The resulting spatial signal is cross correlated with itself and related to the intensity distribution over the source. Thus without averaging in the time domain, a spatial Fourier transform relation is derived between the far-field intensity correlation and the source irradiance, similar to the results of Hanbury Brown and Twiss.

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I. INTRODUCTION

The history of intensity interferometry is rooted in the work of R. Hanbury Brown and R. Q. Twiss. Their earliest investigations[1*] dealt with the problem of resolving stellar radio sources by a technique involving the correlation of the squared outputs of two receivers. The advantages accrued relate to the reduction of certain kinds of experimental constraints as well as the comparative insensitivity of the method to atmospheric scintillation.[2] A preliminary conclusion reached at that time was that this technique of intensity correlation would not be applicable at optical frequencies due to limitations imposed by photon noise. However, in later work,[3] Hanbury Brown and Twiss showed that meaningful intensity correlations could be made at optical frequencies even with highly degenerate sources. The limitations placed on this approach by quantum noise and detector efficiency have been somewhat severe, calling for highly refined experimental technique.

For laser illumination, the situation is very different. The signal-to-noise ratio can be typically increased by six orders of magnitude.[4] However, the statistics of the source must be considered in the measurement. The key to relating intensity correlations to some property involving field correlations lies in the assumption of gaussian statistics,[5] for which all higher moments are determined from the first and second. Single mode lasers, though, are distinctly non-gaussian in their statistics, and, therefore, cannot be described by theory framed for thermal sources. But with the addition of only a few axial modes, the field amplitude becomes nearly gaussian distributed.[6]

The principal formula used by Hanbury Brown and Twiss[7] to infer the diameter of a distant source shows that the time-averaged correlation of intensities at two points is equal to the product of a function involving

References may be found on page 29.

the temporal characteristics of the source with the square of the spatial Fourier transform of the source intensity distribution. If the intent of an intensity correlation experiment is to gain information concerning the source intensity distribution, then the source temporal statistics may be of little interest in themselves. Their consideration is necessary if the product-output of the detectors is averaged in the time domain (as it nearly always is) to overcome the limitations imposed by photon and detector noise and possibly reduce scintillation effects produced by transmission in the atmosphere.

Intensity interferometry can be understood as a two-point correlation of intensities following the squaring of the electric field at the detector. If the source is quasi-monochromatic, each differential element on the object emits a number of temporal modes which interfere with each other at the detector. Assuming the detector has sufficient speed[8] to detect these beat frequencies, it is the amplitude and phase of the incoming intensity envelope which is utilized. It is then commonly argued that the random fluctuations in the temporal statistics from different points of the source cause the beat frequencies from each source point to add on-the-average incoherently at the detector. The time-averaged intensity correlation is then proportional to the squared spatial Fourier transform of the source intensity. This approach results in essentially a squared version of the Van Cittert-Zernike theorem.[9]

We wish to argue that the requirement of surface roughness at the source (to assure spatial incoherence) is sufficient to guarantee the incoherent addition of beat frequencies at the intensity detectors. Thus if temporal noise (photon noise, time-dependent detector noise) is largely absent in a local spatial sense, as might be the case with a multi-axial mode laser used with photographic detection, then the intensity information might be gathered during one resolution time of the detector over a plane section normal to the direction of light propagation. Any noise arising in the process would be spatial in nature, and might be averaged out by taking a sufficiently large area of spatial

correlation. The reduction of atmospheric spatial noise would be similar to a process known as aperture averaging.[10] Film-grain noise would be extremely well averaged due to the relatively large area of averaging.

The relative insensitivity of intensity interferometry to turbulence is due to the assumed dispersionless nature of the propagation medium. Since each temporal frequency sees the same refractive index, the differential (beat) frequencies remain unchanged. However, the spatial Fourier transform relation between the source and the far-field scales as the average frequency, not the beat frequency, and thus the resolution afforded by optical frequencies is maintained.

The concept of examining spatial beat frequencies of second-order correlation is, of course, not new. Many classical field-correlation interferometers, as well as holographic experiments, are built on this principle involving a spatial or time lag between interfering beams of the same source. More difficult is the spatial recording of beats from two independent sources as demonstrated by Magyar and Mandel.[11]

We have examined the problem of relating far-field spatial intensity (fourth-order) correlations to the intensity distribution of the source, without here considering the limitations due to noise. Our approach has been completely classical, drawing on a straightforward generalization of a methodology given recently by Marchand and Wolf.[12] Our notation is similar and we follow closely their development through their Eq. (35).

II. THE INTERMEDIATE-AVERAGE MUTUAL COHERENCE FUNCTION

For a stationary scalar wave field, the mutual coherence function for the correlation of two space-time points is often written

$$\Gamma(P_1,P_2,\tau,T) \equiv \frac{1}{2T} \int_{-T}^{T} V_T(P_1,t+\tau)\, V_T^*(P_2,t)\, dt, \qquad (1)$$

where the limits for the time integration are allowed to approach infinity. For this case, however, we wish to keep the parameter T finite,

and by the subscripts indicate that we assume a knowledge of $V_T(P_1,t+\tau)$ and $V_T^*(P_2,t)$ only over the finite sample length 2T. We wish to call $\Gamma(P_1,P_2,\tau,T)$ the intermediate-average mutual coherence function and carefully stress that, for arbitrary T or shift of origin, it may bear little resemblance to the mutual coherence function defined by the ensemble average.

Following Ref. 12, we represent V(P,t) as the temporal Fourier transform of the complex analytic signal

$$V_T(P,t) = \int_0^\infty v_T(P,\omega)\, \exp(-i\omega t)\, d\omega, \quad \text{for} \quad 0 \leq |t| \leq T \tag{2a}$$

and

$$v_T(P,\omega) = \int_{-T}^{T} V_T(P,t)\, \exp(i\omega t)\, dt\,. \tag{2b}$$

Substituting Eq. (2a) into Eq. (1), interchanging the order of integration and time averaging and performing the time integration, we get

$$\Gamma(P_1,P_2,\tau,T) = \iint_0^\infty W_T(P_1,P_2,\omega_1,\omega_2)\, \exp(-i\omega_1\tau)\, \mathrm{sinc}[(\omega_1-\omega_2)T]\, d\omega_1 d\omega_2, \tag{3}$$

where

$$\mathrm{sinc}\, x \equiv \frac{\sin x}{x} \tag{4}$$

and the function

$$W_T(P_1,P_2,\omega_1,\omega_2) \equiv v_T(P_1,\omega_1)\, v_T^*(P_2,\omega_2), \tag{5}$$

and the subscript T here and later implies a function based on the electric field statistics only for the particular sample 2T in length about the origin. The sinc function of Eq. (3) assumes the role of a low-pass filter. It T is very small, the two frequency variables of Eq. (3) are essentially independent and all cross terms are represented in the product of Eq. (5). These cross terms form a high-frequency spectral content. However, as T tends to infinity, the sinc function assumes the role of a delta function, constraining correlation to occur only between identical frequencies in the transform product and forcing the integral to a one-dimensional form. In the limit of large T, the filtered spectrum of Eq. (3) becomes the mean square value (dc) of each temporal frequency component in the signal.

Following Marchand and Wolf[12] and the earlier lead of Walther,[13] $v_T(P,\omega)$ is represented in the form of an angular (spatial) spectrum of plane waves in Cartesian coordinates where

$$v_T(P,\omega) = \iint_{-\infty}^{\infty} a_T(p,q,\omega) \exp[ik(px+qy+mz)]\, dpdq, \tag{6}$$

$$m = (1 - p^2 - q^2)^{1/2} \qquad \text{if } p^2 + q^2 \leq 1 \tag{7a}$$

$$= i(p^2 + q^2 - 1)^{1/2} \qquad \text{if } p^2 + q^2 > 1, \tag{7b}$$

and

$$k = \omega/c, \tag{8}$$

where c is the vacuum velocity of light. Equation (6) indicates that $v_T(P,\omega)$ is formed by a superposition of homogeneous spatial waves propagating in the half space $z > 0$ for the criterion expressed by Eq. (7a) and a set of evanescent waves propagating parallel to the plane $z = 0$ for the case described by Eq. (7b).

Expressing Eq. (5) in the form of the angular spectrum of plane waves defined by Eq. (6) we get

$$W_T(P_1,P_2,\omega_1,\omega_2) = \int\int\int\int\int_{-\infty}^{\infty} A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2)$$

$$\times \exp[ik_1(p_1x_1 + q_1y_1 + m_1z_1)] \tag{9}$$

$$\times \exp[-ik_2(p_2x_2 + q_2y_2 + m_2z_2)]\, dp_1dq_1dp_2dq_2,$$

where

$$A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2) \equiv a_T(p_1,q_1;\omega_1)\, a_T^*(p_2,q_2;\omega_2). \tag{10}$$

Using Eq. (9), therefore, the intermediate-average mutual coherence function of Eq. (3) can be written

$$\Gamma(P_1,P_2,\tau,T) = \int\int_0^{\infty} \exp(-i\omega_1\tau)\ \mathrm{sinc}[\omega_1 - \omega_2)T]\, d\omega_1d\omega_2$$

$$\times \int\int\int\int\int_{-\infty}^{\infty} A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2) \tag{11}$$

$$\times \exp[ik_1(p_1x_1 + q_1y_1 + m_1z_1)]$$

$$\times \exp[-ik_2(p_2x_2 + q_2y_2 + m_2z_2)]\, dp_1dq_1dp_2dq_2.$$

If T is allowed to approach infinity, the limiting form of the sinc function forces $\omega_1 = \omega_2$, and the intermediate-average mutual coherence function clearly reduces to the form of Eq. (13) of Ref. 12 by the elimination of one of the time-frequency integrals.

III. THE ANGULAR CORRELATION FUNCTION AND OTHER SPATIAL CORRELATION FUNCTIONS

The cross-spectral density function $W_T(P_1,P_2;\omega_1,\omega_2)$ is now expressed as a four-dimensional spatial Fourier integral as

$$W_T(x_1,y_1,z_1;x_2,y_2,z_2;\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int\int\int\int \hat{W}_T(f_1,g_1;z_1;f_2,g_2;z_2;\omega_1,\omega_2) \times \exp[i(f_1x_1 + g_1y_1 + f_2x_2 + g_2y_2)]\, df_1dg_1df_2dg_2. \tag{12}$$

Equation (9) with Eq. (12) therefore implies

$$A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2) = k_1^2k_2^2\, \hat{W}_T(k_1p_1,k_1q_1,z_1;-k_2p_2,-k_2q_2;z_2;\omega_1,\omega_2) \times \exp[-i(k_1m_1z_1 - k_2m_2z_2)], \tag{13}$$

and specifically if $z_1 = z_2 = 0$, Eq. (13) becomes

$$A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2) = k_1^2k_2^2\, \hat{W}_T(k_1p_1,k_1q_1;0;-k_2p_2,-k_2q_2;0;\omega_1,\omega_2). \tag{14}$$

Equation (14) indicates that the angular correlation function $A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2)$ and the four-dimensional spatial Fourier transform of the cross-spectral density function are related at the plane $z = 0$ if $f_1 = k_1p_1$, $g_1 = k_1q_1$, $f_2 = -k_2p_2$, and $g_2 = -k_2q_2$.

Further, the spatial transform of the time spectrum of the field can be represented by

$$v_T(x,y,z;\omega) = \int_{-\infty}^{\infty}\int \hat{v}_T(f,g;z;\omega) \exp[i(fx+gy)]\, dfdg. \tag{15}$$

Comparison of Eqs. (15) and (6) indicates

$$a_T(p,q;\omega) = k^2\, \hat{v}_T(kp,kq,z,\omega)\, \exp(-ikmz). \tag{16}$$

Using Eq. (16) in Eq. (1)) gives

$$\begin{aligned} A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_2) &= k_1^2 k_2^2\, \hat{v}_T(k_1p_1,k_1q_1,z_1;\omega_1) \\ &\quad \times \hat{v}_T^*(k_2p_2,k_2q_2,z_2;\omega_2) \\ &\quad \times \exp[-i(k_1m_1z_1 - k_2m_2z_2)]. \end{aligned} \tag{17}$$

The inter ediate-average spatial correlation function is defined

$$\begin{aligned} &\hat{v}_T(k_1p_1,k_1q_1,z_1;\omega_1)\, \hat{v}_T^*(k_2p_2,k_2q_2,z_2;\omega_2) \\ &\quad \equiv V_T(k_1p_1,k_1q_1;z_1;k_2p_2,k_2q_2;z_2;\omega_1,\omega_2) \qquad (18a) \\ &\quad = V_T(k_1p_1,k_1q_1;0;k_2p_2,k_2q_2;0_2;\omega_1,\omega_2) \\ &\quad \times \exp[i(k_1m_1z_1 - k_2m_2z_2)]\,, \qquad (18b) \end{aligned}$$

where the product form of Eq. (18b) is implied by the independence of the left side of Eq. (17) on z_1 and z_2. If we set $z_1 = z_2 = 0$, Eqs. (17) and (18) imply

$$A_T(p_1,q_1;p_2,q_2;\omega_1,\omega_1) = k_1^2 k_2^2\, V_T(k_1p_1,k_1q_1;0;k_2p_2,k_2q_2,0;\omega_1,\omega_2). \tag{19}$$

Finally using Eqs. (14) and (19), we find

$$V_T(f_1,g_1;0;f_2,g_2;0;\omega_1,\omega_2) = \hat{W}_T(f_1,g_1;0;-f_2,-g_2;0;\omega_1;\omega_2)\,. \tag{20}$$

Thus the relationships between the angular correlation function and the cross-spectral density function are established by Eq. (14) and the angular correlation function and the spatial frequency correlation function by Eq. (19) for the case of intermediate time averaging.

IV. THE INTERMEDIATE-AVERAGE CORRELATION FUNCTION IN THE FAR FIELD

Now the form of the cross-spectral density function is examined in the far field. Defining $r_n = (x_n^2 + y_n^2 + z_n^2)^{1/2}$, we seek the asymptotic forms for the case of P_1 and P_2 tending to infinity in the paths indicated by the direction cosines

$$\frac{x_2}{r_2}, \frac{y_2}{r_2}, \frac{z_2}{r_2} \quad \text{and} \quad \frac{x_1}{r_1}, \frac{y_1}{r_1}, \frac{z_1}{r_1}.$$

Rewriting Eq. (9) using the definition of Eq. (10), we have

$$W_T(P_1,P_2;\omega_1,\omega_2) = \iint_{-\infty}^{\infty} a_T(p_1,q_1;\omega_1) \exp[ik_1(p_1x_1 + q_1y_1 + m_1z_1)]\, dp_1dq_1$$

$$\times \iint_{-\infty}^{\infty} a_T^*(p_2,q_2;\omega_2) \exp[-ik_2(p_2x_2 + q_2y_2 + m_2z_2)]\, dp_2dq_2 . \tag{21}$$

As $k_1r_1 \to \infty$ and $k_2r_2 \to \infty$, the asymptotic form of the two-dimensional integrals is given by Miyamoto and Wolf[14] as

$$W_T(P_1,P_2;\omega_1,\omega_2) \simeq \frac{(2\pi)^2}{k_1k_2} \cos\theta_1 \cos\theta_2 A_T\left(\frac{x_1}{r_1}, \frac{y_1}{r_1}; \frac{x_2}{r_2}, \frac{y_2}{r_2}; \omega_1,\omega_2\right)$$

$$\times \frac{\exp[i(k_1r_1 - k_2r_2)]}{r_1r_2}, \tag{22}$$

where

$$\frac{z_1}{r_1} \equiv \cos\theta_1 \quad , \quad \frac{z_2}{r_2} \equiv \cos\theta_2 \tag{23}$$

and use of Eq. (10) has been made. Finally if Eq. (22) is substituted into Eq. (3), we get for the intermediate-average mutual coherence function

$$\Gamma(P_1,P_2,\tau,T) = \iint_0^\infty \exp(-i\omega_1\tau)\ \mathrm{sinc}[(\omega_1 - \omega_2)T]\ d\omega_1 d\omega_2$$

$$\times \frac{4\pi^2}{k_1 k_2} \cos\theta_1\ \cos\theta_2\ \frac{\exp[i(k_1 r_1 - k_2 r_2)]}{r_1 r_2} \tag{24}$$

$$\times A_T\left(\frac{x_1}{r_1}, \frac{y_1}{r_1}, \frac{x_2}{r_2}, \frac{y_2}{r_2}; \omega_1,\omega_2\right).$$

In addition, because of the relationship given earlier relating the angular correlation function to the cross-spectral density and the spatial-frequency functions (Eqs. (14) and (19)), Eq. (22) can be written in the following forms:

$$W_T(P_1,P_2;\omega_1,\omega_2) \sim 4\pi^2\ \cos\theta_1\ \cos\theta_2\ k_1 k_2\ \frac{\exp[i(k_1 r_1 - k_2 r_2)]}{r_1 r_2} \tag{25a}$$

$$\times \hat{W}_T\left(\frac{k_1 x_1}{r_1}, \frac{k_1 y_1}{r_1}; 0; -\frac{k_2 x_2}{r_2}, -\frac{k_2 y_2}{r_2}; 0; \omega_1,\omega_2\right)$$

$$= 4\pi^2\ \cos\theta_1\ \cos\theta_2\ k_1 k_2\ \frac{\exp[i(k_1 r_1 - k_2 r_2)]}{r_1 r_2} \tag{25b}$$

$$\times V_T\left(k_1 \frac{x_1}{r_1}, k_1 \frac{y_1}{r_1}; 0; k_2 \frac{x_2}{r_2}, k_2 \frac{y_2}{r_2}; 0; \omega_1,\omega_2\right).$$

Equations (25a) and (25b) can also be used in Eq. (3) to provide alternate forms of Eq. (24). Equations (24) and (25) form the modified version of the forms given in Ref. 12, Eqs. (33) through (35). Using these results, we are now in a position to form the self-intensity function in the far field.

V. THE SELF-INTENSITY FUNCTION IN THE FAR FIELD

We now examine the form of the self-intensity in the far field by letting points $P_1 = P_2 = P$ and then letting the time delay, τ, be zero. Under these conditions, the mutual coherence function reduces to the self-intensity,[15] and, using Eq. (25b) in Eq. (3) and the definition given in Eq. (18a), we have

$$I(P,T) = 4\pi^2 \cos^2\theta \iint_0^\infty d\omega_1 d\omega_2 \frac{\omega_1\omega_2}{c^2} \frac{\exp[i(k_1 - k_2)r]}{r^2} \operatorname{sinc}[(\omega_1 - \omega_2)T] \tag{26}$$

$$\times \hat{v}_T(k_1 \frac{x}{r}, k_1 \frac{y}{r} ; 0 ; \omega_1)\, \hat{v}_T^*(k_2 \frac{x}{r}, k_2 \frac{y}{r} ; 0 ; \omega_2) \quad ,$$

where, as indicated earlier, the sinc function acts to suppress temporal frequencies in the cross spectrum higher than $\sim 1/(2T)$ Hz. We now utilize the linear transformation of the time frequency variables (for which the Jacobian is unity) defined by

$$\omega_1 - \omega_2 \equiv \rho \quad \text{and} \quad \frac{\omega_1 + \omega_2}{2} \equiv \sigma . \tag{27}$$

Writing the ω variables in terms of these center-of-mass coordinates we get

$$\omega_1 = \frac{2\sigma + \rho}{2} \quad \text{and} \quad \omega_2 = \frac{2\sigma - \rho}{2} , \tag{28}$$

which when substituted into Eq. (26) gives

$$I(P,T) \approx \left(\frac{2\pi}{cr}\right)^2 \cos^2\theta \int_{\sigma=0}^{\infty} d\sigma\, \sigma^2 \int_{\rho=-\infty}^{\infty} \mathrm{sinc}(\rho T)\, \exp(i\rho r/c)\, d\rho$$

$$\times \hat{v}_T\left(\frac{\sigma+\rho/2}{c}\frac{x}{r}\,,\ \frac{\sigma+\rho/2}{c}\frac{y}{r}\,;\ 0\ ;\ \sigma+\rho/2\right) \tag{29}$$

$$\times \hat{v}_T^*\left(\frac{\sigma-\rho/2}{c}\frac{x}{r}\,,\ \frac{\sigma-\rho/2}{c}\frac{y}{r}\,;\ 0\ ;\ \sigma-\rho/2\right),$$

where the dependence of the amplitude on the difference-frequency coordinate, ρ, has been dropped since for quasimonochromatic radiation $\sigma^2 >> |\rho^2/4|$.

Now using the defining transform relation of Eq. (15), we write the spatial correlation function at the source (z=0) where

$$\hat{v}_T\left(\frac{\sigma+\rho/2}{c}\frac{x}{r}\,,\ \frac{\sigma+\rho/2}{c}\frac{y}{r}\,;\ 0\ ;\ \sigma+\rho/2\right)\ \hat{v}_T^*\left(\frac{\sigma-\rho/2}{c}\frac{x}{r}\,,\ \frac{\sigma-\rho/2}{c}\frac{y}{r}\,;\ 0\ ;\ \sigma-\rho/2\right)$$

$$= \frac{1}{(2\pi)^2} \iiiint_{-\infty}^{\infty} v_T(\xi_1,\eta_1;\ 0\ ;\ \sigma+\rho/2)\ v_T^*(\xi_2,\eta_2;\ 0\ ;\ \sigma-\rho/2) \tag{30}$$

$$\times \exp\left[-i\left(\frac{\sigma+\rho/2}{c}\right)\left(\xi_1\frac{x}{r} + \eta_1\frac{y}{r}\right)\right] \exp\left[i\left(\frac{\sigma-\rho/2}{c}\right)\left(\xi_2\frac{x}{r} + \eta_2\frac{y}{r}\right)\right] d\xi_1 d\xi_2 d\eta_1 d\eta_2.$$

Now the intermediate-average spatial-correlation function,

$$v_T(\xi_1,\eta_1;\ 0\ ;\ \sigma+\rho/2)\ v_T^*(\xi_2,\eta_2;\ 0\ ;\ \sigma-\rho/2)\ ,$$

when considered with the filtering action of the sinc function of Eq. (29) will have an effective contribution only for the low frequency components formed by the difference-frequency terms $\sim 1/(2T)$ Hz or less.

In addition, we assume the mode population to be a slowly-varying function of σ, since $\sigma >> \rho/2$ and thus we write

$$[v_T(\xi_1,\eta_1,0;\sigma+\rho/2)\ v_T^*(\xi_2,\eta_2,0;\sigma-\rho/2)]_{\text{low freq.}}$$

$$= A(\xi_1,\eta_1;\sigma+\rho/2)\ A(\xi_2,\eta_2;\sigma-\rho/2)\ \exp[i\phi(\xi_1-\xi_2,\eta_1-\eta_2;\rho) \qquad (31)$$

$$= A(\underline{\xi}_1,\sigma)\ A(\underline{\xi}_2,\sigma)\ H(\rho)\ \exp[i\phi(\underline{\xi}_1-\underline{\xi}_2;\rho)],$$

where

$$H(\rho) = \begin{cases} 1 & \text{for } A(\underline{\xi};\sigma\pm\rho/2) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Equation (31) acknowledges the loss of the optical-frequency phase, while maintaining the phase of the intensity envelope formed by temporal beat modes. The degree to which the phase of this envelope is detected depends on the bandwidth of the source and the detector resolution, $2T$.

Essentially, these arguments were made by Hanbury Brown and Twiss[16] except for the defining of the H function. Its introduction is brought about by the description of narrow-band sources by terms in $A(\underline{\xi},\sigma)$. For a thermal source of relatively large bandwidth, the maximum difference ρ will extend far beyond the temporal-frequency response of the system (here reflected in the sinc term of Eq. (24)) and be continuous as well. But for a laser source exhibiting a series of axial modes, the complete difference-frequency domain might lie entirely within the system response but be piece-wise continuous in its extent.

Relative to the representation of the intermediate-average by the form of Eq. (31), we wish to reiterate a statement made following Eq.(1) that the intermediate-averaging process may bear little resemblance to the infinite time average, even so far as the detail of the amplitude

terms, $A(\underline{\xi},\sigma)$. This situation would be serious if our intent were to infer, for example, the time-frequency statistics of the source. But in the present concept, we desire only to infer the spatial properties of the source. If we consider a multi-axial-mode laser beam scattered from a spatially-rough surface, the lack of correspondence between the two average is unimportant, for all such mode history is integrated out; all areas of the scatterer see the same mode characteristics. Any mode fluctuation would be seen simply as a variation in total received power from one sample to the next. Here, we simply require for one detector-resolution time over a spatial domain that the process of Eq. (31) exhibit a minimum of two temporal modes (to maintain the phase term $\phi(\underline{\xi}_1 - \underline{\xi}_2,\rho)$ with sufficient mode population (reflected in the amplitude terms $A(\underline{\xi},\sigma)$) such that quantum noise in both the carrier wave and the detector can be ignored.

Using the results of Eq. (31) in Eq. (29) and taking $\theta \ll 1$, we write

$$I(P,T) = \left(\frac{1}{cr}\right)^2 \int_0^\infty \sigma^2 \, d\sigma \int_{-\infty}^{\infty} d\rho \, \exp(i\rho r/c) \, \mathrm{sinc}(\rho T) \, H(\rho)$$

$$\times \iiiint_{-\infty}^{\infty} A(\underline{\xi}_1,\sigma) \, A(\underline{\xi}_2,\sigma) \, \exp[j\phi(\underline{\xi}_1 - \underline{\xi}_2 ;\rho)] \tag{32}$$

$$\times \exp\left[-i\left(\frac{\sigma+\rho/2}{c}\right)(\xi_1 \frac{x}{r} + \eta_1 \frac{y}{r})\right] \exp\left[i\left(\frac{\sigma-\rho/2}{c}\right)(\xi_2 \frac{x}{r} + \eta_2 \frac{y}{r})\right] d\xi_1 d\xi_2 d\eta_1 d\eta_2 .$$

We now have the self intensity in the far field expressed as a double integral over sum- and difference-frequency components as well as two, two-dimensional spatial Fourier transforms over the source. We will next use it to form the fourth-order correlation function.

VI. FOURTH-ORDER INTENSITY CORRELATION IN THE FAR FIELD

Using Eq. (32), the intensity correlation between points P_1 and P_2 in the far field can be written

$$\begin{aligned}
&\mathrm{Re}\, I(P_1;T)\, \mathrm{Re}\, I(P_2;T) \\
&= \frac{1}{2(cr)^4} \int_0^\infty \sigma^2\, d\sigma \int_0^\infty \sigma'^2\, d\sigma' \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\rho'\, \mathrm{sinc}(\rho T)\, \mathrm{sinc}(\rho' T)\, H(\rho) H(\rho') \\
&\times \iiiiiiiint_{-\infty}^{\infty} A(\underline{\xi}_1,\sigma)\, A(\underline{\xi}_1',\sigma')\, A(\underline{\xi}_2,\sigma)\, A(\underline{\xi}_2',\sigma') \\
&\times \Bigg\{ \cos\Bigg[\phi(\underline{\xi}_1-\underline{\xi}_2;\rho) + \phi(\underline{\xi}_1'-\underline{\xi}_2';\rho') + \left(\frac{\sigma-\rho/2}{c}\right)\left(\xi_2 \frac{x_1}{r} + \eta_2 \frac{y_1}{r}\right) \\
&+ \left(\frac{\sigma'-\rho'/2}{c}\right)\left(\xi_2' \frac{x_2}{r} + \eta_2' \frac{y_2}{r}\right) - \left(\frac{\sigma+\rho/2}{c}\right)\left(\xi_1 \frac{x_1}{r} + \eta_1 \frac{y_1}{r}\right) + \frac{r}{c}(\rho+\rho') \\
&- \left(\frac{\sigma'+\rho'/2}{c}\right)\left(\xi_1' \frac{x_2}{r} + \eta_1' \frac{y_2}{r}\right)\Bigg] \\
&+ \cos\Bigg[\phi(\underline{\xi}_1-\underline{\xi}_2;\rho) - \phi(\underline{\xi}_1'-\underline{\xi}_2';\rho') + \left(\frac{\sigma-\rho/2}{c}\right)\left(\xi_2 \frac{x_1}{r} + \eta_2 \frac{y_1}{r}\right) \\
&- \left(\frac{\sigma'-\rho'/2}{c}\right)\left(\xi_2' \frac{x_2}{r} + \eta_2' \frac{y_2}{r}\right) - \left(\frac{\sigma+\rho/2}{c}\right)\left(\xi_1 \frac{x_1}{r} + \eta_1 \frac{y_1}{r}\right) \\
&+ \frac{r}{c}(\rho-\rho') + \left(\frac{\sigma'+\rho'/2}{c}\right)\left(\xi_1' \frac{x_2}{r} + \eta_1' \frac{y_2}{r}\right)\Bigg]\Bigg\}\, d\xi_1 d\xi_1' d\xi_2 d\xi_2' d\eta_1 d\eta_1' d\eta_2 d\eta_2' ,
\end{aligned} \tag{33}$$

where use has been made of the trigonometric identity

$$2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b).$$

We now investigate the phase variation in the above terms. In order to develop the condition of spatial incoherence, we demand that the source surface be spatially rough, specifically that phase changes of π occur $\sim 10^{-6}$ m across its extent. Next we make an order of magnitude calculation of the size of the term $\frac{\sigma-\rho/2}{c}\left(\frac{x}{r}\right)\xi \sim \frac{\sigma x\xi}{cr}$. Choosing some approximate values, $\frac{\sigma}{c} = \frac{2\pi}{\lambda} \sim 10^{7}\ m^{-1}$, $x \sim 1m$, $r \sim 10^{4}$ m, we find $\frac{\sigma x}{cr} \sim 10^{3}\ m^{-1}$. Letting $\pi = 10^{3}\,\xi$ implies ξ must move through $\pi \times 10^{-3}$ m to get π change.

Thus the phase terms of Eq. (33) describing the source spatial roughness have a spatial rate of oscillation of about 10^{3} greater than the remaining terms. On the basis of a somewhat heuristic argument given in the Appendix, we assert that the dominant terms of Eq. (33) are those for which the phase variation is a minimum.

Upon examining Eq. (33), we note that in the first cosine argument, the random phase terms add and thus there is no choice of $\underline{\xi}$ or ρ terms which will slow the phase. Turning to the second cosine expression, we see that there are two general cases for which the phase variation is a minimum. The first is the familiar one for which the parameter ρ is equal to zero. This corresponds to the normal infinite-time average situation. For this case we set $\underline{\xi}_1 = \underline{\xi}_2$ and $\underline{\xi}_1' = \underline{\xi}_2'$ with the result that all the source-spatial information collapses in the cosine argument. This is the normal dc-intensity term in the far field. For convenience, we will suppress this term in following expressions. For $\rho \neq 0$, the minimum phase can be achieved by removing the primes of Eq. (33), halving the number of integrals to give

$$\text{Re } I(P_1;T)\ \text{Re } I(P_2;T) = \frac{1}{(cr)^4}\int_{\sigma=0}^{\infty}\sigma^4\,d\sigma\int_{\rho=0+}^{\infty}d\rho\ \text{sinc}^2(\rho T)\ H(\rho)$$

$$\times \iiint_{-\infty}^{\infty} A^2(\xi_1,\sigma)\, A^2(\xi_2,\sigma)\, \cos\left\{\left(\frac{\sigma-\rho/2}{cr}\right)\left[\xi_2(x_1-x_2)+\eta_2(y_1-y_2)\right]\right. \tag{34}$$

$$\left. +\left(\frac{\sigma+\rho/2}{cr}\right)\left[\xi_1(x_1-x_2)+\eta_1(y_1-y_2)\right]\right\} d\xi_1 d\xi_2 d\eta_1 d\eta_2 \,.$$

The cosine expression of Eq. (34) can be written

$$\cos\left\{\left(\frac{\sigma-\rho/2}{cr}\right)\left[\xi_2(x_1-x_2)+\eta_2(y_1-y_2)\right]-\left(\frac{\sigma+\rho/2}{cr}\right)\left[\xi_1(x_1-x_2)+\eta_1(y_1-y_2)\right]\right\}$$

$$= \cos\left\{\frac{\sigma}{cr}\left[(\xi_2-\xi_1)\,(x_1-x_2)+(\eta_2-\eta_1)\,(y_1-y_2)\right]\right. \tag{35}$$

$$\left. -\frac{\rho}{2cr}\left[(\xi_1+\xi_2)\,(x_1-x_2)+(\eta_2+\eta_1)\,(y_1-y_2)\right]\right\},$$

which is in the form of the identity

$$\cos\,(a-b) = \cos a \cos b + \sin a \sin b\,. \tag{36}$$

If the angular size of the source is small, b is much less than unity, ensuring that we can write, as did Hanbury Brown and Twiss,[16]

$$\cos\,(a-b) \approx \cos\,(a)\,. \tag{37}$$

Using Eq. (37), we write finally Eq. (34), expressing the two-point intensity correlation in the far field of a spatially incoherent source, as:

$$I(P_1;T)\; I(P_2;T) = \frac{1}{(cr)^4}\int_{\omega=0}^{\infty}\omega^4\, d\omega \int_{\rho=0+}^{\infty} H(\rho)\, \mathrm{sinc}^2(\rho T)\, d\rho$$

$$\times \iint_{-\infty}^{\infty} I(\underline{\xi}_2,\omega)\, \exp\left\{-i\frac{k}{r}\left[\xi_2(x_1-x_2) + \eta_2(y_1-y_2)\right]\right\} d\xi_2 d\eta_2 \qquad (38a)$$

$$\times \left[\iint_{-\infty}^{\infty} I(\underline{\xi}_1,\omega)\, \exp\left\{-i\frac{k}{r}\left[\xi_1(x_1-x_2) + \eta_1(y_1-y_2)\right]\right\}\right]^* d\xi_1 d\eta_1$$

$$= \frac{1}{(cr)^4} \int_{\rho=0+}^{\infty} H(\rho)\, \mathrm{sinc}^2(\rho T)\, d\rho \int_{\omega=0}^{\infty} \omega^4\, d\omega \qquad (38b)$$

$$\times \left| \iint_{-\infty}^{\infty} I(\underline{\xi},\omega)\, \exp\left\{-i\frac{k}{r}\left[\xi(x_1-x_2) + \eta(y_1-y_2)\right]\right\} d\xi d\eta \right|^2 ,$$

where the mean intensity has been suppressed.

If the source time-frequency characteristics of the source are such that ρ is continuous over the domain for which $\mathrm{sinc}^2(\rho T)$ is essentially nonzero, then $H(\rho) = 1$ for all ρ and then the difference-frequency term can be integrated easily to give

$$\int_{\rho=0+}^{\infty} \mathrm{sinc}^2(\rho T)\, d\rho = \pi/(2T). \qquad (39)$$

It is evident from Eq. (38b) that the two-point intensity correlation in the far field is proportional to the modulus of the spatial Fourier transform across a spatially-rough source. Only for the case that the intensity distribution on the source is pure even can the phase of the spatial transform be inferred and used to invert uniquely Eq. (38) to get the intensity distribution on the source, $I(\underline{\xi},\omega)$.

VII. SPATIAL AVERAGING

Let us rewrite Eq. (38) in a position-vector notation for the spatial coordinates so that

$$I(\underline{P}_1,T)\ I(\underline{P}_2,T) = \frac{1}{(cr)^4} \int_{\rho=0+}^{\infty} H(\rho)\ \mathrm{sinc}^2(\rho T) \int_{\omega=0}^{\infty} \omega^4\ d\omega \times \left| \int_{-\infty}^{\infty} I(\underline{\xi},\omega)\ \exp(-i\ \frac{k}{r}\ \underline{\xi}\cdot\underline{d})\ d\underline{\xi} \right|^2 , \tag{40}$$

where

$$\underline{d} = (x_1-x_2)\ \hat{i} + (y_1-y_2)\ \hat{j} , \tag{41}$$

and

$$\underline{\xi} = \xi\hat{i} + \eta\hat{j} \quad . \tag{42}$$

By observing Eq. (40) one can define

$$I(\underline{P}_1;T)\ I(\underline{P}_2;T) \equiv C(\underline{P}_1,\underline{P}_2;T) = C(\underline{d};T) \ . \tag{43}$$

It is thus obvious that the two-point intensity correlation for the ideal case of noiseless imaging is spatially dependent only on the coordinate difference in the far field. This property of Eq. (40) reflects its intrinsic spatial stationarity, indicating that the global dependence is absent. If noise introduced in the imaging process is independent of the source character, then it will be spatial in character and related to atmospheric turbulence or detector limitations. Averaging the function of Eq. (40) over an area would increase the signal-to-noise ratio, the area of sufficient size being determined by the correlation interval of the noise statistics.

Averaging Eq. (40) over an area A_o simply, then, multiplies the two-point correlation by a factor A_o giving

$$< I(\underline{P}_1;T)\ I(\underline{P}_2;T) >_{\text{spatial average}} = < C(\underline{d};T) >_{A_o}$$

$$= \frac{A_o}{(cr)^4} \int_{\rho=0+}^{\infty} H(\rho)\text{sinc}^2(\rho T)\ d\rho \int_{\omega=0}^{\infty} \omega^4\ d\omega \left| \int_{-\infty}^{\infty} I(\underline{\xi},\omega)\ \exp(-i\frac{k}{r}\ \underline{\xi}\cdot\underline{d})\ d\underline{\xi} \right|^2 . \quad (44)$$

VIII. SUMMARY AND CONCLUSIONS

Using a methodology of Marchand and Wolf as a starting point, we have developed an expression for the two-point intensity correlation in the far field, independent of time averaging except for the temporal resolution of the detector. This result is derived for narrow-band, high-intensity light scattered from a spatially-rough surface.

To illustrate the above ideas, we wish to describe a simple experiment which embodies these mathematical ideas. Using a helium-neon laser in a single-axial-mode configuration, a symmetrical source is transilluminated. The source has a random-phase character to obtain spatial incoherence. The far-zone intensity pattern is then recorded by film using an exposure time less than the reciprocal of the source bandwidth.[17] This insures that the phase of the beat frequency is recorded.

Next, the film is developed so that it is linear in intensity and used to make two identical positive transparencies. The positives are then placed in a collimated beam, forming the correlated intensity over an averaging area, A_o. The signal transmitted by the transparency pair is optically Fourier transformed, a dc block inserted to remove the unwanted average term, and the total remaining irradiance measured. This signal represents the mathematical expression given by Eq. (44) for the

transparency spatial lag, $\underline{d}$. Since the source is known, a priori, to be symmetrical, the transform of Eq. (44) is pure real. The square root of the correlation signal is proportional to the spectrum which is then known as a function of spatial lag $\underline{d}$. Finally, this one dimensional signal is Fourier transformed by machine to give the scaled source irradiance.

We have therefore shown that, given a symmetrical, spatially incoherent source illuminated by high-intensity light, the far-zone intensity pattern can be used to form the optical image of the source if the signal is recorded with sufficiently-short time resolution.

APPENDIX

We argue heuristically that the form of Eq. (33) for arbitrary $\phi(\underline{\xi}_1 - \underline{\xi}_2 ; \rho)$ approaches a condition expressed by one form of the Riemann-Lebesgue theorem[18] which asserts that

$$\lim_{k \to \infty} \int_a^b F(x) \cos(kx)\, dx = 0 \,. \tag{A1}$$

Here, the terms on the right-hand side of Eq. (33), independent of $\phi(\underline{\xi}_1 - \underline{\xi}_2;\rho)$, assume the role of $F(x)$, while $\cos(kx)$ is represented by terms dependent on $\phi(\underline{\xi}_1 - \underline{\xi}_2;\rho)$. Taking the limit in Eq. (A1) in this context implies a surface roughness of arbitrarily small scale and that the energy will be radiated at these very small spatial frequencies. These waves, though, are evanescent and do not transfer any energy. This kind of contradiction is discussed by Beran and Parrent[19] and can arise similarly when the constraint of extreme spatial incoherence is expressed in the form of a delta function of coordinate differences and interpreted in a strict sense. On the basis of the limit implied by Eq. (A1), we infer that the dominant terms of Eq. (33) are those for which the phase variation is a minimum.

This similarity to second-order spatial incoherence can be realized by a physical argument. If we consider the imaging of a spatially-rough surface illuminated by perfectly monochromatic light under the condition that the surface is sufficiently large to be composed of many independent scatterers, we can see that the light waves from each scatterer will be uncorrelated at a point at the receiver by virtue of the random phase introduced at the source. Further, the intensity at this detector point will simply be the sum of the squares of the individual field contributions at that point. It should then seem reasonable that if we set up a similar situation of random-phase addition in difference-frequency space, the incoherent addition of intensities should hold as well, without appealing to the approach of extending an average over a time history of temporal fluctuations.

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