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SCATTERING FROM AN ELASTIC SPHERICAL SHELL

William F. Fender

Naval Undersea Center
San Diego, California

October 1972

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Sensor and Information Technology Department

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13 ABSTRACT

A mathematical model was developed to predict the scattering of a plane acoustic wave from an elastic spherical shell with different fluid media inside and out. The model was developed from the more convenient standpoint of elastic theory rather than from the acoustic point of view Resulting equations were solved in terms of known functions. Matrix methods were used throughout. A numerical method is presented that takes advantage of some peculiarities of the resulting matrix and is appropriate to computer implementation. Some results of a computer program for the spherical shell are given. A few limiting cases of the exact solutions are compared with previously reported results.

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SUMMARY

PROBLEM

Develop a mathematical model to predict the scattering of a plane acoustic wave from an elastic spherical shell with different fluid media inside and out.

RESULTS

A mathematical model of the spherical shell was developed from the more convenient standpoint of elastic theory rather than from the acoustic point of view. The resulting equations were solved in terms of known functions. Matrix methods are used throughout. A numerical method is presented that takes advantage of some peculiarities of the resulting matrix and is appropriate to computer implementation. Some results of a computer program for the spherical shell are given. A few limiting cases of the exact solutions are compared with previously reported results.

CONTENTS

INTRODUCTION

In this paper, a mathematical model is developed to predict the scattering of a plane acoustic wave from an elastic spherical shell with different fluid media inside and out. The model is developed from the more convenient standpoint of elastic theory rather than from the acoustic point of view. The resulting equations are solved in terms of known functions. Matrix methods are used throughout. A numerical method is presented that is appropriate to computer implementation and takes advantage of some peculiarities of the resulting matrix. Some results of a computer program for a spherical shell are given. A few limiting cases of the exact solutions are compared with previously reported results.

The paper consists of two sections in addition to the Introduction. The first, A Summary of Elastodynamics, summarizes the essential elements of elastodynamics needed to provide a unified treatment of the scattering problems considered in this paper. The second, Scattering from a Spherical Shell, develops the desired mathematical model.

SUMMARY OF LINEAR ELASTODYNAMICS

Problems in elastodynamics are similar to boundary–initial value problems in other areas of mathematical physics and are characterized by a fundamental system of field equations with corresponding initial and boundary conditions. Given a prescribed system of forces and displacements over the surface of an elastic body, we may predict the deformation of the body using a solution of the appropriate elastic problem. Solutions to the elastic problems considered in this paper are constructed using scalar and vector potentials. The basic laws of infinitesimal linear elasticity and the relevant elements of potential theory are presented here for reference.

EQUATIONS OF MOTION

The law of motion in linear elasticity, usually referred to as Cauchy's law, is given by

$$\mathrm{div}T = \rho \ddot{\underset{\sim}{u}} , \tag{1}$$

where T is the stress tensor, $\underset{\sim}{u}$ is the displacement vector, and body forces have been neglected. In this paper we will assume infinitesimal displacement theory is valid so that the strain tensor E is related to the displacement by

$$E = \frac{1}{2}(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T) . \tag{2}$$

In addition, we will restrict our attention to homogeneous and isotropic materials where the constitutive relation is given by

$$T = 2\mu E + \lambda(\mathrm{tr}E)\, I , \tag{3}$$

where λ and μ are the Lame constants and I the identity tensor. Using a few standard identities, equations 1 to 3 may be manipulated to obtain Navier's equation of motion:

$$\mu\nabla^2\underset{\sim}{u} + (\lambda + \mu)\nabla(\nabla\cdot\underset{\sim}{u}) = \rho\ddot{\underset{\sim}{u}} \quad (4)$$

This equation may also be written

$$C_L{}^2\nabla(\nabla\cdot\underset{\sim}{u}) - C_T{}^2 \text{ curl curl } \underset{\sim}{u} = \ddot{\underset{\sim}{u}} , \quad (5)$$

where the longitudinal and transverse elastic wave velocities are given, respectively, by

$$C_L{}^2 = \frac{\lambda + 2\mu}{\rho} , \; C_T{}^2 = \frac{\mu}{\rho} . \quad (6)$$

INITIAL AND BOUNDARY DATA

In the mixed problem of elastodynamics we must specify the following initial and boundary data to insure a unique solution:

Initial conditions

$$\underset{\sim}{u}(\cdot,0) = \underset{\sim}{u}_0 \text{ and } \dot{\underset{\sim}{u}}(\cdot,0) = \underset{\sim}{v}_0 \text{ on B} \quad (7)$$

Displacement boundary condition

$$\underset{\sim}{u} = \hat{\underset{\sim}{u}} \text{ on } S_1 \times (o,t) \quad (8)$$

Traction condition

$$\underset{\sim}{t} = \hat{\underset{\sim}{t}} \text{ on } S_2 \times (o,t) \quad (9)$$

where $\underset{\sim}{u}_0, \underset{\sim}{v}_0, \hat{\underset{\sim}{u}}, \hat{\underset{\sim}{t}}$ are specified data, and B is an open set in Euclidean space & which is the interior of a closed and bounded region $\bar{B}$ (the closure of B) whose boundary ∂B is the union of surfaces $\partial B = S_1 \cup S_2$.

POTENTIAL THEORY

Next, we will briefly outline the basic elements of potential theory.

Theorem A

Let ϕ be a scalar field that is continuous on $\bar{B}$ and class C^N on B. Then there exists a scalar field γ such that

$$\phi = \nabla^2\gamma . \quad (10)$$

Proof: See Kellogg (reference 1)

The above theorem is easily extended to vector fields by applying the result to each component.

Theorem B – Helmholtz Theorem (reference 2)

Let $\underset{\sim}{u}$ be a vector field which is continuous on $\bar{B}$ and class C^N on B. Then there exists a scalar field ϕ and a vector field $\underset{\sim}{\psi}$, both of class C^N, such that

$$\underset{\sim}{u} = \nabla\phi + \nabla \times \underset{\sim}{\psi} \text{ and div } \underset{\sim}{\psi} = 0 . \qquad (11)$$

Proof: From Theorem A, there exists a vector field $\underset{\sim}{v}$ on B such that

$$\underset{\sim}{u} = \nabla^2 \underset{\sim}{v} . \qquad (12)$$

Using the standard identity curl curl $\underset{\sim}{v} = \nabla$div $\underset{\sim}{v} - \nabla^2 \underset{\sim}{v}$, we find

$$\underset{\sim}{u} = \nabla \text{div } \underset{\sim}{v} - \text{curl curl } \underset{\sim}{v} .$$

Therefore, if we let

$$\phi = \text{div } \underset{\sim}{v} \text{ and } \underset{\sim}{\psi} = -\text{ curl } \underset{\sim}{v} , \qquad (13)$$

the desired conclusion follows.

Theorem C – Lamé Solution (reference 2)

Let $\underset{\sim}{u} = \nabla\phi + \nabla \times \underset{\sim}{\psi}$, where ϕ and $\underset{\sim}{\psi}$ are class C^3 fields on B that satisfy

$$\nabla^2 \phi - \frac{1}{C_L^2}\ddot{\phi} = 0 \qquad (14)$$

$$\nabla^2 \underset{\sim}{\psi} - \frac{1}{C_T^2}\ddot{\underset{\sim}{\psi}} = 0 . \qquad (15)$$

Then, $\underset{\sim}{u}$ is an elastic motion which satisfies Navier's equation (equation 5).

Proof:

$$C_L^2 \nabla \text{div } \underset{\sim}{u} - C_T^2 \text{ curl curl } \underset{\sim}{u} - \ddot{\underset{\sim}{u}} = \nabla \left[C_L^2 \nabla^2 \phi - \ddot{\phi}\right] + \text{curl}$$

$$\left[- C_T^2 \text{ curl curl } \underset{\sim}{\psi} - \ddot{\underset{\sim}{\psi}}\right] = \nabla \left[C_L^2 \nabla^2 \phi - \ddot{\phi}\right] + \text{curl} \left[C_T^2 \nabla^2 \underset{\sim}{\psi} - \ddot{\underset{\sim}{\psi}}\right] = 0$$

where curl(curl curl $\underset{\sim}{\psi}$) = – curl($\nabla^2 \underset{\sim}{\psi}$) by the previous identity.

The Lamé solution can be shown to be complete, that is, any solution to Navier's equation can be written as a scalar and vector potential. The completeness theorem and proof are discussed in Sternberg (reference 2).

Notice that the potentials ϕ and ψ are not unique. For example, let

$$\phi' = \phi + \text{constant} \qquad \underset{\sim}{\psi}' = \underset{\sim}{\psi} + \underset{\sim}{\nabla}\gamma .$$

The displacement $\underset{\sim}{u}$ is unchanged using either the primed or unprimed potentials.

FLUID MEDIA

In this paper a fluid will be treated as an elastic material which supports no transverse waves, $C_T = \mu = 0$. In this case Navier's equation becomes

$$C^2 \nabla(\nabla \cdot \underset{\sim}{u}) = \hat{\underline{u}} \qquad C^2 = \frac{\lambda}{\rho} . \tag{16}$$

Using the Lamé solution (Theorem C, $\psi = 0$)

$$\underset{\sim}{u} = \nabla \phi, \text{ whereby equation 14,} \tag{17}$$

$$\nabla^2 \phi - \frac{1}{C^2} \ddot{\phi} = 0 . \tag{18}$$

The stress-displacement relation may be written

$$T = \rho C^2 (\text{div } \underset{\sim}{u}) I = \rho \ddot{\phi} \; I . \tag{19}$$

In many cases the time-dependent solution can be written as a superposition of sinusoidal solutions, for example using Fourier Transforms. If we assume the solution is smooth enough that we can interchange the differentiation and transform operations, we may construct the time-dependent solution frequency by frequency. Throughout this paper we will assume the functions to be smooth and their time dependence to be $e^{-i\omega t}$. For convenience we will summarize the basic equations discussed in this chapter removing the time dependence.

SUMMARY OF SOLID MATERIAL EQUATIONS

Na:.:r's equation

$$\frac{1}{k_L^2} \nabla^2 \underset{\sim}{u} - \frac{1}{k_T^2} \text{curl curl } \underset{\sim}{u} = -\underset{\sim}{u} \tag{20}$$

Lamé solution

$$\underset{\sim}{u} = \nabla \phi + \nabla \times \underset{\sim}{\psi} \tag{21}$$

$$\nabla^2 \phi + k_L^2 \phi = 0 , \qquad k_L^2 = \frac{\omega^2}{C_L^2} \tag{22}$$

$$\nabla^2 \underset{\sim}{\psi} + k_T^2 \underset{\sim}{\psi} = 0, \qquad k_T^2 = \frac{\omega^2}{C_T^2} \tag{23}$$

Stress displacement

$$T = \mu \left[\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T \right] + \lambda (\text{div } \underset{\sim}{u}) I \tag{24}$$

SUMMARY OF FLUID MATERIAL EQUATIONS

Navier's equation

$$\nabla(\nabla \cdot \underset{\sim}{u}) = -k^2 \underset{\sim}{u} \qquad k^2 = \frac{\omega^2}{C^2} \tag{25}$$

Lamé solution

$$\underset{\sim}{u} = \nabla \phi \tag{26}$$

$$\nabla^2 \phi + k^2 \phi = 0 \tag{27}$$

Stress displacement

$$T = -\rho\, \omega^2 \phi\, I \tag{28}$$

SCATTERING FROM A SPHERICAL SHELL

INTRODUCTION

A sphere is one of the few finite geometrical shapes in which the equations of linear elasticity can be solved analytically in terms of known functions. Such a solution is useful not only for direct sonar mathematical modeling applications, such as spherical baffles, domes and acoustic lenses; but serves as a useful check on various approximate techniques. For example, a comparison of resonant frequencies is made with various thin-shell theories in reference 3. Because the solution is exact, experimental verifications may be obtained for a large range of parameters, the limits being upon the experimental apparatus.

In this section, a solution suitable for digital computer implementation is developed for plane wave scattering from an elastic spherical shell with different fluid media inside and out. The problem for identical fluids inside and outside has been done by Goodman and Stern (reference 4). In addition, numerical implementation details and a few limiting cases are discussed.

THEORETICAL DEVELOPMENT

Consider a homogeneous plane wave impinging upon an elastic spherical shell of outer radius a, inner radius b and thickness δ. Let the incident wave come in along the polar axis (with no loss in generality) as shown in figure 1. The inside of the shell contains a fluid of density ρ_I and sound velocity C_I. In general, the outer fluid will be different, and is described by parameters ρ_O, C_O. The elastic shell is a homogeneous, isotropic material of density ρ and Lamé constants λ, μ. The longitudinal and transverse elastic wave velocities are

$$C_L^2 = \frac{\lambda + 2\mu}{\rho} \text{ and } C_T^2 = \frac{\mu}{\rho}\text{, respectively.}$$

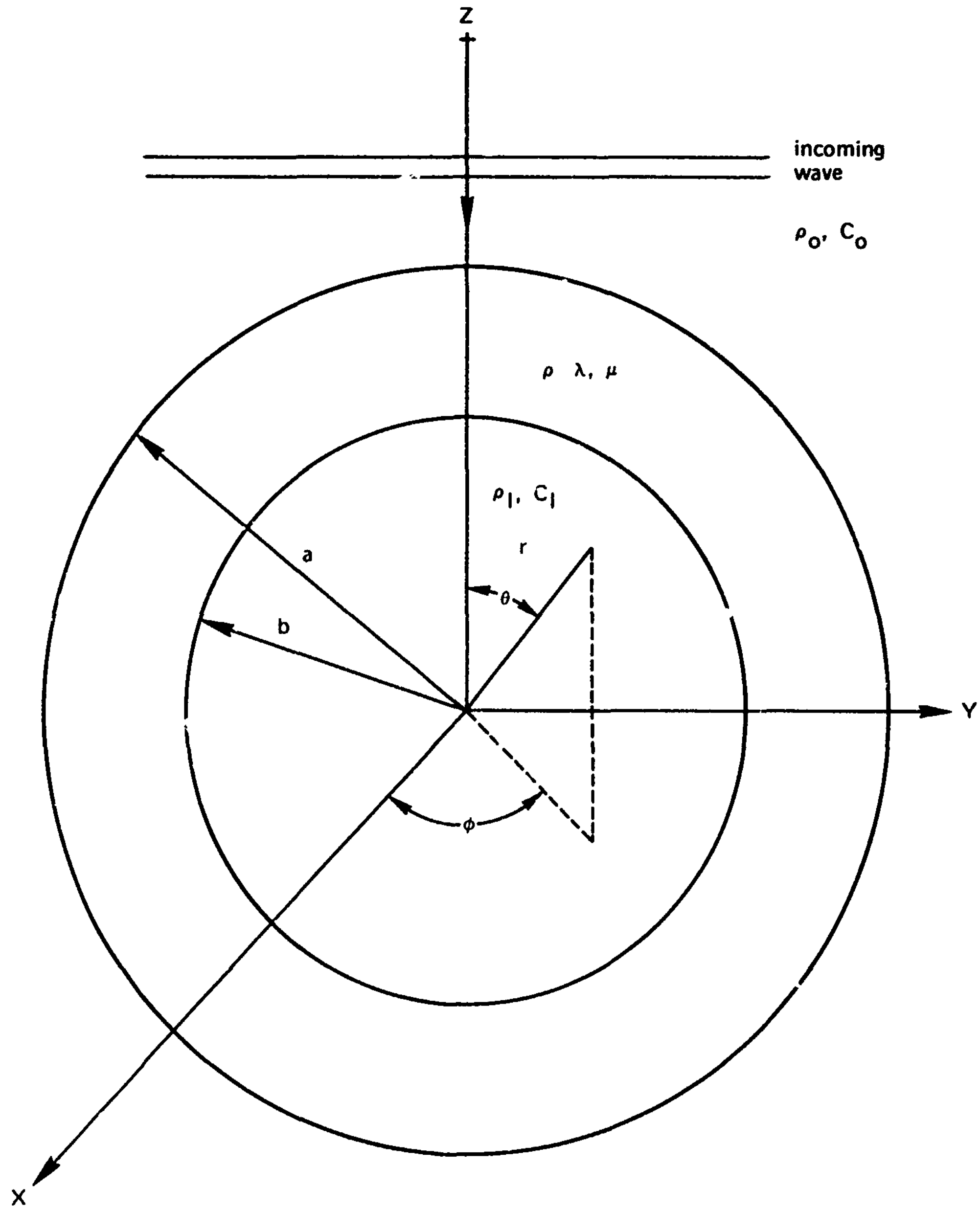

Figure 1. Geometry of an elastic spherical shell.

By symmetry of the incoming wave boundary condition, we need to consider only the axisymmetric problem, i.e., no variables will depend upon the ϕ coordinate of our spherical coordinate system. We will further assume that there is no motion of the elastic shell in the ϕ direction

$$u_\phi = 0 \ . \tag{29}$$

Components of vectors and tensors in this chapter refer to the physical components relative to the natural basis associated with the spherical coordinate system.

The displacement vector in the shell may be represented in terms of potentials as

$$u = \nabla\phi + \nabla \times \underset{\sim}{\psi} \ . \tag{30}$$

Next, we will demonstrate that one component of the vector potential $\underset{\sim}{\psi}$ is sufficient to construct a solution to the axisymmetric sphere problem. Since the potential $\underset{\sim}{\psi}$ is not unique, we can choose many possible potentials that will give a solution to the elastic equation. Recall that from Theorem A in the Summary of Linear Elastodynamics, there exists a vector $\underset{\sim}{v}$ such that

$$\underset{\sim}{u} = \nabla^2 \underset{\sim}{v} \ . \tag{31}$$

We can see that a possible vector v that will satisfy the symmetry assumption $u_\phi = 0$ is chosen by letting $v_\phi = 0$. That is, writing out the Laplacian in spherical coordinates (reference 5. p. 116),

$$u_\phi = (\nabla^2 \underset{\sim}{v})_\phi = \nabla^2 v_\phi + \frac{v_\phi}{r^2 \sin^2\theta} \ ,$$

we can see that $v_\phi = 0$ implies $u_\phi = 0$.

Now, referring to the proof of Theorem B, equation 13 says

$$\underset{\sim}{\psi} = - \operatorname{curl} \underset{\sim}{v} \ .$$

In spherical coordinates, with $v_\phi = 0$,

$$\underset{\sim}{\psi} = 0\, \underset{\sim}{e}_r + 0\, \underset{\sim}{e}_\theta + \frac{\underset{\sim}{e}_\phi}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right] \ . \tag{32}$$

Therefore, a sufficient choice for the vector potential consistent with the symmetry of the problem is $\psi_r = \psi_\theta = 0$, retaining only ψ_ϕ. So

$$\underset{\sim}{\psi} = \psi_\phi \underset{\sim}{e}_\phi \ .$$

Equations 22 and 23 may be expanded in spherical coordinates to give

$$\begin{aligned}
&\left(r^2\phi_{,r}\right)_{,r} + \frac{1}{\sin\theta}(\sin\theta\, \phi_{,\theta})_{,\theta} + k_L^2 r^2 \phi = 0 \\
&\left(r^2 \psi_{\phi,r}\right)_{,r} + \frac{1}{\sin\theta}(\sin\theta\, \psi_{\phi,\theta})_{,\theta} + \left(k_T^2 r^2 - \frac{1}{\sin^2\theta}\right)\psi_\phi = 0 \ .
\end{aligned} \tag{33}$$

Solutions to equation 33 may be obtained using separation of variables. The radial equation has a solution in terms of spherical Bessel functions $j_\ell(\cdot)$, $n_\ell(\cdot)$, and the angular equation has a solution in terms of associated Legendre polynomials of zero and first order, $P_\ell(\cdot)$ and $P_\ell^1(\cdot)$, respectively.

$$\phi = \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\,[j_\ell(k_L r) A_\ell + n_\ell(k_L r) B_\ell]$$

$$\psi_\phi = \sum_{\ell=0}^{\infty} \xi_\ell P_\ell^1(\cos\theta)\,[j_\ell(k_T r) C_\ell + n_\ell(k_T r) D_\ell] \tag{34}$$

where A_ℓ, B_ℓ, C_ℓ, D_ℓ are arbitrary constants to be determined from the boundary conditions and $\xi_\ell = (-i)^\ell (2\ell + 1)$ is included for later convenience. In addition, for ease of writing let

$$Q_\ell(k_L r) = \xi_\ell [j_\ell(k_L r) A_\ell + n_\ell(k_L r) B_\ell]$$

$$R_\ell(k_T r) = \xi_\ell [j_\ell(k_T r) C_\ell + n_\ell(k_T r) D_\ell] \, . \tag{35}$$

Then,

$$\phi = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\, Q_\ell(k_L r)$$

$$\psi_\phi = \sum_{\ell=0}^{\infty} P_\ell^1(\cos\theta)\, R_\ell(k_T r) \, . \tag{36}$$

Before going any further, we need some useful relations for derivatives of Legendre Polynomials and spherical Bessel Functions.

$$P_{\ell,\theta}(\cos\theta) = P_\ell^1(\cos\theta) \tag{37}$$

$$P_{\ell,\theta\theta} + \cot\theta\, P_{\ell,\theta} = -\ell(\ell+1) P_\ell \tag{38}$$

We follow the sign convention of Magnus and Oberhettinger (reference 6) for Legendre polynomials. The second relation follows directly from Legendre's differential equation.

$$j_{\ell,r}(kr) = k\, j_\ell'(kr) \tag{39}$$

$$r^2 j_{\ell,rr}(kr) + 2r j_{\ell,r}(kr) + [k^2 r^2 - \ell(\ell+1)]\, j_\ell(kr) = 0 \tag{40}$$

The first equation is a standard identity and the second is Bessel's equation.

Writing out equation 30 in spherical coordinates, we find that

$$u_r = \phi_{,r} + \frac{1}{r}\psi_{\phi,\theta} + \frac{\cot\theta}{r}\psi_\phi$$

$$u_\theta = \frac{1}{r}\phi_{,\theta} - \psi_{\phi,r} - \frac{1}{r}\psi_\phi \, . \tag{41}$$

Using the solutions shown in equation 34, we find that

$$u_r = \sum_{\ell=0}^{\infty} P_\ell Q_{\ell,r} + \frac{P_{\ell,\theta}}{r} R_\ell + \frac{\cot\theta}{r} P_\ell^1 R_\ell \, .$$

Using equations 37 and 38, we obtain the following expression:

$$u_r = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left[Q_{\ell,r}(k_L r) - \frac{\ell(\ell+1)}{r} R_\ell(k_T r)\right] .$$

For the radial functions let the prime indicate differentiation with respect to the entire argument

$$\{\}_{,r}(kr) = \frac{dx}{dr}\frac{d}{dx}\{\}(x) = k\{\}'(x)$$

$$u_r = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left[k_L Q_\ell'(k_L r) - \frac{\ell(\ell+1)}{r} R_\ell(k_T r)\right] . \tag{42}$$

Similarly,

$$u_\theta = \sum_{\ell=0}^{\infty} P_\ell^1(\cos\theta)\left[\frac{Q_\ell(k_L r)}{r} - k_T R_\ell'(k_T r) - \frac{1}{r} R_\ell(k_T r)\right] . \tag{43}$$

Expanding the stress-displacement relation, equation 24, in spherical coordinates, we find that

$$T^{rr} = (2\mu + \lambda)\, u_{r,r} + \lambda\left[\frac{2}{r} u_r + \frac{1}{r} u_{\theta,\theta} + \frac{\cot\theta}{r} u_\theta\right] \tag{44}$$

$$T^{r\theta} = \mu\left[\frac{1}{r} u_{r,\theta} + u_{\theta,r} - \frac{u_\theta}{r}\right] . \tag{45}$$

Substituting equations 42 and 43 into equation 44 and employing some algebra, one obtains the following expression:

$$T^{rr} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left[k_L^2(2\mu Q_\ell'' - \lambda Q_\ell) - \frac{2\mu\ell(\ell+1)}{r}\left(k_T R_\ell' - \frac{R_\ell}{r}\right)\right] . \tag{46}$$

Similarly, equation 45 may be written

$$T^{r\theta} = \mu \sum_{\ell=0}^{\infty} P_\ell^{\,1}(\cos\theta)\left\{ \frac{2}{r}\left[k_L Q'_\ell - \frac{Q_\ell}{r}\right] - \left[k_T^{\,2} R''_\ell - \frac{R_\ell}{r^2}\left(2-\ell(\ell+1)\right)\right]\right\} . \tag{47}$$

In the outer fluid media the incident plane wave may be written as (reference 5, p. 1466)

$$\phi_{inc} = \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\, j_\ell(k_o r) . \tag{48}$$

The scattered wave potential is a solution to equation 27 for outgoing waves:

$$\phi_s = \sum_{\ell=0}^{\infty} \xi_\ell F_\ell^{\,o} P_\ell(\cos\theta)\, h_\ell(k_o r) , \tag{49}$$

where

$$h_\ell(\cdot) = j_\ell(\cdot) + i\, n_\ell(\cdot) . \tag{50}$$

The total outside potential is

$$\phi_o = \phi_{inc} + \phi_s = \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\left[j_\ell(k_o r) + F_\ell^{\,o} h_\ell(k_o r)\right] . \tag{51}$$

Differentiating, we obtain the outside displacement:

$$u_o^{\,r} = \frac{\partial \phi_o}{\partial r} = \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\left[k_o j'_\ell(k_o r) + F_\ell^{\,o} k_o h'_\ell(k_o r)\right] . \tag{52}$$

For the stress, equation 28 gives

$$T_o^{rr} = -\rho_o \omega^2 \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\left[j_\ell(k_o r) + F_\ell^{\,o} h_\ell(k_o r)\right] . \tag{53}$$

With the requirement that the inside potential be finite at the origin, we can write

$$\phi_I = \sum_{\ell=0}^{\infty} \xi_\ell F_\ell^{\,I} P_\ell(\cos\theta)\, j_\ell(k_I r) . \tag{54}$$

The displacement is

$$u_I^r = \frac{\partial \phi_I}{\partial r} = \sum_{\ell=0}^{\infty} \xi_\ell F_\ell^I P_\ell(\cos\theta) k_I j'_\ell(k_I r) . \tag{55}$$

and the normal stress

$$T_I^{rr} = -\rho_I \omega^2 \sum_{\ell=0}^{\infty} \xi_\ell F_\ell^I P_\ell(\cos\theta) j_\ell(k_I r) . \tag{56}$$

At the boundaries of the shell, we require the normal displacement and stress to be continuous and the tangential stress to vanish. Therefore we must satisfy the following conditions: At the outer radius, r = a.

$$T^{rr}(r{=}a) = T_O^{rr}(r{=}a) \tag{57}$$

$$T^{r\theta}(r{=}a) = 0 \tag{58}$$

$$u^r(r{=}a) = u_O^r(r{=}a) ; \tag{59}$$

at the inner radius, r = b.

$$T^{rr}(r{=}b) = T_I^{rr}(r{=}b) \tag{60}$$

$$T^{r\theta}(r{=}b) = 0 \tag{61}$$

$$u^r(r{=}b) = u_I^r(r{=}b) . \tag{62}$$

For algebraic convenience let

$$\begin{aligned} x &= k_O a \qquad & x_L &= k_L a \qquad & x_T &= k_T a \\ y &= k_I b \qquad & y_L &= k_L b \qquad & y_T &= k_T b . \end{aligned} \tag{63}$$

From equations 46, 53, and 57 we obtain the following expression for normal stress continuity at the outer radius, where r = a:

$$-\rho_O \omega^2 \sum_{\ell=0}^{\infty} \xi_\ell P_\ell(\cos\theta)\left[j_\ell(x) + F_\ell^O h_\ell(x)\right] =$$

$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left\{ k_L^2 \left[2\mu Q''_\ell(x_L) - \lambda Q_\ell(x_L)\right] - \frac{2\mu\ell(\ell+1)}{a}\left[k_T R'_\ell(x_T) - \frac{R_\ell(x_T)}{a}\right]\right\}$$

Equating term by term, we find that

$$-\rho_o\omega^2\xi_\ell\left[j_\ell(x)+F_\ell^o h_\ell(x)\right] = k_L^2\left[2\mu Q_\ell''(x_L)-\lambda Q_\ell(x_L)\right]$$

$$-\frac{2\mu\ell(\ell+1)}{a}\left[k_T R_\ell'(x_T)-\frac{R_\ell(x_T)}{a}\right].$$

Dividing through by $\rho\omega^2$ and using the relations

$$\frac{\mu}{a^2}=\frac{\rho\omega^2}{x_T^2},\; k_L^2=\frac{\rho\omega^2}{2\mu+\lambda},\text{ we find}$$

$$-\frac{\rho_o}{\rho}\xi_\ell\left[j_\ell(x)+F_\ell^o h_\ell(x)\right] = \frac{1}{\lambda+2\mu}\left[2\mu Q_\ell''(x_L)-\lambda Q_\ell(x_L)\right]$$

$$-\frac{2\ell(\ell+1)}{x_T^2}\left[x_T R_\ell'(x_T)-R_\ell(x_T)\right].$$

Using the definition of Q_ℓ and R_ℓ, given by equation 35, we obtain the following expression:

$$\left[\frac{\rho_o}{\rho}h_\ell(x)\right]F_\ell^o+\frac{1}{\lambda+2\mu}\left[2\mu j_\ell''(x_L)-\lambda j_\ell(x_L)\right]A_\ell$$

$$+\frac{1}{\lambda+2\mu}\left[2\mu n_\ell''(x_L)-\lambda n_\ell(x_L)\right]B_\ell-\frac{2\ell(\ell+1)}{x_T^2}\left[x_T j_\ell'(x_T)-j_\ell(x_T)\right]C_\ell$$

$$-\frac{2\ell(\ell+1)}{x_T^2}\left[x_T n_\ell'(x_T)-n_\ell(x_T)\right] = -\frac{\rho_o}{\rho}j_\ell(x) \qquad (64)$$

Define the following matrix coefficients

$$S_{11}=\frac{\rho_o}{\rho}h_\ell(x)$$

$$S_{12}=0$$

$$S_{13}=\frac{1}{\lambda+2\mu}\left[2\mu j_\ell''(x_L)-\lambda j_\ell(x_L)\right]$$

$$S_{14}=\frac{1}{\lambda+2\mu}\left[2\mu n_\ell''(x_L)-\lambda n_\ell(x_L)\right]$$

$$S_{15}=\frac{-2\ell(\ell+1)}{x_T^2}\left[x_T j_\ell'(x_T)-j_\ell(x_T)\right]$$

$$S_{16} = \frac{-2\ell(\ell+1)}{x_T^2}[x_T n_\ell'(x_T) - n_\ell(x_T)]$$

$$\beta_1 = -\frac{\rho_o}{\rho} j_\ell(x) .$$

Continuity of normal stress at the inner boundary, where r = b, is similar; this time we use equation 56 for the fluid stress. By inspection we obtain

$$\frac{\rho_I}{\rho} j_\ell(y) F_\ell^I + \frac{1}{\lambda+2\mu}[2\mu j_\ell''(y_L) - \lambda j_\ell(y_L)] A_\ell$$

$$+ \frac{1}{\lambda+2\mu}[2\mu n_\ell''(y_L) - \lambda n_\ell(y_L)] B_\ell - \frac{2\ell(\ell+1)}{y_T^2}[y_T j_\ell'(y_T) - j_\ell(y_T)] C_\ell$$

$$- \frac{2\ell(\ell+1)}{y_T^2}[y_T n_\ell'(y_T) - n_\ell(y_T)] D_\ell = 0 . \qquad (65)$$

Define the matrix coefficients

$$S_{51} = 0$$

$$S_{52} = \frac{\rho_I}{\rho} j_\ell(y)$$

$$S_{53} = \frac{1}{\lambda+2\mu}[2\mu j_\ell''(y_L) - \lambda j_\ell(y_L)]$$

$$S_{54} = \frac{1}{\lambda+2\mu}[2\mu n_\ell''(y_L) - \lambda n_\ell(y_L)]$$

$$S_{55} = \frac{-2\ell(\ell+1)}{y_T^2}[y_T j_\ell'(y_T) - j_\ell(y_T)]$$

$$S_{56} = -\frac{2\ell(\ell+1)}{y_T^2}[y_T n_\ell'(y_T) - n_\ell(y_T)] .$$

From equations 47 and 58 we derive the following expression for the tangential stress at the outer radius, where r = a:

$$\frac{2}{a}\left[k_L Q_\ell'(x_L) - \frac{Q_\ell(x_L)}{a}\right] - \left[k_T^2 R_\ell''(x_T) - \frac{R_\ell(x_T)}{a^2}(2 - \ell(\ell+1)\right] = 0 .$$

Using the expressions for Q_ℓ and R_ℓ, we find

$$2\,[x_L j_\ell'(x_L) - j_\ell(x_L)]\,A_\ell + 2\,[x_L n_\ell'(x_L) - n_\ell(x_L)]\,B_\ell$$
$$-\left[x_T^2 j_\ell''(x_T) + (\ell+2)(\ell-1)\,j_\ell(x_T)\right] C_\ell$$
$$-\left[x_T^2 n_\ell''(x_T) + (\ell+2)(\ell-1)\,n_\ell(x_T)\right] D_\ell = 0\,. \qquad (66)$$

The matrix elements are

$$S_{31} = S_{32} = 0$$
$$S_{33} = 2\,[x_L j_\ell'(x_L) - j_\ell(x_L)]$$
$$S_{34} = 2\,[x_L n_\ell'(x_L) - n_L(x_L)]$$
$$S_{35} = -\left[x_T^2 j_\ell''(x_T) + (\ell+2)(\ell-1)\,j_\ell(x_T)\right]$$
$$S_{36} = -\left[x_T^2 n_\ell''(x_T) + (\ell+2)(\ell-1)\,n_\ell(x_T)\right].$$

Similarly, at the inner boundary $r = b$, and

$$2\,[y_L j'(y_L) - j_\ell(y_L)]\,A_\ell + 2\,[y_L n_\ell'(y_L) - n_\ell(y_L)]\,B_\ell$$
$$-\left[y_T^2 j_\ell''(y_T) + (\ell+2)(\ell-1)\,j_\ell(y_T)\right] C_\ell$$
$$-\left[y_T^2 n_\ell''(y_T) + (\ell+2)(\ell-1)\,n_\ell(y_T)\right] D_\ell = 0\,. \qquad (67)$$

The matrix coefficients are

$$S_{41} = S_{42} = 0$$
$$S_{43} = 2\,[y_L j_\ell'(y_L) - j_\ell(y_L)]$$
$$S_{44} = 2\,[y_L n_\ell'(y_L) - n_\ell(y_L)]$$
$$S_{45} = -\left[y_T^2 j_\ell''(y_T) + (\ell+2)(\ell-1)\,j_\ell(y_T)\right]$$
$$S_{46} = -\left[y_T^2 n_\ell''(y_T) + (\ell+2)(\ell-1)\,n_\ell(y_T)\right].$$

Using equations 42, 52 and 59 we obtain the following expression for the continuity of normal displacement:

$$\xi_\ell \left[k_o j_\ell'(x) + F_\ell^o k_o h_\ell'(x)\right] = k_\ell Q_\ell'(x_L) - \frac{\ell(\ell+1)}{a} R_\ell(x_T).$$

Multiplying through by a, we obtain

$$\xi_\ell \left[x j_o'(x) + F_\ell^o x h'(x)\right] = x_L Q_\ell'(x_L) - \ell(\ell+1) R_\ell(x_T).$$

Expanding, we obtain

$$- x h_\ell'(x) F_\ell^o + x_L j_\ell'(x_L) A_\ell + x n_\ell'(x_L) B_\ell$$

$$- \ell(\ell+1) [j_\ell(x_T) C_\ell + n_\ell(x_T) D_\ell] = x j_\ell'(x). \quad (68)$$

Define the matrix elements

$$S_{21} = - x h_\ell'(x)$$

$$S_{22} = 0$$

$$S_{23} = x_L j_\ell'(x_L)$$

$$S_{24} = x_L n_\ell'(x_L)$$

$$S_{25} = - \ell(\ell+1) j_\ell(x_T)$$

$$S_{26} = - \ell(\ell+1) n_\ell(x_T)$$

$$\beta_2 = x j_\ell'(x).$$

Similarly, at the inner boundary r = b, and

$$- y j_\ell'(y) F_\ell^I + y_L j_\ell'(y_L) A_\ell + y_L n_\ell'(y_L) B_\ell$$

$$- \ell(\ell+1) [j_\ell'(y_T) C_\ell + n_\ell(y_T) D_\ell] = 0. \quad (69)$$

Define the matrix coefficients

$$S_{61} = 0$$

$$S_{62} = - y j_\ell'(y)$$

$$S_{63} = y_L j_\ell'(y_I)$$

$$S_{64} = y_L n_\ell'(y_L)$$

$$S_{65} = - \ell(\ell+1) j_\ell(y_T)$$

$S_{66} = -\ell(\ell+1)\, n_\ell(y_T)$.

From the boundary conditions, we may write the following matrix relation for the unknown constants $A_\ell, B_\ell, C_\ell, D_\ell, F_\ell^{O}, F_\ell^{I}$. For $\ell \geq 1$,

$$\begin{bmatrix} S_{11} & 0 & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & 0 & S_{23} & S_{24} & S_{25} & S_{26} \\ 0 & 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ 0 & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} F_\ell^{O} \\ F_\ell^{I} \\ A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (70)$$

For $\ell = 0$, we have a special case since the associated Legendre polynomials $P_\ell^{1}(\cos\theta)$ which appear in the tangential stress must vanish. Therefore, the matrix becomes

$$\begin{bmatrix} S_{11} & 0 & S_{13} & S_{14} \\ S_{21} & 0 & S_{23} & S_{24} \\ 0 & S_{52} & S_{53} & S_{54} \\ 0 & S_{62} & S_{63} & S_{64} \end{bmatrix} \begin{bmatrix} F_o^{O} \\ F_o^{I} \\ A_o \\ B_o \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \end{bmatrix} \qquad (71)$$

If we allow the inner and outer fluid media to be the same and rearrange the matrix, we derive Goodman's matrix (reference 4) except for sign convention differences. The coefficients A_ℓ and B_ℓ differ by a minus sign from those of Goodman.

NUMERICAL IMPLEMENTATION

In this section the details appropriate to computer implementation of the previously derived solution are considered. In particular, the matrix is further manipulated and a method of solution which takes advantage of some peculiarities in the matrix is discussed.

The standard recurrence relations for spherical Bessel functions will be used and are repeated here for convenience.

$$j'(x) = \frac{n}{x} j_n(x) - j_{n+1}(x).$$

$$j_n''(x) = \left[\frac{n(n-1)}{x^2} - 1\right] j_n(x) + \frac{2}{x} j_{n+1}(x).$$

The matrix elements S_{ij}, after some algebra, may be written in the following form:

$$
\begin{aligned}
S_{11} &= \frac{\rho_o}{\rho} h_\ell(x) \\
S_{13} &= \frac{1}{x_T^2}\left[4x_L j_{\ell+1}(x_L) + \left(2\ell(\ell-1) - x_T^2\right) j_\ell(x_L)\right] \\
S_{14} &= \frac{1}{x_T^2}\left[4x_L n_{\ell+1}(x_L) + \left(2\ell(\ell-1) - x_T^2\right) n_\ell(x_L)\right] \\
S_{15} &= \frac{-2\ell(\ell+1)}{x_T^2}\left[(\ell-1) j_\ell(x_T) - x_T j_{\ell+1}(x_T)\right] \\
S_{16} &= \frac{-2\ell(\ell+1)}{x_T^2}\left[(\ell-1) n_\ell(x_T) - x_T n_{\ell+1}(x_T)\right] \\
S_{21} &= -\ell h_\ell(x) + x h_{\ell+1}(x) \\
S_{23} &= \ell j_\ell(x_L) - x_L j_{\ell+1}(x_L) \\
S_{24} &= \ell n_\ell(x_L) - x_L n_{\ell+1}(x_L) \\
S_{25} &= -\ell(\ell+1) j_\ell(x_T) \\
S_{26} &= -\ell(\ell+1) n_\ell(x_T) \\
S_{33} &= 2\left[(\ell-1) j_\ell(x_L) - x_L j_{\ell+1}(x_L)\right] \\
S_{34} &= 2\left[(\ell-1) n_\ell(x_L) - x_L n_{\ell+1}(x_L)\right] \\
S_{35} &= \left[x_T^2 - 2(\ell^2-1)\right] j_\ell(x_T) - 2x_T j_{\ell+1}(x_T) \\
S_{36} &= \left[x_T^2 - 2(\ell^2-1)\right] n_\ell(x_T) - 2x_T n_{\ell+1}(x_T) \\
S_{43} &= 2\left[(\ell-1) j_\ell(y_L) - y_L j_{\ell+1}(y_L)\right] \\
S_{44} &= 2\left[(\ell-1) n_\ell(y_L) - y_L n_{\ell+1}(y_L)\right] \\
S_{45} &= \left[y_T^2 - 2(\ell^2-1)\right] j_\ell(y_T) - 2y_T j_{\ell+1}(y_T) \\
S_{46} &= \left[y_T^2 - 2(\ell^2-1)\right] n_\ell(y_T) - 2y_T n_{\ell+1}(y_T) \\
S_{52} &= \frac{\rho_1}{\rho} j_\ell(y)
\end{aligned}
\qquad (72)
$$

$$S_{53} = \frac{1}{y_T^2}\left[4y_L j_{\ell+1}(y_L) + \left(2\ell(\ell-1) - y_T^2\right) j_\ell(y_L)\right]$$

$$S_{54} = \frac{1}{y_T^2}\left[4y_L n_{\ell+1}(y_L) + \left(2\ell(\ell-1) - y_T^2\right) n_\ell(y_L)\right]$$

$$S_{55} = \frac{-2\ell(\ell+1)}{y_T^2}[(\ell-1)j_\ell(y_T) - y_T j_{\ell+1}(y_T)]$$

$$S_{56} = \frac{-2\ell(\ell+1)}{y_T^2}[(\ell-1)n_\ell(y_T) - y_T n_{\ell+1}(y_T)]$$

$$S_{62} = -\ell j_\ell(y) + y j_{\ell+1}(y)$$

$$S_{63} = \ell j_\ell(y_L) - y_L j_{\ell+1}(y_L) \qquad \text{(72 contd)}$$

$$S_{64} = \ell n_\ell(y_L) - y_L n_{\ell+1}(y_L)$$

$$S_{65} = -\ell(\ell+1) j_\ell(y_T)$$

$$S_{66} = -\ell(\ell+1) n_\ell(y_T)$$

$$\beta_1 = \frac{\rho_o}{\rho} j_\ell(x)$$

$$\beta_2 = \ell j_\ell(x) - x J_{\ell+1}(x) .$$

Notice that only two matrix elements are complex, S_{11} and S_{21}. Rather than use a complex matrix solution routine, a solution algorithm is discussed which will allow one to construct the solution of the complex matrix by solving a real matrix with two different right-hand vectors.

Divide the first column of the S_{ij} matrix, equation 70, by S_{21}.

$$\begin{bmatrix} S_{11}/S_{21} & 0 & S_{13} & S_{14} & S_{15} & S_{16} \\ 1 & 0 & S_{23} & S_{24} & S_{25} & S_{26} \\ 0 & 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ 0 & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} F_\ell^o S_{21} \\ F_\ell^I \\ A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (73)$$

Written in vector form,

$$S\underset{\sim}{z} = \underset{\sim}{\beta} . \tag{74}$$

Expanding in terms of real and imaginary parts, we find that

$$(S_R + iS_I)(\underset{\sim}{z}_R + i\underset{\sim}{z}_I) = \underset{\sim}{\beta} ;$$

or

$$S_R \underset{\sim}{z}_R - S_I \underset{\sim}{z}_I = \underset{\sim}{\beta} \tag{75}$$

$$S_R \underset{\sim}{z}_I + S_I \underset{\sim}{z}_R = 0 . \tag{76}$$

Let

$$\alpha_R = \mathrm{Re}\left(\frac{S11}{S21}\right), \alpha_I = \mathrm{Im}\left(\frac{S11}{S21}\right) . \tag{77}$$

Then, the real and imaginary parts of the matrix S are

$$S_R = \begin{bmatrix} \alpha_R & 0 & S_{13} & S_{14} & S_{15} & S_{16} \\ 1 & 0 & S_{23} & S_{24} & S_{25} & S_{26} \\ 0 & 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ 0 & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} . \tag{78}$$

The imaginary part has only one element:

$$S_I = \alpha_I \underset{\sim}{e}_1 \otimes \underset{\sim}{z}_1 . \tag{79}$$

Using this fact, equation 76 may be written

$$S_R \underset{\sim}{z}_I = - S_I \underset{\sim}{z}_R = - \alpha_I z_{R1} \underset{\sim}{e}_1 . \tag{80}$$

Suppose we solve the following equation for a vector x.

$$S_R \underset{\sim}{x} = \underset{\sim}{e}_1 . \tag{81}$$

Then, it is clear that z_I is some scalar multiple of $\underset{\sim}{x}$, so we write

$$\underset{\sim}{z}_I = \gamma \underset{\sim}{x} . \tag{82}$$

where the constant γ is to be determined.

$$
\begin{bmatrix} \alpha_R & 0 & S_{13} & S_{14} & S_{15} & S_{16} \\ 0 & 0 & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ 0 & 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ 0 & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ 0 & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} F_\ell^O \\ F_\ell^I \\ A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{89}
$$

where

$$
\bar{S}_{23} = S_{13} - S_{23}\alpha_R \qquad \bar{S}_{25} = S_{15} - S_{25}\alpha_R
$$
$$
\bar{S}_{24} = S_{14} - S_{24}\alpha_R \qquad \bar{S}_{26} = S_{16} - S_{26}\alpha_R \tag{90}
$$

$$
\bar{F}_\ell^O = F_\ell^O/S_{11}
$$
$$
\bar{\beta}_2 = \beta_1 - \beta_2\alpha_R . \tag{91}
$$

Eliminate the first row and column of equation 89.

$$
\begin{bmatrix} 0 & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} F_\ell^I \\ A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{92}
$$

and

$$
\alpha_R F_\ell^O + S_{13}A_\ell + S_{14}B_\ell + S_{15}C_\ell + S_{16}D_\ell = \beta_1 . \tag{93}
$$

Similarly, multiply the sixth row of equation 92 by $-S_{52}/S_{62}$ and add to the fifth row.

$$\begin{bmatrix} 0 & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ 0 & S_{33} & S_{34} & S_{35} & S_{36} \\ 0 & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ 0 & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} F_\ell^I \\ A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{S}_{63} = S_{53} - S_{63}\left(\frac{S_{52}}{S_{62}}\right) \qquad \bar{S}_{65} = S_{55} - S_{65}\left(\frac{S_{52}}{S_{62}}\right)$$

$$\bar{S}_{64} = S_{54} - S_{64}\left(\frac{S_{52}}{S_{62}}\right) \qquad \bar{S}_{66} = S_{56} - S_{66}\left(\frac{S_{52}}{S_{62}}\right) \tag{94}$$

Then, eliminate the first column and fifth row.

$$\begin{bmatrix} \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ S_{33} & S_{34} & S_{35} & S_{36} \\ S_{43} & S_{44} & S_{45} & S_{46} \\ \bar{S}_{63} & \bar{S}_{64} & \bar{S}_{65} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{95}$$

$$S_{52}F_\ell^I + S_{53}A_\ell + S_{54}B_\ell + S_{55}C_\ell + S_{56}D_\ell = 0 \tag{96}$$

The real matrix solution is therefore reduced to the solution of a 4X4 matrix and the appropriate multiplications to find F_ℓ^I and F_ℓ^O from equations 96 and 93, respectively.

As we saw before, the ℓ=0 matrix is a special case of equation 71. We can follow the same procedure outlined above to solve the full matrix. The real matrix becomes

$$S_R = \begin{bmatrix} \alpha_R & 0 & S_{13} & S_{14} \\ 1 & 0 & S_{23} & S_{24} \\ 0 & S_{52} & S_{53} & S_{54} \\ 0 & S_{62} & S_{63} & S_{64} \end{bmatrix} \tag{97}$$

We may also follow a similar Gaussian elimination procedure to take advantage of the zeroes. Following the same steps as before we must solve a 2X2 matrix.

$$\begin{bmatrix} \bar{S}_{23} & \bar{S}_{24} \\ \bar{S}_{63} & \bar{S}_{64} \end{bmatrix} \begin{bmatrix} A_o \\ B_o \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \end{bmatrix} \tag{98}$$

where

$$\bar{S}_{23} = S_{13} - \alpha_R S_{23}, \bar{S}_{63} = S_{53} - S_{63}\left(\frac{S_{52}}{S_{62}}\right)$$

$$\bar{S}_{24} = S_{14} - \alpha_R S_{24}, \bar{S}_{64} = S_{54} - S_{64}\left(\frac{S_{52}}{S_{62}}\right) \tag{99}$$

and

$$\alpha_R F_o{}^O + S_{13} A_o + S_{14} B_o = \beta_1 \tag{100}$$

$$S_{52} F_o{}^I + S_{53} A_o + S_{54} B_o = 0 . \tag{101}$$

In conclusion, the time-saving methods described in this section have been incorporated into the spherical shell computer program. Comparison of a previous version of the program using a standard LU decomposition of the original 6X6 matrix and the modified version shows a 60-percent savings of the actual inversion time, resulting in an overall computer program time savings of 35 percent.

LIMITING CASES

We will now consider several limiting cases of the elastic shell solution. In the limit of vanishing shell thickness, the solution is shown to reduce to scattering from two dissimilar fluid media. In particular, when the wavelength is much greater than the size of the fluid sphere, we obtain a solution applicable to such bodies as air bubbles. If the shell thickness is non-zero but sufficiently thin, a first order approximation to the solution is obtained using a Taylor series expansion of the matrix elements. This solution is useful to avoid linear dependence problems in the S_{ij} matrix for sufficiently thin shells.

Suppose we let the thickness of the shell be zero, a = b. Then, by inspection of the matrix elements, equation 72,

$$S_{1j} = S_{5j} \qquad j = 3, 4, 5, 6$$

$$S_{2j} = S_{6j} \tag{102}$$

$$S_{3j} = S_{4j}$$

Subtracting the fifth row of the shell matrix, equation 70, from the first row and the sixth row from the second, we obtain a relation which determines the coefficients for the inner and outer fluid potentials:

$$\begin{bmatrix} S_{11} & -S_{52} \\ S_{21} & -S_{62} \end{bmatrix} \begin{bmatrix} F_\ell{}^O \\ F_\ell{}^I \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} . \tag{103}$$

From the definition of the matrix elements this becomes

$$\begin{bmatrix} \rho_o h_\ell(x) & -\rho_1 j_\ell(\eta x) \\ -x h'_\ell(x) & +\eta x j'_\ell(\eta x) \end{bmatrix} \begin{bmatrix} F_\ell^{o} \\ F_\ell^{1} \end{bmatrix} = \begin{bmatrix} -\rho_o j_\ell(x) \\ x j'_\ell(x) \end{bmatrix} . \tag{104}$$

where $\eta = \dfrac{k_1}{k_o} = \dfrac{C_o}{C_1}$.

The inverse of the matrix is easily obtained:

$$\begin{bmatrix} F_\ell^{(o)} \\ F_\ell^{(1)} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} +\eta x j'_\ell(nx) & +\rho_1 j_\ell(\eta x) \\ x h'_\ell(x) & \rho_o h_\ell(x) \end{bmatrix} \begin{bmatrix} -\rho_o j_\ell(x) \\ x j'_\ell(x) \end{bmatrix} . \tag{105}$$

where $\Delta = +\rho_o \eta x h_\ell(x) j'_\ell(\eta x) - \rho_1 x h'_\ell(x) j_\ell(\eta x)$.

The coefficient for the outer potential is easily obtained:

$$F_\ell^{o} = \frac{-\rho_o \eta j_\ell(x) j'_\ell(\eta x) + \rho_1 j'_\ell(x) j_\ell(\eta x)}{+\rho_o \eta h_\ell(x) j'_\ell(\eta x) - \rho_1 h'_\ell(x) j_\ell(\eta x)} .$$

or

$$F_\ell^{o} = -\left[\frac{j'_\ell(x) + i\Gamma_\ell j_\ell(x)}{h'_\ell(x) + i\Gamma_\ell h_\ell(x)} \right] , \tag{106}$$

where

$$\Gamma_\ell = \frac{i\rho_o \eta}{\rho_1} \left[\frac{j'_\ell(\eta x)}{j_\ell(\eta x)} \right] .$$

This result is identical with Morse and Ingard (reference 8) except for sign convention.

When the wavelength is much greater than the size of the sphere, we may use a small-argument limiting form of spherical Bessel Functions (reference 9)

$$\left.\begin{aligned} j_\ell(x) &\to \frac{x^\ell}{1.3 \ldots (2\ell+1)} \\ n_\ell(x) &\to \frac{-1.1.3 \ldots (2\ell-1)}{x^{\ell+1}} \end{aligned}\right\} \quad x \to 0 .$$

In particular,

$$j_0(x) = 1 \qquad j_1(x) = \frac{x}{3}$$

$$n_0(x) = -\frac{1}{x} + \frac{x}{2} \qquad n_1(x) = \frac{1}{x^2} \tag{107}$$

$$h_0(x) = 1 + i\left(\frac{x}{2} - \frac{1}{x}\right) \qquad h_1(x) = \frac{i}{x^2}.$$

The lowest order term for the expansion coefficients, equation 106, becomes

$$\Gamma_0 = \frac{-i\rho_0}{\rho_1}\eta\frac{j_1(\eta x)}{j_0(\eta x)} = \frac{-i\rho_0}{\rho_1}\eta^2\frac{x}{3}$$

$$F_0{}^0 = -\left[\frac{-j_1(x) + i\Gamma_0 j_0(x)}{-h_1(x) + i\Gamma_0 h_0(x)}\right]$$

$$= -\left[\frac{-\frac{x}{3}\left(1 - \frac{\rho_0}{\rho_1}\eta^2\right)}{\frac{i}{x^2} + \left(\frac{\rho_0}{\rho_1}\eta^2\frac{x}{3}\right)\left(1 + \frac{ix}{2} - \frac{i}{x}\right)}\right]$$

$$F_0{}^0 = \frac{\frac{x^3}{3i}\left(1 - \frac{\rho_0}{\rho_1}\eta^2\right)}{1 - \frac{\rho_0\eta^2}{3\rho_1}(x^2 - ix^3)} \quad x \ll 1 . \tag{108}$$

Similarly, for $\ell = 1$

$$\Gamma_1 = \frac{i\rho_0}{\rho_1}\eta\left[\frac{j_1'(nx)}{j_1(nx)}\right] = \frac{i\rho_0}{\rho_1}\eta\left[\frac{j_0(nx)}{j_1(nx)} - \frac{2}{nx}\right]$$

$$= \frac{i\rho_0}{\rho_1 x}$$

$$F_1^{o} = -\left[\frac{j_1'(x) + i\Gamma_1 j_1(x)}{h_1'(x) + i\Gamma_1 h_1(x)}\right] = -\left[\frac{j_0(x) - j_1(x)\left(\frac{2}{x} - i\Gamma_1\right)}{h_0(x) - h_1(x)\left(\frac{2}{x} - i\Gamma_1\right)}\right]$$

$$= -\left[\frac{1 - \frac{x}{3}\left(\frac{2}{x} + \frac{\rho_o}{\rho_1 x}\right)}{1 + i\left(\frac{x}{2} - \frac{1}{x}\right) - \frac{i}{x^2}\left(\frac{2}{x} + \frac{\rho_o}{\rho_1 x}\right)}\right]$$

$$= -\left[\frac{1 - \frac{1}{3}\left(2 + \frac{\rho_o}{\rho_1}\right)}{1 + i\left[\frac{x}{2} - \frac{1}{x} - \frac{i}{x^3}\left(2 + \frac{\rho_o}{\rho_1}\right)\right]}\right]$$

$$= -\frac{x^3}{3}\left[\frac{1 - \frac{\rho_o}{\rho_1}}{x^3 + i\left[\frac{x^4}{2} - x^2 - i\left(2 + \frac{\rho_o}{\rho_1}\right)\right]}\right] .$$

Dropping higher order terms in denominator, we find

$$F_1^{o} = \frac{x^3}{3i}\left[\frac{1 - \frac{\rho_o}{\rho_1}}{2 + \frac{\rho_o}{\rho_1}}\right]$$

$$F_1^{o} = \frac{x^3}{3i}\left[\frac{\rho_1 - \rho_o}{2\rho_1 + \rho_o}\right] . \qquad (109)$$

The higher order coefficients are proportional to $x^{2\ell+1}$ and are therefore negligible.

At the surface of the sphere we can obtain an expression for the total potential using the first two terms of equation 51.

$$\varphi_o(r=a) = 1 + F_o^{\,o} h_o(x) + \xi_1 \cos\theta \left[j_1(x) + F_1^{\,o} h_1(x) \right]$$

$$= 1 + F_o^{\,o}\left[1 - \frac{i}{x}\right] - 3i\cos\theta \left[\frac{x}{3} + F_1^{\,o}\left(\frac{i}{x^2}\right)\right]$$

$$= 1 + F_o^{\,o}\left[1 - \frac{i}{x}\right] - ix\cos\theta\left(\frac{3\rho_1}{2\rho_1 + \rho_o}\right)$$

$$= 1 + \left[\frac{\frac{x^3}{3i}\left(1 - \frac{\rho_o}{\rho_1}\eta^2\right)}{1 - \frac{\rho_o \eta^2}{3\rho_1}(x^2 - ix^3)}\right]\left[1 - \frac{i}{x}\right] - ix\cos\theta\left(\frac{3\rho_1}{2\rho_1 + \rho_o}\right)$$

$$= 1 - \frac{\frac{1}{3}\left(1 - \frac{\rho_o}{\rho_1}\eta^2\right)[x^2 + ix^3]}{1 - \frac{\rho_o \eta^2}{3\rho_1}(x^2 - ix^3)} - ix\cos\theta\left(\frac{3\rho_1}{2\rho_1 + \rho_o}\right)$$

$$\varphi_o(r=a) = 1 - ix\cos\theta\left(\frac{3\rho_1}{2\rho_1 + \rho_o}\right) - \frac{\frac{1}{3}\left(1 - \frac{\rho_o}{\rho_1}\eta^2\right)[x^2 + ix^3]}{1 - \frac{\rho_o \eta^2}{3\rho_1}(x^2 - ix^3)}. \qquad (110)$$

Notice that for a light, compressible sphere like an air bubble, where $\rho_1 \ll \rho_o$, the linear term in x is negligible. Therefore,

$$\varphi_o(r=a) \simeq 1 + \frac{\frac{\rho_o}{3\rho_1}\eta^2(x^2 + ix^3)}{1 - \frac{\rho_o \eta^2}{3\rho_1}(x^2 - ix^3)} = \frac{1 + 2i\frac{\rho_o \eta^2}{3\rho_1}x^3}{1 - \frac{\rho_o \eta^2}{3\rho_1}(x^2 - ix^3)}.$$

There is a resonance at $x^2 = \dfrac{3\rho_1}{\rho_o \eta^2}$.

Letting $\bar{x}^2 = \dfrac{3\rho_1}{\rho_o \eta^2} = \dfrac{3\rho_1 C_1^2}{\rho_o C_o^2}$.

$$\varphi_o(r=a) = \frac{\bar{x}^2 + i2x^3}{\bar{x}^2 - (x^2 - ix^3)}. \qquad (111)$$

This result agrees with that obtained for acoustic scattering from air bubbles in water (references 10, 11).

When the thickness of the shell is small, but non-zero, a first order approximation to the shell matrix may be obtained using a Taylor series expansion of the matrix elements. The procedure does not provide much analytical simplification of the matrix, but avoids numerical difficulties in solution. When the shell is thin, two rows of the shell matrix are approximately equal, creating a linearly dependent matrix.

In order to complement the numerical solution method described in the previous section for thin shells, we will work with the reduced, real matrix, equation 95,

$$\begin{bmatrix} \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ S_{33} & S_{34} & S_{36} & S_{36} \\ S_{43} & S_{44} & S_{45} & S_{46} \\ \bar{S}_{63} & \bar{S}_{64} & \bar{S}_{65} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{112}$$

where the barred matrix elements are defined by equations 90 and 94.

Notice that for a thin shell, the second row is approximately equal to the third row of the matrix equation 111. Expanding the elements in the third row in a Taylor's series about r=a, we obtain, to first order in the thickness δ, these expressions for the matrix elements:

$$\begin{aligned} S_{43}(Y_L) &= S_{33}(x_L) - k_L \delta S'_{33}(x_L) \\ S_{44}(Y_L) &= S_{34}(x_L) - k_L \delta S'_{34}(x_L) \\ S_{45}(Y_T) &= S_{35}(x_T) - k_T \delta S'_{35}(x_T) \\ S_{46}(Y_T) &= S_{36}(x_T) - k_T \delta S'_{36}(x_T) . \end{aligned} \tag{113}$$

Subtracting row 3 from row 4, we find that the matrix equation 112 becomes

$$\begin{bmatrix} \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ S_{33} & S_{34} & S_{35} & S_{36} \\ -k_L \delta S'_{33} & -k_L \delta S'_{34} & -k_T \delta S'_{35} & -k_T \delta S'_{36} \\ \bar{S}_{63} & \bar{S}_{64} & \bar{S}_{65} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{bmatrix} = \begin{bmatrix} \bar{\beta}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{114}$$

The matrix is valid to first order in the thickness δ and eliminates the linear dependence problem when the shell is sufficiently thin.

COMPUTED RESULTS

In this section typical computed results are presented for scattering from an air-filled, aluminum spherical shell. The results shown in figures 2 and 3 were obtained using the following input data. MKS units are used throughout.

$\rho_I = 1.21$	$C_I = 343$	
$\rho_o = 1.026X10^3$	$C_o = 1.5X10^3$	
$\rho = 2.7X10^3$	$C_L = 6.412X10^3$	$C_T = 3.043X10^3$
$\delta/a = .05$	$r/a = 1$	

The total pressure as a function of k_0a at the surface of the elastic shell is shown for angles from the incoming wave of 0 and 180 degrees, respectively. The complexity of the scattered field is due to the vibrational characteristics of the elastic sphere.

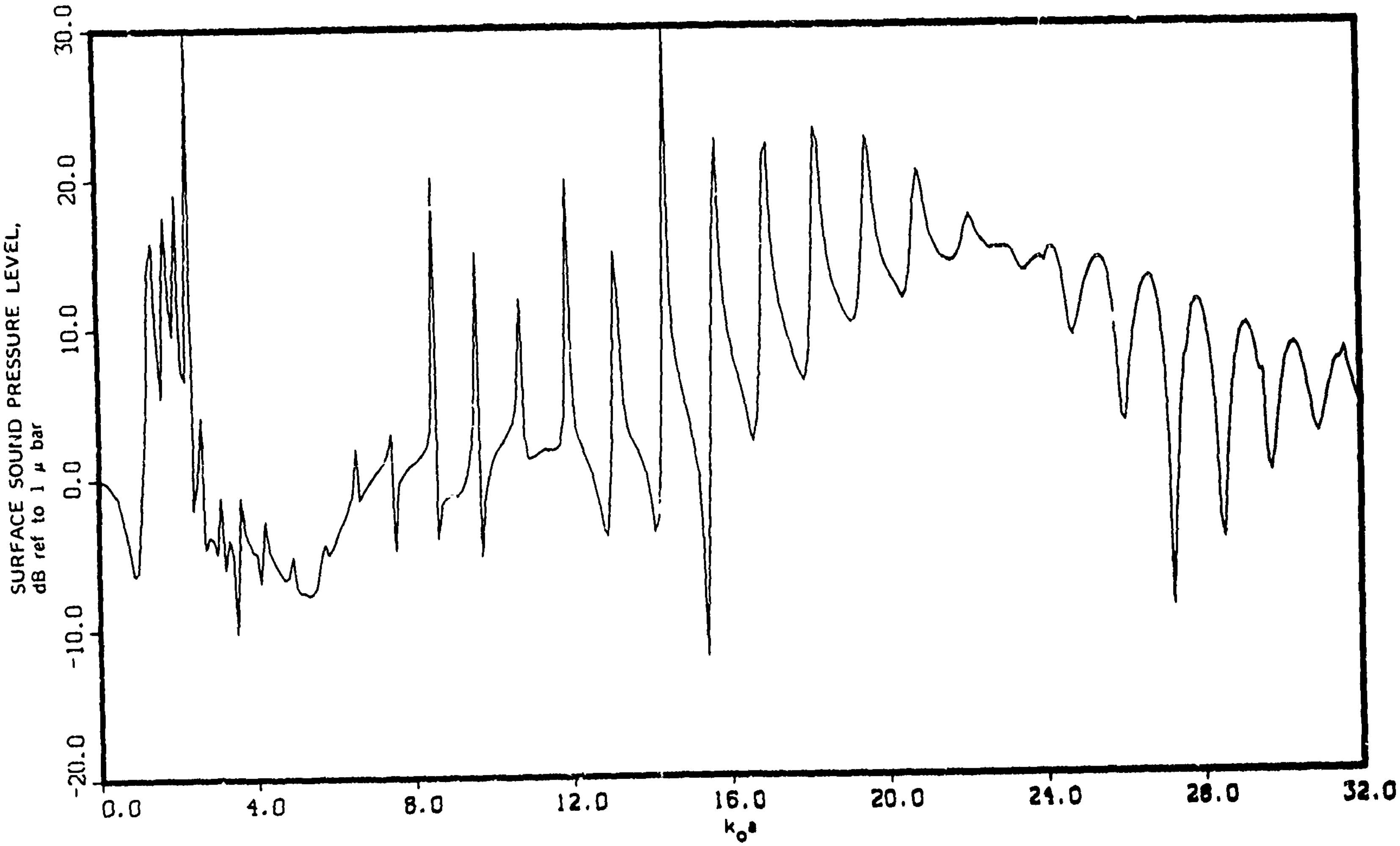

Figure 2. Predicted total pressure as a function of k_0a at the surface of a spherical shell, θ = 0 degrees.

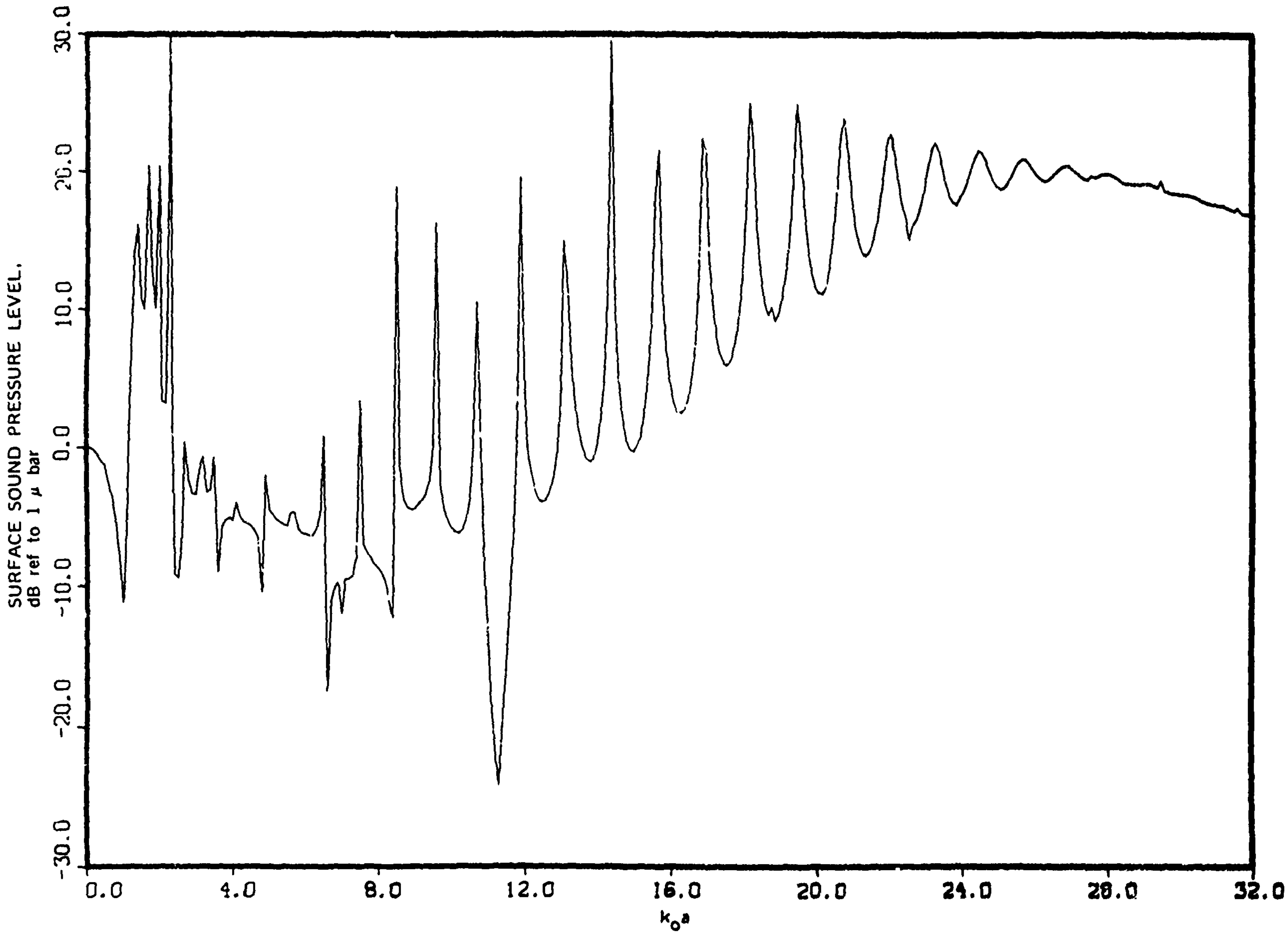

Figure 3. Predicted total pressure as a function of k_0a at the surface of a spherical shell, θ = 180 degrees.

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