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OPTIMUM PASSIVE SIGNAL PROCESSING FOR
ARRAY DELAY VECTOR ESTIMATION

William R. Hahn

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White Oak, Maryland

1 September 1972

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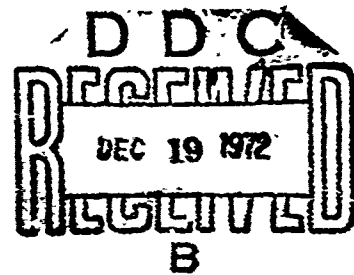
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NAVAL ORDNANCE LABORATORY, WHITE OAK, SILVER SPRING, MARYLAND

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Optimum Passive Signal Processing for Array Delay Vector Estimation

This report briefly summarizes and then further develops the topics treated in the author's Ph.D. thesis and reported in NOLTR 72-120. The work was partially funded under Naval Ship Systems Command Task Number J8692/SF11-121-101.

The report will be of interest to those working on passive sonar target localization, optimum signal processing for passive sonars, and passive array calibration.

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Captain, USN
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By direction

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Chapter 1

Introduction

1.1 In many physical problems of interest, with sonar, radar, and seismology as examples, the time records of the outputs of an array of sensors are observed over some time interval and used to estimate the position of a distant noise source. Inversely, the position of the distant noise source may be known, and the intent may be to estimate the positions of the sensors comprising the array. Typically, the sensor outputs are amplitude scaled and delayed replicas of the waveform from the distant noise source, corrupted by additive noises, usually local in origin. If the amplitude gradient across the array of the waveforms from the distant source is negligible, essentially all of the geometric information is encoded in the set of delays associated with the propagation across the array of the wavefronts from the distant source. This paper discusses the theoretical bounds on the precision with which the set of delays can be measured, and shows that either properly filtered beamformers or properly filtered systems of correlators can be used to obtain estimates that achieve the theoretical bound.

1.2 The theoretical bound discussed is the Cramér-Rao matrix bound [1], which is the appropriate bound to use when large numbers of samples, or equivalently, long observation times, are used.

1.3 For the purposes of this paper, it is more convenient to use its inverse, the Fisher Information Matrix, and to compare the inverses of the matrices for the beamformer and multiple correlator delay measurement schemes to the Fisher Information Matrix.

1.4 The results of Chapters 2 and 3 are developed in greater detail in [2], as is the first half of Chapter 4. The remainder of Chapter 4 and Chapter 5 have not been reported elsewhere.

1.5 The following notation is used: If A is a matrix, A^{-1} is its inverse, A^* is its conjugate, A^T is its transpose, $\text{tr } A$ is its trace, and $\det A$ is its determinant. A square matrix whose elements off the main diagonal are all zero may be written as $\text{diag}(a_1, a_2, \dots, a_n)$, where a_i is the i -th diagonal element. Vectors are column vectors unless otherwise specified. $\underline{1}$ denotes a vector with every element a one (1). $\underline{0}$ is a matrix of zeros, and I is the identity matrix. $\langle \cdot \rangle$ is the expectation operator, and $\text{grad } f$ is the row vector which is the gradient of the scalar f . The gradient of a vector is the matrix in which the i -th row is the gradient of the i -th component of the vector. The Kronecker delta is denoted in the usual way as δ_{ij} . Integrals of the form $\int_{-T/2}^{T/2} f dt$ are written as $\int_T f dt$, and integrals of the form $\int_{-\omega_N}^{\omega_N} f d\omega$ are written as $\int_B f d\omega$. The quantity ω_N is defined as $\omega_N = N\omega_0$, where N and ω_0 are defined in Chapter 2.

1.6 The symbols MLE, FIM, CRMB, and HOT are abbreviations for Maximum Likelihood Estimate, Fisher Information Matrix, Cramér-Rao Matrix Bound, and Higher Order Terms (as in series expansions), respectively. The symbols (CRMB) and (FIM) represent the designated matrices themselves.

Chapter 2

The Fisher Information Matrix

2.1 Assume that the signal wave fronts from a distant noise source propagate across an M element array of sensors and that the signal amplitude gradient across the array is negligible. The signal at the i-th sensor is $s(t-d_i)$, where $s(t)$ is the signal at a reference point near the array, and d_i is the delay at the i-th sensor. Without loss of generality, the reference point is assumed to be at the location of the first element in the array. Thus, $d_1 = 0$. The output of the i-th sensor is

$$x_i(t) = s(t-d_i) + n_i(t), \quad (1)$$

where $n_i(t)$ is the additive sensor noise. The M sensors are observed for T seconds, $-T/2 \leq t \leq T/2$, and the M time records are represented by Fourier coefficients,

$$X_i(\omega_k) = \frac{1}{\sqrt{T}} \int_T x_i(t) \exp\{-jk\omega_0 t\} dt, \quad (2)$$

where $\omega_0 = 2\pi/T$, and $\omega_k = k\omega_0$. The following assumptions will determine the joint density function for the random Fourier coefficients.

- a. The random signal and each of the M additive sensor noises are all stationary zero-mean Gaussian random processes.
- b. All of the random processes are independent.
- c. T is large compared to the correlation times of the random processes, and also to the time needed for the signal wave fronts to transverse the array.

2.2 If X is a vector containing the Fourier coefficients as elements, if $S(\omega)$ and $N_i(\omega)$ are the signal and noise power spectra at the i-th sensor, and if only

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Fourier coefficients up to frequency $N\omega_0$ are to be processed, the density for X can be written as

$$p(X) = \left[\pi^{MN} \prod_{k=1}^N \det R(k) \right]^{-1} \exp \left\{ - \sum_{k=1}^N X^T(k) R^{-1}(k) X^*(k) \right\}, \quad (3)$$

where:

$$\begin{aligned} X(k) &= (X_1(\omega_k), X_2(\omega_k), \dots, X_M(\omega_k))^T \\ V(k) &= (1, e^{j\omega_k d_2}, \dots, e^{j\omega_k d_M})^T \\ N(k) &= \text{diag}(N_1(\omega_k), N_2(\omega_k), \dots, N_M(\omega_k)) \\ R(k) &= N(k) + S(\omega_k) V^*(k) V^T(k) \end{aligned} \quad (4)$$

2.3 In what follows the frequency arguments of the functions discussed will be generally suppressed. \sum_{B+} will indicate a sum over the positive Fourier

frequencies being considered, while \sum_i will be used to denote the array sum $\sum_{i=1}^M$.

When ω appears following a sum \sum_{B+} , it will be understood to stand for $k\omega_0$.

2.4 Since $\det R(k) = (1 + \sum_i S/N_i) \det N(k)$, only the exponential part of the density function will depend on the d_i . Let the signal delay vector D be defined as

$$D = (d_2, d_3, \dots, d_M)^T. \quad (5)$$

2.5 The likelihood function for D is

$$L(D) = \left[\pi^{MN} \prod_{B+} \det R \right]^{-1} \exp \left\{ - \sum_{B+} X^T R^{-1} X^* \right\}. \quad (6)$$

2.6 The CRMB for unbiased estimators of the vector argument of the likelihood function is the inverse of the FIM, denoted by (FIM), where

$$(FIM) = - \langle \text{grad}(\text{grad} \ln L(D))^T \rangle. \quad (7)$$

The gradients in equation (7) are taken with respect to the components of the vector D. If G is defined by

$$G = S / (1 + \sum_i S/N_i), \quad (8)$$

the inverse of the matrix R is

$$R^{-1} = N^{-1} - GN^{-1} V^* V^T N^{-1}. \quad (9)$$

Provided that only the elements of D depend on a and b, the typical element of the FIM has the form

$$\begin{aligned} - \langle \frac{\partial}{\partial a} \frac{\partial}{\partial b} \ln L(D) \rangle &= - \sum_{B+} G \langle X^T N^{-1} \frac{\partial}{\partial a} (\frac{\partial}{\partial b} V^* V^T) N^{-1} X^* \rangle \\ &= \sum_{B+} \omega^2 G \sum_{km} \frac{S}{N_k N_m} \frac{\partial(D_k - D_m)}{\partial a} \frac{\partial(D_k - D_m)}{\partial b}. \end{aligned} \quad (10)$$

2.7 From equation (10) it follows that the FIM pertinent to the estimation of the vector D is

$$(FIM) = \sum_{B+} 2\omega^2 \frac{S^2}{1 + \sum_i \frac{S}{N_i}} [(\text{tr} N^{-1}) N_p^{-1} - N_p^{-1} \underline{1} \underline{1}^T N_p^{-1}]. \quad (11)$$

In equation (11), N_p^{-1} is the N^{-1} matrix with the first row and column partitioned away. Because of the assumed smoothness of all of the spectra relative to the frequency increment $\omega_0 = 2\pi/T$, the FIM can also be written as

$$(FIM) = \frac{T}{2\pi} \int_B \omega^2 \frac{S^2}{1 + \sum_i \frac{S}{N_i}} [(\text{tr} N^{-1}) N_p^{-1} - N_p^{-1} \underline{1} \underline{1}^T N_p^{-1}] d\omega. \quad (12)$$

Chapter 3

The Maximum Likelihood Estimate

3.1 It is well known that when the MLE is based on a large number of independent samples, it is consistent, asymptotically normal, and asymptotically efficient [3]. Since the observation time T is large compared to the process correlation times, there should be, in some sense, a large number of independent samples. The covariance matrix for the error in the MLE for D should be the CRMB, at least to first order.

3.2 The results that follow are independent of the true delay vector, and the equations for the likelihood function and MLE are considerably simplified if the true delay vector is assumed to be 0. The vector D of this chapter is the MLE and is therefore the measurement error. The steering vector corresponding to the error D is

$$V^T = (1, \exp\{j\omega d_2\}, \dots, \exp\{j\omega d_M\}). \quad (13)$$

The MLE vectors D and V satisfy

$$\begin{aligned} 0 &= \text{grad} \ln L(D) \\ &= \text{grad} \sum_{B+} G \sum_{in} \frac{X_i X_n^*}{N_i N_n} \exp\{j\omega(d_n - d_i)\} \\ &= \text{grad}(A + BD + \frac{1}{2} D^T CD + \text{HOT}), \end{aligned} \quad (14)$$

where by expanding $\exp\{j\omega(d_n - d_i)\}$ as a power series, the vector B is seen to be

$$B = \sum_{B+} j\omega G \underline{1}^T N^{-1} [X X_p^{*T} - X^* X_p^T] N^{-1}, \quad (15)$$

while the matrix C is determined by

$$\frac{1}{2} D^T C D = \sum_{B+} \frac{1}{2} (j\omega)^2 G \sum_{ik} \frac{X_i X_k^*}{N_i N_k} (d_k - d_i)^2 \quad (16)$$

In equation (15), X_p is the single frequency data vector with the first element partitioned away. The terms $X_i X_k^*$ in equation (16) are elements of the sample covariance matrix at a single frequency based on T seconds of data. These sample covariance elements do not converge, even if T is arbitrarily long [4]. However, since T is large compared to the process correlation times, the spectra are smooth enough so that the sample covariance can be averaged with samples from nearby frequencies to provide statistical convergence. The \sum_{B+} summation in equation (16) provides such an averaging of the $X_i X_k^*$. Thus, it is assumed that $X_i X_k^*$ can be replaced by $R_{ik} = \langle X_i X_k^* \rangle$ in equation (16), from which it follows that $C = \langle C \rangle$, or

$$C = - \sum_{B+} 2\omega^2 \frac{S^2}{1 + \sum_i \frac{S}{N_i}} [(\text{tr } N^{-1}) N_p^{-1} - N_p^{-1} \mathbf{1} \mathbf{1}^T N_p^{-1}] \quad (17)$$

= - (FIM).

3.3 From equation (15), it follows immediately that $\langle B \rangle = 0$, and not so immediately that

$$\langle B^T B^* \rangle = \sum_{B+} 2\omega^2 \frac{S^2}{1 + \sum_i \frac{S}{N_i}} [(\text{tr } N^{-1}) N_p^{-1} - N_p^{-1} \mathbf{1} \mathbf{1}^T N_p^{-1}] \quad (18)$$

= (FIM).

3.4 Neglecting the HOT, and assuming $C = \langle C \rangle$, the vector D is given by

$$D = - \langle C \rangle^{-1} B^T, \quad (19)$$

so that:

$$\langle D \rangle = \underline{0}, \tag{20}$$

and

$$\begin{aligned} \langle D^* D^T \rangle &= \langle C \rangle^{-1} \langle B^T B^* \rangle^* \langle C \rangle^{-1} \\ &= (\text{FIM})^{-1} \\ &= (\text{CRMB}) \end{aligned} \tag{21}$$

3.5 Thus the MLE is unbiased, and in the limit of large T achieves the CRMB. The MLE processor can be readily implemented as indicated in Figure 1. The MLE processor is just a steered and filtered beamformer followed by a square-law averager. The individual inputs are each steered and then filtered with filters whose frequency response is the inverse of the additive noise spectrum for that particular sensor. The beam sum is then formed and fed to a filter whose squared magnitude response is $|F(j\omega)|^2 = G(\omega)$. The MLE is determined as that set of steering delays that gives the maximum deflection of the output meter.

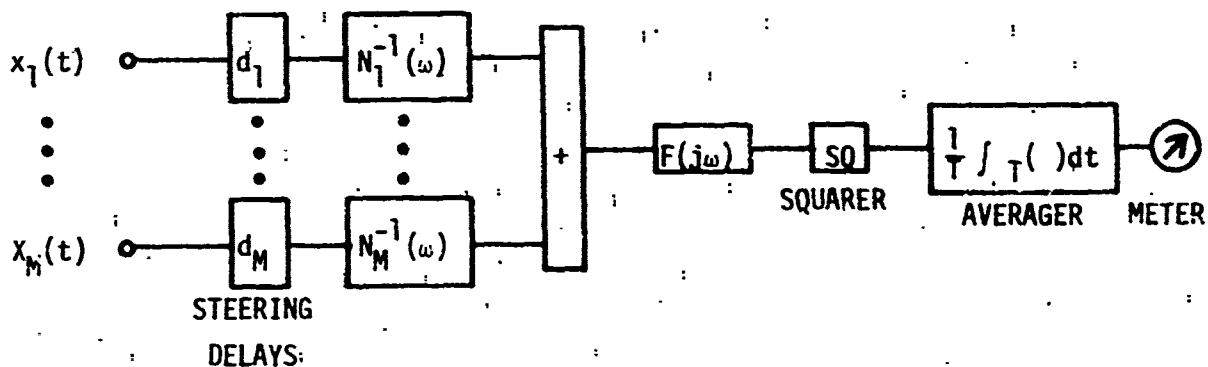


FIGURE 1: Beamformer Implementation of the MLE Processor

Chapter 4

Correlator Delay Measurement Systems

4.1 The following scheme can be used to estimate the unknown delay vector D . Let a system of correlators be used to form all the $M(M-1)/2$ correlograms corresponding to processing all of the M input wave forms taken two at a time. The individual correlators are assumed to have input filters for each channel, and the position of the correlogram peak is used as a signal delay estimate for that sensor pair. If each correlator is to provide an unbiased estimate of the corresponding signal delay, the input filters must have the same phase response, and hence can be taken to be identical filters. A typical correlator is shown in Figure 2. The steering delay is adjusted to give the maximum deflection of the meter, and this defines the delay estimate for the correlator.

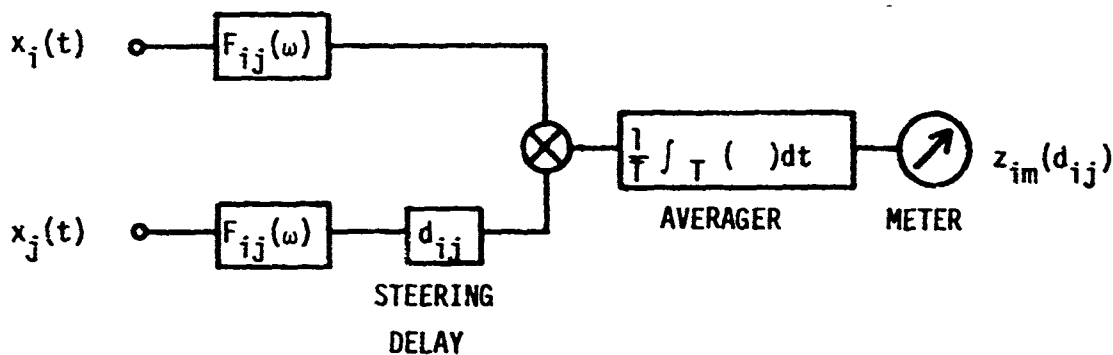


FIGURE 2: A Typical Multiplier Correlator

4.2 Let d_{ij} be the correlator estimate for the signal delay from the i -th to the j -th sensor, based on the correlation of the $X_i(t)$ and $X_j(t)$ time records. Let e_{ij} and F_{ij} be, respectively, the error in the estimate d_{ij} , and the filter used on the inputs to the correlator. Define the following scalars, vectors, and matrices:

$$\begin{aligned}
 G(ij;kl) &= [(S+N_i\delta_{ik})(S+N_j\delta_{jl}) - (S+N_i\delta_{il})(S+N_j\delta_{jk})] \\
 D_C &= (d_{12}, d_{13}, \dots, d_{1M}, d_{23}, \dots, d_{(M-1)M})^T \\
 E &= (e_{12}, e_{13}, \dots, e_{(M-1)M})^T \\
 F &= \text{diag} (|F_{12}|^2, |F_{13}|^2, \dots, |F_{(M-1)M}|^2) \\
 K &= \int_B \omega^2 SF \, d\omega \\
 &= \text{diag} (K_{12}, K_{13}, \dots, K_{(M-1)M}) \\
 G &= [G(ij;kl)]
 \end{aligned} \tag{22}$$

4.3 The square matrix G has for its elements the scalars $G(ij;kl)$ positioned according to the scheme determined by the order of the subscripts in EE^T , where ij is the row designation, and kl the column designation.

4.4 Using equations (22), the covariance matrix for the correlator scheme measurement error vector can be compactly written [2]:

$$\begin{aligned}
 P_E &= \langle EE^T \rangle \\
 &= \frac{2\pi}{T} K^{-1} \left(\int_B \omega^2 FGF \, d\omega \right) K^{-1}.
 \end{aligned} \tag{23}$$

4.5 The correlator delay measurement vector is related to the vector to be estimated, D , by the equation

$$D_C = AD + E, \tag{24}$$

where the matrix A, with its rows and columns labeled with the same sets of ordered subscripts used for the elements of D_c (for the rows) and D (for the columns), has for its element in the ij,k position

$$A(ij;k) = \delta_{jk} - \delta_{ik}. \quad (25)$$

4.6 Since $\langle E \rangle = \underline{0}$, the Gauss-Markov estimate [5] for the vector D based on the correlator measurements is

$$\hat{D} = [A^T P_E^{-1} A]^{-1} A^T D_c, \quad (26)$$

and the covariance matrix for the Gauss-Markov estimate is

$$\langle (\hat{D}-D)(\hat{D}-D)^T \rangle = [A^T P_E^{-1} A]^{-1}. \quad (27)$$

4.7 In [2] it is demonstrated that if $M = 2$ or $M = 3$, the Gauss-Markov estimate for D achieves the CRMB provided that the filters satisfy

$$|F_{ij}|^2 = \frac{S^2/N_i N_j}{1 + \sum_k \frac{S}{N_k}}. \quad (28)$$

It was conjectured in [2] that the choice of filters given by equation (28) was optimum for $M > 3$. The conjecture is in fact valid, and the Gauss-Markov estimate so obtained is in fact efficient, that is, achieves the CRMB. This is demonstrated in what follows.

4.8 From the definitions in equations (22), and with the filters defined by equation (28), the FIM of equation (11) is

$$(FIM) = \frac{T}{2\pi} A^T K A, \quad (29)$$

which suggests investigating the possibility that

$$A^T P_E^{-1} A \stackrel{?}{=} \frac{T}{2\pi} A^T K A. \quad (30)$$

With the filters given by equation (28), the matrix FGF can be written as

$$\begin{aligned}
 \text{FGF} = & \frac{S^2}{1 + \sum_i \frac{S}{N_i}} \text{diag} \left(\frac{1}{N_1 N_2}, \frac{1}{N_1 N_3}, \dots, \frac{1}{N_{M-1} N_M} \right) \\
 & - \sum_{1 \leq \alpha < \beta < \gamma \leq M} \frac{S^3 / N_\alpha N_\beta N_\gamma}{\left(1 + \sum_i \frac{S}{N_i}\right)^2} U_{\alpha\beta\gamma} U_{\alpha\beta\gamma}^T,
 \end{aligned} \tag{31}$$

where $U_{\alpha\beta\gamma}$ is a column vector whose rows are labeled with the same scheme as is used for the elements of D_C , and whose element in the ij row is

$$U_{\alpha\beta\gamma} (ij) = \delta_{\alpha i} \delta_{\beta j} - \delta_{\alpha i} \delta_{\gamma j} + \delta_{\beta i} \delta_{\gamma j}. \tag{32}$$

Then the matrix

$$\int_B \omega^2 \text{FGF} d\omega = K - \sum_{1 \leq \alpha < \beta < \gamma \leq M} H_{\alpha\beta\gamma} U_{\alpha\beta\gamma} U_{\alpha\beta\gamma}^T, \tag{33}$$

where

$$H_{\alpha\beta\gamma} = \int_B \omega^2 \frac{S^3 / N_\alpha N_\beta N_\gamma}{\left(1 + \sum_i \frac{S}{N_i}\right)^2} d\omega. \tag{34}$$

4.9 Recursively applying the matrix inverse lemma [6] to the right side of equation (33), the inverse can be written as

$$\left(\int_B \omega^2 \text{FGF} d\omega \right)^{-1} = K^{-1} + \sum_{1 \leq \alpha < \beta < \gamma \leq M} K^{-1} U_{\alpha\beta\gamma} M_{\alpha\beta\gamma} \tag{35}$$

where for the purposes of this paper it is not necessary to specify further the matrix $M_{\alpha\beta\gamma}$. From the relations defining the matrix A and the vector $U_{\alpha\beta\gamma}$, it follows that

$$A^T U_{\alpha\beta\gamma} = \underline{0}. \tag{36}$$

4.10 Thus, the inverse of the covariance matrix for the Gauss-Markov estimate for D, with the correlator inputs filtered according to equation (28), is

$$\begin{aligned}
A^T P_E^{-1} A &= \frac{T}{2\pi} A^T K \left(\int_B \omega^2 FGF d\omega \right)^{-1} KA \\
&= \frac{T}{2\pi} A^T KA \\
&= (\text{FIM}).
\end{aligned}
\tag{37}$$

4.11 Thus, the correlator system optimally filtered according to equation (28) provides an efficient estimate.

Chapter 5

Suboptimally Filtered Correlator Systems

5.1 For diverse reasons the decision not to use the optimal filters of equation (28) at the input to a correlator delay measurement may be made. It is then relevant to investigate the degree to which the delay estimate is degraded. The question can be answered in a simple way under the following hypothesis:

- a. The ratio S/N_i is the same at each sensor.
- b. Identical filters, F , are used on each input channel.
- c. The suboptimally filtered Gauss-Markov delay estimate is used.

Under these hypotheses, the matrix FGF becomes

$$\begin{aligned}
 FGF &= |F|^4 (N^2 + MNS) I \\
 &\quad - \sum_{1 \leq \alpha < \beta < \gamma \leq M} |F|^4 SN U_{\alpha\beta\gamma} U_{\alpha\beta\gamma}^T,
 \end{aligned}
 \tag{38}$$

where I is the identity matrix and $U_{\alpha\beta\gamma}$ is defined in the same way as in the preceding chapter. If the Gauss-Markov estimate is formed from the suboptimally filtered delay estimates, the inverse of the covariance matrix for the Gauss-Markov estimate is

$$\begin{aligned}
 A^T P_E^{-1} A &= \frac{T}{2\pi} A^T K \left(\int_B \omega^2 FGF d\omega \right)^{-1} KA \\
 &= \frac{T}{2\pi} \left(\int_B \omega^2 |F|^4 (N^2 + MNS) d\omega \right)^{-1} A^T A \\
 &= \frac{T}{2\pi} \frac{\left(\int_B \omega^2 S |F|^2 d\omega \right)^2}{\int_B \omega^2 |F|^4 (N^2 + MNS) d\omega} A^T A.
 \end{aligned}
 \tag{39}$$

The FIM for this case is

$$(\text{FIM}) = \frac{T}{2\pi} \left(\int_B \omega^2 \frac{S^2/N^2}{1 + M \frac{S}{N}} d\omega \right) A^T A. \quad (40)$$

5.2 Thus, the covariance matrices for the optimally and suboptimally filtered estimates differ by a constant factor, and it is easy to take account of the effects of suboptimally filtering the inputs. Equations (39) and (40) can also be used to determine the degradation of the delay estimate due to an imprecise knowledge of either $S(\omega)$ or $N(\omega)$.