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CAUSALITY STRUCTURE OF ENGINEERING SYSTEMS

Romano Mario De Santis

Michigan University

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Romano Mario De Santis

under the direction of
Professor William A. Porter

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13. ABSTRACT

This is a study of the structure of engineering systems with respect to time related behavior. The development is unconventional in not being based on system properties such as linearity, time invariance, etc. The main question of interest concerns the possibility of decomposing a general system into a sum of systems with less complex time related behavior. An analysis of characteristic properties of such systems and their interconnection with other aspects of system behavior is also made. The approach used in the study is "functional analytic," making use of recently developed axiomatic structures called the group resolution space and Hilbert resolution space. This in turn permits the efficient utilization of powerful techniques from the Gohberg-Krein theory of Volterra operators. The most important result of the study is a canonical causality decomposition theorem. Other results include a theorem connecting the temporal properties of an open loop system to the stability of the closed loop feedback system.

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by

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ABSTRACT

CAUSALITY STRUCTURE OF ENGINEERING SYSTEMS

by

Romano Mario De Santis

Chairman: William A. Porter

This study is about the structure of engineering systems with respect to time related behavior. Studies of this type are usually based on such system properties as linearity, time invariance, continuity and physical realizability. A distinguishing character of the present development is that none of these properties is essential. This explains in part the unconventional character of questions, approach and results.

In contrast to a physically realizable system, an engineering system need not be causal and may present a complex time related behavior. The main question of interest concerns then the possibility of decomposing a general system into the sum of systems with a time related behavior which is, in some sense, "simple". The analysis of characteristic properties of such systems and of their interconnections with other aspects of system behavior is also a relevant question of investigation.

The above questions are applicable to many and diverse engineering fields and the desirability of an abstract approach is quite

natural. The approach adopted in this study can be described as "functional analytic" and makes use of recently developed axiomatic structures called group resolution space and Hilbert resolution space. This not only provides a natural framework in which to embed the intended research, but it also permits an efficient utilization of powerful techniques from the Gohberg-Krein theory of Volterra operators.

The most important result of this study is a canonical causality decomposition theorem. This theorem states that under appropriate hypotheses every engineering system can be represented as the sum of "simple" systems. Another relevant theorem connects the temporal properties of an open loop system to the stability of the closed loop feedback system.

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LIST OF SYMBOLS

Symbol	Meaning	Section first used
R	field of real numbers	2.1
ν	linearly ordered set	2.2
G	abelian group	2.2
$S[G, \nu]$	GRS with group G and linear set ν	2.2
GRS	Group resolution space	2.2
P^t, P_t	truncation (projection) operators in a GRS (HRS)	2.2
$\bar{P}^t, \bar{P}_t$	truncation operators in a GRS	2.2
$dP(t)$	evaluation operator in a GRS	2.2
C	causal	2.3
A	anticausal	2.3
M	memoryless	2.3
X	crosscausal	2.3
$\underline{C}$	strongly causal	2.3
$\underline{A}$	strongly anticausal	2.3
$\underline{X}$	strongly crosscausal	2.3
$\bar{C}$	strictly causal	2.3
$\bar{A}$	strictly anticausal	2.3
$\mathcal{A}$	the causality alphabet $\{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$	2.3
$\underline{\mathcal{A}}$	the subset $\{\underline{A}, \underline{C}, M, \underline{X}\}$ of $\mathcal{A}$	2.3

T_α	an operator on a GRS or HRS which is $\alpha \in A$	2.3
$	space of operators on a GRS (or a HRS)	2.5
$\mathbf{T}$	an operator mapping $ into itself	2.5
$(\psi(t), S, \zeta(t))$	state decomposition	3.2
S	state set	3.2
$\psi(t)$	first element of a state decomposition	3.2
$\zeta(t)$	third element of a state decomposition	3.2
$\mathbf{K}(t)$	family of invertible mappings between two state sets	3.3
$\phi(t, q)$	transition operator on a state set	3.3
ℓ_2	space of real sequences $\{x_i\}$ with $\sum x_i^2 < \infty$	4.1
$L_2[0, t_f]$	space of real functions square integrable in $[0, t_f]$	4.1
H	Hilbert space	4.2
$\mathbb{R}$	resolution of the identity in H	4.2
HRS	Hilbert resolution space	4.2
$[H, P^t]$	HRS with Hilbert space H and $\mathbb{R} = \{P^t\}$	4.2
Ω	partition of ν	4.2
t_0, t_∞	minimum and maximum elements of ν	4.2
$\lvert\ \rvert$	norm in a Hilbert space	4.2
T(s)	family of operators on a HRS	4.2
$\int dPT(s)$	integral on a HRS	4.2
$\mathfrak{m}\!\!\int dPT(s)$	"lower type" integral on a HRS	4.2

$M\!\!\!\!\int dPT(s)$	"upper type" integral on a HRS	4.2
$\int dP[\quad]P^{s}$	integral transformator	4.2
HS	Hilbert Schmidt operator	4.7
$\|\quad\|$	Hilbert Schmidt norm	4.7

The results of this thesis are reported under the labels of lemmas, propositions, and theorems. The results appearing as lemmas have been borrowed from the references. Unless otherwise stated, all the results labeled propositions and theorems are original to the best of the author's knowledge.

1. INTRODUCTION

1.1 Introduction

When the concepts of time and its associates - past, present and future - are extended from the world of physical systems into the broader context of engineering systems they become abstract entities. They may then preserve little in common with the original notion of physical "fourth dimension" and the terms real and nonreal time are often adopted.

Heuristically, a system is called causal if "present behavior" of the system is only a function of the past. In this regard a widely accepted axiom is the principle of causality: all physical systems are causal. As engineering systems need not be physical nor do they necessarily function in real time, it is quite natural that for these systems the assumption of causality is no longer valid.

Indeed, in the engineering domain it is not unusual to be faced with problems related to noncausal behavior. Classical filtering and estimation problems in communication theory do not necessarily have a causal solution [14], [35]. In stability theory the instability of a large class of linear time invariant systems can be explained in terms of absence of causality [11], [53]. In optimal control theory noncausal controllers occur regularly in minimum effort, minimum fuel, and optimal time problems [41], [44].

In the implementation of an engineering system the requirement that the system be causal is important. In particular if the implementation has to be physical, then this requirement becomes mandatory. When this happens one either confines himself to consider only causal systems [9], [63], [68], [69], or, in an alternative approach, one lifts the causality constraint in a preliminary phase of the design and reconsiders it in a later phase [30], [66]. In this latter instance, usually an analysis of the causality structure of the system is in order [48].

This then explains one characteristic of the present development: a study about time related behavioral patterns of engineering systems. That is, about the modalities by which past, present and future of the input affect the present of the output. In the context of this study causal and noncausal as well as linear and nonlinear systems are considered.

A proper characterization of the nature of time behavioral patterns may suggest various engineering implications. For instance, the fact that a system has noncausal behavior implies that it cannot be physically implemented. On the other hand, if the noncausal behavior is, in some sense, "almost causal" then, for engineering purposes, the system can still enjoy the property of physical implementation.

To mention other types of implications note that in linear systems theory the fact that a physical realization of a linear time invariant dynamical system defines uniquely the input-output weighting pattern of the system [65] can be viewed as a consequence of the special causality structure of the system. Similarly, in stability theory some special causality properties of an open loop system can imply the stability of the closed loop feedback system [11], [12], [13]*.

A second characteristic of this development will, at this point, appear natural: the study of interconnections between time related behavioral patterns and other aspects of system behavior. In this regard interconnections between state and causality structures are investigated. An additional topic of research is the study of relations between stability and causality properties.

In a broad sense a study of the present type falls into the context of causality theory. On the other hand, emphasis here is on systems which do not necessarily enjoy those properties on which much of the previous research on causality is based, namely: linearity, time invariance, continuity, and most notable causality. It is then not surprising if the classical theory on causality presents limitations in this area.

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*Other significant implications can also be envisioned in less academic and more earthly engineering areas. Here the reader is referred to Appendix A.

In particular, for instance, the technical literature has paid scant attention to the study of noncausal systems. Apparently this situation has been justified by the feeling that noncausal behavior can be treated as causal behavior in reverse time. This suits quite well studies confined to physical systems and, to a certain extent, linear systems [48]. However, such an approach is not applicable to the type of problems considered here.

A more general approach has recently been adopted by Porter [41], [42] and Saeks [48], [49]. These references treat causal and noncausal systems on an equal footing and provide much of the ideas and inspiration for the present development. These efforts are, however, mostly confined to the study of linear systems in a Banach or a Hilbert space context. Many systems of interest, however, not only are not linear but, most important, cannot be fitted in any Banach or Hilbert space framework.

The above discussion allows us to introduce a third and perhaps most relevant characteristic of this study: the abstract approach. The desirability of an abstract treatment is quite natural in view of the many and diverse engineering fields in which the subject applies. To those fields already mentioned we can, for instance, add game theory, decision theory, prediction theory, electric network theory and others. Some of the advantages of abstraction are then evident: the possibility of emphasizing those features which are truly connected to time related behavior; the unification of diverse classes of problems; the possibility of obtaining results with a wide range of applicability; the contribution to studies with an interdisciplinary character.

1.2 Organization

Commensurate with the discussion in the previous section we will treat the following topics: i) identification, analysis and physical interpretation of those systems which, in some sense, are "basic" with respect to time related behavior; ii) consideration of the question whether a general system can be decomposed into the sum of basic systems; iii) investigation of some connections between causality and other system theoretic properties.

The development of topics i) and ii) is carried out in Chapters 2 and 4. More specifically, Chapter 2 considers systems with a mathematical representation (group resolution spaces) which is "minimal" for this type of study and does not possess any topological structure. Chapter 4 deals with linear and nonlinear systems defined on a Hilbert space.

Topic iii) is treated in Chapters 3 and 5. In Chapter 3 some connections between causality and state structures are studied. This study is embedded in a group resolution space context. Chapter 5 investigates some connections between special causality properties of an open loop system and the stability properties of the closed loop feedback system. Here the system is modeled by a mapping on a Hilbert space.

1.3 A General Summary of Results

A general theory of the causality related structure of engineering systems has been developed. The mathematical framework in which this theory is imbedded is called group resolution space. In this setting we identify and study various "basic" systems: causal (strongly, strictly), anticausal (strongly, strictly), crosscausal (strongly, strictly), and memoryless. Relevant properties of these systems are illuminated by Propositions 2.4.1-6 and 2.5.1-6.

The concepts of causal, anticausal, and memoryless systems are in agreement with those already appearing in the technical literature. The concept of crosscausality is new and plays an essential role in the extension of the available theory of causality into a nonlinear systems context. When a group resolution space (GRS) becomes a Hilbert resolution space (HRS), these concepts have a natural and straightforward extension. Some aspects of this extension are illustrated by Definitions 4.3.1-6 and Propositions 4.4.1-4. Moreover if we restrict our interest to linear systems in a Hilbert resolution space, then our framework coincides with the one which was adopted by Saeks as a starting point. This allows one to rediscover results which had already been stated by Saeks (Propositions 4.7.1-3). New results in line with Saek's development are also obtained (Theorem 4.5.1, and Propositions 4.5.1-3 and 4.7.6-10).

In GRS various forms of a causality canonical decomposition Theorem can be obtained (Theorem 2.6.1). A typical statement of this theorem is as follows: every system in a GRS can be decomposed into the sum of causal, anticausal, crosscausal, and memoryless systems. When we confine our attention to weakly additive systems then crosscausal operators disappear and most of our results assume the familiar formulation of those results which have been stated by other authors in the case of linear systems in HRS (Theorem 2.7.1 and Propositions 2.7.1 - 4).

In a HRS context it is found that a causality canonical decomposition theorem cannot in general be stated. However necessary and sufficient conditions for a canonical decomposition to exist can be given (Theorems 4.6.1 and 4.7.1). These conditions are utilized to study the causality structure of various classes of special systems. In particular it is shown that a Hilbert Schmidt system admits a canonical causality decomposition. Moreover the components of such a decomposition are mutually orthogonal (Theorem 4.8.1 and Proposition 4.8.1-3).

The GRS framework is also utilized to study some common features of causality and state concepts. To do this we first generalize the notion of state which was developed by Saeks [49] and illustrate some important aspects of this generalization (Proposition 3.3.1-4). We proceed then to investigate meaningful interconnections between state and causality properties. In this regard Theorem 3.4.1 relates the identification of basic causality components of a system with its state and costate decompositions. Additional results can also be found in the statements of Propositions 3.4.1 - 6.

The final part of this thesis is concerned with the study of interconnections between causality and stability. For this purpose we consider a basic feedback system in HRS and relate some causality properties of an open loop system to the stability of the closed loop system. Important results here are provided by Theorems 5.2.1, 5.3.1 and 5.4.1. Some relevant implications of these theorems are illustrated by Propositions 5.4.1-4.

1.4 A Brief Review of Research on Causality

Questions concerning causality related concepts have been a focus of modern scientific research for a number of decades. Most of this research is directed to the study of the principle of causality and its implications in various aspects of system behavior. These implications range from constraints on the frequency response of single variate linear, time invariant systems, to a number of more involved properties for the so called "collision matrix" of infinite dimensional systems defined by various scattering processes

[55], [56], [61]. The above type of properties are usually categorized with the title of dispersion relations [6], [7].

The investigation of the connections between the principle of causality and dispersion relations has long been the objective of considerable research. Among others this research is marked by the contributions of Paley and Wiener, Foures and Segal, Toll and Youla, Castriota and Carlin.

Paley and Wiener [38] can be considered the pioneers of the modern study on causality. Their famous Paley-Wiener theorem gave necessary and sufficient conditions for the modulus of a frequency response function to represent a causal system. This theorem is the basis for much of the subsequent related classical work on causality.

Using the Paley-Wiener theorem, Toll obtained an additional formulation of necessary and sufficient conditions [55]. Here causality is associated with analytical properties of the frequency response function when its domain is extended from the imaginary axis to the complex plane.

Similar conditions were offered by Foures and Segal [20]. Later on Youla, Castriota and Carlin extended these results to the frequency response matrix function of linear passive n-port networks [63].

The mathematical techniques supporting the above developments are essentially centered on the theory of functions of complex variables and Fourier integrals. In a somewhat different context the properties of causal systems have also been studied by Sandberg [51], Falb and Freedman [19].

Sandberg considered nonlinear systems on a Hilbert space context and obtained some necessary conditions for causality in terms of physically meaningful quantities. Falb and Freedman focused their attention on linear systems in locally compact abelian groups and obtained a generalization of the classical results of Foures and Segal. With the work of these latter authors, the study of causality assumes a more general character. This is marked by two basic features: association of the concept of causality with other system theoretic concepts such as passivity, dissipativity, stability, etc., and the introduction of modern tools of functional analysis.

Recently this new type of approach has been supported by a number of other authors [11], [42], [43], [48], [60]. Here the works of Porter, and of Saeks are the most closely related to the present study and will be a subject of attention later in this development.

2. CAUSALITY ON GROUP RESOLUTION SPACE

2.1 Introduction

The development of this chapter can perhaps be best introduced by a brief discussion of some relevant work of Saeks [48].

Saeks models a system by the use of an axiomatic structure called Hilbert resolution space. In this context he defines and studies various classes of systems with special causality related properties. These classes consist basically of causal, anticausal and memoryless systems. In heuristic terms a memoryless system has the property that the present of the output is only affected by the present of the input. An anticausal system is characterized by the property that it becomes causal if we reverse the direction in which the time flows.

To categorize the causality structure of a linear system, Saeks offers a canonical causality decomposition theorem. This theorem states that every linear system (on a Hilbert resolution space) can be decomposed into the sum of causal, anticausal and memoryless systems. At this point one begins to envision a reasonable approach to the causality problem: to start by defining and studying a set of "basic" systems with a particular causality character; and then to investigate whether a general system can be decomposed into the combination of basic systems. Indeed, these two steps are precisely the core of Saeks' development.

It may thus appear as if the causality problem has been exhausted. To show that this is not the case we introduce the following observations: a) the statement of the canonical causality decomposition theorem, even in the limited form in which it has been presented, is not, in general, valid (a proof of this fact is given in Section 4.7) ; b) the classes of causal, anticausal and memoryless systems appear unable to account for the causality behavior of simple nonlinear systems in communication and control theory; c) the mathematical model proposed by Saeks limits the scope of his results to a very restricted class of systems. This model cannot, for example, be used to discuss causality concepts in relation to even the most elementary sequential machine.

A close examination reveals that the causality problem is far from being solved. Two fundamental questions which seem to characterize this problem are the following: under what circumstances is it possible to decompose a system into the sum of basic systems with a well understood causality structure? What properties should a system have in order to be considered basic?

In the present chapter we address our efforts to the investigation of the above two questions. Clearly a first order of business is the formulation of a mathematical model to represent the systems under consideration. In this regard a setting called group resolution space is developed. Among the various advantages provided by the group resolution space setting the most important are the following: i) it allows a meaningful development of causality which is general enough for modeling most system problems in information, automatic control and automata theory; ii) it is minimal in the sense that every element of this mathematical setting appears to be essential for a reasonable development of causality to be carried out; iii) it permits other system theoretic concepts (such as state) to be analyzed.

At this point an overview of the chapter may be in order. Section 2 deals with some necessary mathematical preliminaries, in particular the characterization of group resolution spaces. In Section 3 we give the definitions of basic causality concepts. Some properties of these concepts are studied in Sections 4 and 5. Section 6 considers the question of causality canonical decomposition. In Section 7 the results of the previous sections are revisited to treat the case of weakly additive systems. Finally, Section 8 gives some conclusive remarks.

2.2 Group Resolution Spaces (GRS)

In this section we introduce the mathematical setting of group resolution spaces on which much of the present causality study will be based.

To start, recall that an abelian group (in the sequel simply a group) is a set G together with a binary operation $(x, y) \rightarrow xy$ mapping

$G \times G \rightarrow G$ such that the following conditions are satisfied:

i) $x(yz) = (xy)z$ for all $x, y, z \in G$ (associative law)

ii) there exists an element $e \in G$ such that $ex = x$ for all $x \in G$ (e is the identity element)

iii) for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = e$ (a^{-1} is called the left inverse of a)

iv) $ab = ba$ for all $a, b \in G$.

A set ν is linearly ordered if it is equipped with a relation "$\leq$", contained in the Cartesian product $\nu \times \nu$, such that for all $t, s, \tau \in \nu$, the following conditions are satisfied:

i) $t \leq t$ (reflexive)

ii) $t \leq s$ and $s \leq t$ imply $t = s$ (antisymmetric)

iii) $t \leq s$ and $s \leq \tau$ imply $t \leq \tau$ (transitive)

iv) $t, \tau \in \nu$, implies $t \leq \tau$ or $\tau \leq t$ (trichotomy).

Given a (abelian) group G and a linearly ordered set ν the space $S[G, \nu]$ is defined as follows: if $x \in S[G, \nu]$ then, to every $t \in \nu$, x associates a well defined element $x(t) \in G$. In other words we have:

$$S[G, \nu] = \{x \mid x \text{ is a mapping } \nu \rightarrow G\}.$$

The space $S[G, \nu]$ comes equipped with a family of truncation operators P^t, P_t. These operators P^t, P_t act on $S[G, \nu]$ as follows: if x, y, z belong to $S[G, \nu]$ and $y = P^t x$, $z = P_t x$, then

$$y(s) = x(s) \text{ for } s \leq t, \; y(s) = \phi \text{ otherwise, and}$$

$$z(s) = x(s) \text{ for } s \geq t, \; z(s) = \phi \text{ otherwise}$$

where the symbol ϕ indicates the null element of G. The space $S[G, \nu]$ together with the family of truncation operators will be called a *group resolution space* (in short *GRS*).

In the sequel we will also use an additional family of operators $dP(t) : S[G,\nu] \to S[G, \nu]$. This family is defined as follows: if x and y belong to $S[G, \nu]$ and $y = dP(t)x$, then $y(s) = x(s)$ if $s = t$ and $y(s) = \phi$ otherwise. Clearly $dP(t) = P^t P_t = P_t P^t$. For later use we denote by $\bar{P}^t$ and $\bar{P}_t$ the operators:

$$\bar{P}^t = P^t - dP(t) \text{ and } \bar{P}_t = P_t - dP(t)$$

Occasionally $dP(t)x$, $\bar{P}^t x$ and $\bar{P}_t x$ are referred to respectively as *present*, *past* and *future* of x at *time* t.

To gain some familiarity with the above definitions and to provide some background for later considerations we introduce the following simple examples.

Example 1. Let R^n be the usual Euclidean space with dimension n. Every element x in R^n is given by an n-tuple of real numbers $x = (x_1, x_2, \ldots, x_n)$ and it can be viewed as a function mapping the set of integers $\{1, 2, \ldots, n\}$ into the set of real numbers R.

Consider the set $\nu = \{1, 2, \ldots, n\}$ to be equipped with the natural order, and the set $G = R$ to be equipped with the familiar notion of addition. Clearly ν is a linearly ordered set, G is a group and we can then talk about the group resolution space $R^n = S[G, \nu]$.

The various families of truncation operators are defined as follows: if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $w = (w_1, \ldots, w_n)$ and $z = (x_1, \ldots, z_n)$ are elements of $S[G, \nu]$ then the relations

$$y = P^j x, \quad w = P_j x, \quad z = dP(j)x \qquad j \in \{1, 2, \ldots, n\}$$

imply the following

$$y_i = x_i \text{ for } j \leq i \text{ and } y_i = 0 \text{ for } j > i$$

$$w_i = 0 \text{ for } j < i \text{ and } w_i = x_i \text{ for } i \geq j$$

$$z_i = x_i \text{ for } j = i \text{ and } z_i = 0 \text{ for } i \neq j. \qquad \triangleleft$$

Example 2. Denote by $[0, \infty)$ the set of positive real numbers and by R the set of all real numbers.

If we consider the elements in $[0, \infty)$ to be ordered in the natural way and the set R to be again equipped with the usual notion of addition, then we can view $[0, \infty)$ as a linearly ordered set and R as a group. Using the notation $\nu = [0, \infty)$ and $G = R$, we can then consider the group resolution space

$$S[G, \nu] = \{x \mid x \text{ is a function mapping } [0, \infty) \text{ into } R\}.$$

The various families of truncation operators are now defined as follows: if $x, y, w, z \in S[G, \nu]$ and

$$y = P^t x, \quad w = P_t x, \quad z = dP(t)x, \quad t \in [0, \infty)$$

then we have the following relations

$$y(s) = x(s) \text{ for } s \leq t, \; y(s) = 0 \text{ for } s > t$$

$$w(s) = 0 \text{ for } s < t, \; w(s) = x(s) \text{ for } s \geq t$$

$$z(s) = x(s) \text{ for } s = t \text{ and } z(s) = 0 \text{ for } s \neq t. \qquad \triangleleft$$

A GRS can be easily given a group structure. To this purpose the addition of two elements is defined as follows: if $x, y, z \in S[G, \nu]$ then z is the addition of x and y, $z = x + y$, when $z(s) = x(s) + y(s)$ for all $s \in \nu$. Similarly z is the difference between x and y, $z = x - y$ if $y(s) - x(s) = z(s)$. For later use it is observed that by virtue of these definitions, any element $x \in S[G, \nu]$ can be represented as follows: $x = \sum_{t \in \nu} dP(t)x$. Similarly $P^t x$ and $P_t x$ can be represented by

$$P^t x = \sum_{s \leq t} dP(s)x, \quad P_t x = \sum_{s \geq t} dP(s)x.$$

Consider now operators T, T', T'' all mapping $S[G, \nu] \rightarrow S[G, \nu]$. The operator T is the addition of T' and T'', $T = T' + T''$, if for every $x \in S[G, \nu]$ the following relation holds: $Tx = T'x + T''x$. T is the difference between T' and T'', $T = T' - T''$, if $Tx = T'x - T''x$. T is the composition of T' and T'', $T = T'T''$ is for every $x \in S[G, \nu]$ the following relation holds: $Tx = T'(T''x)$.

An operator T on a GRS is unbiased if $T[\phi] = \phi$ where ϕ indicates the element x in $S[G, \nu]$ such that $x(s) = \phi$ (the null element in G) for all $s \in \nu$. In the sequel only unbiased operators will be considered. This is not a serious limitation. If T is not unbiased, the development will still be valid for the operator $T - T[\phi]$.

2.3 Causality Concepts in GRS

In this section we introduce and discuss some basic concepts related to causality. The notation T is used to indicate an operator (unbiased) on $S[G, \nu]$ and x and y indicate two elements in $S[G, \nu]$.

Definition 1. T is causal (anticausal) if $P^t x = P^t y$ implies $P^t Tx = P^t Ty$ ($P_t x = P_t y$ implies $P_t Tx = P_t Ty$).

Definition 2. T is memoryless if it is simultaneously causal and anticausal.

Definition 3. T is strongly causal (strongly anticausal) if T is causal (anticausal) and $TdP(t)x = \phi$.

Definition 4. T is strictly causal (strictly anticausal) if $\bar{P}^t x = \bar{P}^t y$ implies $P^t Tx = P^t Ty$ ($\bar{P}_t x = \bar{P}_t y$ implies $P_t Tx = P_t Ty$).

Definition 5. T is crosscausal if $TdP(t)x = \phi$; $P^t x = \phi$ implies $P^t Tx = \phi$ and $P_t x = \phi$ implies $P_t Tx = \phi$.

Definition 6. T is strongly crosscausal if $\bar{P}^t x = \phi$ implies $P^t Tx = \phi$ and $\bar{P}_t x = \phi$ implies $P_t Tx = \phi$.

The concepts of causal, anticausal, and memoryless systems are in formal agreement with those proposed by Porter [42], [43] and Saeks [48]. It will be apparent later that this agreement extends also to the concepts of strictly causal and strictly anticausal systems.

The concepts of crosscausality and strong causality find no correspondence in previous work of other authors. It will become clear with the unfolding of the present development, that these concepts are essential for the study of the causality structure of nonlinear systems. In Section 7 it is shown that in the case of additive systems the concept of crosscausality vanishes and strong causality coincides with strict causality. This explains in part the absence of these concepts in the classical literature.

The attention of the reader is called to the fact that in definitions 1-6 the domain of the operator T is assumed to be the whole space $S[G, \nu]$. When the operator T is defined only on a particular subset E of $S[G, \nu]$ the concepts of causal, anticausal, and memoryless systems can be easily extended. In Definition 1 for example, it would be sufficient to add the sentence "for all x and y in E".

To extend the concepts of crosscausality, strong causality, and strict causality appropriate and less evident modifications are required.

Usually these modifications must be varied in order to suit each specific situation. This is a rather serious limitation and will again present itself when the question of canonical causality decomposition is considered. This limitation should be viewed as the price being paid to the inevitable compromise: to obtain a development which is restricted enough as to lead to meaningful results and which at the same time is general enough as to provide ground for further investigation when these results are not directly applicable.

In the sequel it will be helpful to refer to the causality related concepts through an alphabetic code. For this purpose the following shorthand notations will be adopted.

A = anticausal	X = crosscausal
C = causal	$\underline{X}$ = strongly crosscausal
M = memoryless	$\underline{A}$ = strongly anticausal
$\bar{A}$ = strictly anticausal	$\underline{C}$ = strongly crosscausal
$\bar{C}$ = strictly causal	

The alphabet $\mathcal{A} = \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$ will be used extensively and in various ways. On some occasions, for instance, a sentence of the type "the operator T is either causal, or anticausal, or memoryless" is abbreviated as follows "the operator T is $\alpha \in \{C, A, M\}$. On other occasions to remind the reader of the causality character of the operator T, the symbols T_C, T_A, T_M are used.

As an illustration of the notational convenience provided by the proposed causality alphabetic code, it is helpful to consider the statement of the following principle of causal duality*: "Let a statement or equality be phrased using relations involving concepts associated to the alphabet $\mathcal{A} = \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$ and families of truncation operators such as $\{P^t, P_t, \bar{P}^t, \bar{P}_t\}$. Then the statement or equality remains valid if the following interchange in symbols occurs:

$$\begin{array}{llll} P^t \to P_t, & \bar{P}^t \to \bar{P}_t, & P_t \to P^t, & \bar{P}_t \to \bar{P}^t \\ C \to A, & \underline{C} \to \underline{A}, & A \to C, & \underline{A} \to \underline{C} \\ \bar{C} \to \bar{A}, & \bar{A} \to \bar{C}, & \underline{X} \to \underline{X}, & X \to X. \end{array}$$

At this point the formulation aspect of this study has been laid down and the above machinery can be used to answer some of the basic questions of interest. First, to gain some familiarity with the various concepts it is helpful to discuss a simple example.

Example 1. Consider the GRS described in Example 2.1. Every element x in this GRS is given by a function $x(\cdot)$ mapping every $t \in [0, \infty)$ into an element $x(t) \in R$. Consider the operator T defined as follows: if $y, x \in S[G, \nu]$ and $y = Tx$ then

$$y(t) = x(t + \tau_1)x(t + \tau_2)$$

where τ_1 and τ_2 are real numbers and we have used the notation $x(s) = 0$ for $s < 0$.

*A formal proof is omitted for brevity. The reader should have no difficulty to verify by direct inspection that this principle holds in all future situations where we invoke its validity.

The causality character of T is determined by the location of the point (τ_1, τ_2) in the R^2 plane. For instance, if (τ_1, τ_2) coincides with the origin then T is memoryless. When $(\tau_1, \tau_2) \neq (0,0)$ various situations can occur and are illustrated in Figure 2.1.

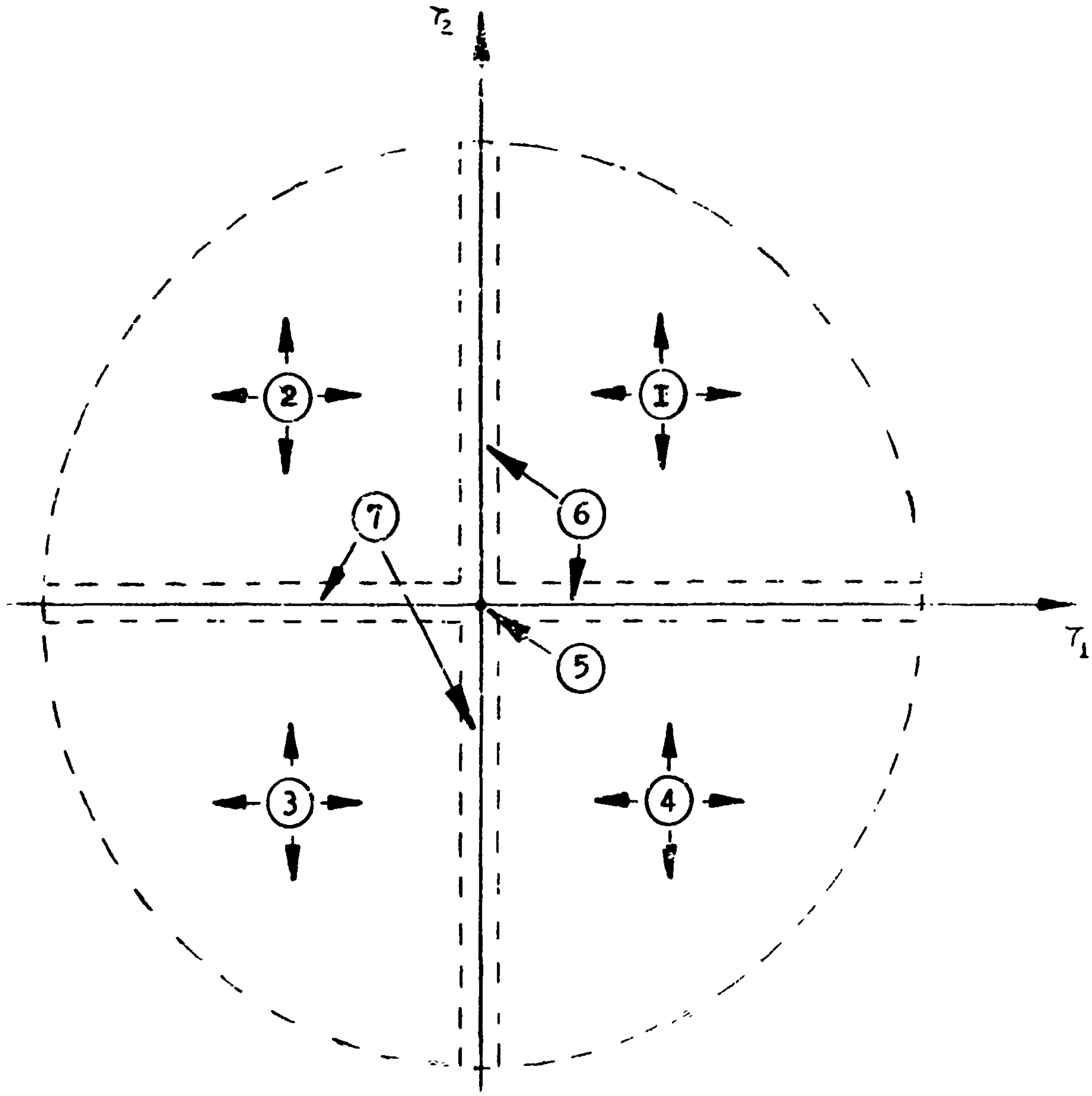
In particular, if the point (τ_1, τ_2) is in the third quadrant then T is, in general, causal. More specifically, if both τ_1 and τ_2 are negative and different from zero then T is simultaneously strictly causal and strongly causal; if τ_1 is negative and τ_2 is equal to zero then T is strongly causal but not strictly causal.

Similarly if the point (τ_1, τ_2) is located in the second or fourth quadrant then T is, in general, a crosscausal operator. Finally, if τ_1 is positive and τ_2 is negative then T is strongly crosscausal. ◁

2.4 Causality Properties of Basic Systems in GRS

In this section we elaborate on some meaningful physical interpretations of the causality concepts formulated in Section 2.3. This will also lead us to a number of mathematical implications which will play a key role later in the development.

To help motivate our discussion recall that, in heuristic terms, a system T is causal if the present of the output does not depend on the future of the input. This can be expressed by saying that the past of the output is only a function of the past of the input. This property is in turn equivalent to a special "sum-type" representation.



1 anticausal, strongly anticausal, strictly anticausal
2 crosscausal, strongly crosscausal
3 causal, strongly causal, strictly causal
4 crosscausal, strongly crosscausal
5 causal, anticausal, memoryless
6 anticausal, strongly anticausal, crosscausal
7 causal, strongly causal, crosscausal

Figure 2.1: Causality character of the operator T as a function of (τ_1, τ_2).

Proposition 1. The following statements are equivalent:

a) T is causal

b) $P^t T = P^t T P^t$ for all $t \in \nu$

c) T has the following representation $T = \sum_{s \in \nu} dP(s)\, T P^s$.

Proof. a) $\Rightarrow$ b). Suppose that T is causal. If b) is not true, then there exists an $x \in S[G, \nu]$ and $t \in \nu$ such that $P^t Tx \neq P^t T P^t x$. If we now use the notation $y = P^t x$, we obtain that $P^t x = P^t y$ and $P^t Tx \neq P^t Ty$. This is a contradiction to the hypothesis that T is causal.

b) $\Rightarrow$ c). If b) holds then for every $s \in \nu$ we have: $P^s T = P^s T P^s$ and since $dP(s) = dP(s) P^s$ we obtain $dP(s)\, T = dP(s)\, T P^s$. Hence we can write $T = \sum_{s \in \nu} dP(s)\, T P^s$.

c) $\Rightarrow$ a). Suppose that x and y belong to $S[G, \nu]$ and $P^t x = P^t y$. Since $P^t dP(s) = \phi$ when $t < s$, then if c) holds we can write

$$P^t Tx = \sum_{s \leq t} dP(s)\, T P^s x.$$

But for $s < t$ we have $P^s P^t = P^s$. It follows

$$P^t Tx = \sum_{s \leq t} dP(s)\, T P^s P^t x = \sum_{s \leq t} dP(s)\, T P^s P^t y = P^t Ty. \quad \triangleleft$$

Using techniques similar to those adopted above, we obtain the next proposition. This proposition states that a memoryless system is characterized by the property that past or future of the output are a function of, respectively, only the past or the future of the input. In other words, the present of the output is determined only by the present of the input. This property can also be put into a 1-1 correspondence with a special "sum-type" representation.

Proposition 2. The following statements are equivalent

a) T is memoryless

b) $P^tT = P^tTP^t$ and $P_tT = P_tTP_t$

c) $T = \sum_{s \in \nu} dP(s)\,TdP(s)$.

A meaningful physical interpretation can also be assigned to the concepts of strong and strict causality. In particular, a strongly causal system is a causal system with the property that the present of the input cannot affect the present of the output when the past of the input is null. The behavior of a strictly causal system satisfies a more severe requirement. In addition to being causal this type of system has the property that the present of the input can in no way affect the present of the output. These considerations are formalized and made more precise by the following Propositions 3 and 4.

Proposition 3. The following statements are equivalent:

a) T is strongly causal

b) $P^tT = P^tTP^t$ and $dP(t)\,TdP(t) = \phi$

c) T is causal and $\sum_{s \in \nu} dP(s)\,TdP(s) = \phi$.

Proof. a) $\Rightarrow$ b). In view of Proposition 1, all we have to show is that $dP(t)\,TdP(t) = \phi$. Indeed from $P^tT = P^tTP^t$ it follows $\bar{P}^tT = \bar{P}^tT\bar{P}^t$ and from here we obtain

$$P^tTdP(t) = (\bar{P}^t + dP(t))\,TdP(t) = \bar{P}^tTdP(t) + dP(t)\,Tdp(t)$$

$$= \bar{P}^tT\bar{P}^tdP(t) + dP(t)\,TdP(t) = dP(t)\,TdP(t).$$

This implies that if $TdP(t) = \phi$ then we also must have $dP(t)\,TdP(t) = \phi$.

b) $\Rightarrow$ c). If b) is true then, from Proposition 1, T is causal. Moreover, since we must have $dP(s)\,TdP(s) = \phi$ for every $s \in \nu$ then it follows that we must also have $\sum_{s\in\nu} dP(s)\,TdP(s) = \phi$.

c) $\Rightarrow$ a). If c) holds then T is causal and we must have $dP(t)\,TdP(t) = \phi$. This implies that $P^tTdP(t) = \phi$ and T must then be strongly causal. ◁

Proposition 4. The following statements are equivalent:

a) T is strictly causal

b) $P^tT = P^tT\bar{P}^t$

c) $T = \sum_{s\in\nu} dP(s)\,T\bar{P}^s$ and $\sum_{s\in\nu} dP(s)\,TdP(s) = \phi$.

Proof. a) $\Rightarrow$ b). If b) is not true then we can find a pair $x, y \in S[G,\nu]$ such that $\bar{P}^tx = \bar{P}^ty$ and $P^tTx \neq P^tTy$. This is a contradiction to a).

b) $\Rightarrow$ c). If b) holds then for every $t \in \nu$ we have

$$dP(t)\,T = dP(t)\,P^tT = dP(t)\,P^tT\bar{P}^t = dP(t)\,T\bar{P}^t$$

and

$$dP(t)\,TdP(t) = dP(t)\,T\bar{P}^tdP(t) = \phi.$$

From here it follows that

$$T = \sum_{s \in \nu} dP(s)\, T = \sum_{s \in \nu} dP(s)\, T\bar{P}^s \text{ and } \sum_{s \in \nu} dP(s)\, T dP(s) = \phi.$$

c) $\rightarrow$ a). Suppose that c) holds and let $x, y \in S[G, \nu]$ be such that $\bar{P}^t x = \bar{P}^t y$. Then we have

$$P^t Tx = \sum_{s \leq t} dP(s)\, T\bar{P}^s x = \sum_{s \leq t} dP(s)\, T\bar{P}^s \bar{P}^t x$$

$$= \sum_{s \leq t} dP(s) T\bar{P}^s \bar{P}^t y = P^t Ty.$$

We can then conclude that T is strictly causal. ◁

The causality behavior of a crosscausal system has a physical interpretation which is similar to those seen in the case of causal, anticausal or memoryless systems. In this case the past or the future of the output are null whenever, respectively, past or future of the input are null. Moreover, if past and future of the input are simultaneously null then the present of the input has no effect on the output.

Analogously, a strongly crosscausal system is a crosscausal system with the additional property that the present of the output is null whenever the past or the future of the input is null.

To further illustrate the nature of this type of system we introduce Propositions 5 and 6.

<u>Proposition 5.</u> **The following statements are equivalent:**

a) T is crosscausal

b) $T\bar{P}^t = \bar{P}^t T \bar{P}^t$, $T\bar{P}_t = \bar{P}_t T \bar{P}_t$ **and** $TdP(t) = \phi$

c) $\sum_{s\in\nu} dP(s)\,T\bar{P}^s = \sum_{s\in\nu} dP(s)\,T\bar{P}_s = \sum_{s\in\nu} dP(s)\,TdP(s) = \phi.$

<u>Proof.</u> a) ⟶ b). Suppose that T is crosscausal. If $T\bar{P}^t \neq \bar{P}^t T \bar{P}^t$ then there exists an element $x \in S[G,\nu]$ such that $T\bar{P}^t x \neq \bar{P}^t T \bar{P}^t x$. Using $\bar{P}^t = I - P_t$ we have $\bar{P}^t T \bar{P}^t = T\bar{P}^t - P_t T \bar{P}^t$ and hence from the above inequality $P_t T \bar{P}^t x \neq \phi$ holds. Letting $y = \bar{P}^t x$ we have

$$P_t Ty = \phi \text{ and } P_t y = \phi$$

Since, by hypothesis, T is crosscausal this is impossible and we must then have $T\bar{P}^t = \bar{P}^t T \bar{P}^t$. Invoking the principle of causal duality we also obtain $\bar{P}_t T = \bar{P}_t T \bar{P}_t$*. Finally suppose $TdP(t) \neq \phi$. Then there exists an element $x \in S[G,\nu]$ such that $TdP(t)\,x \neq \phi$. This is again a contradiction to a).

b) ⟶ c). Suppose that b) holds, then we have

$$T\bar{P}^s = \bar{P}^s T \bar{P}^s \text{ and } T\bar{P}_s = \bar{P}_s T \bar{P}_s.$$

From these relations and the fact that $TdP(s) = \phi$ we obtain

$$TdP(s) = \phi = \bar{P}^s TdP(s) + dP(s)\,TdP(s) + \bar{P}_s TdP(s)$$

---

*For a direct proof the reader is invited to apply the change in symbols indicated by the principle of causal duality and to repeat "verbatim" the argument used to obtain $T\bar{P}^t = \bar{P}^t T \bar{P}^t$.

It follows

$$\phi = \overline{P}^s T P_s dP(s) + dP(s) T dP(s) + \overline{P}_s T P^s dP(s)$$

$$= \overline{P}^s P_s T P_s dP(s) + \overline{P}_s P^s T P^s dP(s) + dP(s) T dP(s)$$

$$= dP(s) T dP(s) = \phi$$

and consequently

$$\sum_{s \in \nu} dP(s) T dP(s) = \phi.$$

Moreover, using the fact that $dP(s)\overline{P}^s = dP(s)\overline{P}_s = \phi$ we obtain

$$\sum_{s \in \nu} dP(s) T\overline{P}^s = \sum_{s \in \nu} dP(s)\overline{P}^s T\overline{P}^s = \phi$$

and

$$\sum_{s \in \nu} dP(s) T\overline{P}_s = \sum_{s \in \nu} dP(s)\overline{P}_s T\overline{P}_s = \phi.$$

We can then conclude that c) holds.

c) $\longrightarrow$ a). Suppose that c) holds. For every $x \in S[C, \nu]$ we have

$$TdP(t)x = \sum_{s \in \nu} dP(s) TdP(t)x$$

$$= \sum_{s<t} dP(s) TdP(t)x + dP(t) TdP(t)x + \sum_{s>t} dP(s) TdP(t)x.$$

Noting that $\overline{P}_s dP(t) = dP(t)$ for $s < t$, $\overline{P}^s dP(t) = dP(t)$ for $s > t$, $dP(t)dP(s) = dP(t)$ for $s = t$ and $dP(t)dP(s) = \phi$ for $s \neq t$, we obtain

$$TdP(t)x = \sum_{s<t} dP(s)T\bar{P}_s dP(t)x + \sum_{s>t} dP(s)T\bar{P}^s dP(s)x +$$

$$dP(t) \sum_{s\in\nu} dP(s)TdP(s)dP(t)x = \phi.$$

Moreover if $P_t x = \phi$, then we also have

$$P_t Tx = P_t \sum_{s\geq t} dP(s)T\bar{P}^t x = \sum_{s\geq t} dP(s)T\bar{P}^s P^t x = \phi.$$

From the above and the principle of causal duality it follows that $P^t y = \phi$ implies $P^t Ty = \phi$. It can then be concluded that T is cross-causal. ◁

Proposition 6. The following statements are equivalent:

a) T is strongly crosscausal

b) $TP^t = \bar{P}^t TP^t$ and $TP_t = \bar{P}_t TP_t$

c) $\sum_{s\in\nu} dP(s)TP^s = \sum_{s\in\nu} dP(s)TP_s = \sum_{s\in\nu} dP(s)TdP(s) = \phi$

Proof. a) ⟶ b). Suppose that a) holds. If $TP^t \neq \bar{P}^t TP^t$, then there exists an $x \in S[G, \nu]$ such that $TP^t x \neq \bar{P}^t TP^t x$. This implies $P_t TP^t x \neq \phi$. Using the notation $P^t x = y$ we then obtain that $\bar{P}_t y = \phi$ and $P_t Ty \neq \phi$. This is a contradiction to a). We must then have $TP^t = \bar{P}^t TP^t$. Invoking the principle of causal duality this argument gives also $TP_t = \bar{P}_t TP_t$.

b) ⟶ c). If b) holds then for each $t \in \nu$ we must have

$$dP(t)TP^t = dP(t)\bar{P}^t TP^t = \phi, \quad dP(t)TP_t = dP(t)\bar{P}_t TP_t = \phi$$

and

$$dP(t)TdP(t) = dP(t)\bar{P}^t TdP(t) = \phi.$$

It follows

$$\sum_{s\in\nu} dP(s)\,TP^s = \sum_{s\in\nu} dP(s)\,TP_s = \sum_{s\in\nu} dP(s)\,TdP(s) = \phi.$$

c) $\Rightarrow$ a). Suppose that c) is true. Then for every $x \in S[G,\nu]$ such that $\bar{P}_t x = \phi$ we must have

$$P_t Tx = \sum_{s\geq t} dP(s)\,TP^t x = \sum_{s\geq t} dP(s)\,TP^s P^t x = \phi.$$

By the principle of causal duality this also says that if $y \in S[G,\nu]$ and $\bar{P}^t y = \phi$ then $P^t Ty = \phi$. It can then be concluded that T is strongly cross-causal. ◁

2.5 Causality and the Composition of Systems in GRS

In this section we continue our investigation on some causality related aspects and properties. In particular Propositions 1-5 illustrate some properties connected with the addition and composition of systems with a special causality structure. Two disjointedness properties of these systems are given in Propositions 6 and 7.

In most nontrivial applications, basic subsystems are combined through the operations of addition and composition. A first natural question then is whether the class of systems which are $\alpha \in \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$ is closed under such operations. The following proposition shows that the answer to this question is affirmative.

Proposition 1. If T and T' are $\alpha \in \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$ then T + T' and TT' are also α.

Proof. In view of the similarity among the various cases, we will only consider $\alpha = C$ and $\alpha = \underline{X}$. Suppose then that T and T' are C. From Proposition 4.1 we have

$$P^t T = P^t T P^t \text{ and } P^t T' = P^t T' P^t.$$

From here we obtain

$$P^t(T + T') = P^t T + P^t T' = P^t T P^t + P^t T' P^t = P^t (T + T') P^t$$
$$P^t TT' = P^t T P^t T' = P^t T P^t T' P^t = P^t TT' P^t.$$

Applying again Proposition 4.1 it follows that T + T' and TT' are C.

Suppose now that T and T' are $\underline{X}$. From Proposition 4.6 we have

$$TP^t = \bar{P}^t T P^t \text{ and } TP^t = \bar{P}_t T P_t.$$

This implies

$$(T + T') P^t = TP^t + T'P^t = \bar{P}^t T P^t + \bar{P}^t T' P^t = \bar{P}^t (T + T') P^t$$

and by the principle of causal duality

$$(T + T') P_t = \bar{P}_t (T + T') P_t.$$

By Proposition 4.6 it follows that T + T' is strongly crosscausal. Similarly for TT' we have

$$TT'P^t = TP^t T' P^t = \bar{P}^t T P^t T' P^t = \bar{P}^t TT' P^t$$

$$TT'P_t = TP_t T' P_t = \bar{P}_t T P_t T' P_t = \bar{P}_t TT' P_t$$

and consequently TT' is also strongly crosscausal. ◁

The causality properties of a system given by the composition of systems with diverse causality structures can usually be determined only by a case by case analysis. Under special circumstances some general statements are, however, possible. The next proposition, for instance, indicates that whenever a system T is composed with a memoryless system T' then the causality character of TT' and T'T coincides with that of T.

Proposition 2. If T is memoryless and T' is $\alpha \in \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$ then TT' and T'T are also α.

Proof. This proof follows a pattern which is similar for the various choices of α. It will then be sufficient to treat the case in which T' is $\underline{X}$. From Propositions 2 and 6 we obtain the following relations

$$P^tT = P^tTP^t \text{ and } P_tT = P_tTP_t$$

$$T'P^t = \bar{P}^tT'P^t \text{ and } T'P_t = \bar{P}_tT'P_t.$$

From here we also obtain

$$TP^t = P^tTP^t + \bar{P}_tTP^t = P^tTP^t + \bar{P}_tT\bar{P}_tP^t = P^tTP^t$$

$$TP_t = P_tTP_t + \bar{P}^tTP_t = P_tTP_t + \bar{P}^tT\bar{P}^tP_t = P_tTP_t.$$

It then follows that

$$TT'P^t = T\bar{P}^tT'P^t = \bar{P}^tT\bar{P}^tT'P^t = \bar{P}^tTT'P^t$$

$$TT'P_t = T\bar{P}_tT'P_t = \bar{P}_tT\bar{P}_tT'P_t = \bar{P}_tTT'P_t$$

and similarly,

$$T'TP^t = \bar{P}^t T'TP^t$$

$$T'TP_t = \bar{P}_t T'TP_t.$$

From these relations and Proposition 4.6 we can conclude that TT' and T'T are indeed $\underline{X}$. ◁

The next Propositions 3, 4 and 5 illustrate an interesting reproducing property of strongly causal, strictly causal and strongly crosscausal systems. According to this property, the classes of strictly causal and strongly causal systems can be viewed as a "symmetric ideal" in the class of causal systems. Similarly the class of strongly crosscausal systems can be viewed as a symmetric ideal in the larger class of crosscausal systems. For brevity we will only prove Proposition 3.

Proposition 3. If T is strongly causal and T' is causal then TT' and T'T are also strongly causal.

Proof. From Proposition 4.1 and 4.3 we have the following relations

$$P^tT = P^tTP^t \text{ and } dP(t)\,TdP(t) = \phi$$

$$P^tT' = P^tT'P^t.$$

These relations also imply

$$TP_t = P_tTP_t \text{ and } T'P_t = P_tT'P_t.$$

It follows

$$P^tTT' = P^tTP^tT' = P^tTP^tT'P^t = P^tTT'P^t$$

and similarly

$$P^tT'T = P^tT'TP^t.$$

Moreover,

$$dP(t)\,TT'dP(t) = dP(t)\,TP^tP_tT'dP(t) = dP(t)\,TdP(t)\,T'dP(t) = \phi$$

and consequently

$$dP(t)\,T'TdP(t) = \phi.$$

From the above relations and Proposition 4.3 we can then conclude that T'T and TT' are strongly causal. ◁

Proposition 4. If T is strictly causal and T' is causal then TT' and T'T are also strictly causal.

Proposition 5. If T is strongly crosscausal and T' is crosscausal then TT' and T'T are strongly crosscausal.

To conclude our review of causality properties, we observe that the various causality classes under consideration are, in general, overlapping. For example a memoryless system is also causal and anticausal. Similarly a system which is crosscausal can, at the same time, be causal or anticausal. The next result shows that the classes of strongly causal, strongly anticausal, strongly crosscausal and memoryless systems, however, are disjoint.

Proposition 6. If T is simultaneously α_1 and α_2 with $\alpha_1, \alpha_2 \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ then either $\alpha_1 = \alpha_2$ or T is the null operator.

Proof. For the various choices of α_1 and α_2 the arguments to be used turn out to be similar. It will be then sufficient to treat a special case. Suppose, for instance, that T is simultaneously $\underline{X}$ and $\underline{C}$. Then from Propositions 4.1 and 4.6 we have

$$T = \sum_{s \in \nu} dP(s) TP^s \text{ and } \sum_{s \in \nu} dP(s) TP^s = \phi.$$

It follows that T must be the null operator. ◁

The next result is similar to Proposition 6 and states that also the classes of strictly causal, strictly anticausal, memoryless and crosscausal systems are disjoint.

Proposition 7. If T is simultaneously α_1 and α_2 with $\alpha_1, \alpha_2 \in \{\bar{A}, \bar{C}, M, X\}$ then either $\alpha_1 = \alpha_2$ or T is the null operator.

The above discussion provides a clear picture of the relations of containment among the various causality classes. These relations are further illustrated in Figure 2.2.

2.6 Canonical Causality Decomposition in GRS

The development in the previous sections provides insight into the causality structure of those systems which are basic, that is

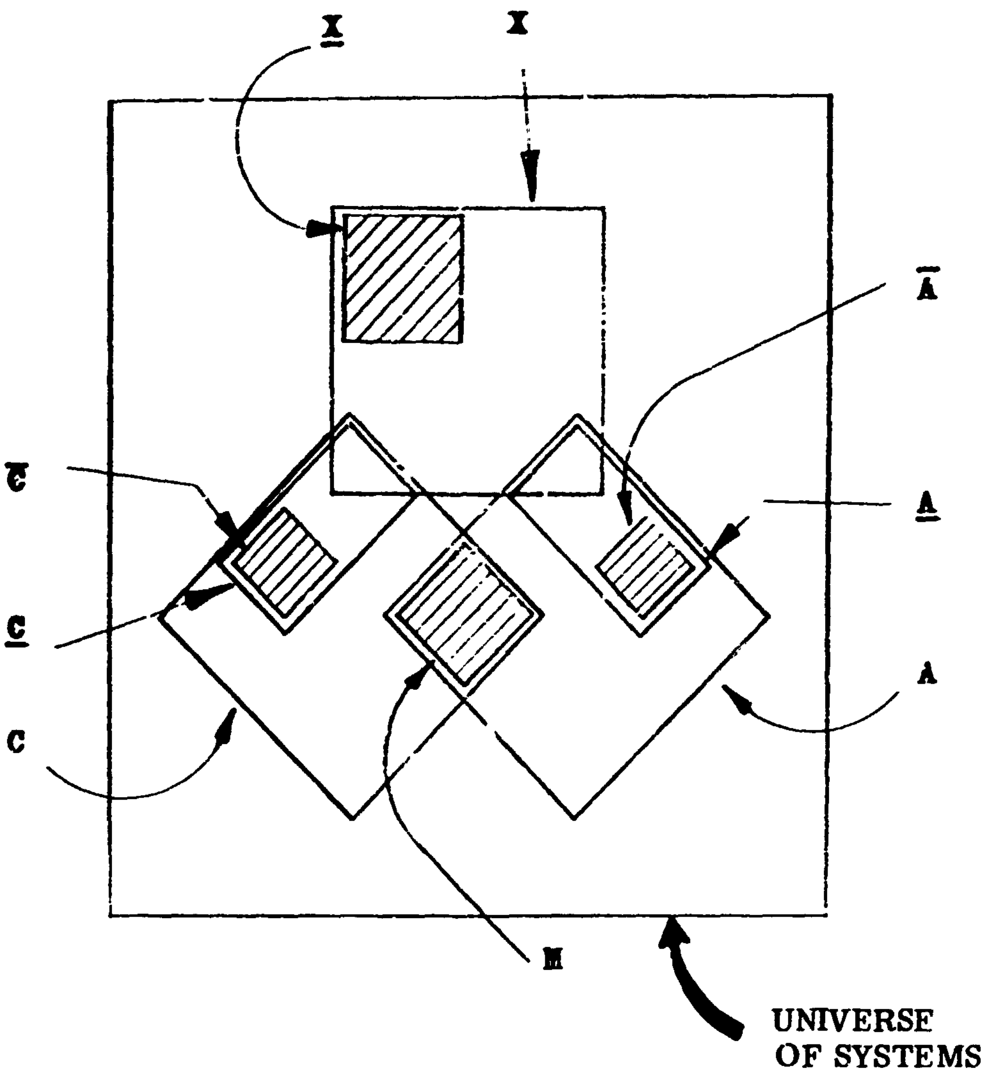


Figure 2. 2: Relations of containment among various classes of systems.

systems which are $\alpha \in \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$. The purpose of this section is to extend that insight to the case of more general systems.

To help motivate the development, observe that our insight into the causality structure of basic systems can be readily used to characterize the causality structure of those systems which are formed by the addition of basic systems. The natural question then arises as to whether it is possible to represent a general system as the addition of basic systems. If this representation does exist it is also of interest to know whether it is unique.

Such questions as those above have been considered in the technical literature and are known collectively as the canonical causality decomposition problem [48]. However, as the systems considered in [48] are linear and defined on a Hilbert space while our systems are neither linear nor are defined in a linear space, the answer given in [48] is of little help to this development.

Concerning the matter of uniqueness suppose, for example, that $T = T_A + T_C$. Then we can also write $T = T'_C + T'_A$, where $T'_A = T_A - T_M$, $T'_C = T_C - T_M$ and T_M is any memoryless operator. This implies that the decomposition of T is not unique.

A moment of reflection shows that the above argument is successful because the various classes of basic systems are overlapping. On the other hand we have seen in Proposition 5.6 that the classes of systems $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ are disjoint. From here it appears natural to modify the causality decomposition question with the following more specific formulation: "is it possible to decompose a general system into the sum of systems which are $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$? If a decomposition of this type exists it will be called a canonical causality decomposition*.

To begin our study, we settle the uniqueness question with the following proposition.

Proposition 1. If a canonical causality decomposition exists then it is unique.

Proof. Suppose that a system T has two canonical decompositions. Then we can write

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}} = T'_{\underline{A}} + T'_{\underline{C}} + T'_M + T'_{\underline{X}}$$

From here it follows that

$$T_M - T'_M = T'_{\underline{A}} - T_{\underline{A}} + T'_{\underline{C}} - T_{\underline{C}} + T'_{\underline{X}} - T_{\underline{X}} \tag{1}$$

and since $T_M - T'_M$ is a memoryless operator, from Proposition 4.2 we must have

$$T_M - T'_M = \sum_{s \in \nu} dP(s)\,(T_M - T'_M)\,dP(s)\,.$$

This leads to the following

*Note that an alternative formulation of a causality decomposition is also suggested by Proposition 5.7. While attention is focused on the $\{\underline{A}, \underline{C}, M, \underline{X}\}$ decomposition, a decomposition based on $\{\bar{A}, \bar{C}, M, X\}$ can be handled with similar techniques and results.

$$T_M - T'_M = \sum_{s\in\nu} dP(s)\,(T'_{\underline{A}} - T_{\underline{A}})\,dP(s) + \sum_{s\in\nu} dP(s)\,(T'_{\underline{C}} - T_{\underline{C}})\,dP(s)$$

$$+ \sum_{s\in\nu} dP(s)\,(T'_{\underline{X}} - T_{\underline{X}})\,dP(s)\,.$$

Applying Propositions 4.3 and 4.6 we obtain that each term on the right hand side of the above equation is null. We can then conclude that $T_M = T'_M$.

Let us now show that $T_{\underline{C}} = T'_{\underline{C}}$. Since $T'_M - T_M = \phi$, Equation (1) leads to

$$T_{\underline{C}} - T'_{\underline{C}} = T'_{\underline{A}} - T_{\underline{A}} + T'_{\underline{X}} - T_{\underline{X}}.$$

Since $T_{\underline{C}} - T'_{\underline{C}}$ is strongly causal we can apply Proposition 4.3 and obtain

$$T_{\underline{C}} - T'_{\underline{C}} = \sum_{s\in\nu} dP(s)\,(T_{\underline{C}} - T'_{\underline{C}})\,P^s.$$

This leads to the following

$$T_{\underline{C}} - T'_{\underline{C}} = \sum_{s\in\nu} dP(s)\,(T'_{\underline{A}} - T_{\underline{A}})\,P^s + \sum_{s\in\nu} dP(s)\,(T'_{\underline{X}} - T_{\underline{X}})\,P^s$$

Observe that the second term on the right hand side of the above equation is null by Proposition 4.6 and the fact that $T'_{\underline{X}} - T_{\underline{X}}$ is strongly crosscausal. The first term is also null as can be seen by using the strong anticausality of $T'_{\underline{A}} - T_{\underline{A}}$ together with the dual of Proposition 4.3, namely

$$\sum_{s \in \nu} dP(s)(T'_{\underline{A}} - T_{\underline{A}})P^s = \sum_{s \in \nu} dP(s)P_s(T'_{\underline{A}} - T_{\underline{A}})P_sP^s$$

$$= \sum_{s \in \nu} dP(s)(T'_{\underline{A}} - T_{\underline{A}})dP(s) = \phi.$$

We can then conclude that $T'_{\underline{C}} = T_{\underline{C}}$, and from the principle of causal duality $T'_{\underline{A}} = T_{\underline{A}}$. Finally from Equation (1) and the fact that $T'_M - T_M = T'_{\underline{C}} - T_{\underline{C}} = T'_{\underline{A}} - T_{\underline{A}} = \phi$, we have $T_{\underline{X}} - T_{\underline{X}} = \phi$. Hence $T_{\underline{X}} = T'_{\underline{X}}$ and the proof is complete. ◁

From the above proposition if T has the property that $T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}$ it is meaningful to talk about the α component of T, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$. In this regard we will use the following definition.

<u>Definition 1.</u> If $T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}$, then $T_{\underline{A}}, T_{\underline{C}}, T_M$ and $T_{\underline{X}}$ are called respectively strong anticausal, strongly causal, memoryless and strongly crosscausal <u>components</u> of T.

In general the question of existence of a canonical decomposition can be viewed in terms of the existence of a special set of mappings on the space of operators. This point of view offers some technical advantages and it is convenient to clarify it.

To begin with, let $\$$ indicate the space of all operators T on a GRS, and $\mathbf{T}$ an operator on $\$$. To emphasize that $\mathbf{T}$ maps a space of operators into itself, $\mathbf{T}$ is called a <u>transformator</u>*. A transformator

*This terminology is inspired from Gohberg and Krein [25].

$\mathbf{T}$ will be indicated by $\mathbf{T}_\alpha$, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ if for each $T \in \$$ we have that $\mathbf{T}_\alpha\,[T]$ is an α operator. For example the range of $\mathbf{T}_{\underline{X}}$ consists of the class of operators which are $\underline{X}$. Occasionally we will also use the notation I to indicate the identity transformator, that is the transformator with the property that $I\,[T] = T$ for every T in $\$$.

The following proposition shows the equivalence between existence of a canonical decomposition and existence of a special set of transformators.

<u>Proposition 2.</u> A canonical causality decomposition always exists if and only if there exists a set of transformators $\mathbf{T}_\alpha$, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\} = \underline{A}$, such that the following properties are satisfied:

pi) $\mathbf{T}_\alpha$ is defined on $\$$

pii) For each pair $(\alpha, \gamma) \in \underline{A}^2$ we have that

$$\mathbf{T}_\alpha \mathbf{T}_\gamma = \phi \text{ if } \alpha \neq \gamma \text{ and } \mathbf{T}_\alpha \mathbf{T}_\gamma = \mathbf{T}_\alpha \text{ if } \alpha = \gamma$$

piii) $\sum_{\alpha \in \underline{A}} \mathbf{T}_\alpha = I.$

<u>Proof.</u> "only if". If each T has a causality canonical decomposition we can define the transformators $\mathbf{T}_\alpha$ according to the following rule: $\mathbf{T}_\alpha\,[T] = T_\alpha$ where T_α is the α component of T. Then properties pi) and piii) are clearly satisfied. To prove pii) suppose that $\mathbf{T}_\alpha \mathbf{T}_\gamma \neq \phi$ for $\alpha \neq \gamma$. Then we can find an operator T such that T is α and the γ component of T is different from zero. It would follow that T has at

least two distinct causality canonical decompositions. This is a contradiction to Proposition 1.

"if". Suppose that there exists a set of transformators $\mathbf{T}_\alpha$ such that pi), pii) and piii) are satisfied. Then for every operator $T \in \$$ we can compute the operators $T_\alpha = \mathbf{T}_\alpha[T]$, for each $\alpha \in \underline{A}$, and from piii) we can write

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}.$$

This means that T has a causality canonical decomposition. ◁

With the above result and the development in Section 4, it is now easy to state and prove that a canonical causality decomposition always exists.

Proposition 3. Every system on a GRS can be decomposed into the sum of strongly causal, strongly anticausal, strongly crosscausal and memoryless components.

Proof. Suppose that T is an operator on $S[G, \nu]$ and define a transformator $\mathbf{T}_M$ according to the following rule:

$$\mathbf{T}_M[T] = \sum_{s \in \nu} dP(s)\, T\, dP(s).$$

From Proposition 4.2, $\mathbf{T}_M[T]$ is a memoryless operator. Define now the transformators $\mathbf{T}_{\underline{C}}$, $\mathbf{T}_{\underline{A}}$ as follows

$$\mathbf{T}_C[T] = \sum_{s \in \nu} dP(s)\,[T - \mathbf{T}_M[T]]\,P^s$$

$$\mathbf{T}_A[T] = \sum_{s \in \nu} dP(s)\,[T - \mathbf{T}_M[T]]\,P_s.$$

From Proposition 4.1 and its dual, $\mathbf{T}_{\underline{C}}[T]$ and $\mathbf{T}_{\underline{A}}[T]$ are respectively strongly causal and strongly anticausal. Finally, consider the transformator $\mathbf{T}_{\underline{X}}$ defined by the following expression

$$\mathbf{T}_{\underline{X}}[T] = T - \mathbf{T}_{M}[T] - \mathbf{T}_{\underline{C}}[T] - \mathbf{T}_{\underline{A}}[T]$$

Observe that the operator $\mathbf{T}_{\underline{X}}[T]$ satisfies the relations in Proposition 4.6. Hence, it is strongly crosscausal. At this point verify that the above set of transformators $\mathbf{T}_M$, $\mathbf{T}_{\underline{C}}$, $\mathbf{T}_{\underline{A}}$, and $\mathbf{T}_{\underline{X}}$ satisfies pi-ii-iii) in Proposition 2. It can then be concluded that T has a causality canonical decomposition. ◁

Combining Propositions 1 and 3, it is now possible to state the following Canonical Causality Decomposition Theorem.

Theorem 1. Every system on a GRS has a canonical causality decomposition and this decomposition is unique.

To illustrate the meaning of the above result we now pause to discuss a simple example.

Example 1. Suppose that $S[G, \nu]$ is described as in Example 2.1. Then every element $x \in S[G, \nu]$ is given by an n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$. For simplicity we will assume that $n = 3$. In this GRS we can consider the (unbiased) system T defined as follows: if $x, y \in S[G, \nu]$ and $y = Tx$, then

$$\begin{cases} y_1 = (x_1)^2 + x_1x_3 + 2x_1x_2 \\ y_2 = 2(x_2)^2 + 4x_1x_3 + 3x_2x_3 + 5(x_1)^2 \\ y_3 = x_1x_2 + 3(x_1)^2 + x_1x_2. \end{cases}$$

It is easily verified that T does not satisfy anyone of the requirements in Definitions 2.1 - 2.6. Therefore T is not a basic system. Theorem 1 says, however, that T can be represented by the addition of basic systems. If these basic systems are chosen to be α, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ then the representation is unique. In this case we can write

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}.$$

Moreover according to the proof of Proposition 3, $T_{\underline{A}}$, $T_{\underline{C}}$, T_M and $T_{\underline{X}}$ can be computed by the expressions

$$T_M = \sum_{i=1}^{3} dP(i)\, T dP(i)$$

$$T_{\underline{A}} = \sum_{i=1}^{3} dP(i)\, TP^i - T_M$$

$$T_{\underline{C}} = \sum_{i=1}^{3} dP(i)\, TP_i - T_M$$

$$T_{\underline{X}} = T - T_{\underline{A}} - T_{\underline{C}} - T_M.$$

The meaning of these expressions is that $T_{\underline{A}}, T_{\underline{C}}, T_M$ and $T_{\underline{X}}$ are described as follows: if $x, y \in S[G, \nu]$ and $y = Tx$, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ then we write:

$$T_{\underline{M}}: \begin{cases} y_1 = (x_1)^2 \\ y_2 = 2(x_2)^2 \\ y_3 = (x_3)^2 \end{cases} \qquad T_{\underline{A}}: \begin{cases} y_1 = 2x_2x_1 + 3x_3x_1 \\ y_2 = 3x_3x_2 \\ y_3 = 0 \end{cases}$$

$$T_{\underline{C}}: \begin{cases} y_1 = 0 \\ y_2 = 5(x_1)^2 \\ y_3 = 2x_1x_2 + 3(x_1)^2 \end{cases} \qquad T_{\underline{X}}: \begin{cases} y_1 = 0 \\ y_2 = 4x_3x_1 \\ y_3 = 0. \end{cases}$$

◁

Often the question of canonical causality decomposition arises in a more complicated setting than the one considered here. In general the complications are due to the following: the operator T may not be defined over all the GRS; the operator T may belong to a particular subset $\$_1$ of $\$$ and it is of interest to determine whether a canonical decomposition exists such that the components of T belong to $\$_2$, another subset of $\$$.

In both such situations the statement of Proposition 1 remains valid. When the domain of T is a proper subset of the GRS it is best to proceed on a case by case basis. An illustration of the type of procedures which might be used is given in [42] and [48] where operators T defined on Hilbert spaces are considered. In regard to the second situation it may be helpful to rephrase the result of Proposition 1 in a slightly more convenient form.

Proposition 4. **The following statements are equivalent:**

a) **Every operator $T \epsilon \$_1$ has a canonical decomposition**

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}} \text{ with } T_{\underline{A}}, T_{\underline{C}}, T_M, T_{\underline{X}} \text{ in } \$_2.$$

b) **There exists a set of transformators $\mathbf{T}_\alpha$, $\alpha \epsilon \{\underline{A}, \underline{C}, M, \underline{X}\}$ such that the following properties are satisfied:**

pi) $\mathbf{T}_\alpha$ is defined over $\$_1$ and maps $\$_1$ into $\$_2$

pii) and piii) identical to pii) and piii) in Proposition 1.

From a conceptual point of view it is important to underline the resemblance between the properties of the set of transformators $\mathbf{T}_\alpha$ considered in Propositions 2 and 4 and the familiar properties of a set of orthogonal "projection-operators". In particular in Section 4.8, it is shown that, in the case of a very special subset $\$_1$, the transformators $\mathbf{T}_\alpha$ can indeed be identified as orthogonal projectors. In general the following proposition holds.

Proposition 5. Suppose that a set of transformators $\mathbf{T}_\alpha$ has the properties pi-ii-iii) considered in Proposition 1. Then the transformators $\mathbf{T}_\alpha$ are additive and idempotent.

Proof.

Let us start to prove that the transformators $\mathbf{T}_\alpha$ are additive. To this purpose consider two operators T and T' and let Y be the operator defined by the following

$$Y = \sum_{\alpha\in\underline{A}} \mathbf{T}_\alpha[T + T'] - \sum_{\alpha\in\underline{A}} \mathbf{T}_\alpha[T] - \sum_{\alpha\in\underline{A}} \mathbf{T}_\alpha[T']$$

where $\underline{A} = \{\underline{A}, \underline{C}, M, \underline{X}\}$. From piii) we have that the operator Y is null. Compute now $\mathbf{T}_\alpha[Y]$. In view of pi-ii) we obtain

$$\mathbf{T}_\alpha[Y] = \mathbf{T}_\alpha[T + T'] - \mathbf{T}_\alpha[T] - \mathbf{T}_\alpha[T'] = \phi .$$

It follows that

$$\mathbf{T}_\alpha[T + T'] = \mathbf{T}_\alpha[T] + \mathbf{T}_\alpha[T'].$$

Let us now show that $\mathbf{T}_\alpha^{\;2} = \mathbf{T}_\alpha$. Suppose that T is an operator in \$. Then by piii) we have

$$\mathbf{T}_{\underline{A}}[T] + \mathbf{T}_{\underline{C}}[T] + \mathbf{T}_M[T] + \mathbf{T}_{\underline{X}}[T] = T.$$

Apply now $\mathbf{T}_\alpha$ to both members of the above equation. The desired result is an immediate consequence of pii). ◁

As we observed in a footnote at the beginning of this section, a canonical causality decomposition can also be envisioned in terms of systems which are $\alpha \in \{\bar{A}, \bar{C}, M, X\}$. Using techniques similar to those illustrated, all of the previous results could be rephrased and proved in order to accommodate this alternative formulation. In particular, for instance, in correspondence to Theorem 1 we would obtain the following.

Proposition 6. Every system can be uniquely decomposed into the sum of strictly causal, strictly anticausal, memoryless and crosscausal components.

Example 2. **Consider the system discussed in Example 1. According to Proposition 6 this system can be uniquely represented by the expression**

$$T = T_{\bar{A}} + T_{\bar{C}} + T_M + T_X.$$

In particular $T_{\bar{A}}, T_{\bar{C}}, T_M$ and T_X are defined as follows: if $x, y \in S[G, \nu]$ and $y = T_\alpha x$, $\alpha \in \{\bar{A}, \bar{C}, M, X\}$, then

$$T_M: \begin{cases} y_1 = (x_1)^2 \\ y_2 = 2(x_2)^2 \\ y_3 = (x_3)^2 \end{cases} \qquad T_{\bar{A}}: \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 = 0 \end{cases}$$

$$T_{\bar{C}}: \begin{cases} y_1 = 0 \\ y_2 = 5(x_1)^2 \\ y_3 = 2x_1x_2 + 3(x_1)^2 \end{cases} \qquad T_X: \begin{cases} y_1 = 2x_2x_1 + 3x_3x_1 \\ y_2 = 3x_3x_2 + 4x_3x_1 \\ y_3 = 0. \end{cases}$$

◁

2.7 Causality and Weakly Additive Operators in GRS

In this section the results of Sections 4, 5 and 6 are specialized for the case in which the systems under consideration are weakly additive.

Definition 1. An operator T on $S[G, \nu]$ is weakly additive if it has the following property

$$T[x] = T[P^t x] + T[\bar{P}_t x] = T[\bar{P}^t x] + T[P_t x]$$

for all $x \in S[G, \nu]$ and every $t \in \nu$.

In control and communication theory a weakly additive operator is a convenient system model in those situations where the sources of nonlinear behavior have a memoryless character. Operators with the weak additivity property have already been considered in the technical literature [23], [35], [40], [69]. Their importance will be further emphasized in Chapter 5 where some important implications of this property will be studied. In the sequel it is shown that most of the causality properties which are usually stated for the class of linear systems, hold "verbatim" for the more general case of weakly additive systems.

Proposition 1. For a weakly additive system T the following statements are equivalent:

a) T is causal

b) $P^t x = \phi$ implies that $P^t Tx = \phi$

c) $TP_t = P_t TP_t$

d) $P^t T = P^t TP^t$

e) $T = \sum_{s \in \nu} dP(s)\, TP^s$.

Proof. a) $\Longrightarrow$ b). Suppose that T is causal. Then for every $t \in \nu$ and all $x, y \in S[G, \nu]$ such that $P^t x = P^t y$, we have $P^t Tx = P^t Ty$. In particular if $P^t x = \phi$ then $P^t Tx = P^t T\phi = \phi$.

b $\rightarrow$ c). Suppose that $TP_t x \neq P_t TP_t$. Then there exists an x in $S[G, \nu]$ such that $TP_t x \neq P_t TP_t x$. This implies that $\overline{P}^t TP_t x = \phi$. If we use the notation $P_t x = y$, then we have $\overline{P}^t Ty \neq \phi$ and $\overline{P}^t y = \phi$. This is a contradiction to b).

c) $\rightarrow$ d). If $P^t T \neq P^t TP^t$ then by the weak additivity of T we have $P^t T\overline{P}_t \neq \phi$. It follows $T\overline{P}_t = P^t T\overline{P}_t + \overline{P}_t TP_t \neq \overline{P}_t T\overline{P}_t$. This is a contradiction to c).

The proof of d) $\rightarrow$ e) and e) $\rightarrow$ a) is identical to that given in Proposition 4.1. ◁

The attention of the reader is called to the formal equivalence between the above result and Proposition 2.3 in [48]. From the principle of causal duality and Proposition 1 we obtain the next proposition.

<u>Proposition 2.</u> If T is weakly additive the following statements are equivalent:

a) T is memoryless

b) $P^t x = \phi$ implies $P^t Tx = \phi$ and $P_t x = \phi$ implies $P_t Tx = \phi$

c) $TP^t = P^t TP^t$ and $TP_t = P_t TP_t$

d) $P^t T = P^t TP^t$ and $P_t T = P^t TP_t$

e) $T = \sum_{s \in \nu} dP(s)\, T dP(s)$.

The following two propositions indicate that in the case of weakly additive systems the concepts of strong and strict causality are equivalent and the concept of crosscausality vanishes.

Proposition 3. For a weakly additive system the concepts of strong and strict causality are equivalent.

Proof. Since every strictly causal system is also strongly causal it is sufficient to show that if a system is weakly additive and strongly causal then it is also strictly causal. Suppose then that T is weakly additive and strongly causal. Applying Proposition 4.3 we obtain

$$T = \sum_{s \in \nu} dP(s)\, TP^{s}$$

and from the weak additivity of T

$$T = \sum_{s \in \nu} dP(s)\, T\bar{P}^{s} + \sum_{s \in \nu} dP(s)\, TdP(s)\,.$$

Applying again Proposition 4.3 we must have $\sum_{s \in \nu} dP(s)\, TdP(s) = \phi$. Hence T is strictly causal. ◁

Proposition 4. If T is weakly additive and crosscausal then T is the null operator.

Proof. If T is a weakly additive operator on $S[G, \nu]$ then for every $x \in S[G, \nu]$ we can write

$$Tx = \sum_{s \in \nu} dP(s)\, TdP(s)\, x + \sum_{s \in \nu} dP(s)\, T\bar{P}^{s} x + \sum_{s \in \nu} dP(s)\, T\bar{P}_{s} x.$$

Suppose that T is also crosscausal. Then we can apply Proposition 4.5 and obtain that all the terms on the right hand side of the above equation are null. This implies that T is the null operator. ◁

Observe that in view of Propositions 3 and 4 the containment relations indicated in Figure 2.2 can be simplified as in Figure 2.3.

In regard to a canonical decomposition theorem for weakly additive operators, we have the following result.

Theorem 1. Every weakly additive system can be uniquely decomposed into the sum of strictly causal, strictly anticausal and memoryless weakly additive systems.

Proof. Suppose that T is an operator on $S[G, \nu]$. By Theorem 6.1 T has a unique canonical decomposition $T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}$. If T is weakly additive then for each $s \in \nu$ the operator $dP(s)\,T$ is weakly additive. This implies that $dP(s)\,T_{\underline{A}}$ and $dP(s)\,T_{\underline{C}}$ are also weakly additive. Suppose for instance, that $dP(s)\,T_{\underline{C}}$ is not. Then we would have a point $t \in \nu$, $t < s$, and an element $x \in S[G, \nu]$ such that $dP(s)\,T_{\underline{C}}P^s \neq dP(s)\,T_{\underline{C}}P^t x + dP(s)\,T_{\underline{C}}(P^s - P^t)x$. This, however, is not possible because by the weak additivity of $dP(s)\,T$ we have

$$dP(s)\,T_{\underline{C}}P^s x = dP(s)\,TP^s x = dP(s)\,TP^t x + dP(s)\,T(P^s - P^t)x$$

$$= dP(s)\,T_{\underline{C}}P^t x + dP(s)\,T_{\underline{C}}(P^s - P^t)x.$$

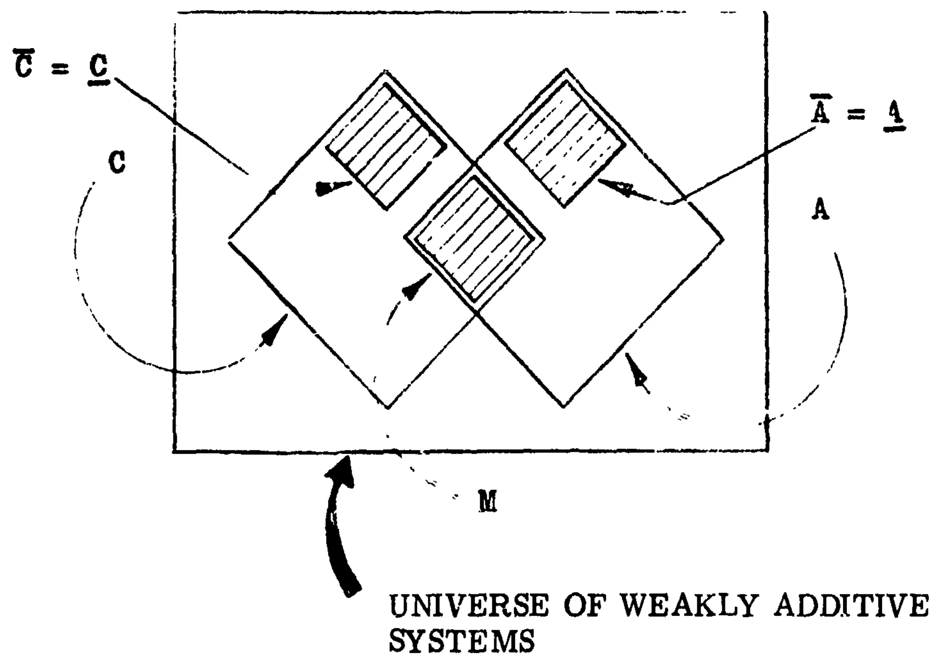


Figure 2. 3 : Relations of containment among various causality classes of weakly additive systems.

From the weak additivity of $dP(s)\,T_{\underline{C}}$ and $dP(s)\,T_{\underline{A}}$ we obtain the weak additivity of $T_{\underline{C}}$ and $T_{\underline{A}}$. Moreover since every memoryless operator is weakly additive so is T_M. It follows that $T_{\underline{X}}$ must also be weakly additive. We can now apply Propositions 3 and 4 and obtain $T_{\underline{X}} = \phi$, $T_{\underline{C}} = T_{\overline{C}}$ and $T_{\underline{A}} = T_{\overline{A}}$. We can then conclude that if T is weakly additive then T has the unique decomposition $T = T_{\overline{A}} + T_{\overline{C}} + T_M$. ◁

2.8 Summary

This chapter is characterized by the development of a mathematical setting in which some basic causality questions may be studied. It contains two novel ideas: the use of the structure of group resolution spaces and the introduction of the class of crosscausal operators. These two ideas are exploited to discuss with a certain mathematical rigor various important properties related to system causality. These properties are emphasized by a number of propositions and theorems in Sections 4, 5, 6 and 7.

The most important theoretical results consist of Theorems 6.1 and 7.1. Theorem 6.1 gives a general canonical decomposition theorem for nonlinear operators on nonlinear spaces. It can be stated as follows: every system in a group resolution space can be decomposed into the sum of strongly causal, strongly anticausal, memoryless and strongly crosscausal systems. When the system under consideration has a "weak additivity" property then the statement of Theorem 6.1 can be improved to the form of Theorem 7.1. This latter theorem also provides a canonical decomposition of the type conjectured by Saeks [48]. To paraphrase this result: every weakly additive system in a group resolution space is given by the sum of strictly causal, strictly anticausal and memoryless weakly additive systems.

3. CONNECTIONS BETWEEN THE CONCEPTS OF CAUSALITY AND STATE

3.1 Introduction

In the previous chapter the structure of a system with respect to time related behavior was investigated. This included the study and formulation of various causality concepts.

It should be observed that the concept of state is also related to the temporal behavior of a system. Heuristically speaking, causality concepts reflect the modalities by which past, present and future of the input affect the present of the output, while the concept of state indicates how past and present of the input affect the future of the output.

From the above observation the question arises as to whether the knowledge of the causality structure of a system supplies some information about the system state structure and, conversely, whether the state structure might have ramifications concerning the causality structure. To investigate this question it is convenient to model a system as an operator on a group resolution space (GRS). A first order of business is to properly define in this context the concept of state.

The format of the present chapter is a natural realization of this line of reasoning. In Section 2 we establish the notion of state in GRS. Section 3 deals with some characterizations of the structure of a system with respect to state. In Section 4 interconnections between state and causality concepts are developed. Section 5 closes the chapter with some concluding remarks.

3.2 On the Notion of State in GRS

In developing the concept of state we are faced with satisfying two partially competing requirements: the notion of state has to have sufficient generality to encompass systems in the various engineering fields of interest; the notion has to be flexible enough as to lead the investigation to meaningful results.

A concept of state based on equivalence classes would be sufficiently general. Such an approach was used by Mesarovich [33], Windernecht [60] and Zadeh [64] among others, and could easily be formalized in our mathematical framework. Such a development does not appear, however, to be sufficiently flexible. The notion of state via the definition of a state decomposition as recently proposed by Saeks in [48], is another potential alternative. Translated in the framework of group resolution spaces (GRS) the following definition would be obtained: a state decomposition of a system T on $S[G, \nu]$ is a triple $(\psi(t), S, \zeta(t))$ where S is a set (state set) and $\psi(t)$, $\zeta(t)$ are families of mappings such that

$$\psi(t) : S[G, \nu] \rightarrow S,\ \psi(t) = \psi(t) P^t ;\ \zeta(t) : S \rightarrow S[G, \nu],\ \zeta(t) = \bar{P}_t \zeta(t) \quad (3.1)$$

and

$$\zeta(t)\psi(t) = \bar{P}_t T P^t \quad (3.2)$$

for every $t \in \nu$.

This latter definition is quite satisfactory in a linear system context. It is not fully satisfactory, however, in the more general framework under consideration. The deficiency of this definition resides in part in the fact that this concept does not satisfy the "consistency conditions" required by Zadeh [65]. In particular it can happen that the knowledge of the state of a system T at a certain time, t, is not sufficient to describe the input-output behavior of T after t. This can perhaps be best illustrated by introducing a simple example.

Example 1. Consider the system T which was discussed in Example 2.3.1. Recall that if x and y belong to $S[G, \nu]$ then x and y are real functions defined on the interval $[0, \infty)$. Moreover if $y = Tx$, then y is given by the following expression:

$$y(t) = x(t+\tau_1)\, x(t+\tau_2) .$$

If we now suppose that $\tau_1 = -\frac{1}{2}$ and $\tau_2 = \frac{1}{2}$, then the triple $(\psi(t), S, \zeta(t))$, with S consisting of a null element, ϕ, and $\zeta(t)$ mapping S into the null element of $\bar{P}_t\, S[G, \nu]$, satisfies Equations (3.1) and (3.2). Therefore according to the definition of state introduced by Saeks, this triple is an admissable state decomposition of T. In this case, however, the state would fail to give an adequate description of the past configuration of T so that the future behavior of T can be satisfactorily characterized. For instance, if $t = 2$ is the present time, then from the state of T at t and the future of x, we cannot in general evaluate the future of y = Tx. Indeed, the future of y depends on the values of x in the interval (1.5, 2] and no record of these values can be found in the state of T. ◁

In the present study attention is focused on a generalization of the state decomposition format. In stating this generalization S_1, S_2 are sets and $S = S_1 \times S_2$ plays the role of the state set. Similarly $\psi_1(t), \psi_2(t), \zeta_1(t), \zeta_2(t)$ are parameterized families of mappings while $\psi(t) = (\psi_1(t)\psi_2(t))$ and $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ play the role of input-to-state and state-to-output mappings. Our format definition is the following.

Definition 1. A state decomposition of an operator T on $S[G, \nu]$ is a triple $(\psi(t), S, \zeta(t))$ such that:

i) $S = S_1 \times S_2$ where S_1, S_2 are sets

ii) $\psi(t) = (\psi_1(t), \psi_2(t))$, where domain $[\psi_i(t)] = P^t S[G, \nu]$ and range $\psi_i(t) \subseteq S_i$, $i=1, 2, t \in \nu$

iii) $\zeta(t) = (\zeta_1(t), \zeta_2(t))$, where domain $\zeta_i(t) = S_i$, range $\zeta_1(t) \subseteq \bar{P}_t S[G, \nu]$ and the range of $\zeta_2(t)$ is contained in the space of operators on $\bar{P}_t S[G, \nu]$

iv) For every $x, y \in S[G, \nu]$ the following relations are satisfied

$$\zeta_1(t)\psi_1(t)P^t x = \bar{P}_t T P^t x$$

$$\zeta_2(t)\left[\left[\psi_2(t)|P^t x|\right]\bar{P}_t y\right] = \delta T_{P^t x}|\bar{P}_t y|$$

where the symbol $\delta T_x[y]$ stands for $T[x+y] - Tx - Ty$.

By the substitution of the term "state" with "costate" and by a subsequent substitution of symbols as indicated in the principle of causal duality, Definition 1 provides a concept which is dual to that of a state decomposition. As this concept will also be useful, we introduce the following additional definition.

<u>Definition 2.</u> A <u>costate decomposition</u> of an operator T on $S[G, \nu]$ is a triple $(\psi(t), S, \zeta(t))$ such that:

i) $S = S_1 \times S_2$, where S_1, S_2 are sets

ii) $\psi(t) = (\psi_1(t), \psi_2(t))$, where domain $[\psi_i(t)] = P_t S[G, \nu]$ and range $\psi_i(t) \subseteq S_i$, $i=1,2$, $t \in \nu$

iii) $\zeta(t) = (\zeta_1(t), \zeta_2(t))$, where domain $\zeta_i(t) = S_i$, range $\zeta_1(t) \subseteq \bar{P}^t S[G, \nu]$ and the range of $\zeta_2(t)$ is contained in the space of operators on $\bar{P}^t S[G, \nu]$

iv) For every $x, y \in S[G, \nu]$ the following relations are satisfied

$$\zeta_1(t)\psi_1(t) P_t x = \bar{P}^t T P_t x$$

$$\zeta_2(t)\Big[[\psi_2(t)[P_t x]]\bar{P}^t y\Big] = \delta T_{P_t x}[\bar{P}^t y].$$

In Definition 1, a state decomposition can be viewed as given by two components:

i) the first component $(\psi_1(t), S_1, \zeta_1(t))$ reflects the influence of the past of the input over the future of the output with the constraint that the future of the input is null;

ii) the second component $(\psi_2(t), S_2, \zeta_2(t))$ indicates how the past of the input influences the effect of the future of the input on the future of the output.

The proposed state decomposition formulation is a natural extension of the formulation of Saeks. The main difference between the two consists of the appearance of $(\psi_2(t), S_2, \zeta_2(t))$ the second component of the state. This inclusion allows the present definition of state to satisfy the forementioned consistency conditions of Zadeh [65]. The concept of state proposed by Saeks satisfies these conditions only in the particular case in which the system is weakly additive (Proposition 2.4). Thus, by the introduction of the component $(\psi_2(t), S_2, \zeta_2(t))$ the gap existing between the definitions given by Saeks and Zadeh appears to be bridged.

3.3 State Related Concepts in GRS

The structure of systems with respect to the notion of state can be diverse. To describe the variety of situations which can occur, it is convenient to introduce the concepts of controllability, observability and minimality. For details on the genesis of these concepts the interested reader is referred to [18].

Definition 1. The state decomposition $(\psi(t), S, \zeta(t))$ is completely controllable with respect to the first (second) component if $\psi_1(t)$ $(\psi_2(t))$ is onto.

Definition 2. The state decomposition $(\psi(t), S, \zeta(t))$ is completely observable with respect to the first (second) component if $\zeta_1(t)$ $(\zeta_2(t))$ is 1-1.

Definition 3. The state decomposition is minimal with respect to the first (second) component if it is completely controllable and completely observable with respect to the first (second) component.

Definition 4. A state decomposition is strictly minimal (strictly completely controllable, strictly completely observable) if it is minimal (completely controllable, completely observable) with respect to both first and second components.

Definition 5. Two state decompositions $(\psi(t), S, \zeta(t))$ and $(\psi'(t), S', \zeta'(t))$ are strictly equivalent if there exists a unique family of invertible mappings $\mathbb{K}(t)$ such that the following conditions are satisfied

$$\psi(t) = \mathbb{K}(t)\,\psi'(t) \quad \text{and} \quad \zeta(t) = \zeta'(t)\,\mathbb{K}^{-1}(t).$$

In what follows we give some results related to the above state concepts. These results allow a better appreciation of our definitions and consequently a better understanding of the results of the next section.

They are, however, not essential to the main development and can be omitted at first reading.

As a start, the following proposition gives a connection between the mathematical concept of minimal state decomposition and the intuitive notion of a state space which contains only "essential" elements.

<u>Proposition 1.</u> Suppose that the completely observable $(\psi(t), S, \zeta(t))$ and the strictly minimal $(\underline{\psi}(t), \underline{S}, \underline{\zeta}(t))$ are state decompositions of T. Then there exists a unique family of mappings $\mathbb{K}(t)$ taking S onto $\underline{S}$ and such that

$$\underline{\psi}(t) = \mathbb{K}(t)\,\psi(t) \quad \text{and} \quad \zeta(t) = \underline{\zeta}(t)\,\mathbb{K}(t)\,. \tag{3.3}$$

<u>Proof.</u> First we show that the family $\mathbb{K}(t)$ exists and satisfies Equation (3.3). For this purpose suppose that, at index t, v is the state of T in S. Let U be the set in $S[G, \nu]$ with the following property:

$$v = \psi(t)\,u \text{ for any } u \in U.$$

For any pair u_1, u_2 in U we have

$$\zeta(t)\,\psi(t)\,u_1 = \zeta(t)\,\psi(t)\,u_2$$

and this implies

$$\zeta_i(t)\,\psi_i(t)\,u_1 = \zeta_i(t)\,\psi_i(t)\,u_2, \quad i=1,2.$$

From this equation and the definition of state decomposition it follows

$$\bar{P}_t T P^t u_1 = \bar{P}_t T P^t u_2 \quad \text{and} \quad \bar{P}_t \delta T_{P^t u_1} [\quad] = \bar{P}_t \delta T_{P^t u_2} [\quad],$$

and from here

$$\underline{\zeta}(t)\, \underline{\psi}(t)\, u_1 = \underline{\zeta}(t)\, \underline{\psi}\, u_2 .$$

Moreover, from the minimality of $(\underline{\psi}(t), \underline{S}, \underline{\zeta}(t))$, $\underline{\zeta}(t)$ must be 1-1 and therefore we must also have

$$\underline{\psi}(t)\, u_1 = \underline{\psi}(t)\, u_2 .$$

Note, at this point, that according to the above construction at each $t \in \nu$ every element v in S defines a unique set U in $S[G, \nu]$; such a set U, in turn, defines a unique element $\underline{v}$ in $\underline{S}$. Thus we have built $\mathcal{K}(t)$ a family of mappings $S \rightarrow \underline{S}$. Clearly for each $u \in S[G, \nu]$ we have

$$\underline{v} = \underline{\psi}(t)\, u = \underline{\zeta}(t)\, \psi(t)\, u = \mathcal{K}(t)\, v$$

and therefore $\underline{\psi}(t) = \mathcal{K}(t)\, \psi(t)$. We also have $\zeta(t) = \mathcal{K}(t)\, \underline{\zeta}(t)$. This follows from

$$\underline{\zeta}(t)\ \mathcal{K}(t)\, \psi(t) = \underline{\zeta}(t)\, \underline{\psi}(t) \quad \zeta(t)\, \psi(t)$$

and the fact that $\psi(t)$ is onto.

It remains to be proved that each mapping $\mathcal{K}(t)$ is onto and that the family $\mathcal{K}(t)$, $t \in \nu$, is unique. If $\mathcal{K}(t)$ is not onto then there exists a state $\underline{v} \in \underline{S}$ such that

$$\mathbb{K}(t)\, v \neq \underline{v} \quad \text{for every } v \in S. \tag{3.4}$$

From the strict minimality of $(\underline{\psi}(t), \underline{S}, \underline{\zeta}(t))$, $\underline{\psi}(t)$ is onto. Hence there exists an input $u \in S[G, \nu]$ such that $\underline{v} = \underline{\psi}(t)\, u_o$. Let $v = \psi(t)\, u_o$, and let U be a subset of $S[G, \nu]$ such that

$$v = \psi(t)\, u_o = \psi(t)\, u, \text{ for any } u \in U.$$

Then the following relation holds

$$\underline{v} = \underline{\psi}(t)\, u \text{ for any } u \in U$$

and from here, $\underline{v} = \mathbb{K}(t)\, v$. This is a contradiction to Equation (3.4). Hence $\mathbb{K}(t)$ must be onto. Finally, suppose that the family $\mathbb{K}(t)$ is not unique. Then for some $t \in \nu$, there would exist at least two mappings $\mathbb{K}_1(t)$ and $\mathbb{K}_2(t)$ such that $\mathbb{K}_1(t) \neq \mathbb{K}_2(t)$ and Equation (3.3) holds. This implies that there would exist an element $v \in S$ such that

$$\mathbb{K}_1(t)\, v \neq \mathbb{K}_2(t)\, v \tag{3.5}$$

and

$$\zeta(t)\, v = \underline{\zeta}(t)\, \mathbb{K}_1(t)\, v = \underline{\zeta}(t)\, \mathbb{K}_2(t)\, v.$$

It would follow

$$\underline{\zeta}(t)\, \mathbb{K}_1(t)\, v - \underline{\zeta}(t)\, \mathbb{K}_2(t)\, v = \phi.$$

But by hypothesis $\underline{\zeta}(t)$ is 1-1. Hence it must be that

$$\mathbb{K}_1(t)\, v = \mathbb{K}_2(t)\, v.$$

This is a contradiction to Equation (3.5). ◁

The next proposition shows that all strictly minimal state decompositions of an operator T are strictly equivalent. This is a generalization of a similar well known result in the context of linear dynamical systems [42], [65]. A result of this type was also given by Saeks for linear systems in Hilbert space [49].

<u>Proposition 2.</u> All strictly minimal state decompositions are strictly equivalent.

<u>Proof.</u> Suppose that $(\psi(t), S, \zeta(t))$ and $(\underline{\psi}(t), \underline{S}, \underline{\zeta}(t))$ are strictly minimal state decompositions for the operator T. We have to show that there exists a family of invertible mappings $\mathbb{K}(t)$ such that the equations in Definition 5 are satisfied.

From Proposition 1, we know that there exists a family of onto mappings $\mathbb{K}(t): S \rightarrow \underline{S}$ such that

$$\underline{\psi}(t) = \mathbb{K}(t)\,\psi(t) \text{ and } \underline{\zeta}(t) = \zeta(t)\;\mathbb{K}(t).$$

Therefore all we have to show is that $\mathbb{K}(t)$ is 1-1. If $\mathbb{K}(t)$ is not 1-1 there would exist two elements v, $v_0 \in S$ such that $v \neq v_0$ and $\mathbb{K}(t)\, v = \mathbb{K}(t)\, v_0$. It would follow that

$$\zeta(t)\, v = \underline{\zeta}(t)\,\mathbb{K}(t)\, v = \underline{\zeta}(t)\,\mathbb{K}(t)\, v_0 = \zeta(t)\, v_0,$$

and this equation is a contradiction to the fact that by the strict minimality of the state decomposition $\zeta(t)$ is 1-1. ◁

An additional property of a strictly minimal state decomposition is that it allows to define the notion of a "transition operator" (Saeks [49]).

<u>Proposition 3</u>. Suppose that $(\psi(t), S, \zeta(t))$ is a strictly minimal state decomposition. For $q \leq t$, $t, q \in \nu$, assume that no input to the system occurs between q and t. Then there exists a well defined mapping $\Phi(t,q)$ such that

$$v(t) = \Phi(t,q)\,v(q)$$

where v is the state of the system.

<u>Proof</u>. To every element $t \in \nu$ and each element $v \in S$, associate U(t), a subset of $P^t S[G,\nu]$, such that U(t) is maximal with respect to the following property: if $u \in U(t)$, then $v = \psi(t)u$. This defines the family of mappings $M(t): S \to \mathbb{P}\{S[G,\nu]\}$ where $\mathbb{P}\{S[G,\nu]\}$ indicates the power set of $S[G,\nu]$. Conversely, to every subset $U(t) \subseteq P^t S[G,\nu]$ with the property that there exists a $v \in S$ such that

$$v = \psi(t)u, \text{ any } u \in U(t),$$

associate the element v. This defines a family of mappings

$$[\![M(t)]\!]^{-1}: \mathbb{P}\{S[G,\nu]\} \to S.$$

Let $t, q \in \nu$, $t \geq q$ and suppose that is an input to T such that $[P^t - P^q]\underline{u} = \phi$. Consider the states v(t) and v(q) given by the following equations

$$v(t) = \psi(t)\, P^t \underline{u}, \quad v(q) = \psi(q)\, P^q \underline{u}. \tag{3.6}$$

In what follows it is shown that

$$v(t) = [\![M(t)]\!]^{-1} M(q)\, v(q)\,, \tag{3.7}$$

thus $\Phi(t, q) = [\![M(t)]\!]^{-1} M(q)$ is the mapping whose existence was to be proved.

The proof of Equation (3.7) is equivalent to the proof of the following equation

$$M(q)\, v(q) \subseteq M(t)\, v(t)\,.$$

Suppose that $u \in U(q) = M(q)\, v(q)$. From the definition of state decomposition and Equation (3.6) we have the following

$$\zeta(q)\, v(q) = \begin{pmatrix} \zeta_1(q)\, v_1(q) \\ \zeta_2(q)\, v_2(q) \end{pmatrix} = \begin{pmatrix} \bar{P}_q T P^q \underline{u} \\ \bar{P}_q \delta T_{P^q \underline{u}} [\quad] \end{pmatrix} = \begin{pmatrix} \bar{P}_q T P^q u \\ \bar{P}_q \delta T_{P^q u} [\quad] \end{pmatrix}$$

for all $u \in U(q)$. It follows

$$\zeta(t)\, v(t) = \begin{pmatrix} \zeta_1(t)\, v_1(t) \\ \zeta_2(t)\, v_2(t) \end{pmatrix} = \begin{pmatrix} \bar{P}_t T P^t \underline{u} \\ \bar{P}_t\, \delta T_{P^t \underline{u}} [\quad] \end{pmatrix}$$

$$= \begin{pmatrix} \bar{P}_t T P^q \underline{u} \\ \bar{P}_t \delta T_{P^q \underline{u}} [\quad] \end{pmatrix} = \begin{pmatrix} \bar{P}_t \bar{P}_q T P^q u \\ \bar{P}_t \bar{P}_q \delta T_{P^q u} [\quad] \end{pmatrix}$$

$= \zeta(t)\, \psi(t)\, P^q u$, all $u \in U(q)$.

From the above equation and the fact that $\zeta(t)$ is 1-1 we have

$$v(t) = \psi(t) P^q u, \text{ all } u \in U(q)$$

and therefore $U(q) = U(t)$. The proof is thus complete. ◁

The operator $\Phi(t,q)$ is called the Transition operator of the state decomposition. An interesting corollary to the above result is the following.

Corollary 1. Let $\Phi(t,q)$ be the transition operator of a strictly minimal state decomposition $(\psi(t), S, \zeta(t))$. Then

$$\Phi(t,t) = I$$

$$\Phi(t,r) = \Phi(t,q)\Phi(q,r), \; t \geq q \geq r.$$

In Section 2.7 we observed that, when a system is weakly additive, the causality definitions presented in this development coincide with those proposed by Saeks. The natural question then is whether a similar situation occurs for the definition of state decomposition. In more precise terms: is it true that in the case of weakly additive systems Saeks' concept of a state decomposition coincides with the concept formalized in Definition 1.1? The following proposition shows that the answer to this question is affirmative.

Before stating this result it is convenient to pause for a moment and clarify some additional shorthand notations. It will be said that a state decomposition $(\psi(t), S, \zeta(t))$ has its second component given by the triple (ϕ, ϕ, ϕ) if S_2 has only one element (again indicated by ϕ) and $\zeta_2(t)$ maps this element into the null operator on $\bar{P}_t S[G, \nu]$. Finally a state decomposition $(\psi(t), S, \zeta(t))$ is given by the triple (ϕ, ϕ, ϕ) if both its first and second components are given by the triple (ϕ, ϕ, ϕ).

Proposition 4. If T is weakly additive then it has a state decomposition whose second component is given by the triple (ϕ, ϕ, ϕ).

Proof. Choose a triple $(\psi(t), S, \zeta(t))$ where $\psi(t) = (\psi_1(t), \phi)$, $S = (S_1, \phi)$ $\zeta(t) = (\zeta_1(t), \phi)$ and $(\psi_1, S_1, \zeta_1(t))$ is any admissable first component for a state decomposition of T. To prove that $(\psi(t), S, \zeta(t))$ is admissable it is sufficient to show that $(\psi_2(t), S_2, \zeta_2(t)) = (\phi, \phi, \phi)$ satisfies the requirements of Definition 1.1. That this is indeed the case follows from the relation

$$\bar{P}_t \delta T_{P^t u}[\bar{P}_t x] = \bar{P}_t T[\bar{P}_t x + P^t u] - \bar{P}_t T[\bar{P}_t x] - \bar{P}_t T[P^t u] = \phi$$

which is valid for all $\bar{P}_t x \in \bar{P}_t S[G, \nu]$ and $P^t u \in P^t S[G, \nu]$. ◁

3.4 Some Connections Between Causality and State Properties in GRS

Heuristically, if a system is anticausal, then its state space should be vacuous. This result is given mathematical rigor in the following proposition.

Proposition 1:* If T is anticausal then it has a state decomposition given by the triple (ϕ, ϕ, ϕ). Conversely, if the triple (ϕ, ϕ, ϕ) is an admissable state decomposition for T, then T must be anticausal.

Proof. Suppose that T is anticausal. Then for every element $u \in S[G, \nu]$ and $t \in \nu$ we have $\bar{P}_t Tu = \bar{P}_t T \bar{P}_t u$ and therefore $\bar{P}_t T P^t u = \phi$. Similarly, for any $x \in \bar{P}_t S[G, \nu]$ and $t \in \nu$ we have

$$\bar{P}_t \delta T_{P^t u}[x] = \bar{P}_t T[P^t u + \bar{P}_t x] - \bar{P}_t T P^t u - \bar{P}_t T P_t x = \phi$$

and, from here, $\bar{P}_t \delta T_{P^t u}[\ ] = \phi$. Choose now the triple $(\psi(t), S, \zeta(t))$ such that $S = (\phi, \phi)$, $\psi(t) = (\phi, \phi)$ and $\zeta(t) = (\phi, \phi)$. In view of the above equations it is trivial to verify that this triple satisfy the requirements of Definition 1.1 and therefore it is an admissable state decomposition for T. Conversely, suppose that the triple (ϕ, ϕ, ϕ) is an admissable state decomposition for T. Then from Definition 1.1, we have

$$P_t T P^t = \phi \text{ and } \bar{P}_t T[P^t u + \bar{P}_t x] - \bar{P}_t T P^t u - \bar{P}_t T \bar{P}_t x = \phi.$$

From this it follows that $P_t T = P_t T P_t$ and appl[illegible] the dual of Proposition 3.1 we can conclude that T is anticausal. ◁

In the case of memoryless systems (which are anticausal) a straightforward application of the above proposition and of the principle of causal duality generates the following corollary

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*A result of this type was first established [illegible] Sacks in a linear system context (see [48]).

Corollary 1. A necessary and sufficient condition for T to be memoryless is that the triple (ϕ, ϕ, ϕ) be simultaneously an admissable state and costate decomposition for T.

Corollary 2. If the system T is anticausal then it admits a strictly minimal state decomposition.

When a system is crosscausal a result similar to Proposition 1 can be stated.

Proposition 2. If T is crosscausal then there exists a state decomposition with first component given by the triple (ϕ, ϕ, ϕ). Conversely, if T has a state decomposition with first component given by the triple (ϕ, ϕ, ϕ), then T is given by the sum of a crosscausal plus an anticausal operator.

Proof. First we show that the triple $(\psi_1(t), S_1, \zeta_1(t)) = (\phi, \phi, \phi)$ is an admissible first component for a state decomposition of T, where T is crosscausal. Beginning with Proposition 2.2.3, we have $dP(s)TP^t = \phi$ for each $s > t$. This implies that

$$\bar{P}_t TP^t = \bigvee_{s > t} dP(s)TP^t = \phi.$$

From here it follows easily that $(\psi_1(t), S_1, \zeta_1(t)) = (\phi, \phi, \phi)$ satisfies the requirements of Definition 1.1 and therefore is an admissible first component for a state decomposition of T.

Suppose now that the first component of a state decomposition of T is given by the triple (ϕ, ϕ, ϕ). This implies that we must have

$$\bar{P}_t T P^t = \phi \text{ for every } t \in \nu. \tag{3.9}$$

Note that from Proposition 2.6.6 we can write

$$T = T_A + T_X + T_{\bar{C}}, \tag{3.10}$$

where T_A, T_X and $T_{\bar{C}}$ are uniquely defined anticausal, crosscausal and strictly causal operators. From the dual of Proposition 2.4.1 and Proposition 2.4.5 we have

$$\bar{P}_t T_A P^t = \bar{P}_t P^t T_A P^t = \phi \quad \text{and} \quad \bar{P}_t T_X P^t = \bar{P}_t P^t T_X P^t = \phi.$$

From these relations and Equations (3.9) and (3.10) it follows that $\bar{P}_t T_{\bar{C}} P^t = \phi$. This implies $dP(s)\, T_{\bar{C}} P^t = \phi$ whenever $s > t$ and from the fact that $T_{\bar{C}}$ is strictly causal it follows that $T_{\bar{C}}$ is a null operator. From this and Equation (3.10) we can conclude that T is given by the addition of a crosscausal plus an anticausal component. ◁

<u>Corollary 3.</u> If T is crosscausal then it admits a state decomposition which is minimal with respect to its first component.

Before proceeding further it is convenient to introduce the following result which is a useful tool in extending Propositions 1 and 2.

<u>Proposition 3.</u> Let $(\psi'(t), S', \zeta'(t))$ and $(\psi''(t), S'', \zeta''(t))$ be two state decompositions for T' and T'' respectively. Then an admissable state

decomposition for $T = T' + T''$ is given by the triple $(\psi(t), S, \zeta(t))$, where

$$(*)\begin{cases} \psi_i(t) = (\psi_i'(t), \psi_i''(t)), \ i=1,2, \ t \in \nu \\ S_i = S_i' \times S_i'', \ i=1,2, \ t \in \nu \\ \zeta_i(t)\, s_i = \zeta_i'(t)\, s_i'' + \zeta_i''(t)\, s_i'', \ i=1,2, \ t \in \nu \end{cases}$$

where $\quad s_i = (s_i', s_i'') \in S_i' \times S_i'', \ i=1,2.$

Proof. To show that the triple $(\psi_1(t), S_1, \zeta_1(t))$, defined by (*) for $i=1$, is an admissible first component for a state decomposition of $T = T' + T''$, it is sufficient to show that the following holds

$$\zeta_1(t)\,\psi_1(t) = \bar{P}_t T P^t.$$

Since $(\psi'(t), S', \zeta'(t))$ and $(\psi''(t), S'', \zeta''(t))$ are state decompositions of T' and T'' respectively, we must have

$$\zeta_1'(t)\,\psi_1'(t) = \bar{P}_t T' P^t \quad \text{and} \quad \zeta_1''(t)\,\psi_1'(t) = \bar{P}_t T'' P^t .$$

It follows then that

$$\begin{aligned} \zeta_1(t)\,\psi_1(t) &= \zeta_1'(t)\,\psi_1'(t) + \zeta_1''(t)\,\psi_1'(t) = \bar{P}_t T' P^t + \bar{P}_t T'' P^t \\ &= \bar{P}_t (T' + T'')\, P^t = \bar{P}_t T P^t. \end{aligned}$$

This implies that $(\psi_1(t), S_1, \zeta_1(t))$ is an admissable first component for a state decomposition of T.

Using an identical argument it can be shown that $(\psi_2(t), S_2, \zeta_2(t))$ defined by (*) is an admissable second component for a state decomposition of T. ◁

Using Proposition 3, the results in Propositions 1 and 2 can be applied in more general situations. This is illustrated by the following Propositions 4, 5 and 6.

Proposition 4. For every operator T there exists a state decomposition with the property that the first component depends only on the strictly causal part of T.

Proof. Applying Proposition 2.6.6, T can be represented as follows

$$T = T_{\bar{A}} + T_{\bar{C}} + T_M + T_X$$

Suppose that $(\psi_\alpha(t), S_\alpha, \zeta_\alpha(t))$, $\alpha \in \{\bar{A}, \bar{C}, M, X\}$, are state decompositions for $T_{\bar{A}}$, $T_{\bar{C}}$, T_X and T_M respectively. By Propositions 1 and 2, $(\psi_\alpha(t), S_\alpha, \zeta_\alpha(t))$, $\alpha \in \{\bar{A}, M, X\}$, can be chosen in such a way that the first component is given by the triple (ϕ, ϕ, ϕ). A state decomposition $(\psi(t), S, \zeta(t))$ for T can now be obtained by the procedure indicated in Proposition 3. The first component of this state decomposition depends clearly only on $T_{\bar{C}}$. ◁

Proposition 5. If the strongly causal component of T is weakly additive, then there exists a state decomposition with the property that the first component depends only on the strictly causal part of T and the second component depends only on the strongly crosscausal part of T.

Proof. Consider again the representation $T = T_{\bar{A}} + T_{\bar{C}} + T_M + T_X$. By the application of Proposition 1 and 2.4 it is possible to find state decompositions for $T_{\bar{A}}$, T_M and $T_{\bar{C}}$ with the property that the second component is given by the triple (ϕ, ϕ, ϕ). The rest of the proof can be obtained through an argument identical to that used in the proof of Proposition 4. ◁

Proposition 6. A necessary and sufficient condition for T to admit a state decomposition with a first component given by the triple (ϕ, ϕ, ϕ) is that the strictly causal part of T is null.

The proof of Proposition 6 uses techniques similar to those seen in Propositions 4 and 5 and will be omitted for brevity.

To help motivate the next result, observe that a system can clearly have more than one state decomposition. Conversely, Propositions 1 and 2 indicate that a state decomposition can correspond simultaneously to many systems. The natural question that arises is concerned with what must be in common to two systems in order that they can both have the same state decomposition. The answer to this question is given by the following theorem.

Theorem 1. A state decomposition of T defines uniquely the strongly causal and strongly crosscausal components of T.

Proof. We have to show that if two systems T and T' admit an identical state decomposition $(\psi(t), S, \zeta(t))$, then $T_{\underline{C}} = T'_{\underline{C}}$ and $T_{\underline{X}} = T'_{\underline{X}}$, where $T_{\underline{C}}, T_{\underline{X}}$ and $T'_{\underline{C}}, T'_{\underline{X}}$ are strongly causal and strongly crosscausal components respectively of T and T'. Consider the operator T - T'. By the application of Proposition 3, a state decomposition of T - T' is given by $(\tilde{\psi}(t), \tilde{S}, \tilde{\zeta}(t))$, where, using the notation of Proposition 3,

$$\tilde{\psi}_i(t) = (\psi_i'(t), \psi_i''(t)) \; i=1,2; \; \tilde{S}_i \subseteq S_i \times S_i$$

i=1, 2 and $\zeta(t)$ is defined as follows: if $\tilde{v} = (v_i, v_i) \in \tilde{S}_i$ then $\tilde{\zeta}_i(t)\tilde{v}_i =$ $\zeta_i v_i - \zeta_i v_i$. Clearly $(\tilde{\psi}(t), \tilde{S}, \tilde{\zeta}(t))$ has the property that $\tilde{\zeta}(t)\tilde{\psi}(t) = \phi$, for i=1, 2. From Definition 1.1 it follows that $\bar{P}_t(T - T')P^t = \phi$ and $\bar{P}_t \delta T_P t_x[\quad] = \phi$ for every $x \in S[G, \nu]$ and $t \in \nu$. This implies that the triple (ϕ, ϕ, ϕ) is an admissable state decomposition for T - T'. From Proposition 1 it follows that T - T' must be anticausal. This means that $T_{\underline{C}} + T_{\underline{X}} - T'_{\underline{C}} - T'_{\underline{X}} = \phi$. From Proposition 2.4.1 it then follows that $T_{\underline{C}} = T'_{\underline{C}}$ and $T_{\underline{X}} = T'_{\underline{X}}$, which completes the proof. ◁

Interesting corollaries of the above result are the following.

Corollary 4. A state and a costate decompositions of T define strongly causal, strongly anticausal and strongly crosscausal components of T.

Corollary 5. If T is given by the sum of a strongly causal and a strongly crosscausal system, then T is completely defined by any one of its state decompositions.

Corollary 6. Suppose that $(\psi(t), S, \zeta(t))$ and $(\psi'(t), S', \zeta'(t))$ are state decompositions of, respectively, T and T'. If $(\psi(t), S, \zeta(t))$ and $(\psi'(t), S', \zeta'(t))$ are strictly equivalent, then T and T' have identical strongly causal and strongly crosscausal components.

Note that the above statements provide r[illegible]lts which in the context of linear dynamical systems are very famil[illegible]. In particular from Corollary 7 we obtain that two strictly equivalent dynamical systems have the same weighting pattern [65].

3.5 Summary

In this chapter we have studied some interconnections between the causality concepts developed in Chapter 2 and the concept of state. Motivated by the causality development of Chapter 2, it was natural to embed this study in the framework of group resolution space. A first order of business was then to formalize a meaningful concept of state in this general context.

For this purpose an extension of the state decomposition format of Saeks was introduced. As a partial result of this extension our notion of state satisfies the consistency conditions required by Zadeh. Moreover it permits to generalize most of the available state-related results which are usually formulated for linear systems. This leads, for instance, to a generalization in a nonlinear context of Proposition 2.4, 2.5 and Theorem 2.7 in [48].

Meaningful connections between state and causality properties were also established. These results give a rigorous formulation of the heuristic idea that whenever a system has something special with respect to its state structure then it has also something special with respect to its causality structure. In particular, this is illustrated by Propositions 2.3.1-4. The converse is also true.

The most significant result of the chapter is given by Theorem 3.1. This theorem states that the strongly causal and strongly crosscausal components of a system are uniquely defined by any one of its state decompositions.

4. CAUSALITY PROPERTIES IN HILBERT RESOLUTION SPACE

4.1 Introduction

Many system problems in communication and automatic control have natural models as operators on a Banach or a Hilbert space. As these spaces cannot be viewed, in general, as group resolution spaces (GRS), the causality results in Chapter 2 are not directly applicable. By applying appropriate modifications, most of the development in GRS can, however be extended as to cover operators on a Banach or a Hilbert space. The main objective of this chapter is to discuss the nature of these modifications and to elaborate on the type of results which can be stated in the new context.

To achieve this objective we will focus attention on the case of systems in Hilbert space. This will permit us to utilize some results recently developed by mathematicians of the Russian school, [24], [25]. At the same time we will be able to illustrate basic ideas and techniques which can also be used in a Banach space context. In this latter case, however, a more cumbersome and complex development would be required [42].

There are several procedures by which the causality development of Chapter 2 can be made compatible with a Hilbert space context. To motivate the specific character of our approach it is helpful to first consider two simple examples

<u>Example 1.</u> Let R^n be the Hilbert space defined by ordered n-tuples of real numbers with the usual inner product. Every element $x = (x_1, x_2, \ldots, x_n) \in R^n$ can be viewed as an element of the GRS described in Example 2.2.1 and conversely every element of that GRS can be viewed as an element of R^n. It follows then that all the causality concepts and results presented in Chapter 2 are valid and meaningful in the context of operators on the Hilbert space R^n. ◁

<u>Example 2.</u> Let ℓ_2 be the familiar Hilbert space defined by ordered sequences of real numbers $(x_1, x_2, \ldots)$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$. The space ℓ_2 can be viewed as a subset of the group resolution space $S[G, \nu]$ which is obtained by setting $G = R$ and $\nu = \{1, 2, \ldots\}$. Moreover, the truncation operators P^i and P_i associated with $S[G, \nu]$ map $\ell_2 \to \ell_2$ and can be viewed as projection-operators on ℓ_2.

The causality concepts of Chapter 2 can be applied in a natural way to systems in ℓ_2. For example a system T mapping $\ell_2 \to \ell_2$ can be called causal if,

$$P^i x = P^i y \text{ implies } P^i Tx = P^i Ty \text{ for all } x, y \in \ell_2 \text{ and } i \in \nu.$$

The other causality concepts can be similarly defined. Note however that ℓ_2 is not a GRS. This implies that the causality results in Chapter 2 do not necessarily apply. For example the proof of Theorem 3.1 is not valid in ℓ_2 and the question of existence of a canonical causality decomposition has to be reconsidered ◁

The discussion of the above examples illustrates a typical "case by case" procedure for carrying the causality study into a Hilbert space context. This procedure is quite satisfactory in many instances. It implies, however, the possible need for a new approach to suit each specific Hilbert space. The desirability of a unified treatment motivates a more sophisticated and structural approach. This leads in turn to the adoption of a mathematical setting called Hilbert resolution space and to a causality development in this setting.

A few comments about the organization of this development are in order. Section 2 is used to review some mathematical preliminaries and in particular to define Hilbert resolution spaces and some related notions. In Sections 3 and 4 causality c[illegible]s and properties are discussed. Some closure properties of systems with a special causality structure are considered in Section 5. Section 6 deals with the question of causality canonical decomposition. In Section 7 the treatment is specialized to weakly additive systems in Hilbert space. In Section 8 Hilbert Schmidt operators are discussed. Finally, Section 9 reviews some highlights of the chapter.

4.2 Hilbert Resolution Spaces

In this section some basic mathematical concepts and definitions are briefly reviewed. The purpose of this review is twofold: to clarify the notational machinery used in the following sections and to make the treatment reasonably self contained. A unified treatment of those standard mathematical concepts and definitions which are needed in this chapter can be found in [27], [41], [62]. For the concepts related to Hilbert resolution spaces and integrals in these spaces the reader is referred to [24], [25], [42], [50].

The reader is assumed to be familiar with the notions of metric spaces, linear spaces, normed spaces, inner product spaces and the concepts of linear and nonlinear mappings in these spaces (see for Example [27], [41], [62]). A Banach space is a normed linear space which is complete under the metric induced by the norm. If x is an element of a Banach space, the norm of x is indicated by the symbol $|x|$. A Hilbert space is a Banach space with the norm induced by an inner product.

Suppose that T is a mapping on a Banach space B. T is bounded if $|T| = \sup_{0 \neq x \in B} \frac{|Tx|}{|x|} < \infty$. If T is bounded, the number $|T|$ is called the norm of T. T is called compact if $T(F)$, the closure of the image under T of a bounded set F is a compact set. T is unbiased if $T(0) = 0$. As in the case of the GRS development, we will only consider unbiased operators.

Suppose that H is a Hilbert space and ν an ordered set with t_0 and t_∞ respectively minimum and maximum elements* of ν. A family $\mathbb{R} = \{P^t\}$, $t \in \nu$, of orthogonal projectors on H is called a Resolution of the Identity if:

*If t_0 and t_∞ do not exist the definition has a natural extension (see, for instance, [42] p. 37).

Ri) $P^k H \supseteq P^\ell H$ when $k > \ell$; $k, \ell \in \nu$

Rii) $P^{t_o} H = 0$, $P^{t_\infty} H = H$.

The projector $I - P^t$ will be denoted by P_t. A pair of projectors $P^s, P^k \in \mathbb{R}$, $s > k$, is called a gap in IR if there is no element $\ell \in \nu$ such that $P^k \subset P^\ell \subset P^s$. IR is continuous if it does not contain any gap. IR is closed if P belongs to IR whenever there exists a sequence of projectors $\{P^i\}$ in IR such that $\{P^i\}$ converges strongly* to P. The Hilbert space H equipped with IR is called a Hilbert resolution space (in short: HRS) and is denoted by the symbol $[H, P^t]$. In the sequel we suppose that IR is closed.

The parallelism between GRS and HRS appears quite natural; indeed using "spectral multiplicity" theory (see Halmos [26]), a HRS can be viewed as a modified GRS with some additional structure.

In GRS it was found that "sum representation" of the type $\sum_{s \in \nu} dP(s) T P^s$, etc..., were essential to the causality development. In a HRS context the notion of an integral can play an equivalent role.

Suppose that $T(s)$, $s \in \nu$, is a family of operators on a HRS indexed by $s \in \nu$, and consider the following operations:

i) Choose a partition Ω of ν, $\Omega = \{\xi_0, \xi_1, \ldots, \xi_N\}$ where $\xi_o = t_o$, $\xi_N = t_\infty$ and $\{\xi_j\}$ is a finite set of elements in ν such that $\xi_j < \xi_{j+1}$, $j = 1, 2, \ldots, N-1$.

*A sequence of operators $\{T_i\}$ is strongly convergent to T, if for every element $x \in H$ we have $\{T_i x\} \to Tx$.

ii) Consider the partial sum

$$I^{\Omega} = \sum_{k=4}^{N} \Delta P(k)\, T(s_k) \tag{4.1}$$

where $\Delta P(k) = P^{\xi_k} - P^{\xi_{k-1}}$ and s_k is any element of υ, such that $\xi_{k-1} \leq s_k \leq \xi_k$.

iii) On the set of all partitions Ω of ν, define a partial order as follows: $\Omega_1 \supset \Omega_2$ if every element of Ω_2 is contained in Ω_1.

iv) Suppose that there exists an operator T such that in correspondence to any $\epsilon > 0$ there is a partition Ω of ν with the following property: the operator norm $|T - I^{\Omega}|$ is less than ϵ if $\Omega \supset \Omega_\epsilon$.

The operator T obtained through operations i-iv) is called the integral of the family T(s) with respect to IR and it is denoted by

$$T = \int dPT(s).$$

Slight variations of the above concept of integral are also of interest. To this purpose we will denote by

$$T = \text{𝔪} \, dP(s) \text{ or } T = \text{𝔐} \, dPT(s)$$

the integral which is obtained by choosing s_k in operation ii) respectively as follows: $s_k = \xi_{k-1}$ or $s_k = \xi_k$. Similarly the operator $T = \int dPT(s)\, dP$ will denote the integral which is obtained by replacing Equation (4.1) in operation ii) by the following

$$I^{\Omega} = \sum_{k=1}^{N} \Delta P(\xi_k)\, T(s_k)\, \Delta P(\xi_k)\,.$$

Often it is convenient to relax the condition on the type of convergence of the partial sums in Equation (4.1). In particular it may be sufficient to require that in step iv) uniform convergence be replaced by "strong convergence". In this case step iv) would have to be modified as follows: "There exists an operator T such that in correspondence to any $x \in H$ and any real positive $\epsilon > 0$, there exists a partition Ω_ϵ of ν with the following property: if $\Omega > \Omega_\epsilon$ then $|(T - I^{\Omega})x| < \epsilon$".

In the course of this development it will be natural to associate with an operator T on a HRS $[H, P^t]$ the families TP^S and TP_S. These families will lead to integrals such as

$$\int dPTP^S, \qquad ⨏ \; dPTP^S, \qquad ⨍ \; dPTP^S$$

and similar.

In a causality context this type of integrals were first applied by Saeks, [48]. Their rigorous study is, however, due to mathematicians of the Russian school ([24], [25]). These studies are documented by a large number of results. Only a few of these results will be needed here and they are listed in Appendix B.

For later use and to gain some familiarity with the above definitions, it is convenient to pause and consider some examples of HRS.

Example 1. Suppose that H is given by $L_2[0,\infty)$, the Hilbert space of Lebesgue square integrable complex functions. In $L_2[0,\infty)$ we can define a family of orthogonal projection operators $\mathbb{R} = \{P^t\}$, $t \in [0,\infty)$, according to the following rule: if x and y $\in L_2[0,\infty)$ and $y = P^t x$, then $y(s) = x(s)$ a.e. in $[0,t]$ and $y(s) = 0$ a.e. in $[t,\infty)$. The family $\mathbb{R} = \{P^t\}$ is a resolution of the identity because it enjoys properties Ri) and Rii). We can then consider the HRS $[L_2[0,\infty), P^t]$. The resolution of the identity in this example is easily shown to be maximal and continuous. $\triangleleft$

Example 2. Suppose that H is given by ℓ_2, the Hilbert space of square summable sequences of complex numbers $(x_1, x_2, \ldots)$. In ℓ_2 we can consider the family of orthogonal projectors $\mathbb{R} = \{P^i\}$, $i \in \upsilon = 1, 2, \ldots$, where each P^i is defined as follows: if $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ $\in \ell_2$ and $y = P^i x$, then $y_j = x_j$ for $j \geq i$ and $y_j = 0$ for $j > i$. The elements of the family $\mathbb{R}$ satisfy the properties Ri) and Rii) and consequently $\mathbb{R} = \{P^i\}$ is a resolution of the identity in ℓ_2. We can then consider the HRS $[\ell_2, P^i]$. Observe that here the resolution of the identity is maximal but not continuous.

Example 3. Suppose that H is given by $L_2[0,\infty) \times \ell_2$. This means that a typical element of H is given by a pair (x_1, x_2), where x_1 is an element of $L_2[0,\infty)$ and $x_2 = (x_{21}, x_{22}, x_{23}, \ldots)$ is an element of ℓ_2. In this Hilbert space we can consider the family of projection operators $\{P^i\}$

defined as follows: if (x_1, x_2) and $(y_1, y_2) \in L_2[0,\infty) \times \ell_2$ and $(y_1, y_2) = P^j(x_1, x_2)$, then

$$y_1(s) = x_1(s) \text{ a.e. in } [0,j],\ y_1(s) = 0 \text{ a.e. in } [j,\infty)$$

and

$$y_{2i} = x_{2i} \text{ for } i \leq j \text{ and } y_{2i} = 0 \text{ for } i > j.$$

Again the family IR satisfies properties Ri) and Rii) and therefore is a resolution of the identity in $L_2 \times \ell_2$. We can then consider the HRS $[L_2 \times \ell_2, P^i]$. Notice that in this case the resolution of the identity is neither maximal nor continuous. ◁

From the above examples we can observe that Hilbert resolution spaces can simultaneously represent those continuous time, sampled data and hybrid systems which are usually considered in automatic control theory (see [41]). More complicated and less familiar systems in communication theory can also be embedded in this framework. In this regard the reader is referred to [17].

4.3 Causality Concepts in HRS

The basic causality concepts which were introduced and discussed in Section 2.3 will now be extended into the HRS context. In the case of causality, anticausality and memorylessness this extension is straightforward. To this purpose let T denote an (unbiased) operator on $[H, P^t]$ and let x and y indicate two elements in $[H, P^t]$.

Definition 1. T is causal (anticausal) if $P^t x = P^t y$ implies $P^t Tx = P^t Ty$ ($P_t x = P_t y$ implies $P_t Tx = P_t T_y$).

Definition 2. T is memoryless if it is simultaneously causal and anticausal.

The task of extending the concepts of crosscausality and strict and strong causality is not as straightforward. The difficulty is caused by the fact that in GRS these concepts were based on the notion of the evaluation operator dP(t). Usually a HRS does not come equipped with this type of operator. This is the case, for instance, for the spaces considered in Examples 2.1 and 2.3.

To overcome the impasse, observe that from Propositions 2.3.3-4-5-6 the causality concepts under consideration can also be based on the notion of special "sum-type" representations. This, together with the heuristic similarity between "sum-type" and "integral-type" representations, leads in a natural way to the following definitions.

Definition 3. T is strongly causal (strongly anticausal) if T is causal (anticausal) and

$$\int dPTdP = 0 \ .$$

Definition 4. T is strictly causal (strictly anticausal) if

$$T = \mathfrak{m}\!\!\int dPTP^s \ (= M\!\!\!\int dPTP_s) \text{ and } \int dPTdP = 0.$$

<u>Definition 5.</u> T is <u>crosscausal</u> if $\int dPTdP = 0$, $P^t x = 0$ implies $P^t Tx = 0$ and $P_t x = 0$ implies $P_t Tx = 0$.

<u>Definition 6.</u> T is <u>strongly crosscausal</u> if

$$\mathfrak{M} dPTP^s = \mathfrak{m} dPTP_s = \int dPTdP = 0.$$

In view of the similarity between causality concepts in GRS and in HRS, most of the considerations and physical interpretations in Section 2.3 can be applied in the HRS context. In particular it is again helpful to adopt the causality alphabetic code $\mathcal{A} = \{A, C, M, X, \underline{A}, \underline{C}, \underline{X}, \bar{A}, \bar{C}\}$. The principle of causal duality also remains valid and will often play a key role in the simplification of the various proofs.

<u>Principle of Causal Duality in HRS</u>: Let a statement or equality be phrased using relations involving concepts associated to the alphabet $\mathcal{A}$, families of projection-operators such as P^t, P_t and integral transformators of the type $\mathfrak{M}[\quad]$, $\mathfrak{m}[\quad]$. Then the statement or equality remains valid if the following interchange in symbols occurs:

$$C \to A,\ \underline{C} \to \underline{A},\ \bar{C} \to \bar{A},\ A \to C,\ \underline{A} \to \underline{C},\ \bar{A} \to \bar{C}$$

$$P^t \to P_t,\ P_t \to P^t,\ \mathfrak{M} \to \mathfrak{m},\ \mathfrak{m} \to \mathfrak{M},\ \underline{X} \to \underline{X},\ X \to X.$$

A brief discussion on the implications connected with the choice between strong and uniform convergence for the integral in the above definitions is in order. As it will become clear in the course of our development, uniform convergence allows greater simplicity in the proofs of certain results. Occasionally it also plays an essential role for the very existence of other important results. Strong convergence, however, is often preferable from a physical point of view. This can perhaps be best illustrated by a simple example.

Example 1. Consider the HRS described in Example 2.1. A system T on this HRS maps every element $x \in L_2[0, \infty)$ into an element $y \in L_2[0, \infty)$. Suppose that T is defined as follows: if $y = Tx$ then

$$y(t) = x(t + \tau_1)x(t + \tau_2)$$

where τ_1 and τ_2 are real numbers and $x(s) = 0$ whenever $s < 0$.

The system under consideration is formally identical to that considered in Example 2.3.1. As in that example, the causality character of T is a function of the location of the point (τ_1, τ_2) in the Cartesian plane R^2. In particular, for $\tau_1 \leq 0$ and $\tau_2 \leq 0$ it is easy to verify that T is causal. Similarly for $\tau_1 \geq 0$ and $\tau_2 \geq 0$ T is anticausal, and for $\tau_1 = 0$, $\tau_2 = 0$ T is simultaneously causal and anticausal, hence memoryless.

When the convergence of the integrals in definitions 3-6 is intended in the strong sense then the causality structure of T as a function of (τ_1, τ_2) can be a subject of further analysis. For instance, if $\tau_1 < 0$ and $\tau_2 > 0$, then in addition to $TP^t = P^tTP^t$ and $TP_t = P_tTP_t$, we have

$\int dPTdPx = 0$ for every $x \in L_2[0,\infty)$. This implies that T is crosscausal. More specifically T is strongly crosscausal because we also have that

$$\mathfrak{m}\!\!\int dPTP^{s}x = \mathfrak{m}\!\!\int dPTP_{s}x = 0.$$

Similarly for $\tau_1 < 0$ and $\tau_2 < 0$ we can verify that

$$Tx = \mathfrak{m}\!\!\int dPTP^{s}x$$

and therefore T is strictly causal. Proceeding in this way we obtain that the causality character of T is identical to that described in Example 2.3.1. In particular the results illustrated in Figure 2.1 are again true.

It is important to observe, however, that in the case of uniform convergence not all of the above results hold and the analysis becomes somewhat less responsive to our expectations. For instance, for $\tau_1 < 0$ and $\tau_2 < 0$ T would not be strictly causal because (in the case of uniform convergence) it is no longer true that $T = \mathfrak{m}\!\!\int dPTP^{s}$. Similarly when $\tau_1 < 0$ and $\tau_2 < 0$ T would not be crosscausal. $\lhd$

To conclude the section the following remark is in order. Results and considerations in this chapter remain formally identical whether strong or uniform convergence is adopted. Thus, the adoption of uniform convergence has to be considered as a matter of mere technical convenience. The situation is somewhat different in the next chapter. In that context the use of uniform convergence is essential and, if strong convergence is adopted instead, then the results of that chapter do not hold.

4.4 Causality Properties of Basic Systems in HRS

From the similarity between the definitions of causality concepts in GRS and those in HRS, it is natural to expect that properties similar to those already seen in GRS might also be stated in the HRS context. To illustrate that this is the case, we will revisit some of the results in Sections 2.4 and 2.5. For brevity our attention will be confined to those results which will be needed later in the development.

To start with, the following Propositions 1 and 2 are similar to Propositions 2.4.1 and 2.4.2. These results have already appeared in the technical literature [43] [48]. As the present mathematical setting is slightly different from that adopted in those references, a brief review of the proofs is included.

Proposition 1. The following statements are equivalent:

a) T is causal

b) $P^t T = P^t T P^t$

c) $T = \mathcal{M}\!\!\int dPTP^s$.

Proof. a) $\longrightarrow$ b). The argument is formally identical to that seen in the proof of Proposition 2.3.1. b) $\longrightarrow$ c). Suppose that T is causal. Then for any partition $\Omega = \{t_o = \xi_o, \xi_1, \xi_2, \ldots, \xi_N = t_\infty\}$ of ν the following relations hold

$$T = \sum_{i=1}^{N} \Delta P(\xi_i) T = \sum_{i=1}^{N} \Delta P(\xi_i) P^{\xi_i} T = \sum_{i=1}^{N} \Delta P(\xi_i) T P^{\xi_i}.$$

Hence $T = ⨍ dPTP^s$. c) $\longrightarrow$ a). Suppose that c) holds and choose any $x, y \in H$ and $t \in \nu$ such that $P^t x = P^t y$. Then we have

$$P^t Tx = P^t ⨍ dPTP^s x = ⨍ P^t dPP^s P^t x = ⨍ P^t dPTP^s P^t y = P^t Ty.$$

This implies that T is causal. $\lhd$

Proposition 2. The following statements are equivalent:

a) T is memoryless

b) $P^t T = P^t T P^t$ and $P_t T = P_t T P_t$

c) $T = \int dPTdP$.

Proof. The equivalence of a) and b) is a direct consequence of Proposition 1 and its dual. It is then sufficient to prove the equivalence between b) and c).

b) $\Longrightarrow$ c). Suppose that b) holds. Observe that for every partition $\Omega = \{t_o = \xi_o, \xi_1, \ldots, \xi_N = t_\infty\}$ we can write

$$T = \sum_{i=1}^{N} \Delta P(\xi_i) T .$$

Applying b) we obtain that for each i the following relation holds

$$\Delta P(\xi_i) T = \Delta P(\xi_i) P^{\xi_i} P_{\xi_{i-1}} T = \Delta P(\xi_i) T P^{\xi_i} P_{\xi_{i-1}}$$

$$= \Delta P(\xi_i) T \Delta P(\xi_i) .$$

It follows that for every partition Ω of N we can write

$$T = \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) .$$

This implies c).

c) $\Longrightarrow$ b). For any $t \in \nu$ and any partition $\Omega = \{t_o = \xi_o, \xi_1, \ldots, \xi_N = t_\infty\}$ such that $t \in \Omega$, we have

$$P^t \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) = P^t \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) P^t$$

and

$$P_t \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) = P_t \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) P_t .$$

If c) holds, the above equation implies

$$P^t T = P^t T P^t \text{ and } P_t T = P_t T P_t$$

thereby completing the proof. $\lhd$

The next proposition is analogous to Proposition 2.4.5. The proof is based on techniques similar to those adopted in Propositions 1 and 2 and will be omitted for brevity.

Proposition 3. The following statements are equivalent:

a) T is crosscausal

b) $TP_t = P_t TP_t$, $TP^t = P^t TP^t$ and $\int dPTdP = 0$

c) $\int dPTP^s = \int dPTP_s = \int dPTdP = 0$.

Techniques similar to those adopted in the proofs of Propositions 1 and 2 can also be applied to obtain results of the type illustrated in Section 2.5. In this regard we introduce the following two propositions.

Proposition 4. If T is strongly causal (strictly causal) and T'' is causal then T'T'' and T''T' are strongly causal (strictly causal).

Proof. If T' is strongly causal then by definition T' is causal and

$$\int dPTdP = 0 \,. \tag{4.2}$$

Since T'' is also causal we can apply Proposition 1 and obtain

$$P^t T'T'' = P^t T'P^t T'' = P^t T'T''P^t.$$

Applying again Proposition 1 this implies that T'T'' is causal.

To prove that T'T'' is strongly causal it remains to be shown that $\int dPT'T''dP = 0$.

Given any partition Ω of ν, $\Omega = \{t_0 = \xi_0, \xi_1, \ldots, \xi_N = t_\infty\}$, from the causality of T' and T" the following relations hold

$$\Delta P(\xi_i) T' = \Delta P(\xi_i) T' P^{\xi_i}$$

$$T'' \Delta P(\xi_i) = P_{\xi_{i-1}} T'' \Delta P(\xi_i)$$

$i = 1, 2, \ldots, N$.

It follows that for each $i = 1, 2, \ldots, N$ we have

$$\Delta P(\xi_i) T' T'' \Delta P(\xi_i) =$$

$$= \Delta P(\xi_i) T' P^{\xi_i} P_{\xi_{i-1}} T'' \Delta P(\xi_i) =$$

$$= \Delta P(\xi_i) T' \Delta P(\xi_i) T'' \Delta P(\xi_i)$$

and from here

$$\sum_{i=1}^{N} \Delta P(\xi_i) T' T'' \Delta P(\xi_i) = \sum_{i=1}^{N} \Delta P(\xi_i) T' \Delta P(\xi_i) T'' \Delta P(\xi_i) .$$

It follows that

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T' \Delta P(\xi_i) T'' \Delta P(\xi_i) \right| \leq (\sup_i |\Delta P(\xi_i) T' \Delta P(\xi_i)| \quad |T''| \tag{4.3}$$

and from Equations (4. 2) and (4.3) it follows $\int dPT'T''dP = \phi$. Similarly for T"T'.

Suppose now that T' is strictly causal. It has to be shown that T'T" is strictly causal. From the definition of strict causality and Proposition 1 the following relations hold

$$T' = ∯ dPT'P^S \text{ and } T'' = ∯ dPT''P^S.$$

From the definition of integral, given any $\epsilon > 0$, two partitions Ω'_ϵ and Ω''_ϵ can be found such that for all partitions Ω, $\Omega \supset \Omega'_\epsilon \cup \Omega''_\epsilon$, $\Omega = \{\xi_0, \xi_1, \ldots, \xi_N\}$ the following relations hold

$$|T' - \tilde{T}'| < \epsilon \quad \text{and} \quad |T'' - \tilde{T}''| < \epsilon \tag{4.4}$$

where

$$\tilde{T}' = \sum_{i=1}^{N} \Delta P(\xi_i)\, T'P^{\xi_{i-1}} \text{ and } \tilde{T}'' = \sum_{i=1}^{N} \Delta P(\xi_i)\, T''P^{\xi_i}.$$

From Equation (4.4) it follows that

$$|T'T'' - \tilde{T}'T''| \leq |T' - \tilde{T}'|\ |T''| \leq \epsilon\ |T''|$$

and

$$|\tilde{T}'T'' - \tilde{T}'\tilde{T}''| \leq |\tilde{T}'|\, \epsilon \leq (|T'| + \epsilon)\, \epsilon.$$

From here we obtain

$$|T'T'' - \tilde{T}'\tilde{T}''| \leq (|T'| + |T''| + \epsilon)\, \epsilon. \tag{4.5}$$

But we also have

$$\tilde{T}'\tilde{T}'' = \sum_{i=1}^{N} \Delta P(\xi_i)\, T'T''P^{\xi_{i-1}} \tag{4.6}$$

and from Equations (4.5) and (4.6) and the definition of integral it follows

$$T'T'' = ∯ dPT'T''P^S.$$

It can then be concluded that T'T" is strictly causal. A similar argument would show that T"T' is also strictly causal. ◁

<u>Proposition 5.</u> If T is simultaneously α_1 and α_2, with $\alpha_1, \alpha_2 \in \{\underset{-}{A}, \underset{-}{C}, M, \underset{-}{X}\}$, then either $\alpha_1 = \alpha_2$ or T is the null operator.

<u>Proof.</u> Suppose, for instance, that T is simultaneously $\underset{-}{A}$ and $\underset{-}{C}$. From Definition 2.4.3 and Proposition 1 we must simultaneously have

$$P^tT = P^tTP^t \tag{4.7}$$

$$T = \mathfrak{M}\!\!\int dPTP_s \tag{4.8}$$

$$\int dPTdP = 0. \tag{4.9}$$

From Equation (4.8) it follows that for any $\epsilon > 0$, there exists a partition Ω_ϵ such that for any other partition $\Omega = \{\xi_0 = t_0, \xi_1, \ldots, \xi_N = t_\infty\}$, $\Omega \supseteq \Omega_\epsilon$, the following relation holds

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i)\, TP_{\xi_{i-1}} - T \right| < \epsilon .$$

But from Equation (4.7) we obtain

$$\sum_{i=1}^{N} \Delta P(\xi_i)\, TP_{\xi_{i-1}} = \sum_{i=1}^{N} \Delta P(\xi_i)\, T\Delta P(\xi_i) .$$

This means that given any $\epsilon > 0$ we can find a partition Ω_ϵ such that for any other partition $\Omega \supseteq \Omega_\epsilon$ we have

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i)\, T\Delta P(\xi_i) - T \right| < \epsilon .$$

This implies $T = \int dPTdP$ and from Equation (4.9), T must be the null operator. The proof for the other cases is similar and will be omitted. ◁

In view of the above results the containment relations among the various classes of systems with a basic causality structure are identical to those already discussed in a GRS context. In particular the pictorial representation in Figure 2.2 is also valid for systems defined in HRS.

4.5 Some Closure Properties

The set of all bounded operators mapping a HRS into itself can, in a natural way, be viewed as a linear and normed space $\tilde{\$}$. Moreover, it is easy to verify that $\tilde{\$}$ is complete under the metric induced by the operator norm.

The various sets of systems with a basic causality structure can also be viewed as linear normed spaces. These spaces are subspaces of $\tilde{\$}$ and the natural question arises whether they too enjoy the completeness property. From the completeness of $\tilde{\$}$ this is equivalent to determine whether the limit of a convergent sequence of operators which are $\alpha \epsilon A$ is a α operator. In this regard we can state the following results.

Proposition 1.* The subspace of bounded causal operators is complete.

Proof. Suppose that $\{T_i\}$ is a convergent sequence of bounded and causal operators and $\{T_i\} \rightarrow T$. We have to show that T is causal.

---

*A result of this type has been already established in the technical literature (see for example [48]).

By Proposition 4.1 this is equivalent to show that given any pair (y,t), $y \in H$ and $t \in \nu$, the following relation holds

$$P^t T y = P^t T P^t y.$$

This can be done by showing that for any positive $\epsilon > 0$ the following expression holds

$$|P^t T y - P^t T P^t y| < \epsilon.$$

To this purpose choose an integer N such that

$$|T - T_N| < \frac{\epsilon}{2|y|}. \tag{4.10}$$

Since T_N is causal it follows

$$P^t T_N y = P^t T_N P^t y. \tag{4.11}$$

From Equations (4 10) and (4.11) the following relation holds

$$|P^t T y - P^t T P^t y| = |P^t T y - P^t T_N y + P^t T_N P^t y - P^t T P^t y| \leq$$

$$\leq |P^t (T - T_N) y| + |P^t (T_N - T) P^t y| \leq \frac{\epsilon}{2} \frac{|y|}{|y|} + \frac{\epsilon}{2} \frac{|y|}{|y|} = \epsilon.$$

It can then be concluded that T is causal and the proof of the theorem is thus complete. ◁

Proposition 2. The subspace of bounded strongly causal operators is complete.

<u>Proof.</u> Suppose that $\{T_i\}$ is a convergent sequence of bounded and strongly causal operators and $\{T_i\} \rightarrow T$. By Proposition 1 T is causal and it remains to be proved that $\int dPTdP = 0$. To this purpose it has to be shown that for any $\epsilon > 0$, a partition Ω_ϵ of ν can be found such that it enjoys the following property: If $\Omega = \{t_o = \xi_o, \xi_1, \ldots, \xi_N = t_\infty\}$ is any other partition of ν and $\Omega \supset \Omega_\epsilon$, then

$$\left| \sum_{j=1}^{N} \Delta P(\xi_j) T \Delta P(\xi_j) \right| < \epsilon .$$

Let i_o be an integer such that

$$|T_{i_o} - T| < \frac{\epsilon}{2} . \tag{4.12}$$

Since T_{i_o} is strongly causal there exists a partition of ν Ω_ϵ such that if $\Omega = \{t_o = \xi_o, \xi_1, \ldots, \xi_N = t_\infty\}$ and $\Omega \supset \Omega_\epsilon$ then the following relation holds

$$\left| \sum_{j=1}^{N} \Delta P(\xi_j) T_{i_o} \Delta P(\xi_j) \right| < \frac{\epsilon}{2} . \tag{4.13}$$

From this equation it follows that

$$\begin{aligned} \left| \sum_{j=1}^{N} \Delta P(\xi_j) T \Delta P(\xi_j) \right| &= \left| \sum_{j=1}^{N} \Delta P(\xi_j)(T - T_{i_o}) \Delta P(\xi_j) + \sum_{j=1}^{N} \Delta P(\xi_j) T_{i_o} \Delta P(\xi_j) \right| \leq \\ &\leq \left| \sum_{j=1}^{N} \Delta P(\xi_j)(T - T_{i_o}) \Delta P(\xi_j) \right| + \left| \sum_{j=1}^{N} \Delta P(\xi_j) T_{i_o} \Delta P(\xi_j) \right| . \end{aligned} \tag{4.14}$$

Moreover

$$\Big|\sum_{j=1}^{N} \Delta P(\xi_j)(T - T_{i_o})\Delta P(\xi_j)\Big| = \sup_j \big|\Delta P(\xi_j)(T - T_{i_o})\Delta P(\xi_j)\big| \leq |T - T_{i_o}|.$$

From here and Equations (4.12), (4.13) and (4.14) it follows that for $\Omega \supseteq \Omega_\epsilon$ the following holds

$$\Big|\sum_{j=1}^{N} \Delta P(\xi_j) T \Delta P(\xi_j)\Big| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

At this point it can be concluded that T is strongly causal and the proof is thus complete. ◁

Proposition 3. The subspace of bounded and crosscausal operators is complete.

Proof. We have to show that if $\{T_i\}$ is a sequence of bounded crosscausal operators and $\{T_i\} \to T_o$, then T_o is also a crosscausal operator. Since each element of the sequence $\{T_i\}$ is crosscausal, from Definition 3.5 we have that whenever $T = T_i$ the following equations are satisfied

$$TP^t = P^t T P^t \qquad (4.15)$$

$$TP_t = P_t T P_t \qquad (4.16)$$

$$\int dP T dP = 0. \qquad (4.17)$$

We have to prove that Equations (4.15), (4.16) and (4.17) are also satisfied for $T = T_o$. Let us start to show that for any $\epsilon > 0$ we have

$$|T_0 P^t - P^t T_0 P^t| < \epsilon.$$

To this purpose choose an integer i_0 such that

$$|T - T_{i_0}| < \frac{\epsilon}{2} .$$

We can then write

$$|T_0 P^t - P^t T_0 P^t| \leq |T_0 P^t - T_{i_0} P^t| + |T_{i_0} P^t - P^t T_0 P^t| \leq 2|T - T_{i_0}| \leq \epsilon .$$

From the arbitrarity of ϵ it follows that Equation (4.15) is true for $T = T_0$. By the principle of causal duality also Equation (4.16) is true for $T = T_0$. To complete the proof we have to show that $\int dPT_0 dP = 0$. Given any $\epsilon > 0$ consider the integer i_0 such that $|T - T_{i_0}| < \frac{\epsilon}{2}$. Since T_{i_0} is crosscausal there exists a partition Ω_ϵ of ν such that for any other partition $\Omega = \{\xi_0 = t_0, \xi_1, \ldots, \xi_N = t_\infty\}$, $\Omega \supseteq \Omega_\epsilon$, we have

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T_{i_0} \Delta P(\xi_i) \right| < \frac{\epsilon}{2} .$$

This implies

$$\max_i |\Delta P(\xi_i) T_{i_0} \Delta P(\xi_i)| < \frac{\epsilon}{2} .$$

From here we obtain the following

$$\max_i |\Delta P(\xi_i) T_0 \Delta P(\xi_i)| \leq \max_i |\Delta P(\xi_i)(T_0 - T_{i_0}) \Delta P(\xi_i)| +$$

$$+ \max_i |\Delta P(\xi_i) T_{i_0} \Delta P(\xi_i)| \leq \epsilon .$$

It follows that for $\Omega \supseteq \Omega_\epsilon$ we have

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) \right| < \epsilon .$$

We can then conclude that $\int dPTdP = 0$ thereby completing the proof. $\triangleleft$

From the above results we can conclude that each one of the subspaces under consideration can be viewed as a Banach space. As these subspaces also enjoy the property of being closed under the operation of composition of operators, we can state the following theorem.

Theorem 1. The subspace of bounded causal (strongly causal, cross-causal) operators is a Banach algebra.

Observe that in the particular case of linear and causal operators, Theorem 1 gives a result which was already stated by Saeks in [48] (see Proposition 3. 1).

4 6 Canonical Causality Decomposition in HRS

In analogy with the causality development for systems defined on a GRS it is natural at this point to ask whether a system on a HRS might be represented by the sum of systems of type $\alpha \epsilon \{\underline{A}, \underline{C}, M, \underline{X}\}$. This representation will be again called a canonical causality decomposition and to investigate its existence we will use an approach similar to that adopted in Section 2.6. To start consider the following result.

<u>Proposition 1.</u> If T has a canonical decomposition then this decomposition is unique.

<u>Proof</u>. Suppose that

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}} = T'_{\underline{A}} + T'_{\underline{C}} + T'_M + T'_{\underline{X}}$$

holds where $\{T_{\underline{A}}, T_{\underline{C}}, T_M, T_{\underline{X}}\}$ and $\{T'_{\underline{A}}, T'_{\underline{C}}, T'_M, T'_{\underline{X}}\}$ are two distinct canonical decompositions. Then it also holds that

$$T_M - T'_M = T'_{\underline{A}} - T_{\underline{A}} + T'_{\underline{C}} - T_{\underline{C}} + T'_{\underline{X}} - T_{\underline{X}}.$$

Since $T_M - T'_M$ is memoryless we can apply Proposition 4.2 and obtain

$$T_M - T'_M = \int dP[T'_{\underline{A}} - T_{\underline{A}} + T'_{\underline{C}} - T_{\underline{C}} + T'_{\underline{X}} - T_{\underline{X}}]dP$$

$$= \int dP[T'_{\underline{A}} - T_{\underline{A}}]dP + \int dP[T'_{\underline{C}} - T_{\underline{C}}]dP + \int dP[T'_{\underline{X}} - T_{\underline{X}}]dP.$$

By Definitions 4.3.3 and 4.3.4, each term on the right hand side of the above equation is null. It then follows that $T_M = T'_M$.

Proceeding in a similar way

$$T_{\underline{C}} - T'_{\underline{C}} = T'_{\underline{A}} - T_{\underline{A}} + T'_{\underline{X}} - T_{\underline{X}}$$

and applying Proposition 4.1 we obtain

$$T_{\underline{C}} - T'_{\underline{C}} = \not{M} dP[T'_{\underline{A}} - T_{\underline{A}}]P^S + \not{M} dP[T'_{\underline{X}} - T_{\underline{X}}]P^S.$$

Observe that by Definition 4.3.4, $\not{M} dP[T'_{\underline{X}} - T_{\underline{X}}]P^S = 0$ and by inspection $\not{M} dP[T'_{\underline{A}} - T_{\underline{A}}]P^S = \int dP[T'_{\underline{A}} - T_{\underline{A}}]dP = 0$. It follows then that $T_{\underline{C}} = T'_{\underline{C}}$.

Applying the principle of causal duality it follows that $T_{\underline{A}} = T'_{\underline{A}}$ and from this it follows also that $T_{\underline{X}} = T'_{\underline{X}}$. We can then conclude that $\{T_{\underline{A}}, T_{\underline{C}}, T_M, T_{\underline{X}}\}$ and $\{T'_{\underline{A}}, T'_{\underline{C}}, T'_M, T'_{\underline{X}}\}$ cannot be distinct. This implies that if T has a canonical causality decomposition then this decomposition must be unique. $\lhd$

In analogy to Definition 2.6.1, when T has a causality canonical decomposition $T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}$, we will refer to $T_{\underline{A}}, T_{\underline{C}}, T_M, T_{\underline{X}}$ as the canonical components of T.

To deal with the question of existence of such a decomposition it is convenient to use again the notion of "transformator". Indicating by $ the space of all operators on a HRS we will say that $\mathbf{T}$ is a <u>transformator</u>* if $\mathbf{T}$ is an operator mapping $ into itself. $\mathbf{T}$ will be denoted by $\mathbf{T}_\alpha$, $\alpha \in \{\underline{A}, \underline{C}, M, \underline{X}\}$ if for each T ∈ $ we have that $\mathbf{T}_\alpha[T]$ is an α operator. As examples of transformators we can mention some of the integrals defined in Section 4.3. In this case the terminology of <u>integral transformators</u> is adopted. With the above setting we can now state a result similar to Proposition 2.6.

<u>Proposition 2.</u> T has a canonical causality decomposition if and only if T is in the domain of the set of transformators defined as follows

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*This terminology is borrowed from Gohberg and Krein [25].

$$\mathbf{T}_M[T] = \int dP T dP$$

$$\mathbf{T}_{\underline{A}}[T] = \mathrm{m}\!\int dP[T - \mathbf{T}_M[T]]P^s$$

$$\mathbf{T}_{\underline{C}}[T] = \mathrm{M}\!\int dP[T - \mathbf{T}_M[T]]P^s$$

$$\mathbf{T}_{\underline{X}}[T] = T - \mathbf{T}_M[T] - \mathbf{T}_{\underline{A}}[T] - \mathbf{T}_{\underline{C}}[T].$$

Proof. "Only if". Suppose that T has a causality canonical decomposition. Then we can write

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M + T_{\underline{X}}$$

From here it follows that

$$T_M = T - T_{\underline{A}} - T_{\underline{C}} - T_{\underline{X}}$$

and applying Proposition 4.2 we obtain

$$T_M = \int dP T dP - \int dP T_{\underline{A}} dP - \int dP T_{\underline{C}} dP$$
$$- \int dP T_{\underline{X}} dP.$$

Applying Definitions 4.3.3-.6 we obtain that $T_M = \mathbf{T}_M[T]$ and therefore T is in the domain of the transformator $\mathbf{T}_M$. Proceeding in a similar fashion we can also obtain that T is in the domain of $\mathbf{T}_{\underline{A}}$, $\mathbf{T}_{\underline{C}}$ and $\mathbf{T}_{\underline{X}}$.

"if". Suppose that T is in the domain of the above integral transformators. Then it is immediate to verify that

$$T = \mathbf{T}_{\underline{A}}[T] + \mathbf{T}_{\underline{C}}[T] + \mathbf{T}_{\underline{M}}[T] + \mathbf{T}_{\underline{X}}[T].$$

By a direct application of the causality definitions we can now recognize that $\mathbf{T}_{\underline{A}}[T]$, $\mathbf{T}_{\underline{C}}[T]$, $\mathbf{T}_{M}[T]$ and $\mathbf{T}_{\underline{X}}[T]$ are respectively $\underline{A}$, $\underline{C}$, M and $\underline{X}$. This completes the proof. ◁

The statements of Propositions 1 and 2 can be combined in the following theorem.

<u>Theorem 1.</u> If T has a canonical decomposition then this decomposition is unique and its components can be computed as follows:

$$T_M = \int dPT\Delta P, \quad T_{\underline{A}} = m\!\!\int dP[T - T_M]P_s$$
$$T_{\underline{C}} = M\!\!\int dP[T - T_M]P^s, \quad T_{\underline{X}} = T - T_{\underline{A}} - T_{\underline{C}} - T_M.$$

To illustrate some implications of Theorem 1 we pause for a moment and discuss two simple examples.

<u>Example 1.</u> Consider $[L_2[0,\infty), P^t]$, the Hilbert resolution space described in Example 4.2.1 and suppose that T is described as follows: if $y, x \in L_2[0,\infty)$ and $y = Tx$ then

$$y(t) = \int_0^{\infty} \int_0^{\infty} K(t, s_1, s_2)\, x(s_1)\, x(s_2)\, ds_1 ds_2$$

where $k(t_1, s_1, s_2)$ is a real function such that

$$\|K\|^2 = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} K^2(t_1, s_1, s_2)\, dt ds_1 ds_2 < \infty$$

Note that from a simple application of Fubini's theorem [27], we have

$$|y|^2 \leq \|K\|^2\, |x|^2 \tag{4.18}$$

Observe also that T can be written as $T = T_1 + T_2 + T_3$, where T_1, T_2 and T_3 are described as follows:

$$y = T_1x \text{ implies } \quad y(t) = \int_0^t \int_0^t K(t, s_1, s_2)x(s_1)x(s_2)\,ds_1ds_2$$

$$y = T_2x \text{ implies } \quad y(t) = \int_t^\infty \int_t^\infty K(t, s_1, s_2)x(s_1)x(s_2)\,ds_1ds_2$$

$$y = T_3x \text{ implies } \quad y(t) = \int_0^t \int_t^\infty K(t, s_1, s_2)x(s_1)x(s_2)\,ds_1ds_2 + \int_t^\infty \int_0^t K(t, s_1, s_2)x(s_1)x(s_2)\,ds_1ds_2.$$

From standard arguments of real analysis based on the validity of Equation (4.18) it is not difficult to verify that

$$\int dPTdP = 0, \quad T_1 = \not\!\int dPTP^s, \quad T_2 = \not\!\int dPTP_s$$

where the integrals are intended in the strong sense. Applying Theorem 1 we can conclude that T has a unique canonical causality decomposition. The basic components of this decomposition are as follows: $T_M = 0$, $T_{\underline{C}} = T_1$, $T_{\underline{A}} = T_2$ and $T_{\underline{X}} = T_3$. ◁

<u>Example 2</u>. Consider again $[L_2[0, \infty), P^t]$ and suppose that T is described as follows: if $y, x \in L_2[0, \infty)$ and $y = Tx$ then

$$y(t) = \int_0^\infty K_1(t, s)x(s)\,ds +$$

$$\int_0^\infty \int_0^\infty K_2(t, s_1, s_2)x(s_1)x(s_2)\,ds_1ds_2 +$$

$$\cdots$$

$$\int_0^\infty \cdots \int_0^\infty K_n(t, s_1, \ldots s_n)x(s_1)x(s_2)\ldots x(s_n)\,ds_1ds_2\ldots ds_n$$

where $K_i(t, s_1, \ldots, s_i)$ is a real function such that

$$\int_0^\infty \cdots \int_0^\infty K^2(t, s_1, s_2, \ldots, s_i)\, ds_1 ds_2 \ldots ds_i < \infty$$

for i=1, 2, ... n. Note that this type of operators are called Frechet-Volterra and have been extensively utilized in the information and automatic control literature [1], [28], [35], [46].

Using a line of reasoning similar to that in Example 1, we can again see that T has a canonical causality decomposition. In this decomposition the memoryless component is null and $T_{\underline{C}}$, $T_{\underline{A}}$ and T , respectively strongly causal, strongly anticausal and strongly crosscausal components of T can be identifed as follows. Let

$y_{\underline{C}} = T_{\underline{C}}x$, $y_{\underline{A}} = T_{\underline{A}}x$ and $y_{\underline{X}} = T_{\underline{X}}x$, then

$$y_c(t) = \sum_{i=1}^{n} \int_0^t \cdots \int_0^t K_i(t, s_1, s_2, \ldots, s_i)\, x(s_1)\, x(s_2) \ldots x(s_i)\, ds_1 ds_2 \ldots ds_i$$

$$y_{\underline{A}}(t) = \sum_{i=1}^{n} \int_t^\infty \cdots \int_t^\infty K_i(t, s_1, s_2, \ldots, s_i)\, x(s_1)\, x(s_2) \ldots x(s_i)\, ds_1 ds_2 \ldots ds_i$$

$$y_{\underline{X}} = Tx - T_{\underline{A}}x - T_{\underline{C}}x. \qquad \triangleleft$$

Theorem 1 links the study of existence of a causality canonical decomposition to the study of the convergence properties of a special class of integral transformators. This latter study has recently received attention from a number of mathematicians from the Russian school. Of particular interest here is the work of Gohberg and Krein [24], [25].

A result of this study with negative implications for the canonical decomposition is that these integral transformators suffer from convergence problems. To emphasize this the following proposition is noted.

Proposition 3. There exists systems T on a HRS which do not have a canonical decomposition.

Proof. It suffices to present an example operator for which at least one of the integral transformators in question fails to exist. The reader is referred to Appendix B for some preliminary results in this direction. In particular from Lemma B. 5, we have that for every HRS with a continuous resolution of the identity there exists a compact self-adjoint operator T such that $\int P^S T dP$ is not defined. From Lemma B. 4 it follows that $\int dPTP^S$ is also not defined. Moreover applying Lemma B. 1

$$T_M = \int dPTdP = 0$$

and applying Lemma B. 2

$$\int dPTP^S = \int\!\!\!\!\!\mathrm{M}\, dPTP^S = \int\!\!\!\!\!\mathrm{M}\, dP[T - T_M]P^S.$$

From the fact that $\int dPTP^s$ is not defined, it can be concluded that $\int dP(T - T_M) P^s$ is also not defined. It follows that such a T does not have a canonical decomposition. ◁

4.7 Causality and Weakly Additive Operators in HRS

The importance of weakly additive operators was already outlined in Section 2.7. In the sequel the development of that section will be revisited in a HRS context.

Definition 1. An operator T on $[H, P^t]$ is weakly additive if for every $x \in H$ and all $t \in \nu$ the following relation is satisfied

$$T[x] = T[P^t x] + T[P_t x].$$

Before proceeding with a causality study of weakly additive operators, it may be helpful to consider some examples. To this purpose we introduce the following proposition.

Proposition 1. The following operators are weakly additive.

i) every linear operator

ii) every memoryless operator

iii) the linear combination of weakly additive operators

iv) the composition T"T' where T' is weakly additive and T" is linear

v) the composition T"T' where T' is memoryless and T" is weakly additive.

Proof. i) and iii) are trivial. ii) Suppose that T is memoryless. Then for any $t \in \nu$ and $x \in H$ the following relations hold:

$$P_tTx = P_tTP_tx, \text{ that is } P^tTP^t = 0, \text{ and}$$

$$P^tTx = P^tTP^tx, \text{ that is } P^tTP_t = 0.$$

It follows that for all $t \in \nu$ and $x \in H$, we have

$$Tx = P^tTx + P_tTx = P^tTP^tx + P_tTP_tx = TP^tx + TP_tx.$$

iv) For all $t \in \nu$ and $x \in H$ the following relations hold

$$T''T'x = T''[T'P^tx + T'P_tx] = T''T'P^tx + T''T'P_tx.$$

v) Again for all $t \in \nu$ and $x \in H$ we have

$$T''T'x = T''[T'P^tx + T'\bar{P}_tx] = T''[P^tT'P^tx + \bar{P}_tT'\bar{P}_tx]$$

$$= T''[P^tT'x + \bar{P}_tT'x] = T''P^tT'x + T''\bar{P}_tT'x =$$

$$= T''P^tT'P^tx + T''\bar{P}_tT'\bar{P}_tx = T''T'P^tx + T''T'\bar{P}_tx.$$

We can conclude that T''T' is weakly additive. $\triangleleft$

Most of the causality properties which were stated in the case of weakly additive operators in a GRS are also valid in the case of weakly additive operators in a HRS. More specifically, the statements of Propositions 1, 2, 3, and 4 in Section 2.7 hold "almost verbatim" in the HRS setting.

In particular, Propositions 2.7.1 and 2.7.2 provide results which were already stated by Porter [42] and Saeks [48] for the case of linear operators.

<u>Proposition 2.</u> If T is weakly additive the following statements are equivalent:

a) T is causal

b) $P^t x = 0$ implies $P^t T x = 0$

c) $TP_t = P_t T P_t$

d) $P^t T = P^t T P^t$

e) $T = \int M dPTP^{s}$.

<u>Proposition 3.</u> If T is weakly additive the following statements are equivalent:

a) T is memoryless

b) $P^t x = 0$ implies $P^t T x = 0$ and $P_t x = 0$ implies $P_t T x = 0$

c) $TP^t = P^t T P^t$ and $TP_t = P_t T P_t$

d) $P^t T = P^t T P^t$ and $P_t T = P_t T P_t$

e) $T = \int dPTd\Gamma$.

In regard to Propositions 2.7.3 and 2.7.4 it is of interest to see the modifications needed in order to make those results applicable in a HRS context. This leads to the following Propositions 4 and 5.

<u>Proposition 4.</u> In the context of weakly additive systems the concepts of strong and strictly causality are equivalent.

Proof. As in Proposition 2.7.3 it is sufficient to show that if a system is weakly additive and strongly causal then it is also strictly causal. Suppose then that T is weakly additive and strongly causal. By the definition of strong causality and Proposition 4.4.1 we obtain

$$T = \mathcal{M}\!\!\!\!\int dPTP^{s} \text{ and } \int dPTdP = 0. \qquad (4.19)$$

Moreover if $\Omega = \{t_0 = \xi_0, \xi_1, \ldots, \xi_N = t_\infty\}$ is any partition of ν then from the weak additivity of T it follows

$$\sum_{i=1}^{N} \Delta P(\xi_i)\, TP^{\xi_i} = \sum_{i=1}^{N} \Delta P(\xi_i)\, TP^{\xi_{i-1}} + \sum_{i=1}^{N} \Delta P(\xi_i)\, T\Delta P(\xi_i)\,.$$

This implies that

$$\mathcal{M}\!\!\!\!\int dPTP^{s} = m\!\!\!\!\int dPTP^{s} + \int dPTdP$$

and from Equation (4.19) it follows that

$$T = m\!\!\!\!\int dPTP^{s}.$$

We can then conclude that T is strictly causal. $\triangleleft$

Proposition 5. If T is weakly additive and crosscausal, then T is null.

Proof. Suppose that T is crosscausal. Then

$$\int dPTdP = 0, \quad \mathcal{M}\!\!\!\!\int dPTP^{s} = 0 \text{ and } m\!\!\!\!\int dPTP_{s} = 0.$$

Given any $\epsilon > 0$ there exists a partition Ω_ϵ of ν such that for any other partition Ω of ν, $\Omega = \{t_0 = \xi_0, \xi_1, \ldots, \xi_N = t_\infty\}$, $\Omega \supseteq \Omega_\epsilon$, the following relations hold

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) \right| < \frac{\epsilon}{3},$$

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T P^{\xi_i} \right| < \frac{\epsilon}{3},$$

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T P_{\xi_{i-1}} \right| < \frac{\epsilon}{3}.$$

By the weak additivity of T we have

$$T = \sum_{i=1}^{N} \Delta P(\xi_i) T = -\sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) + \sum_{i=1}^{N} \Delta P T P^{\xi_i} + \sum_{i=1}^{N} \Delta P(\xi_i) T P_{\xi_{i-1}}.$$

It follows that $|T| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ and from the arbitrarity of ϵ, T must be the null operator. $\lhd$

The next three propositions illustrate some additional properties of weakly additive operators on a HRS. These properties will turn out to be useful in the course of the next chapter. In regard to the proofs, the reader no doubt recognizes that Proposition 6 is a direct consequence of Proposition 4 and Theorem 5.1. Similarly, Proposition 7 follows from Proposition 4, and the proof of Proposition 8 is based on Proposition 5 and Lemma B.3 in the Appendix.

Proposition 6. The space of bounded weakly additive and strictly causal systems is a Banach Algebra.

Proposition 7. A necessary and sufficient condition for a causal and weakly additive system to be strictly causal is that $\int dPTdP = 0$.

<u>Proposition 8.</u> If T is a linear, causal and compact operator on $[H, P^t]$ and $\{P^t\}$ is a continuous resolution of the identity then T is strictly causal.

In regard to the question of canonical causality decomposition, unfortunately a result equivalent to Theorem 2.7.1 cannot be stated. In fact the causality canonical decomposition of an arbitrary weakly additive operator may not exist. Results in this direction are limited to the following.

<u>Theorem 1.</u> If T is weakly additive and has a canonical decomposition then T has the unique representation $T = T_{\underline{A}} + T_{\underline{C}} + T_M$, where

$$T_{\underline{C}} = m\!\!\int dPTP^s, \quad T_{\underline{A}} = M\!\!\!\int dPTP_s \text{ and } T_M = \int dPTdP. \qquad (4.20)$$

<u>Proof.</u> In view of Propositions 2, 3, 4 and 5, we have to show that if a canonical decomposition exists, then $T_{\underline{C}}$ and $T_{\underline{A}}$ can be computed through Equations (4.20). To this purpose, observe that from Theorem 6.1 the following relation holds

$$T_{\underline{C}} = M\!\!\!\int dPTP^s - T_M$$

and from the weak additivity of T

$$M\!\!\!\int dPTP^s = m\!\!\int dPTP^s + \int dPTdP.$$

It follows that

$$T_{\underline{C}} = m\!\!\int dPTP^s + T_M - T_M = m\!\!\int dPTP^s$$

and from the causality principle of duality

$$T_{\underline{A}} = \not\!\!\int dPTP_s .$$

The proof is complete. ◁

The statement of the above theorem is valid also for linear operators and shows that even in this context the canonical decomposition question is not as simple as it was thought in a previous study ([48]). To shed some further light, two additional propositions are introduced.

Proposition 7. Suppose that T is a bounded weakly additive operator and that $T_M = \int dPTdP$ is well defined. Then, given any $\epsilon > 0$, there exists an operator $\tilde{T}$ such that $|T - \tilde{T}| < \epsilon$ and $\tilde{T}$ has a causality canonical decomposition.

Proof. Choose a partition $\Omega_\epsilon = \{t_0 = \xi_0, \xi_1, \ldots, \xi_N = t_\infty\}$ such that

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) - T_M \right| < \epsilon. \tag{4.21}$$

Consider the operator $\tilde{T}$ defined by the following expression

$$\tilde{T} = \sum_{i=1}^{N} \Delta P(\xi_i) TP^{\xi_{i-1}} + T_M + \sum_{i=1}^{N} \Delta P(\xi_i) TP_{\xi_i} . \tag{4.22}$$

From the above equation $\tilde{T}$ has a canonical decomposition. Moreover

$$|T - \tilde{T}| = \Big| \sum_{i=1}^{N} \Delta P(\xi_i) T - \Big(\sum_{i=1}^{N} \Delta P(\xi_i) TP^{\xi_{i-1}} + T_M$$

$$+ \sum_{i=1}^{N} \Delta P(\xi_i) TP_{\xi_i} \Big) \Big| . \tag{4.23}$$

From here and Equations (4.24) and (4.22) it follows

$$|T - \tilde{T}| = |\sum_{i=1}^{N} \Delta P(\xi_i) T \Delta P(\xi_i) - T_M| < \epsilon$$

and the proof is thus complete. ◁

Proposition 10. Suppose that a HRS has a continuous resolution of the identity. Then given any operator T and any real $\epsilon > 0$, there exists an operator $\tilde{T}$ such that $|T - \tilde{T}| < \epsilon$ and $\tilde{T}$ does not have a causality canonical decomposition.

To conclude this section it is of interest to observe that when an operator T on a HRS is linear then all of the above results are automatically valid. In those cases in which T has also some additional specific analytic properties various other interesting causality results can be stated. This is done partly in the next section where Hilbert Schmidt operators are considered and partly in Appendix C where compact and (or) self-adjoint operators are considered.

4.8 Causality and Hilbert Schmidt Operators

In this section some causality properties related to Hilbert Schmidt operators on a HRS are presented. This is done for two main reasons: the importance of this type of operators in control and communication theory ([35], [46], [47]), the fact that the causality properties of these operators enjoy a completeness which is not present in the general case (for example the canonical decomposition theorem as stated in [48] is true for Hilbert Schmidt operators).

Suppose that T is a bounded linear operator on $[H, P^t]$ and let $\{x_\beta\}$, $\beta \in E$, be a complete orthonormal set in H.

Definition 1. A bounded linear operator T is a Hilbert Schmidt (in short HS) operator if the quantity $||T||$

$$||T|| = \{ \sum_{\beta \in E} |Tx_\beta|^2 \}^{\frac{1}{2}}$$

is finite. The number $||T||$ is called the Hilbert Schmidt norm of T.

The following two lemmas state the two special properties of HS operators which determine their particular structure with respect to causality.

Lemma 1. ([16] p. 1010). The Hilbert Schmidt norm is independent of the orthonormal basis used in its definition. The set of all HS operators is a Banach space under the Hilbert Schmidt norm.

Lemma 2. The Banach space of HS operators is a Hilbert space. The inner product between two elements T, S $\in$ HS can be defined by

$$\langle S, T \rangle = \sum_{\beta \in E} \langle Sx_\beta, Tx_\beta \rangle .$$

The causality structure of HS operators can now be clarified by the following propositions.

Proposition 1. The class of causal (anticausal, memoryless) HS operators is a Hilbert space.

Proof. Let CHS (AHS, MHS) denote the space of causal (anticausal, memoryless) HS operators. Clearly from Lemma 2, CHS is a linear inner product space. To show that CHS is a Hilbert space it has to be proved that CHS is complete under the norm induced by the inner product. Suppose then that $\{T_i\}$ is a Cauchy sequence in CHS. Since HS is complete, the sequence $\{T_i\}$ converges to an operator $T \in HS$ and it remains to be shown that T is causal. This follows from Proposition 5.1 and the well known fact that the operator norm is smaller than the Hilbert Schmidt norm. Applying the principle of causal duality, it follows that AHS is a Hilbert space. From the fact that $MHS = CHS \cap AHS$ it is obtained that MHS is also a Hilbert space. The proof is thus complete. ◁

Proposition 2. The space of HS operators is given by the direct sum of the three orthogonal subspaces of strictly causal, strictly anticausal and memoryless operators.

Proof. Using he notation adopted in the proof of Theorem 1, let $\underline{C}HS = CHS \cap (MHS)^{\perp}$ and $\underline{A}HS = AHS \cap (MHS)^{\perp}$ where $(MHS)^{\perp}$ indicates the complement of MHS in HS. Note that $\underline{C}HS$, $\underline{A}HS$ and MHS are orthogonal subspaces.

Let us now prove that every $T \in HS$ has a canonical decomposition

$$T = T_{\underline{C}} + T_{\underline{A}} + T_M.$$

In view of Theorem 6.1 this means that it has to be shown that the integrals $m\!\!\!\int dPTP^S$, $M\!\!\!\!\int dPTP_S$, $\int dPTdP$ are well defined. From Lemma B.3 in the Appendix, it is obtained that $T_M = \int dPTdP$ is well defined. By simple inspection the following relation holds

$$m\!\!\!\int dPTP^S = \int dP[T - T_M]P^S,$$

where $\int dP[T - T_M]dP = 0$. From Lemma B.6 and Lemma B.4 it follows that $m\!\!\!\int dPTP^S$ is well defined. By the principle of duality $M\!\!\!\!\int dPTP_S$ is also well defined. If we now observe that $T_{\underline{C}} \in \underline{C}HS$, $T_{\underline{A}} \in \underline{A}HS$ and $T_M \in MHS$, then we obtain that $T_{\underline{C}}$, $T_{\underline{A}}$ and T_M are mutually orthogonal.

To complete the proof it remains to be shown that the spaces $\underline{C}HS$ and $\underline{A}HS$ are made respectively of strictly causal and strictly anticausal operators. To this purpose suppose that $T \in \underline{C}HS$. If T is not strictly causal then we can write $T = T_{\underline{C}} + T_M$, where $T_{\underline{C}} \in \underline{C}HS$ and $T_M \neq 0 \in MHS$. This is a contradiction to the fact that $\underline{C}HS$ is orthogonal to MHS. We can then conclude that $\underline{C}HS$ consists of strictly causal operators and, from the principle of causal duality, that $\underline{A}HS$ consists of strictly anticausal operators. The proof of the proposition is thus complete. ◁

Proposition 3. Suppose that T is a HS operator. Then there exists a unique HS operator T_α, $\alpha \epsilon \{A, C, M, \bar{A}, \bar{C}\}$ such that

$$||T - T_\alpha|| = \min_{\alpha} ||T - \tilde{T}_\alpha||$$

where $\tilde{T}_\alpha$ is any α HS operator and T_α is the α component of T.

The proof of Proposition 3 is a direct consequence of Proposition 2 and is omitted for brevity. From the combination of the statements of Propositions 1 and 2 we have the following theorem.

Theorem 1. Every Hilbert Schmidt operator has a causality canonical decomposition. The components of this decomposition are mutually orthogonal.

4.9 Summary

The contents of this chapter can be summarized from two points of view: refinements of the GRS results in Chapter 2 or generalization of the causality study developed by Porter [43] and Sacks [48].

From the first point of view, the present development shows how concepts and techniques in GRS can be applied to systems defined on a HRS. In the transition, the validity of most of the results can be maintained. This is illustrated by a number of propositions.

The "sum" type representations which were so helpful in GRS become "integrals" in HRS. This allows an efficient utilization of the results on integral transformators due to mathematicians of the Russian school ([24], [25]). Results from this latter study were used to show that the existence of a canonical causality decomposition does not carry over in HRS. This is illustrated by Theorem 6.1 for the case of general operators and by Theorem 7.1 for the case of weakly additive and linear operators. The situation is somewhat different for linear operators with a special analytic structure. In particular every Hilbert Schmidt operator has a canonical causality decomposition. Moreover, the components of this decomposition can be viewed as mutually orthogonal (Theorem 8.1).

In regard to the second point of view, it should be emphasized that in the case of linear operators the present development coincides "verbatim" with that in [48]. The bridge between the two approaches is provided here by Propositions 5.1-2. It is important to observe however that the main result in [48] (that is Theorem 2.3.5) is in contradiction with our Theorem 7.1. This is because the properties of integral transformators are not as extensive as it was assumed in that study.

5. CAUSALITY, STRICT CAUSALITY AND STABILITY

5.1 Introduction*

To motivate the development of the present chapter, let us start to consider the following linear integral Volterra equation

$$y(s) = x(s) - \int_0^s K(s,t)x(t)\,dt, \quad s \in [0,\infty) \qquad (5.1)$$

where $y(.)$ is a known element of the Hilbert space $L_2[0,\infty)$, while $K(s,t)$ is a known kernel of the equation and $x(.)$ is an unknown function to be computed.

As it is well known†, if the following condition is satisfied

$$\int_0^\infty \int_0^\infty K^2(s,t)\,dsdt < \infty \qquad (5.2)$$

then Equation (5.1) has a unique solution for every $y \in L_2[0,\infty)$. Moreover this solution can be computed by the formula

$$x(s) = y(s) + \int_0^s \mathbb{K}(s,t)y(t)\,dt, \quad t \in [0,\infty). \qquad (5.3)$$

Here $\mathbb{K}(s,t)$ is called the resolving kernel of the Volterra equation and it can be computed by the following expression

$$\mathbb{K}(s,t) = \sum_{n=1}^{\infty} K^n(s,t) \qquad (5.4)$$

where $K^1(s,t) = K(s,t)$ and $K^n(s,t) = K(s,\cdot) * K^{n-1}(\cdot,t)$, $n=2,3,\ldots$.

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*Most of the mathematical concepts used in this chapter are defined in Section 4.2.

† See, for example, reference [34].

In relation to Equation (5.1), Gohberg and Krein [25] observed that the kernel $K(s,t)$, viewed as an operator K on the Hilbert space $L_2[0,\infty)$, has the following properties:

a) K is linear

b) K is compact

c) $K = \oint dPKP^s$.

The integral of c) has to be interpreted in the uniform convergence sense defined in Section 4.1 and is computed with respect to the family $\{P^t\}$ of truncation operators on $L_2[0,\infty)$.

The above observation of Gohberg and Krein led to the study of a "generalized linear Volterra equation" of the type

$$y = x - Kx \tag{5.5}$$

where y is an assigned element of a given Hilbert space H, K is a known operator on H and enjoys properties a), b), c) and x is an unknown element to be computed. The result of this study was the following: under hypotheses a), b), c) the solution to Equation (5.5) exists, is unique and can be computed by the formula

$$x = y + \mathbf{K}y \tag{5.6}$$

where the operator $\mathbf{K}$ is given by the following expression

$$\mathbf{K} = + \sum_{n=1}^{\infty} K^n \tag{5.7}$$

where $K^1 = K$ and, for $n=2,3,\ldots,$ K^n is given by the composition of K with K^{n-1}.

The above results can be given a meaningful engineering interpretation. To this purpose a brief review of current technical terminology is in order. To Equation (5. 5) we can associate a closed loop basic feedback system of the type in Figure 5. 1. The operator K is called the open loop system and the problem of investigating some of the characteristics of the solution x in Equation (5. 5) is generally called the feedback system stability problem.

The stability problem for systems of this type, or equivalent types, has been studied by a number of authors (see for example [11], [40], [52], [59], [67], [68]). In particular (Damborg [11]) the basic feedback system is called stable if the operator (I-K) is invertible, and its inverse is bounded causal and continuous*.

Using the terminology introduced above, the result offered by Gohberg and Krein can be rephrased as follows: "A sufficient condition for a basic feedback system to be stable is that the open loop system be linear compact and strictly causal".

This result is very attractive in its technical conciseness. Ho ever in stability theory the operator K is in general not compact. The following natural questions then arise:

Q1: Can the above result be extended to the case in which K is linear bounded and strictly causal but not necessarily compact?

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*An operator T on a Banach space B is continuous if for any $x \in B$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|Tx - Ty| < \epsilon$ when $|x - y| \leq \delta$.

Q2: If this extension is feasible can it be carried further so as, for example, to apply to some special class of nonlinear systems?

The following sections are devoted to the task of answering these type of questions. A brief comment on their contents may be helpful. The central results of the chapter are stated in Section 2 and are related to a special class of weakly additive systems. In Section 3 these results are specialized to the case of linear systems. Further results are introduced in Section 4. Finally Sections 5 and 6 discuss respectively some applications and concluding remarks.

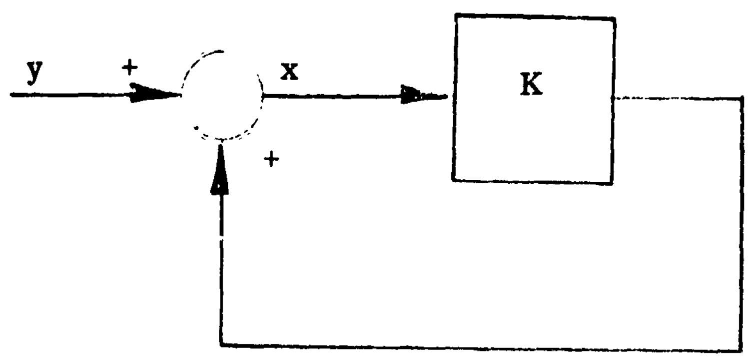

Figure 5.1 : Basic Feedback System

5.2 Stability and Causality

In this section we consider the questions Q1 and Q2 which were posed in the introduction. As it turns out, it is possible to give a completely affirmative answer to Q1. The proof of this result depends very little on the linearity of K. This fact is rather remarkable if it is considered that the proof offered by Gohberg and Krein for their result was based in an essential way on the spectral theory of bounded linear operators (see Theorem 6.1, p. 27 in [25]).

According to the proof proposed in this section, what seems to be essential is the property of weak additivity which was already considered in Sections 2.7 and 4.7. In view of this fact it is convenient to reverse the order of consideration of Q1 and Q2. Thus while the following results can be considered as related to Q2, a formal answer to Q1 is obtained as a corollary.

To start consider the following technical result.

Proposition 1. Suppose that K is a causal Lipschitz continuous* operator with Lipschitz norm less than 1. Then the basic feedback system is stable.

Proof i) (I-K is invertible. We have to show that for every $y \in H$ the equation

$$y = x - Kx \tag{5.8}$$

has a unique solution. This is clearly equivalent to establishing existence and uniqueness of a fixed point x for the operator $T_y : H \rightarrow H$ defined by

$$T_y(x) = y + Kx.$$

---

*An operator T on a Banach space B is called Lipschitz continuous if there exists a positive real number L such that $\sup_{x \neq y \in B} |Tx - Ty| / |x - y| \leq L$. L is called the Lipschitz norm of T.

Note that the operator T_y is such that for any pair $x_1, x_2 \in H$ the following relation holds

$$|T_y(x_1) - T_y(x_2)| = |Kx_1 - Kx_2| \leq L\,|x_1 - x_2|$$

where $L < 1$ is the Lipschitz norm of K. We can then apply the Banach contraction mapping theorem (see for example [11], Lemma 4.1, p. 84). This theorem states that the desired fixed point x exists and it is unique.

ii) $(I-K)^{-1}$ is bounded and continuous*. It will clearly be sufficient to show that $(I-K)^{-1}$ is Lipschitz continuous. To this purpose observe that for any pair $x_1, x_2 \in H$, we have

$$|x_1 - Kx_1 - (x_2 - Kx_2)| = |x_1 - x_2 + Kx_2 - Kx_1|$$
$$\geq |x_1 - x_2| - |Kx_2 - Kx_1| \geq (1 - L)\,|x_1 - x_2| \tag{5.9}$$

where, once again, $L < 1$ is the Lipschitz norm of K. Set

$$y_1 = (I-K)x_1$$
$$y_2 = (I-K)x_2 \tag{5.10}$$

From Equations (5.9) and (5.10) and the fact that (I-K) is invertible, it follows that for each $y_1, y_2 \in H$, we have

$$|y_1 - y_2| \geq (1-L)\,|(I-K)^{-1}y_1 - (I-K)^{-1}y_2|. \tag{5.11}$$

This implies

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*This result has been noted also by Zarantonello [70].

$$\frac{|(I-K)^{-1}y_1 - (I-K)^{-1}y_2|}{|y_1 - y_2|} \leq 1/(1-L). \qquad (5.12)$$

We can then conclude that $(I-K)^{-1}$ is Lipschitz continuous.

iii) $(I-K)^{-1}$ is causal. Suppose that $y_1, y_2, x_1, x_2 \in H$ and

$$x_1 = (I-K)^{-1}y_1, \; x_2 = (I-K)^{-1}y_2. \qquad (5.13)$$

To prove that $(I-K)^{-1}$ is causal, it is sufficient to show that if $P^t y_1 = P^t y_2$, then $P^t x_1 = P^t x_2$.

Note that from Equation (5.13) and the hypothesis that K is causal, we can write

$$P^t y_1 = P^t x_1 - P^t K P^t x_1$$

$$P^t y_2 = P^t x_2 - P^t K P^t x_2.$$

For $P^t y_1 = P^t y_2$ it follows that

$$|P^t x_1 - P^t x_2| = |P^t K P^t x_2 - P^t K P^t x_1|. \qquad (5.14)$$

From this equation we must have $P^t x_2 = P^t x_1$ because if $P^t x_2 \neq P^t x_1$ then we would obtain a contradiction to the hypothesis that the Lipschitz norm of K is less than 1. $\triangleleft$

The above proposition allows to state and prove the central theorem of this chapter.

<u>Theorem 1.</u> Suppose that K has the following properties:

a) $K = K_{\underline{C}} N_\ell + K_C$, where N_ℓ is a memoryless Lipschitz continuous operator and $K_{\underline{C}}$, K_C are bounded linear operators;

b) $K_{\underline{C}}$ is strictly causal and K_C is causal;

c) the norm of K_C is less than 1.

Then the basic feedback system is stable.

<u>Proof.</u> I-K is invertible. We have to show that for any element $y \in H$ we can find a unique element $x \in H$ such that the following equation is satisfied

$$y = x - Kx. \tag{5.15}$$

To start, let L be the Lipschitz norm of N_ℓ and suppose that $|K_C| = 1 - \delta$, where $0 < \delta < 1$. Then for every pair $x_1, x_2 \in H$ we have

$$|N_\ell x_1 - N_\ell x_2| \leq L |x_1 - x_2| \tag{5.16}$$

$$|K_C x_1 - K_C x_2| \leq (1 - \delta) |x_1 - x_2|. \tag{5.17}$$

Note that from the strict causality of $K_{\underline{C}}$ we can find a partition Ω of the ordered set ν, $\Omega = \{t_o = \xi_o, \xi_1, \xi_2, \ldots, \xi_N = t_\infty\}$, such that

$$\left| \sum_{i=1}^{N} \Delta P(\xi_i)\, K_{\underline{C}}\, \Delta P(\xi_i) \right| \leq \delta/2L.$$

This implies that

$$|\Delta P(\xi_i) K_{\underline{C}} \Delta P(\xi_i)| < \delta/2L \qquad (5.18)$$

for every i=1, 2, . . . , N. From Equations (5.16), (5.18) and the fact that N_ℓ is memoryless, it follows that for each i=1, 2, . . . , N and any pair $x_1, x_2 \in H$, we have

$$|\Delta P(\xi_i) K_{\underline{C}} N_\ell \Delta P(\xi_i) x_1 - \Delta P(\xi_i) K_{\underline{C}} N_\ell \Delta P(\xi_i) x_2| =$$

$$= |\Delta P(\xi_i) K_{\underline{C}} \Delta P(\xi_i) N_\ell \Delta P(\xi_i) x_1 - \Delta P(\xi_i) K_{\underline{C}} \Delta P(\xi_i) N_\ell \Delta P(\xi_i) x_2|$$

$$\leq \frac{\delta}{2L} |N_\ell \Delta P(\xi_i) x_1 - N_\ell \Delta P(\xi_i) x_2| \leq \frac{\delta}{2} |\Delta P(\xi_i) x_1 - \Delta P(\xi_i) x_2|.$$

This equation together with Equation (5.17) implies that for each i=1, 2, . . . , N and any pair $x_1, x_2 \in H$ we have

$$\frac{|\Delta P(\xi_i) K \Delta P(\xi_i) x_1 - \Delta P(\xi_i) K \Delta P(\xi_i) x_2|}{|\Delta P(\xi_i) x_1 - \Delta P(\xi_i) x_2|} < 1. \qquad (5.19)$$

Observe now that to solve Equation (5.15) is equivalent to finding an $x \in H$ such that the following equation is satisfied

$$\Delta P(\xi_i) y = \Delta P(\xi_i) x - \Delta P(\xi_i) K P^{\xi_i} x \qquad (5.20)$$

where i=1, 2, . . . , N. Indeed, if $x \in H$ satisfies Equation (5.15) then, from the causality of K, $\Delta P(\xi_i) x$ must clearly satisfy Equation (5.20). Conversely, suppose that the element $x \in H$ is such that $\Delta P(\xi_i) x$ does satisfy Equation (5.20). Then we have

$$\sum_{i=1}^{N} \Delta P(\xi_i) y = \sum_{i=1}^{N} \Delta P(\xi_i) x - \sum_{i=1}^{N} \Delta P(\xi_i) K P^{\xi_i} x.$$

To see that this last equation coincides with Equation (5.15), it suffices to note that from the causality of K we can write

$$\sum_{i=1}^{N} \Delta P(\xi_i) K P^{\xi_i} x = \sum_{i=1}^{N} \Delta P(\xi_i) K x = K x.$$

For i=1, Equation (5.20) becomes

$$\Delta P(\xi_1) y = \Delta P(\xi_1) x - \Delta P(\xi_1) K \Delta P(\xi_1) x. \qquad (5.21)$$

By Equation (5.19), $\Delta P(\xi_1) K \Delta P(\xi_1)$, the restriction of K to the Hilbert space $\Delta P(\xi_1) H$, is Lipschitz continuous with Lipschitz norm less than 1. Proposition 1 can then be applied and there exists a bounded continuous causal operator T_1 such that the element

$$\Delta P(\xi_1) x = T_1 \Delta P(\xi_1) y \qquad (5.22)$$

(and only this element) satisfies Equation (5.21).

For i=2, Equation (5.20) becomes

$$\Delta P(\xi_2) y = \Delta P(\xi_2) x - \Delta P(\xi_2) K P^{\xi_2} x. \qquad (5.23)$$

From the fact that K is weakly additive (see Proposition 4.7.1) we can rewrite this equation as follows

$$\Delta P(\xi_2) y + \Delta P(\xi_2) K \Delta P(\xi_1) x = \Delta P(\xi_2) x - \Delta P(\xi_2) K \Delta P(\xi_2) x.$$

Again by Equation (5.19), $\Delta P(\xi_2) K \Delta P(\xi_2)$, the restriction of K to the Hilbert space $\Delta P(\xi_2) H$, is Lipschitz continuous with Lipschitz norm less than 1. Proposition 1 can again be applied and there exists a bounded continuous and causal operator T_2 such that it provides the following unique solution to Equation (5.23)

$$\Delta P(\xi_2) x = T_2 \, \Delta P(\xi_2) [y + K \Delta P(\xi_1) x]$$

where $\Delta P(\xi_1) x$ is defined by Equation (5.22). By induction, having computed $\Delta P(\xi_1) x$, $\Delta P(\xi_2) x, \ldots, \Delta P(\xi_{i-1}) x$, the solution to Equation (5.20). $\Delta P(\xi_i) x$, can be computed as follows

$$\Delta P(\xi_i) x = T_i \, \Delta P(\xi_i) [y + K \sum_{j=1}^{i-1} \Delta P(\xi_j) x] \tag{5.24}$$

where T_i is a bounded continuous and causal operator.

The above recursive relations define the element $x = \sum_{i=1}^{N} \Delta P(\xi_i) x$ and this element is the unique solution to Equation (5.15). We can then conclude that I-K is indeed invertible.

ii) $(I-K)^{-1}$ is bounded. Suppose that $x = (I-K)^{-1} y$. Note that for each $i = 1, 2, 3, \ldots, N$, $\Delta P(\xi_i) \, (I-K)^{-1}$ is defined by Equation (5.24). Using the boundedness of K, and applying Proposition 1, it is easy to verify that there exists a set of real positive numbers $M_1, M_2', M_2'', \ldots, M_i', M_i''$ such that

$$|\Delta P(\xi_1) (I-K)^{-1} y| \leq M_1 |y|$$

and for i=2, 3, 4, . . . , N

$$|\Delta P(\xi_i)(I-K)^{-1}y| \leq M_i' |y| + M_i'' |P^{\xi_{i-1}}x|.$$

From this equation it follows that there exists a positive M such that

$$|(I-K)^{-1}y| \leq M|y|.$$

It can then be concluded that $(I-K)^{-1}$ is bounded.

iii) $(I-K)^{-1}$ is continuous. Observe that the continuity of $(I-K)^{-1}$ is equivalent to the continuity of $\Delta P(\xi_j)(I-K)^{-1}$, j=1, 2, . . . , N, where $\{\xi_j\}$ is, once again, the partition of ν considered in part i). The transformation $\Delta P(\xi_j)(I-K)^{-1}$ is defined by Equation (5.24) where $P^{\xi-1}x$ is a function of y. By Proposition 1, this transformation is continuous with respect to y and $P^{\xi_{j-1}}x$. It follows that $\Delta P(\xi_j)(I-K)^{-1}$ is continuous on y. But from Equation (5.22) and Proposition 1, the element $P^{\xi_1}x$ depends continuously on y. Then also $P^{\xi_2}x$ depends continuously on y and, by induction, $P^{\xi_{j-1}}x$ depends continuously on y for each j=1, 2, . . . , N. We can then conclude that $(I-K)^{-1}$ is continuous.

iv) $(I-K)^{-1}$ is causal. By virtue of Proposition 4.4.1 it suffices to prove that for any $t \in \nu$ and any $y \in H$, the following relation holds

$$P^t(I-K)^{-1}y = P^t(I-K)^{-1}P^t y. \qquad (5.25)$$

Consider the partition Ω' of ν given by

$$\Omega' = \{t_0 = \xi_0, \xi_1, \xi_2, \ldots, \xi_{i-1}, t, \xi_i, \ldots, \xi_N = t_\infty\}$$

where $\{\xi_o, \xi_1, \ldots, \xi_N\}$ is the partition Ω considered in part i) of this proof and it has been assumed, without any loss of generality, that $t \in [\xi_{i-1}, \xi_i]$. Use the following notations

$$x^1 = (I-K)^{-1}y^1, x^2 = (I-K)^{-1}y^2$$

where $y^1 = y$ and $y^2 = P^t y$. From Equation (5. 24) the following relation holds

$$x^q = (I-K)^{-1}y^q = \sum_{j=1, j\neq 1}^{N} T_j[\Delta P(\xi_j)[y^q + KP^{\xi_{j-1}}x^q]]$$

$$+ T_t[(P^t - P^{\xi_{i-1}})[y^q + KP^{\xi_{i-1}}x^q]]$$

$$+ T_i[(P^{\xi_i} - P^t)[y^q + KP^{\xi_t}x^q]]$$

where q=1, 2. By inspection, from this equation we have

$$P^{\xi_1}x^1 = P^{\xi_1}x^2$$
$$P^{\xi_2}x^1 = P^{\xi_2}x^2$$
$$\text{-----------}$$
$$P^{\xi_i}x^1 = P^{\xi_i}x^2$$
$$P^t x^1 = P^t x^2$$

This implies the validity of Equation (5. 25). The proof of the theorem is at this point complete.

5.3 Stability and Causality for Linear Systems

All of the resu..s in Section 2 are valid in particular when the open loop system under consideration is linear. In this case however those results can be further improved. The following theorem not only guarantees the stability of the basic feedback system but it also gives an expression for the computation of the feedback operator $(I-K)^{-1}$. Before stating this theorem it is convenient to revisit Proposition 2.1.

Proposition 1. If K is linear bounded and causal and it has norm less than 1, then the basic feedback system is stable. Moreover, $(I-K)^{-1}$ can be computed by the following Neumann series

$$(I-K)^{-1} = I + \sum_{n=1}^{\infty} K^n. \tag{5.26}$$

Proof. Observe that from the fact that $|K| \leq 1$ we have that

$$\lim_{N\to\infty} \left| \sum_{n=N}^{\infty} K^n \right| \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} |K|^n = 0.$$

and therefore the right hand side of Equation (5.26) represents a well defined operator. In view of Proposition 2.1, it remains then to be proved that Equation (5.26) is true. This is equivalent to show that for any element $y \in H$ the element x,

$$x = (I + \sum_{n=1}^{\infty} K^n) y$$

satisfies the equation

$$y = x - Kx.$$

This is easily done by inspection

$$y = y + \sum_{n=1}^{\infty} K^n y - K[y + \sum_{n=1}^{\infty} K^n y]$$

$$= y + \sum_{n=1}^{\infty} K^n y - Ky - \sum_{n=2}^{\infty} K^n y = y. \qquad \triangleleft$$

Theorem 1. Suppose that K has the following properties:

a) $K = K_{\underline{C}} + K_C$, where $K_{\underline{C}}$ is strictly causal and K_C is causal

b) $K_{\underline{C}}$ and K_C are linear and bounded

c) the norm of K_C is less than 1.

Then the basic feedback system is stable. Moreover $(I-K)^{-1}$ can be computed by the following Neumann series

$$(I-K)^{-1} = I + \sum_{n=1}^{\infty} K^n. \tag{5.27}$$

Proof. Clearly K satisfies the hypotheses of Theorem 2.1 and therefore the stability of the basic feedback system follows from that theorem.

It remains to be shown that Equation (5.27) holds. To this purpose, choose a partition $\Omega = \{t_0 = \xi_0, \xi_1, \ldots, \xi_N = t_\infty\}$ of ν such that for all $i=1, 2, \ldots, N$ we have*

$$|\Delta P(\xi_i) K \Delta P(\xi_i)| < 1 \tag{5.28}$$

and consider the result of the next proposition. Note that from Proposition 2.1 and Equation (5.28) it follows that Equation (5.29) holds for $n=1$. Moreover, by virtue of Proposition 2, Equation (5.29) holds also for $n=2, 3, \ldots, N$. The proof can then be concluded by observing that for $n=N$, Equation (5.29) coincides with Equation (5.27). ◁

*Note that in the development of the proof of Theorem 2.1 it was shown that this choice of Ω is always possible.

Proposition 2. Suppose that the partition Ω satisfies Equation (5. 28) and that the hypothesis of Theorem 1 are satisfied. If for n= 1, 2, . . . , N, we have

$$P^{\xi_n}(I-P^{\xi_n}KP^{\xi_n})^{-1}P^{\xi_n} = P^{\xi_n} + \sum_{j=1}^{\infty}(P^{\xi_n}KP^{\xi_n})^j \qquad (5.29)$$

then we have also

$$P^{\xi_{n+1}}(I-P^{\xi_{n+1}}KP^{\xi_{n+1}})^{-1}P^{\xi_{n+1}} = P^{\xi_{n+1}} + \sum_{j=1}^{\infty}(P^{\xi_{n+1}}KP^{\xi_{n+1}})^j . \qquad (5.30)$$

Proof. Let us start by introducing the following simplification in notations

$$P = P^{\xi_n},\ Q = P^{\xi_{n+1}},\ \Delta = Q - P$$

and assume that Equation (5. 29) holds. We have to prove that Equation (5. 30) is also valid. To this purpose consider any element y in H and let Qx and Px be defined by the following expressions

$$Qx = Q(I - QKQ)^{-1}Qy$$

$$Px = P(I - PKP)^{-1}Py.$$

Note that

$$Qx = Px + \Delta x$$

where, applying respectively Proposition 2. 1 and Equation (5. 29) we have:

$$\Delta x = \Delta y + \Delta KPx + \sum_{j=1}^{\infty}[\Delta K\Delta]^j(y + \Delta KPx)$$

and

$$Px = Py + \sum_{j=1}^{\infty} (PKP)^j y.$$

It follows

$$Qx = Qy + \sum_{j=1}^{\infty} (PKP)^j y$$

$$+ \Delta K[Py + \sum_{j=1}^{\infty} (PKP)^j y] + \sum_{j=1} [\Delta K\Delta]^j y$$

$$+ \sum_{j=1}^{\infty} [\Delta K\Delta]^j K[Py + \sum_{j=1}^{\infty} (PKP)^j y].$$

From the linearity of K and a convenient rearrangements of the terms on the right hand side of this equation we obtain

$$Qx = Qy + \sum_{j=1}^{\infty} [(PKP)^j + (\Delta K\Delta)^j$$

$$+ \sum_{q+m=j-1} (\Delta K\Delta)^q K(PKP)^m] y \tag{5.32}$$

where $q, m \in \{0, 1, 2, \ldots\}$ and the following notations have been used

$$(\Delta K\Delta)^o = \Delta \text{ and } (PKP)^o = P.$$

Suppose now that for each $j = 1, 2, \ldots$ we have

$$(PKP)^j + (\Delta K\Delta)^j + \sum_{q+m=j-1} (\Delta K\Delta)^q K(PKP)^m = (QKQ)^j. \tag{5.33}$$

This equation and Equation (5.32) would imply

$$Qx = Qy + \sum_{j=1}^{\infty} (QKQ)^j$$

and with an obvious change of notations this last equation coincides with Equation (5.30). To complete the proof of the proposition it remains then to check the validity of Equation (5.33) for j=1, 2, 3, ... To this purpose verify by direct inspection that Equation (5.33) holds for j = 1, that is

$$PKP + \Delta K\Delta + \Delta KP = QKQ. \tag{5.34}$$

From this equation and an induction argument it is easy to verify that

$$(PKP)^{j+1} + (\Delta K\Delta)^{j+1} + \sum_{q+m=j} (\Delta K\Delta)^q K(PKP)^m$$

$$= [(PKP)^j + (\Delta K\Delta)^j + \sum_{q+m=j-1} (\Delta K\Delta)^q K(PKP)^m]QKQ \tag{5.35}$$

where j=1, 2, ... This equation and Equation (5.34) imply that Equation (5.33) is valid for j = 2, that is

$$(PKP)^2 + (\Delta K\Delta)^2 + \sum_{q+m=1} (\Delta K\Delta)^q K(PKP)^m = (QKQ)^2.$$

From this and Equation (5.35) it follows that Equation (5.33) is also valid for j = 3. Proceeding by induction we can conclude that Equation (5.33) is valid for all j=1, 2, 3, ... ◁

As an immediate consequence of Theorem 1 we have the following corollary. This corollary provides a direct answer to question Q1 in the introduction. It also has a meaningful physical interpretation: under certain circumstances the stability of the feedback system can be secured by an appropriate reduction of the gain of its memoryless part.

Corollary 1. Suppose that K has the following properties:

a) $K = K_{\underline{C}} + K_M$ where $K_{\underline{C}}$ is strictly causal and K_M is memoryless

b) $K_{\underline{C}}$ and K_M are linear and bounded

c) the operator norm of K_M is less than 1.

Then the basic feedback system is stable. Moreover $(I-K)^{-1}$ can be computed by the following Neumann series

$$(I-K)^{-1} = I + \sum_{h=1}^{\infty} K^n.$$

Other interesting implications of Theorem 1 are established by the following two propositions. Proposition 3 can often be applied in practical situations. Proposition 4 states some conditions for the input-output equivalence of the systems in Figure 5.2.

Proposition 3. If K is a linear, compact and causal operator on a HRS with a continuous resolution of the identity, then the basic feedback system is stable.

Proof. From the hypotheses and Lemma B.3 in the appendix, we obtain that $\int dPKdP = 0$. From Proposition 4.7.7 it follows that k is strictly causal. At this point the desired result follows from Theorem 1. ◁

Proposition 4. If T' and T" are linear bounded and strictly causal, and T'T" = T"T' = 0, then

$$(I-T'-T'')^{-1}(T'+T'') = (I-T')^{-1}T' + (I-T'')^{-1}T''.$$

Proof. Applying Theorem 1 we obtain the following equations:

$$(I-T'-T'')^{-1} = I + \sum_{n=1}^{\infty} (T'+T'')^n$$

$$(I-T')^{-1} = I + \sum_{n=1}^{\infty} (T')^n$$

$$(I-T'')^{-1} = I + \sum_{n=1}^{\infty} (T'')^n.$$

Using the fact that T'T" = T"T' = 0 we have.

$$(T'+T'')^n = (T')^n + (T'')^n.$$

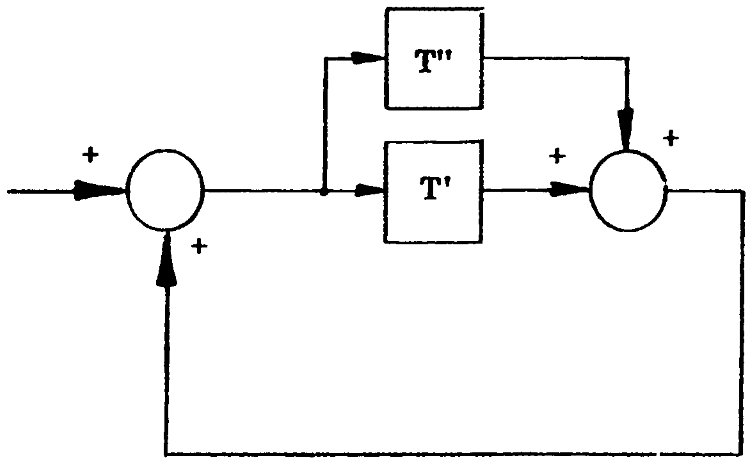
Hence:

$$(I-T'-T'')^{-1}(T'+T'') = \sum_{n=1}^{\infty} (T'+T'')^n$$

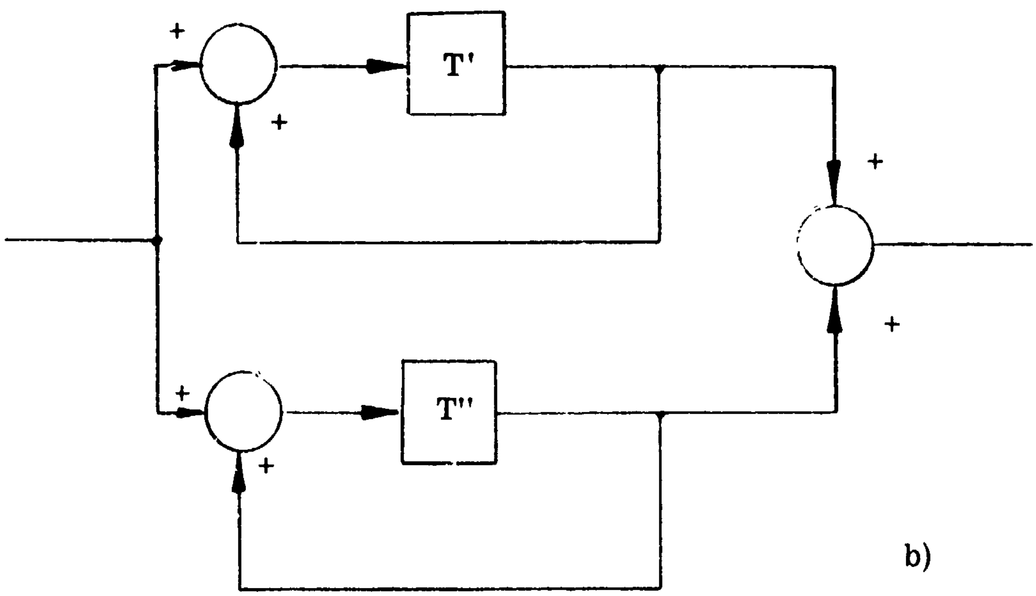
$$= \sum_{n=1}^{\infty} (T')^n + \sum_{n=1}^{\infty} (T'')^n$$

$$= (I-T')^{-1}T' + (I-T'')^{-1}T''.$$

The proof is thus complete. ◁



a)



b)

Figure 5. 2 : Fquivalent Systems considered in corollary 3

5.4 Some further results

The results in the previous two sections provide a clear indication of the technical convenience which derives by associating the mathematical development of the theory of non-self-adjoint operators of Gohberg and Krein ([24], [25]), with the study of causality in Hilbert resolution spaces (Chapter 4 and [41], [42], [48]) and with the stability theory framework proposed by Damborg ([11]). This convenience is further substantiated by the fact that Damborg's development can be directly rephrased in terms of Hilbert resolution spaces. When this is done a substantial gain in generality is obtained. It suffices to note that most of Damborg's results become immediately and simultaneously applicable to continuous time, sampled data and hybrid systems.

It is to be expected that the extent of the results arising from the above three independent developments is not limited to those already discussed. To justify at least in part this expectation, some further results are introduced.

The next theorem applies when the hypotheses of Theorems 2.1 and 3.1 are not readily verified. In this case it is often possible that the invertibility of (I-K) and the continuity of $(I-K)^{-1}$ can be established through subsidiary physical or mathematical arguments (see for instance [11], [12]). To insure the stability of the basic feedback system it remains then to verify that $(I-K)^{-1}$ is causal. The following theorem states that, under some appropriate hypotheses, a sufficient condition for this verification to be successful is that the open loop system is "almost" strictly causal.

<u>Theorem 1.</u> Suppose that T satisfies the following conditions:

a) $T = T_{\underline{C}} + T_C$, where $T_{\underline{C}}$ and T_C are respectively strictly causal and causal operators

b) $T_{\underline{C}}$ is bounded and T_C is Lipschitz continuous

c) T_C is weakly additive and has Lipschitz norm less than 1

d) I-T is invertible and $(I-T)^{-1}$ is Lipschitz continuous.

Then $(I-T)^{-1}$ is causal.

<u>Proof.</u> Since $T_{\underline{C}}$ is strictly causal, there exists a sequence of operators

$$I^{\Omega_i} = \sum_{j=1}^{N} \Delta P(\xi_j)\ T P^{\xi_{j-1}}$$

such that $\{I^{\Omega_i}\} \rightarrow T_{\underline{C}}$. Clearly $\{I^{\Omega_i} + T_C\} \rightarrow T$. For each operator

$I^{\Omega_i} + T_C$ and any element $y \in [H, P^t]$ the equation

$$y = x - I^{\Omega_i} x - T_C x$$

has the unique solution x defined by the following recursive relations

$$\Delta P(\xi_1) x = \Delta P(\xi_1)(I - T_C)^{-1} \Delta P(\xi_1) y$$

$$\Delta P(\xi_2) x = \Delta P(\xi_2)(I - T_C)^{-1} [\Delta P(\xi_2) y + $$
$$+ \Delta P(\xi_2) T_C \, \Delta P(\xi_1) x + \Delta P(\xi_2) \, I^{\Omega_i} \Delta P(\xi_1) x]$$

$$\Delta P(\xi_j) = \Delta P(\xi_j)(I - T_C)^{-1} [\Delta P(\xi_j) y + $$
$$+ \Delta P(\xi_j) T_C P^{\xi_{j-1}} x + \Delta P(\xi_j) I^{\Omega_i} P^{\xi_{j-1}} x]$$

$$\Delta P(\xi_N) = \Delta P(\xi_N)(I - T_C)^{-1} [\Delta P(\xi_N) y + $$
$$+ \Delta P(\xi_N) T_C \, P^{\xi_{N-1}} x + \Delta P(\xi_N) I^{\Omega_i} P^{\xi_{N-1}} x]. \qquad (5.36)$$

These relations follow from the hypotheses that the Lipschitz norm of T_C is less than 1 and T_C is weakly additive plus the application of Proposition 2.1. From Equation (5.36) we can see that $(I - I^{\Omega_i} - T_C)$ is invertible and its inverse $(I - I^{\Omega_i} - T_C)^{-1}$ is causal. From Proposition 4.4.1, to prove that $(I-T)^{-1}$ is causal it is sufficient to show that if y is any element of $[H, P^t]$ and $t \in \nu$, then

$$P^t (I-T)^{-1} y = P^t (I-T)^{-1} P^t y.$$

Assume for a moment that for every $y \in [H, P^t]$ the following relation holds

$$\left|(I-I^{\Omega_i}- T_C)^{-1}y + (I-T)^{-1}y\right| \to 0 \tag{5.37}$$

Then it follows

$$\{P^t(I-I^{\Omega_i}- T_C)^{-1}y\} \to P^t(I-T)^{-1}y \text{ and}$$

$$\{P^t(I-I^{\Omega_i}- T_C)^{-1}P^ty\} \to P^t(I-T)^{-1}P^ty.$$

But $(I-I^{\Omega_i}- T_C)^{-1}$ is causal for each $i=1, 2, \ldots$. Hence

$$P^t(I-I^{\Omega_i}- T_C)^{-1}y = P^t(I-I^{\Omega_i}- T_C)^{-1}P^ty$$

and consequently

$$P^t(I-T)^{-1}y = P^t(I-T)^{-1}P^ty.$$

It remains to be shown that Equation (5.37) is true. Assume the contrary. Then there would exist a positive real ϵ and $\{T_n\}$ a subsequence of $\{I^{\Omega_i}+ T_C\}$ such that

$$\left|(I-T_n)^{-1}y - (I-T)^{-1}y\right| > \epsilon. \tag{5.38}$$

In the sequel we will show that this equation cannot hold. This will mean that our assumption is absurd and hence that Equation (5.37) must be valid.

To start, denote $\{(I-T_n)^{-1}y\}$ and $(I-T)^{-1}y$ respectively by $\{x_n\}$ and x and observe that

$$y = x_n + T_nx_n = x + Tx. \tag{5.39}$$

Note also that from the boundedness of $(I-T)^{-1}$, it follows that $|(I-T_n)^{-1}|$ is uniformly bounded and consequently the sequ·nce $\{x_n\}$ is also uniformly bounded, that is

$$|x_n| = |(I-T_n)^{-1}y| \leq M|y| \tag{5.40}$$

where M is a positive number conveniently chosen.

The proof of the uniform boundedness of $\{(I-T_n)^{-1}\}$ goes briefly as follows: by the boundedness of $(I-T)^{-1}$ there exists a positive real m such that for every $x \in [H, P^t]$ the following relation holds

$$\frac{|(I+T)x|}{|x|} \geq 2m.$$

If we now choose an integer N such that $n \geq N$ implies $|T-T_n| \leq m$, then for $n \geq N$ we have

$$\frac{|(I-T_n)x|}{|x|} \geq \frac{|(I-T)x|}{|x|} - \frac{|(T-T_n)x|}{|x|}.$$

It follows $\frac{|(I-T_n)x|}{|x|} \geq m$ and this implies $|(I-T_n)^{-1}| \leq \frac{1}{m} = M$. This means that $\{(I-T_n)^{-1}\}$ is uniformly bounded.

To proceed with the proof of Equation (5.37), consider now the sequence $\{y_n\}$ given by $y_n = (I-T)x_n$. Observe that from Equations (5.39) and (5.40) we have

$$\begin{aligned} |y_n-y| &= |(I-T)x_n - (I-T_n)x_n| \\ &= |(T+T_n)x_n| \leq M|T+T_n|\;|v|. \end{aligned} \tag{5.41}$$

Note also that we can write

$$(I-T)^{-1}y - (I-T)^{-1}y_n = (I-T)^{-1}y - (I-T_n)^{-1}y.$$

From this equation and Equation (5.38) it follows that

$$|(I-T)^{-1}y - (I-T)^{-1}y_n| > \epsilon$$

and applying Equation (5.41) we obtain the following relation

$$\frac{|(I-T)^{-1}y - (I-T)^{-1}y_n|}{|y-y_n|} \geq \frac{\epsilon}{M|T-T_n|\,|y|} .$$

Since the sequence $\{T_n\}$ converges to T we have that $|T-T_n|$ can be as small as desired. As a consequence, the above equation implies that $(I-T)^{-1}$ is not Lipschitz continuous. This is a contradiction to hypothesis d) and it can then be concluded that Equation (5.38) cannot indeed hold. ◁

Corollary 1. If T is strictly causal and (I-T) is invertible with $(I-T)^{-1}$ Lipschitz continuous, then $(I-T)^{-1}$ is causal.

The rest of this section illustrates the fact that the results of Sections 2 and 3 can often be applied even when the hypotheses on the operator K are different from those assumed in the proofs. In particular, Proposition 1 considers a situation where K is not weakly additive (Figure 5.3). Proposition 2 considers a double loop feedback system (Figure 5.4). Finally Proposition 3 treats a case where K is not strictly causal.

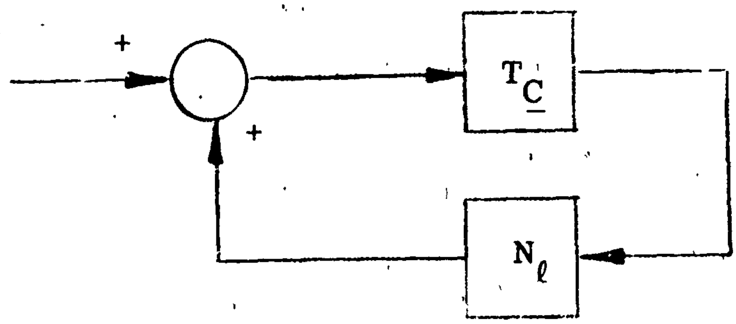


Figure 5.3 A Basic Feedback System With Non-weakly Additive Open Loop

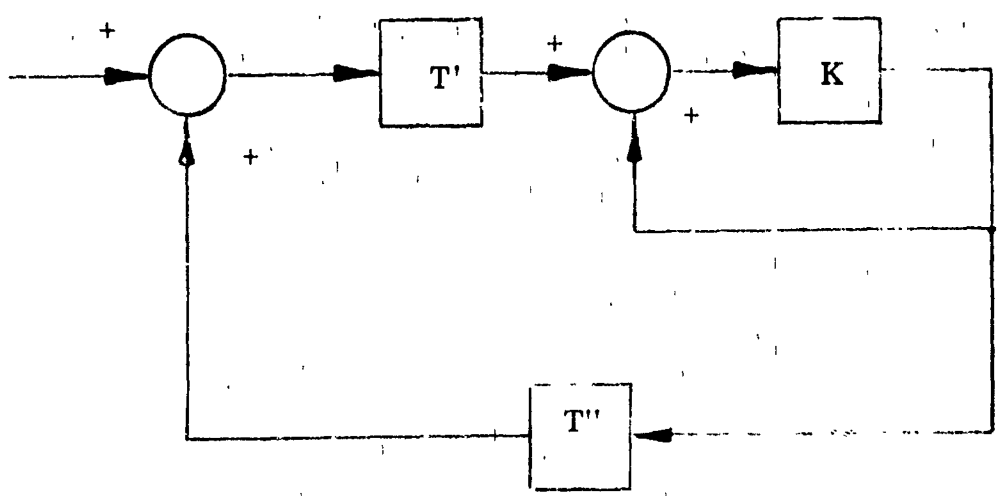


Figure 5.4 A Double Loop Feedback System

Proposition 1. Suppose that $K = N_\ell T$ where N_ℓ and T satisfy the following conditions:

a) T is linear bounded and strictly causal

b) N_ℓ is memoryless invertible and both N_ℓ and $(N_\ell)^{-1}$ are Lipschitz continuous.

Then the basic feedback system is stable.

Proof. Observe that K is not necessarily weakly additive and that Theorem 2.1 is not directly applicable. From Theorem 2.1 it follows however that the operator $(I-TN_\ell)$ is invertible and $(I-TN_\ell)^{-1}$ is bounded causal and continuous. From here it is easy to verify that the operator $Y = N_\ell(I-TN_\ell)^{-1}(N_\ell)^{-1}$ is also bounded causal and continuous. Moreover Y is invertible and its inverse is given by the following relation

$$Y^{-1} = [N_\ell(I-TN_\ell)^{-1}(N_\ell)^{-1}]^{-1} = N_\ell(I-TN_\ell)(N_\ell)^{-1} = (I-N_\ell T).$$

From the above equation it follows that $(I-N_\ell T)$ is invertible and its inverse is bounded causal and continuous. This means that the feedback system under consideration is stable. The proof of the Proposition is thus complete. ◁

Proposition 2. Consider the system in Figure 5.3 and suppose the following:

a) K and T'' are linear and bounded, T' is Lipschitz continuous

b) K is strictly causal, T' is memoryless and T'' is causal.

Then the system under consideration is stable.

Proof. By Theorem 3.1 the system in Figure 5.4 is equivalent to the system in Figure 5.5, where

$$\mathbf{K} = \sum_{n=1}^{\infty} K^n.$$

Observe that the system in Figure 5.5 is equivalent to the system in Figure 5.6. Moreover from the strict causality of K and Theorem 4.4.1, the operator IKKT'' + KT' is strictly causal. The desired result follows at this point from a direct application of Theorem 2.1.

Proposition 3. Suppose that T is linear and the basic feedback system is stable for K = T. Then the basic feedback system is also stable for $K = T + \tilde{K}$, where $\tilde{K} = K_{\underline{C}} N_{\ell}$ and $K_{\underline{C}}, N_{\ell}$ satisfy the hypotheses of Theorem 2.1.

Proof. Observe that, denoting y and x respectively as input and output, the systems in Figures 5.7 and 5.8 are equivalent. This can be seen by means of the following equations

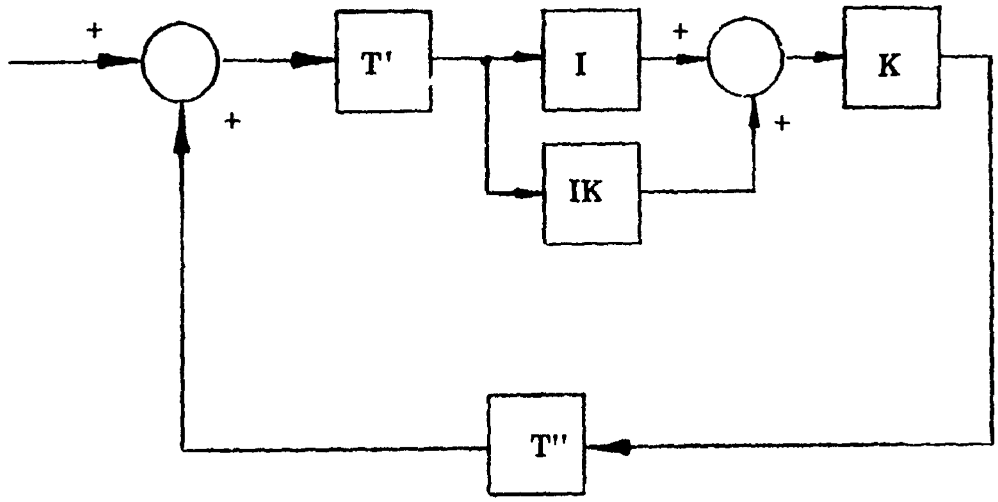


Figure 5. 5 Basic Feedback System Equivalent to the Double Loop Feedback in Figure 5. 4

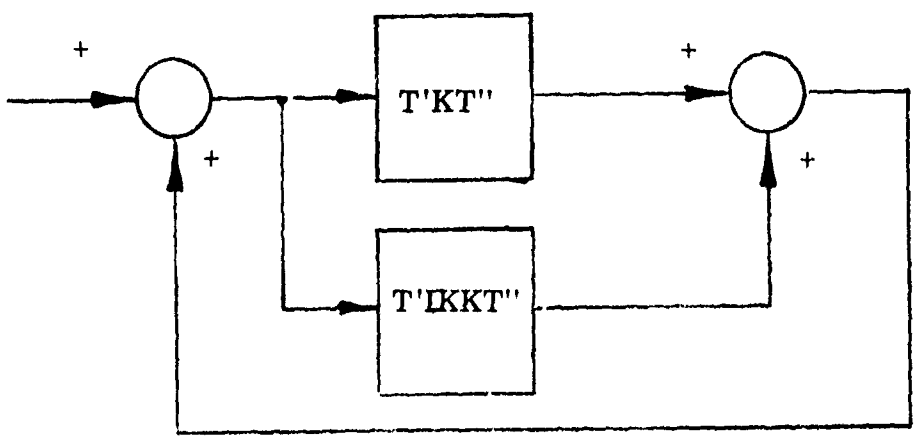


Figure 5. 6 Feedback System Equivalent to the System in Figure 5. 4

$$x = y + \tilde{K}x + Tx$$

$$(I-T)x = y + \tilde{K}x$$

$$x = (I-T)^{-1}y + (I-T)^{-1}Kx$$

By hypothesis $(I-T)^{-1}$ is linear causal and bounded. Applying Proposition 4.4.4 it is immediate to verify that $(I-T)^{-1}\tilde{K}$ satisfies the hypotheses of Theorem 2.1. It follows that the inverse of $(I-(I-T)^{-1}\tilde{K})$ is well defined. Moreover $(I-(I-T)^{-1}\tilde{K})^{-1}$ is bounded causal and continuous.

By the equivalence of the systems in Figures 5.7 and 5.8 we obtain the following relation

$$I - \tilde{K} - T = (I-T)(I-(I-T)^{-1}\tilde{K})$$

It follows

$$(I-\tilde{K}-T)^{-1} = (I-(I-T)^{-1}\tilde{K})^{-1}(I-T)^{-1}.$$

Hence the inverse of $(I-\tilde{K}-T)$ is well defined and it is bounded causal and continuous. The proof of the Proposition is thus complete. ◁

5.5 Applications

The theoretical development in this chapter has a wide range of applicability in the area of functional analysis and stability theory.

In classical functional analysis Theorems 2.1 and 3.1 not only offer a unified proof and a physical interpretation for various and

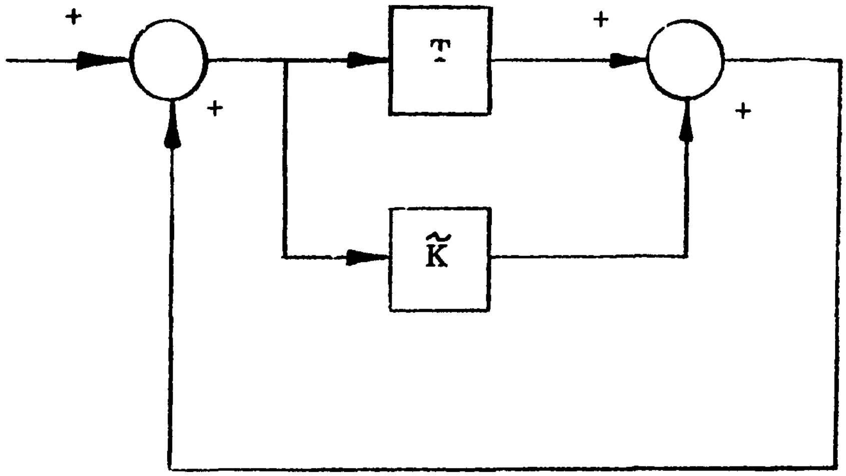


Figure 5.7 Feedback System Considered in Proposition 3

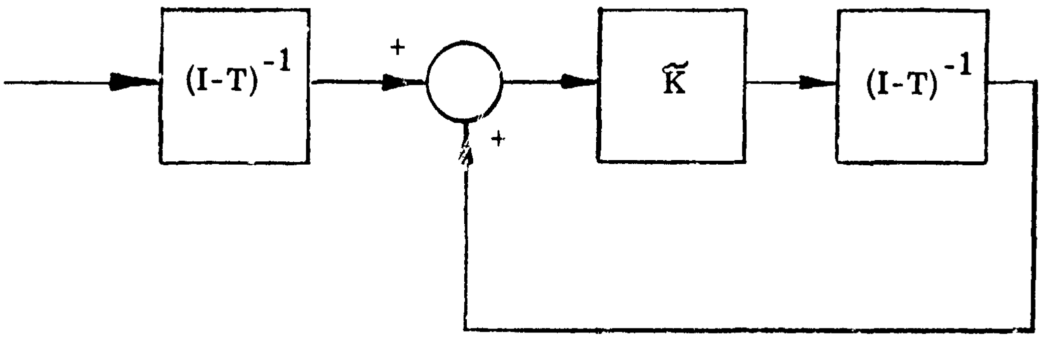


Figure 5.8 A Feedback System Equivalent to that in Figure 5.7

familiar results related to linear and nonlinear integral Volterra equations, but they also allow a ready extension of some of those results.

In modern functional analysis, Corollary 4.4 gives, by a proper reformulation, an important result which was found by Brodskii, Gohberg and Krein. This is Theorem 6.1 a), b) p. 27 in [25]. Here we emphasize that while the proof offered by those authors was based on a sophisticated result from the theory on the stability of the spectrum of bounded linear operators, no use of that theory was made in this development.

In regard to stability theory, this is a most fertile area of application. It should be noted that Theorem 4.1 appears to fit nicely in the general theoretical framework which was proposed by Damborg in [11]. As an illustration of some aspects of the applicability of Theorems 2.1 and 3.1, it is helpful to discuss the stability problem for some simple examples.

Example 1. Consider the system in Figure 5.9. Suppose that this system is described as follows:

i) N_ℓ, T', T" are Lipschitz continuous operators on the space of square integrable functions $L_2[0, \infty)$;

ii) N_ℓ is memoryless, T' and T" are linear and causal

iii) T' (or T") is compact.

The theory of the previous sections can be applied to show that this system is stable. This result does not depend on the particular description of N_ℓ, T' and T'' as long as they satisfy i-ii-iii. However the reader with a special taste for concreteness may suppose the following:

N_ℓ is the nonlinear gain of a memoryless amplifier with characteristic curve indicated in Figure 5.10

T' represents a linear time invariant stable dynamical system with a transfer function of the type

$$G_1(s) = \sum_{n=0}^{N} a_n s^n \Big/ \sum_{m=0}^{M} b_m s^m \qquad N \leq M$$

T'' is an operator defined as follows: if y = T''x then

$$y(t) = \int_0^t K(t,s)x(s)\,ds, \quad \text{where} \int_0^\infty \int_0^\infty K^2(t,s)\,dtds < \infty.$$

Consider the Hilbert resolution space (HRS) $[L_2[0,\infty), P^t]$ which was described in Example 4.2.1 and recall that the resolution of the identity in this case is continuous. Clearly, the problem of the study of the stability of the system under consideration can be embedded in the stability study of a system in this HRS.

The operators N_ℓ, T', T'' can be viewed as operators on $[L_2[0,\infty), P^t]$. Since T'' is linear compact and causal and $\mathbf{R}$ is continuous, Proposition 4.7.8 can be applied and it is obtained that T'' is strictly causal. From

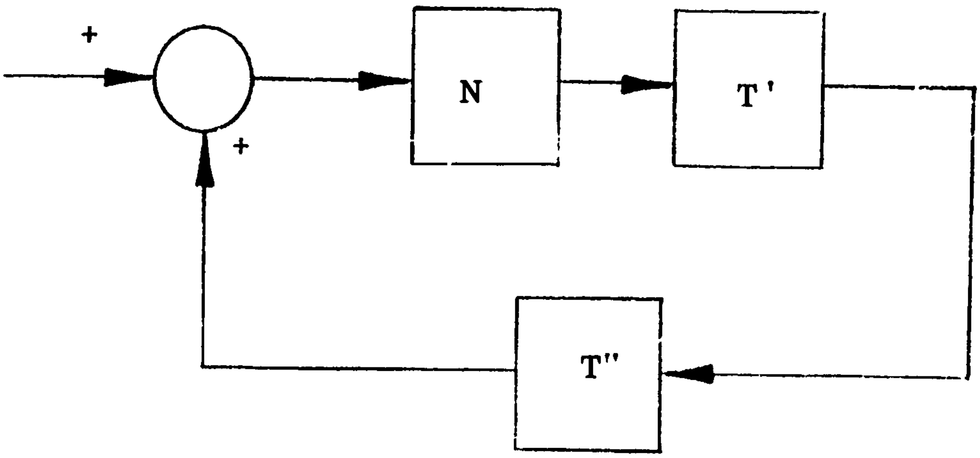


Figure 5. 9 : Feedback System considered in Example 1.

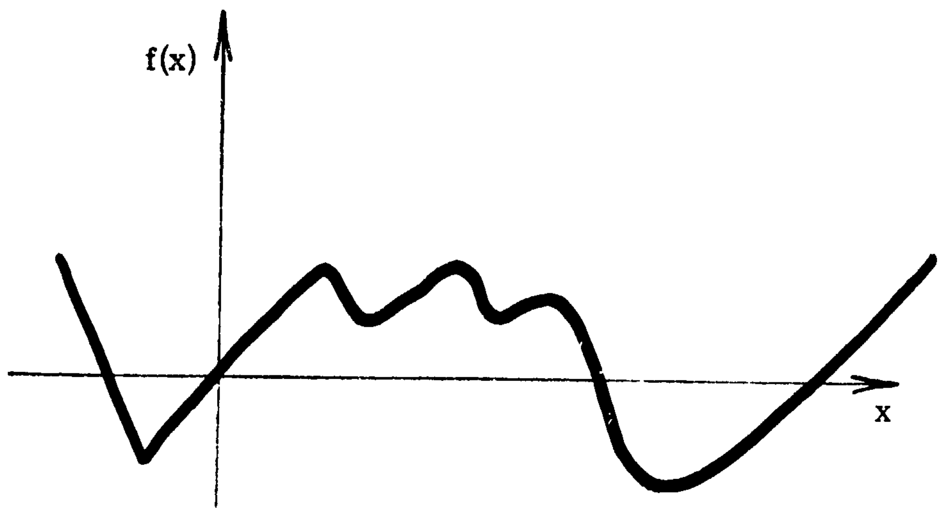


Figure 5. 10 : An admissable nonlinearity.

the application of Proposition 4.4.4 the operator T'T" is strictly causal. We can then apply Theorem 2.1.

From here it is obtained that the system under consideration is indeed stable.

Example 2. Consider the system in Figure 5.1.1. Suppose that this system is described as follows:

i) N_ℓ, T', T", T are operators on the space of real square integrable functions $L_2[0, \infty)$;

ii) N_ℓ, T', T" and T are described by the following relations:

a) $(N_\ell x)(t) = f(x(t))$ where f(.) is a Lipschitz continuous real function

b) $(T'x)(t) = \sum_{n=0}^{\infty} g_n x(t-\Delta t_n)$ where $\Delta t_n > 0$, $\Delta t_n \geq \Delta t_{n-1}$ and $\{g_n\} \in \ell_1$;

c) $T''x)(t) = \int_0^t g(t-\tau)x(\tau)\,d\tau$ where $g(t) \in L_1[0, \infty)$;

d) $(Tx)(t) = h(t)x(t)$ where $g(t) \in L_2[0, \infty)$.

As in example 1, the theory of the previous sections can be applied to show that this system is stable.

Consider again the Hilbert resolution space $[L_2[0, \infty), P^t]$ defined in Example 4.2.1. Observe that the operator N_ℓ is memoryless and the operator $T(G_1 + G_2)$ is linear, causal and bounded. Assume, for a moment, that

$$\int dPT(T' + T'')\,dP = 0. \tag{5.42}$$

Then by Theorem 4.4.1 the operator $T(T'+T'')$ is strictly causal. Corollary 2.2 can then be applied and it can be concluded that the system under consideration is stable.

In the sequel the validity of Equation (5.42) will be proved. To this purpose it will have to be shown that given any $\epsilon > 0$ a partition Ω_ϵ of $\upsilon = [0, \infty)$, $\Omega_\epsilon = \{0 = \xi_0, \xi_1, \xi_2, \ldots, \xi_N = \infty\}$, can be found such that for any other partition $\Omega' = \{0 = \xi'_0, \xi'_1, \xi'_2, \ldots, \xi'_{n'} = \infty\}$, with

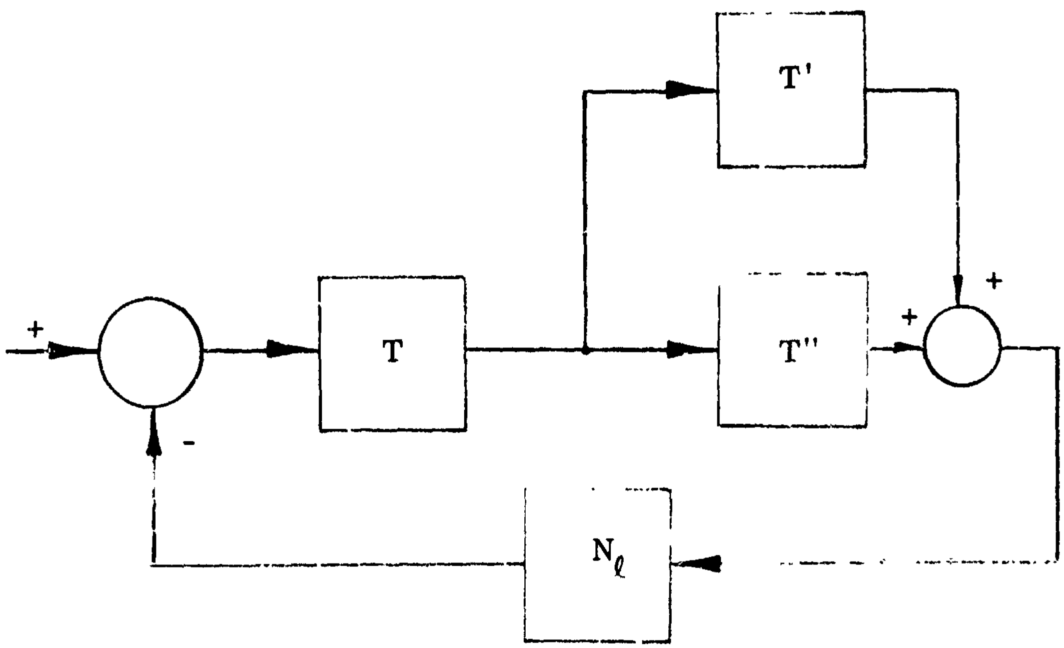


Figure 5.11 : Feedback System considered in Example 2.

the property $\Omega' \geq \Omega_\epsilon$, the following relation holds

$$\left| \sum_{i=1}^{n} \Delta P(\xi'_i)\, T(T' + T'')\, \Delta P(\xi'_i) \right| < \epsilon .$$

Choose an element $\bar{\xi} \in [0, \infty)$ such that

$$\left|(I-P^{\bar{\xi}})\, T(I-P^{\bar{\xi}})\right| \leq \epsilon/2 \left|T' + T''\right| .$$

Clearly then

$$\left|(I-P^{\bar{\xi}})\, T(T' + T'')\, (I-P^{\bar{\xi}})\right| < \epsilon/2. \qquad (5.43)$$

Consider now the operator $P^{\bar{\xi}}\, T'\, T''\, P^{\bar{\xi}}$. If $x, y \in L_2[0, \infty)$ and $y = P^{\bar{\xi}} T\, T''\, P^{\bar{\xi}} x$, then

$$y(t) = \int_0^t h(t)\, g(t-s)\, x(s)\, ds = \int_0^t K(t, s)\, x(s)\, ds$$

where $k(t, s) = h(t)\, g(t-s)$ for $t < \xi$ and $s \leq t$

$= 0$ for $t \geq \bar{\xi}$ or $s \geq t$.

The kernel $K(t, s)$ has the following property

$$\int_0^\infty \int_0^\infty K^2(t, s)\, dt ds \leq \left|h(\cdot)\right|^2 \cdot (\text{ess sup } g(\cdot))^2 \cdot \bar{\xi} < \infty .$$

It follows that the operator $P^{\bar{\xi}} T\, T''\, P^{\bar{\xi}}$ is Hilbert Schmidt. As it is well known, Hilbert Schmidt operators are compact [16].

By Proposition 4.6.5 it then follows

$$\int dP\, P^{\bar{\xi}}\, T\, T''\, P^{\bar{\xi}} dP = 0.$$

Then there exists a partition $\Omega_{0\epsilon} = \{\xi_0^0, \xi_1^0, \xi_2^0, \ldots, \xi_n^0 = \infty\}$ such that for $\Omega' = \{0 = \xi_0, \xi'_1, \xi'_2, \ldots, \xi'_{n'} = \infty\}$, $\Omega' \geq \Omega_{0\epsilon}$, the following relation holds

$$\left| \sum_{i=1}^{h'} P(\xi'_i) P^{\overline{\xi}} T'' P^{\overline{\xi}} P(\xi'_i) \right| < \epsilon/2. \qquad (5.44)$$

Construct now a partition $\Omega_\epsilon = \{0 = \xi_0, \xi_1, \xi_2, \ldots, \xi_n = \infty\}$ with the following properties:

$$\Omega_\epsilon \supset \Omega_{0\epsilon}, \ \overline{\xi} \in \Omega_\epsilon, \text{ and } |\xi_i - \xi_{i-1}| < \Delta t_0 \text{ if } \xi_i < \overline{\xi}.$$

Suppose that $\Omega' = \{0 = \xi'_0, \xi'_1, \xi'_2, \ldots, \xi'_{n'} = \infty\}$ is such that $\Omega' \geq \Omega_\epsilon$. Then from Equations (5.43) and (5.44) it follows

$$\left| \sum_{i=1}^{n'} \Delta P(\xi'_i) T(T'+T'') \Delta P(\xi'_i) \right| \leq \left| \sum_{i=1}^{n'} \Delta P(\xi'_i) P^{\overline{\xi}} T T' P^{\overline{\xi}} \Delta P(\xi'_i) \right| + \left| (I-P^{\overline{\xi}}) T T''(I-P^{\overline{\xi}}) \right| \leq \epsilon$$

where it has been taken into account the fact that

$$\left| \sum_{i=1}^{n'} \Delta P(\xi'_i) P^{\overline{\xi}} T T'' P^{\overline{\xi}} \Delta P(\xi'_i) \right| = 0.$$

At this point it can be concluded that Equation (5.42) is indeed true. ◁

5.6 Summary

Some connections betw en stability and causality have been investigated. To this purpose an "abstract" closed loop feedback system has been considered. The approach adopted is based on the axiomatic framework of Chapter 4. This allows a development which is simultaneously applicable to continuous time, sampled data and hybrid systems.

Particular attention has been given to some implications of the "strict" causality" property of an open loop system over the stability of the closed loop system. The basic results are contained in Theorems 2.1 and 4.1. Theorem 2.1 represents the main contribution of this chapter and states that strict causality of an open loop system plus some other "reasonable" conditions, guarantees the stability of the closed loop system. Some of its practical applications are illustrated by two concrete examples. At the present time, Theorem 2.1 is confined to a special class of "weakly additive" systems. Applications to more general classes of systems have however also been considered.

Theorem 4.1 establishes, under appropriate conditions, a connection between causality of an open loop system and causality of the closed loop system. This result appears to fit nicely in the stability problem theoretical framework which was proposed by Damborg in [11].

APPENDICES

A. Some Examples of Time Related Behavioral Patterns and Engineering Applications

In this appendix we discuss at some length three selected engineering examples. The objective is to supply the reader with some sort of "proving ground" for some of the ideas and abstract considerations in Section 1. 1.

Example 1. Consider the electroacoustic system consisting of a loudspeaker and a microphone located in a reverberant chamber (Figure A. 1). A voltage V_i is applied to the coil of the loudspeaker and the voltage V_o is measured at the microphone. Ignoring transients, this device can be described as a one input-one output linear time invariant system.

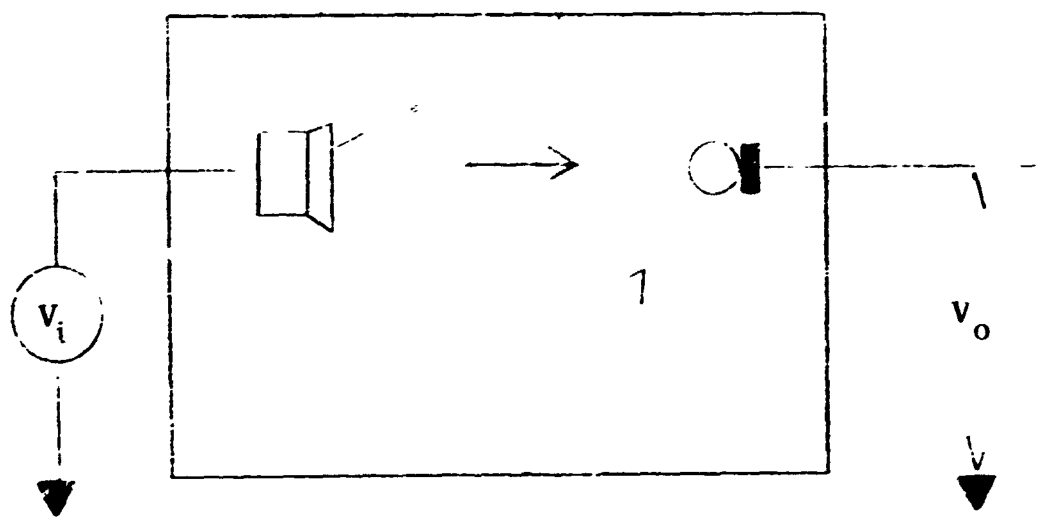


Figure A. 1

The output V_o can be viewed as the combination of two signals. One signal is due to the direct acoustic wave from the loudspeaker; the other is due to cumulative reflections from the walls. Thus F, the frequency response of the overall system, can be expressed as:

$$F(jw) = F_1(jw) + F_2(jw) \qquad w \in (-\infty, \infty) \tag{1}$$

where F_1 represents the idealized direct transmission and $F_2(jw)$ takes into account the influence of the walls.

The frequency response F can be obtained by a combination of experimental and analytical results. For example if the speaker-microphone complex is located in an anechoic chamber, then to a good approximation $F_2 = 0$ and $F = F_1$. Thus the identification of F_1 by experimentation is feasible. Suppose however that an anechoic chamber is not available or that the chamber available is not completely anechoic. Then $F_2 \neq 0$ and the problem of extracting F_1 from F arises [15].

Under normal experimental conditions, the following assumptions are satisfied: the loudspeaker has a finite memory, Δt_1; $\Delta t_2 = x_2 - x_1/c \geq \Delta t_1$, where: x_2 is the length of the path of the direct wave from the loudspeaker to the microphone and x_1 is the shortest path length of the various reflected waves. The constant c is the speed of sound. In the following we show that the problem under consideration can be viewed as one of identifying the anticausal part of a linear system.

Suppose that F has been experimentally evaluated. Multiply it by $e^{jw\Delta t_o}$, with $\Delta t_1 \leq \Delta t_o \leq \Delta t_2$. From Equation (1) we obtain $\tilde{F}(jw) = e^{jw\Delta t_o} F(jw) = e^{jw\Delta t_o} F_1(jw) + e^{jw\Delta t_o} F_2(jw) = \tilde{F}_1(jw) + \tilde{F}_2(jw)$. It is not difficult to convince ourselves that by virtue of the above assumptions, $\tilde{F}_1$ is the Fourier transform of an anticausal operator and $\tilde{F}_2$ is the Fourier transform of a causal operator. Clearly then the problem of computing F_1 can be viewed as the problem of identifying the anticausal component of $\tilde{F}$.

More specifically, the theoretical questions related to canonical decomposition become of paramount importance.

Example 2. Consider an operator, T, on the space of square integrable functions, $L_2[a, b]$. To compute $y(t) = [Tx](t)$, for a given $x \in L_2[a, b]$, one might consider the use of a hybrid computer. If T is causal it may be advantageous to simulate T on the analog part of the hybrid computer as this generally leads to a greater speed of computation. If T is not causal then this procedure is not open to us. However, suppose one could decompose T into a causal component T_C and an anticausal component T_A. Note that every anticausal system can be viewed as causal if we change the direction in which the time flows. It follows then that use of the analog part of the hybrid computer might be salvaged by a decomposition and a partial time reversal.

To clarify suppose that $y = K * f$, where K and f are represented in figure A. 2. Our system is not causal; however, we can write:

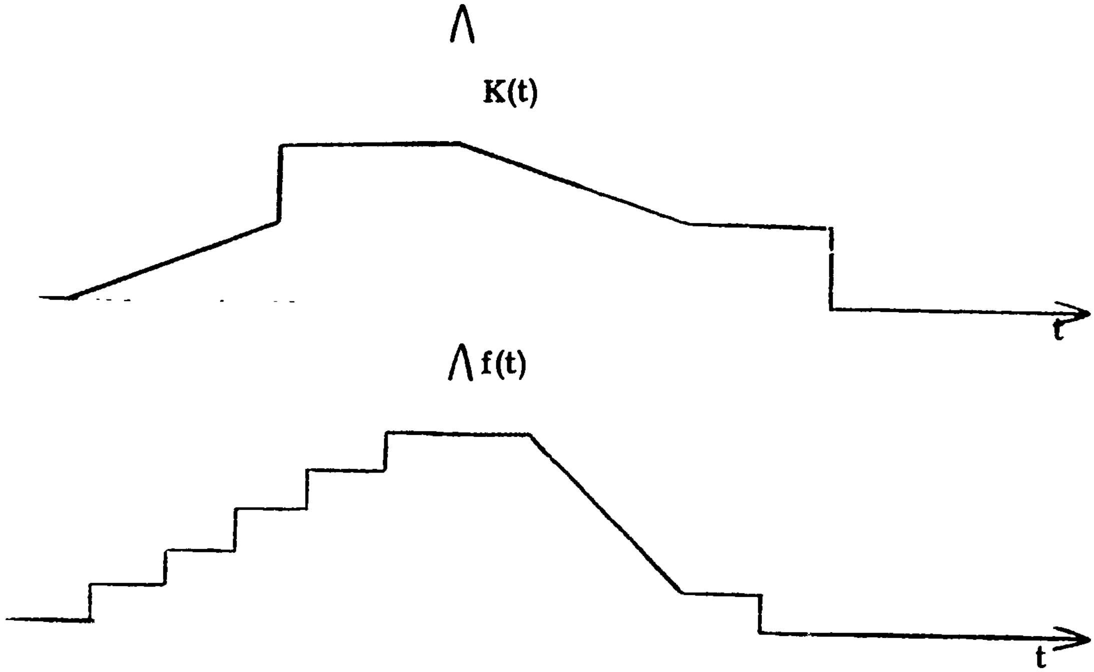


Figure A. 2

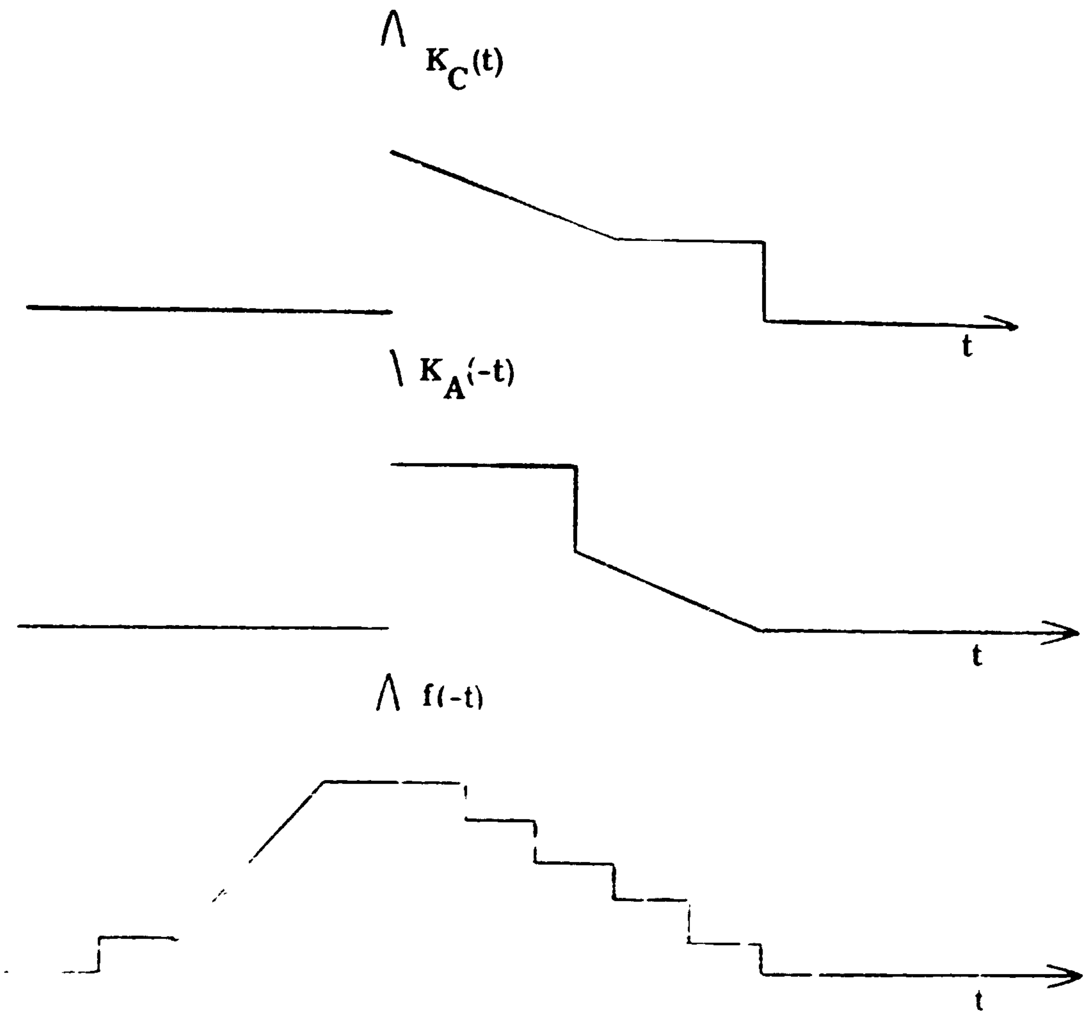


Figure A. 3

$$K(\tau) = K_A(\tau) + K_C(\tau)$$

where

$$K_C(\tau) = K(\tau) \text{ for } \tau \geq 0, \; K(\tau) = 0 \text{ for } \tau < 0$$

$$K_A(\tau) = 0 \quad \text{for } \tau \geq 0, \; K(\tau) = K(\tau) \text{ for } \tau < 0.$$

Clearly K_A is anticausal and K_C is causal (see figure A.3) and the output y can be expressed by

$$y = K_A * f + K_C * f$$

To compute y taking full advantage of the hybrid computer we can now proceed as follows: i) compute the outputs $y_1(t)$ and $y_2(t)$ of the two causal systems $K_C(\tau)$ and $K_A(-\tau)$ with inputs respectively f(t) and f(-t). ii) recognize that $y_1(t) = (K_C * f)(t)$ and $y_2(-t) = (K_A * f)(t)$ and hence compute $y(t) = y_1(t) + y_2(-t)$.

Example 3. Let us consider a quality control problem related to the acceptance or rejection of brake lining material in the automotive industry [10], [22]. The quality control procedure consists of the following:

i) Some laboratory tests are run on the brake lining material

ii) The results are compared with some nominal results

iii) A decision is made concerning the continued use of the material in the production line.

The variations on this quality control procedure depend on a number of factors. Most important among these factors are the choice of the type of tests to be run and the criteria of interpretation to be assigned to the results. A simplified version of one of the most widely adopted solutions consists basically of three steps. A small sample of the brake lining material is inserted in a chromatographic machine (first step). This test generates a response voltage, x(t). The function x, which is recorded as a chart, has a behavior which depends on the physical and chemical properties of the sample. Typically, x appears as in figure A. 4 where t_0 and t_f indicate respectively the initial and terminal time of the test.

To analyze the test result, a straight line, y(t), (called the drift line) is superimposed on the graph of x (second step). The purpose of y is to take into account undesirable effects which are due to a number of sample independent sources. In order to achieve this purpose the drift line must have the following properties: a) the difference x(t)-y(t) has to be positive for every $t \in [t_0, t_f]$; b) the area A between the curve x(t) and the drift-line y(t) ($A = \int_{t_0}^{t_f} [x(t) - y(t)]\, dt$) is minimized over all straight lines satisfying condition a).

In the final test step, the areas of well-defined peaks, i.e., $P_1, P_2, \ldots, P_N$, are computed and it is established whether each peak area is within a certain predetermined tolerance (third step). From a functional analysis point of view, the overall data reduction operation can be viewed as a transformation, π, mapping a special class of continuous functions in $C[t_0, t_f]$ into functions in $\ell_2(n)$. Note that according to extensive experimental results (which indeed constitute the essential foundations of the overall approach) the second step can be performed correctly only after all of the curve x is known. From this observation it follows that π is not a causal system.

Until recently the conventional test procedure was to initiate second and third steps after the test was run. In this way the delay time, t_0, between test initiation and test conclusion is given by $t_0 = t_1 + t_2 + t_3$, where t_1 is the time necessary to run the test, t_2 is the time required to draw the drift line, and t_3 is the time needed to compute $P_1, P_2, \ldots, P_N$. Using standard laboratory techniques, it turns out that t_3 is usually neglectable and that t_1 and t_2 are both of the order of 50-60 minutes. Hence t_0 is in general of the order of 100-120 minutes. In the mean time the production lines continue using the material under test. As an alternative procedure, observe that the mapping π can be viewed as given by the composition of two mappings π_1 and π_2. The mapping π_1 transforms x into $\{P'_i\}$ and the mapping π_2 transforms x into $\{P''_i\}$ where P'_i and P''_i indicate the areas in figure A. 4. Clearly $\{P_i\} = \{P'_i\} - \{P''_i\}$, hence $\pi = \pi_1 - \pi_2$; furthermore, it is not difficult to recognize π_1 as a causal mapping (according to our development π_2 can be easily identified as a "crosscausal" mapping).

To implement the quality control test, we can now adopt the following scheme: compute $\{P'_i\}$ <u>while</u> we run the test, then <u>after</u> the test is run compute $\{P''_i\}$ and finally compute the difference $\{P'_i\} - \{P''_i\}$.

The amount of time required for the overall data reduction operation in this procedure is given by $\tilde{t} = \tilde{t}_1 + \tilde{t}_2$ where: i) $\tilde{t}_1$ is equal to t_1; $\tilde{t}_2$ is the time required to compute $\{P''_i\}$ and to evaluate the difference $\{P'_i\} - \{P''_i\}$. In general it turns out that $\tilde{t}_2$ is of the order of 10-12 minutes. It follows that this second scheme provides a consistent economy of time with respect to the original one.

Area P_i = Area $A_iB_iE_iF_i$

Area P_i' = Area $A_iC_iD_iF_i$

Area P_i'' = Area $B_iC_iD_iE_i$

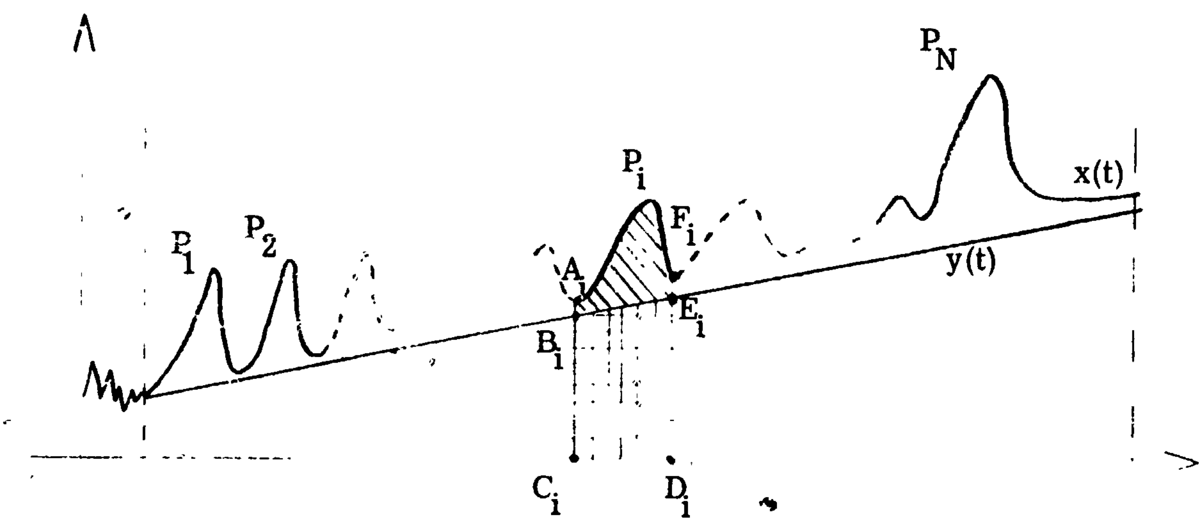


Figure A. 4

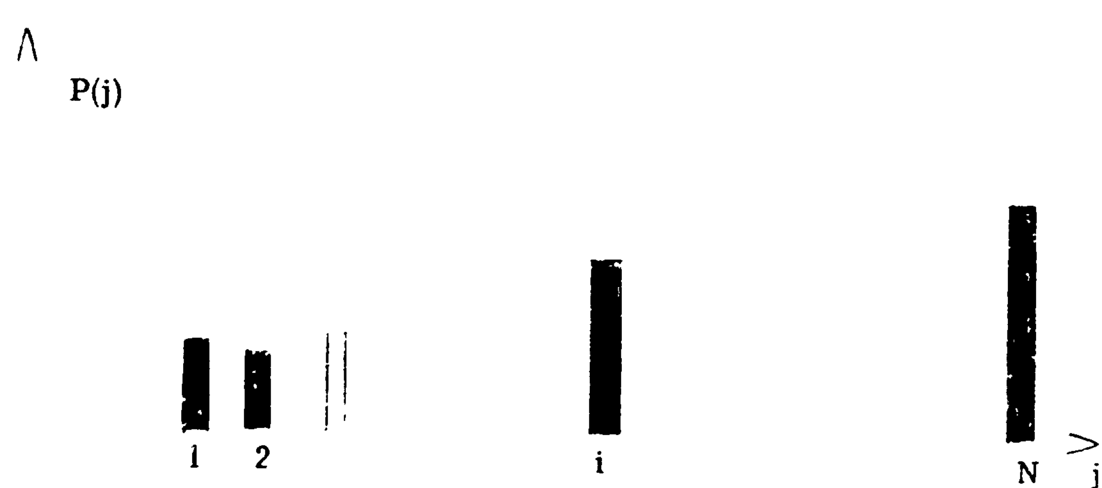


Figure A.5

B. Some Selected Results on Integral Transformators

In this appendix some selected results on integrals on Hilbert resolution spaces are presented and briefly discussed. These results can be found in [25] and the reader is referred to that reference for proofs and further details. A word of warning may be in order: the various original lemmas and theorems have not been reproduced in a faithful form. Terminology and notations have been regularly modified and, what is more serious, range and scope of applicability of some results have, on occasion, been arbitrarily limited. The main justification for doing this is the desire to make these results best suitable for a direct use in the various sections of this study.

To start, the first two results establish some useful connections among various types of integrals.

Lemma 1. ([25] p. 20). For the convergence of the integral $\int dPT(s)$, it is necessary that for any gap $P^{s_2} - P^{s_1}$ in $\mathbf{R}$ the following relation holds

$$(P^{s_2} - P^{s_1})\,[T(s_2) - T(s_1)] = 0.$$

Moreover if the integral $\int dPT(s)$ is well defined then

$$\int dPT(s) = m\!\int dPT(s) = M\!\int dPT(s).$$

Lemma 2. Suppose that T is a linear bounded operator on $[H, P^t]$. If $M\!\int dPTP^s$ and $\int dPTdP$ are well defined then $m\!\int dPTdP$ is also well defined. Moreover the following relation holds

$$M\!\int dPTP^{s} = m\!\int dPTP^{s} + \int dPTdP.$$

The next lemma shows that in the case of linear compact operators, the transformator $\int dP[\ \ ]dP$ is always well defined.

Lemma 3. ([25], p. 23 . Let T be a linear compact operator on $[H, P^t]$. Then $\int dPTdP$ is well defined and equal to

$$\sum_{i\in\omega} (P_i^+ - P_i^-)\,T(P_i^+ - P_i^-)$$

where (P_i^+, P_i^-) is the sequence of gaps in $\mathbf{R}$. In particular if $\mathbf{R}$ is continuous then $\int dPTdP = 0$.

Lemma 4 establishes the fundamental connection between transformators type $\int P^sTdP$ (which were used by Saeks) and transformators type $\int dPTP^s$ which are adopted in the present development.

Lemma 4. ([25], p. 26). If T is a bounded linear operator on $[H, P^t]$ and the integral $\int P^sTdP$ is well defined then the integral $\int dPTP^s$ is also well defined. Moreover, the following relation holds

$$\int dPTP^{s} = T - \int P^{s}TdP.$$

The next two Lemmas 5 and 6 play a key role in regard to the question of canonical causality decomposition.

Lemma 5. ([25], Lemma 4 1, p. 104). For any orthogonal system Φ_j ($j=0, \pm 1, \pm 2, \ldots$) in a Hilbert space H, there exists a resolution of the identity $\mathbf{R}$ such that for every self-adjoint operator

$$T = \sum_{j=-\infty}^{+\infty} \lambda_j (\cdot, \Phi_j)\Phi_j$$

satisfying the conditions

$$\lambda_j \geq 0 \ (j=1,2,\ldots), \ \lambda_j \leq 0 \ (j=0,-1,\ldots)$$

$$\sum_{j=-\infty}^{+\infty} \lambda_j / 2j-1 = \infty,$$

the integral $Y = \int PTdP$ diverges in the strong sense. Conversely for any continuous resolution of the identity $\mathbf{R} = \{P^t\}$ there exists an orthonormal system Φ_j $(j=0,\pm 1,\ldots)$ such that the above conclusions hold for the specified class of self-adjoint operators.

Before stating Lemma 6 it is convenient to introduce the following definition.

Definition 1. An operator T is called a G_ω operator if it satisfies the following conditions:

i) T is compact

ii) the spectrum of $(TT^*)^{1/2}$, $\{s_j(T)\}$, satisfies the condition

$$\sum_{j=-\infty}^{+\infty} (2j-1)^{-1} s_j(T) < \infty.$$

Definition 2. If $T \in G_\omega$ then the expression

$$|T|_\omega = \sum_{j=-\infty}^{+\infty} (2j-1)^{-1} s_j(T)$$

is called G_ω norm of T.

Lemma 6. ([25] Theorem 4.1, p. 106). Suppose that $T \in G_\omega$ is an operator on a $[H, P^t]$ and

$$(P^+ - P^-)\,T(P^+ - P^-) = 0$$

for every gap (P^+, P^-) in $\mathbb{R}$. Then the integral $\int PTdP$ is well defined and it has the precise bound

$$\left|\int PTdP\right|_\omega \leq |T|_\omega.$$

C. Some Additional Results on the Causality Structure of Linear Systems in HRS

This appendix can be viewed as an additional section to Chapter 4 and gives some connections between some causality properties of linear systems in a HRS and some of their analytic properties. In particular, Propositions 1, 2 and 3 illustrate some interesting relations between causality properties and compactness. Propositions 4 and 5 regard the existence of a canonical causality decomposition for G_ω operators.*

Proposition 1. For every linear compact operator T on a Hilbert space H, there exists a "maximal" resolution of the identity $\mathbb{R} = \{P^t\}$ such that T is strictly causal with respect to the HRS $[H, P^t]$.

The proof of this proposition is a direct consequence of a theorem due to Aronszajn and Smith and is omitted for brevity (see [25], Theorem 3.1, p. 15).

Proposition 2. Suppose that T is a self-adjoint operator on a HRS. Then, if T is causal or anticausal, T is memoryless. Moreover, if $T = T_{\underline{A}} + T_{\underline{C}} + T_M$ then $T_M^* = T_M$, $T_C^* = T_A$, $T_A^* = T_C$.

Proof. Suppose that T is a self-adjoint operator on $[H, P^t]$. If T is causal then $P^tT = P^tTP^t$. Taking the adjoint of both members of this equation it follows $T^*P^t = P^tT^*P^t$. Using the fact that T is self-adjoint

*See Definition B. 1.

it follows $TP^t = P^tTP^t$. From this equation and from the dual of Proposition 4.6.3, it can be concluded that T is anticausal. Suppose now that $T = T_{\underline{A}} + T_{\underline{C}} + T_M$. Then $T^* = T^*_{\underline{A}} + T^*_{\underline{C}} + T^*_M$. From Proposition 4.6.1 and 4.6.3 it follows $T_{\underline{C}} = T^*_{\underline{A}}$, $T_{\underline{A}} = T^*_{\underline{C}}$, $T_M = T^*_M$.

<u>Proposition 3.</u> For every compact self-adjoint operator T on a Hilbert space H, there exists a maximal resolution of the identity $\mathbf{R} = \{P^t\}$ such that T is memoryless with respect to the HRS $[H, P^t]$.

<u>Proof.</u> From the compactness of T and Proposition 1, there exists a maximal resolution of the identity $\mathbf{R} = \{P^t\}$ such that T is causal with respect to $[H, P^t]$. From the fact that T is self-adjoint and Proposition 2i) it follows that T is memoryless.

<u>Proposition 4.</u> Suppose that T is an operator on a HRS and that $T \in G_\omega$. Then a canonical causality decomposition

$$T = T_{\underline{A}} + T_{\underline{C}} + T_M$$

always exists and it is unique. Furthermore the operator norms of $T_{\underline{A}}, T_{\underline{C}}, T_M$ are bounded by the number

$$|T|_\omega = \sum_{j=-\infty}^{+\infty} (2j-1)^{-1} s_j(T).$$

<u>Proposition 5.</u> Suppose that T is an operator on a Hilbert space H and that T does not belong to G. Then we can find a resolution of the identity $\mathbf{R} = \{P^t\}$ such that T does not have a canonical causality decomposition on $[H, P^t]$.

The proofs of Propositions 4 and 5 are based on Lemmas B. 5 and B. 6 and are once more omitted for brevity. It is important to note that according to these two propositions the C_ω operators constitute the largest class of compact operators for which a canonical causality decomposition can be guaranteed in general.

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