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ASSEMBLY OF SYSTEMS HAVING MAXIMUM
RELIABILITY

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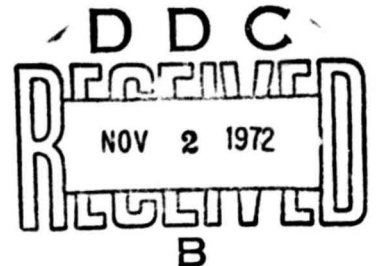
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[0] SUMMARY

The first problem considered in this paper is concerned with the assembly of independent components into parallel systems so as to maximize the expected number of systems that perform satisfactorily. Associated with each component is a probability of it performing successfully. It is shown that an optimal assembly is obtained if the reliability of each assembled system can be made equal. If such equality is not attainable, then bounds are given so that the maximum expected number of systems that perform satisfactorily will lie within these stated bounds; the bounds being a function of an arbitrarily chosen assembly. An improvement algorithm is also presented.

A second problem treated is concerned with the optimal design of a system. Instead of assembling given units, there is an opportunity to "control" their quality, i.e., the manufacturer is able to fix the probability, p , of a unit performing successfully. However, his resources are limited so that a constraint is imposed on these probabilities. For (1) series systems, (2) parallel systems, and (3) k out of n systems, results are obtained for finding the optimal p 's which maximize the reliability of a single system, and which maximize the expected number of systems that perform satisfactorily out of a total assembly of J systems.

1. INTRODUCTION

In a previous paper [1], the authors considered the following reliability problem: A system has n different types of components. Associated with each component is a numerical value. Let $\{a^m\}$ ($m = 1, 2, \dots, n$) denote the set of numerical values of the n components. Let $R(a^1, a^2, \dots, a^n)$ denote the probability that the system will perform satisfactorily, i.e., $R(a^1, a^2, \dots, a^n)$ is the reliability of the system. Now suppose $a_1^m \leq a_2^m \leq \dots \leq a_J^m$ are J components of type m ($m = 1, 2, \dots, n$). Then J systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the J systems are assembled. The results obtained show that if $R(a^1, a^2, \dots, a^n)$ has the properties of a joint cumulative distribution function then of all different ways in which the J systems can be assembled, $E(N)$ is maximized if these J systems have reliability $R(a_j^1, a_j^2, \dots, a_j^n)$ ($j = 1, 2, \dots, J$), i.e., assemble the best of each type, the next best of each type, ..., and finally the worst of each type.

The aforementioned results are applicable to series systems of independent components, but are not applicable to parallel systems of independent components. Section 2 of this paper treats the problem of parallel systems of independent components, and shows that an optimal assembly is obtained if the reliability of each assembled system can be made equal. Furthermore, if equal reliability for each assembled system is not possible, then bounds are obtained so that the maximum expected number of systems that perform satisfactorily will lie

within these stated bounds, the bounds being a function of an arbitrarily chosen assembly. An algorithm will be presented which will produce assemblies that result in an improvement in these bounds, although will not necessarily lead to the optimal assembly. The results are not only applicable to the aforementioned problem but are also applicable to the assembly of parallel systems of independent units, where the units in a system are interchangeable (only one type of unit is in the system). The results are also valid for the interchangeable and non-interchangeable cases when the number of units of a given type that must appear in a given system is not fixed in advance.

Section 3 is concerned with a variation of the assembly problem dealing with system design. Suppose a single system is to be constructed containing n units. Attached to the m^{th} unit is a positive value p_m , which will denote the probability that the m^{th} unit will perform satisfactorily, and these probabilities are assumed to be independent. The n units are to be "manufactured", and the manufacturer has sufficient control of his process that he is able to produce at an aimed at "p" level, but his resources are limited so that there is a constraint imposed on the values of the p 's, namely

$$\sum_{m=1}^n p_m = A$$

For (1) series systems, (2) parallel systems, and (3) k out of n systems (the system operates satisfactorily if at least k out of n units operate satisfactorily), the problem considered is to find the desired p 's so as to maximize the reliability of the system. For a series system the optimal solution is to make all the p 's equal to A/n . For the parallel system the optimal solution is to make one of the p 's equal to A and the rest 0. For the k out of n

system, the results are more complex.

Section 4 is concerned with the same problem considered in Section 3, except that nJ components are to be "manufactured" and assembled into J systems, each being a k out of n system. Again, it is desired to maximize the expected number of systems that perform satisfactorily, subject to constraints on the p 's. A characterization of the optimal solution is given.

2. OPTIMAL ASSEMBLY OF PARALLEL SYSTEMS OF INDEPENDENT COMPONENTS

The following problem will be considered first. A set of M units are given which are numbered $1, 2, \dots, M$. The M units are to be partitioned into J disjoint systems. After completion of a partition, the number of units contained in the j^{th} system ($j = 1, 2, \dots, J$) will be denoted by n_j , with the added restriction that $\sum_{j=1}^J n_j = M$.^{1/} A system will perform satisfactorily if at least one of the n_j units in the system performs satisfactorily, i.e., it is a parallel system. Attached to the m^{th} unit, $m = 1, 2, \dots, M$, is a positive value p_m which will denote the probability that the m^{th} unit will perform satisfactorily, and these probabilities are assumed to be independent. For a given partition, the reliability of system j ($j = 1, 2, \dots, J$), R_j , is the probability that the system will perform satisfactorily, and can be expressed as^{1/}

^{1/} A partition will allow for one or more systems to contain no units so long as $\sum_{j=1}^J n_j = M$. The reliability of a system containing no units will be taken to be zero.

$$R_j = 1 - \prod_{\substack{\text{all } m \text{ in} \\ \text{system } j}} (1 - p_m). \quad (1)$$

Let N denote the number of systems that perform satisfactorily, so that N is a random variable whose distribution will depend upon a given partition. For a given partition, the expected number of systems that perform satisfactorily, $E(N)$, is then seen to be

$$E(N) = \sum_{j=1}^J [1 - \prod_{\substack{\text{all } m \text{ in} \\ \text{system } j}} (1 - p_m)] = J - \sum_{j=1}^J \prod_{\substack{\text{all } m \text{ in} \\ \text{system } j}} (1 - p_m). \quad (2)$$

The problem treated in this section is to find the partition which maximizes (2), or alternatively, to find the partition that minimizes the expression

$$\sum_{j=1}^J \prod_{\substack{\text{all } m \text{ in} \\ \text{system } j}} a_m, \quad (3)$$

where $a_m = (1 - p_m)$. Henceforth, the a_m will be referred to as the positive value attached to the m^{th} unit.

In a given partition, denote the set of n_j units appearing in system j ($j = 1, 2, \dots, J$) by A_j and define $|A_j|$ to equal the product of the values of the units in the set A_j . If A_j is the null set, $|A_j|$ is defined to be 1. Define

$$T = \text{Minimum} \sum_{j=1}^J |A_j|, \quad (4)$$

where the minimum is taken over all possible partitions of the M units into J disjoint sets. The major result of this section shows that for any partition A_1, A_2, \dots, A_J , T satisfies the following inequalities,

$$J \text{ Min } |A_j| \leq T \leq \sum_{j=1}^J |A_j|. \quad (5)$$

In order to prove this result, two lemmas must be verified.

Lemma 1: Let a_m and b_m ($m = 1, 2, \dots, s$) be positive numbers.

If
$$\prod_{m=1}^s a_m = \prod_{m=1}^s b_m,$$

then
$$\sum_{m=1}^s (a_m/b_m) \geq s. \quad (6)$$

Proof: Since $a_s/b_s = \prod_{m=1}^{s-1} b_m / \prod_{m=1}^{s-1} a_m$, the conclusion of the lemma will follow if it can be shown that

$$\sum_{m=1}^{s-1} (a_m/b_m) + \prod_{m=1}^{s-1} b_m / \prod_{m=1}^{s-1} a_m \geq s. \quad (7)$$

Now, define the function

$$f(b_1, b_2, \dots, b_{s-1}) = \sum_{m=1}^{s-1} (a_m/b_m) + \prod_{m=1}^{s-1} b_m / \prod_{m=1}^{s-1} a_m.$$

Differentiating this function with respect to b_k , yields

$$\frac{\partial f(b_1, b_2, \dots, b_{s-1})}{\partial b_k} = -a_k/b_k^2 + \prod_{\substack{m=1 \\ m \neq k}}^{s-1} b_m / \prod_{m=1}^{s-1} a_m, \quad k = 1, 2, \dots, s-1.$$

Equating these to zero, results in

$$a_k/b_k = \prod_{m=1}^{s-1} b_m / \prod_{m=1}^{s-1} a_m = c, \quad k = 1, 2, \dots, s-1. \quad (8)$$

This set of equations implies that each a_k/b_k equals the constant, c .

If all of the $(s-1)$ terms of the left hand side of (8) are multiplied together, then it follows from (8) that

$$\prod_{k=1}^{s-1} (a_k/b_k) = c^{(s-1)}. \quad (9)$$

However, from (8), the left hand side of (9) is also equal to c . Hence, c must equal 1.

Since $f(b_1, b_2, \dots, b_{s-1})$ approaches infinity as one or more of the b 's approach zero or infinity, $f(b_1, b_2, \dots, b_{s-1})$ is minimized when its partial derivatives vanish. Thus, it attains its minimum when $a_m = b_m$ ($m = 1, 2, \dots, s-1$), and $f(b_1, b_2, \dots, b_{s-1}) \geq s$ so that the lemma is proved.

Lemma 2: Let A_1, A_2, \dots, A_J be a partition such that

$|A_1| = |A_2| = \dots = |A_J|$. Then $T = J|A_1|$, i.e., this partition is optimal.

Proof: Consider any other partition - say B_1, B_2, \dots, B_J . Using set theoretic identities, it follows that

$$|B_j| = \frac{|A_j| |B_j \bar{A}_j|}{|A_j \bar{B}_j|}, \quad \text{where}$$

\bar{A}_j and \bar{B}_j are the complements of A_j and B_j , respectively. Hence, to prove the lemma it is sufficient to show that

$$\sum_{j=1}^J (|B_j \bar{A}_j| / |A_j \bar{B}_j|) \geq J. \quad (10)$$

Inequality (10) follows from Lemma 1 by noting that

$$\prod_{j=1}^J |B_j \bar{A}_j| = \prod_{j=1}^J |A_j \bar{B}_j|,$$

because both terms are equal to

$$\left| \left(\bigcup_{j=1}^J A_j B_j \right) \right|.$$

Theorem 1.

For any partition A_1, A_2, \dots, A_J , T satisfies the following

inequalities

$$J \min |A_j| \leq T \leq \sum_{j=1}^J |A_j|.$$

Proof:

Suppose that $|A_k| = \min |A_j|$. It then follows that the units in the sets A_j , $j \neq k$ have values, a_m , associated with them such that $|A_j| \geq |A_k|$. Now, consider a new set of M units. These units are partitioned in the same manner as the original units. The values associated with the units in set A_k are the same as those in the original set. However, the values associated with the units in set A_j , $j \neq k$ are all less than or equal to the corresponding a 's in the original set, but are such that $|A_j|$ now equals $|A_k|$ for all $j \neq k$. For this new set of M units, let T_0 denote the minimum $\sum_{j=1}^J |A_j|$, where again the minimum is taken over all possible partitions of the M units into J disjoint sets. However, from Lemma 2, the given partition must be the optimal one for the new set of units, and furthermore, $T_0 = J|A_k|$. Since, the function T is obviously a monotone increasing function of the set of values $\{a_1, a_2, \dots, a_M\}$, it follows that

$$T \geq T_0 = J|A_k| = J \min |A_j|.$$

The other inequality, i.e., $T \leq \sum_{j=1}^J |A_j|$, is obvious. Hence, the theorem is proved.

The theorem indicates that the maximum expected number of systems that perform satisfactorily will lie within the stated bounds, these

bounds being a function of the chosen partition. Furthermore, if a partition can be found that makes each system have the property that the product of the probabilities of each unit failing, i.e., the a 's, equal, then this partition is optimal in that $E(N)$ is maximized. These results lead to questioning whether or not an algorithm can be obtained which will determine the optimal partition. Unfortunately, the authors have been unable to find one, but an algorithm will be presented which should lead to a "good" solution; but not necessarily an optimal one.

A given partition results in a sequence of sets A_1, A_2, \dots, A_J (each set representing a system) and a corresponding $|A_1|, |A_2|, \dots, |A_J|$. It can be assumed that the $|A|$'s are not equal; otherwise, an optimal partition has been obtained. Choose any two systems whose $|A|$'s are not equal and without loss of generality, denote them by A_1 and A_2 , with $|A_2| > |A_1|$. It will be shown that under certain conditions units of one can be interchanged with units of the other, thereby resulting in a new partition with sharper bounds than given in (5). Let α_1 denote the product of the values which are attached to those units which are to be removed from A_1 and placed into A_2 . Similarly, let α_2 denote the product of the values which are attached to those units which are to be removed from A_2 and placed into A_1 . Thus, in the new partition, the $|A|$'s are given by $(\alpha_2/\alpha_1)|A_1|, (\alpha_1/\alpha_2)|A_2|, |A_3|, \dots, |A_J|$. The main result to be obtained is that if $|A_2| > |A_1|$ and $\|(\alpha_1/\alpha_2)|A_2| - (\alpha_2/\alpha_1)|A_1|\| < \| |A_2| - |A_1| \|$,^{1/} then

^{1/} The symbol $\|Z\|$ reads the absolute value of Z .

$$|A_1| + |A_2| > (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|. \quad (11)$$

Before proving this result, two lemmas are required.

Lemma 3: If $|A_2| > |A_1|$, then $|A_1| + |A_2| > (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|$

$$\Leftrightarrow |A_1|/|A_2| < \alpha_1/\alpha_2 < 1.$$

Proof: It is evident that $|A_1| + |A_2| > (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|$
 $\Leftrightarrow (\alpha_2 - \alpha_1)[|A_2|/\alpha_2 - |A_1|/\alpha_1] > 0$. If $|A_1|/|A_2| < \alpha_1/\alpha_2 < 1$, then
 $(\alpha_2 - \alpha_1)[|A_2|/\alpha_2 - |A_1|/\alpha_1] > 0$.

Now suppose that $(\alpha_2 - \alpha_1)[|A_2|/\alpha_2 - |A_1|/\alpha_1] > 0$. This implies that
if $\alpha_2 - \alpha_1 < 0$, then $[|A_2|/\alpha_2 - |A_1|/\alpha_1] < 0$. However, this cannot
hold because $\alpha_2 - \alpha_1 < 0$ and $|A_2| > |A_1|$ implies that
 $[|A_2|/\alpha_2 - |A_1|/\alpha_1] > 0$. Hence, $(\alpha_2 - \alpha_1) > 0$ and consequently,
 $[|A_2|/\alpha_2 - |A_1|/\alpha_1] > 0$. Therefore, $|A_1|/|A_2| < \alpha_1/\alpha_2 < 1$, and the
Lemma is proved.

Lemma 4: If $|A_2| > |A_1|$ and $|A_1| + |A_2| > (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|$,
then $\| |A_2| - |A_1| \| > \| (\alpha_1/\alpha_2)|A_2| - (\alpha_2/\alpha_1)|A_1| \|$.

Proof: Define the function $F(x) = x|A_2| - (1/x)|A_1|$, $0 \leq x \leq 1$,
and note that it is monotone increasing from $F(0) = -\infty$ to
 $F(1) = |A_2| - |A_1|$, with $F(|A_1|/|A_2|) = |A_1| - |A_2|$. Therefore, for

all x such that $|A_1|/|A_2| < x < 1$,

$$\|x|A_2| - (1/x)|A_1|\| < \| |A_2| - |A_1| \|. \quad (12)$$

However, from Lemma 3, $|A_1|/|A_2| < \alpha_1/\alpha_2 < 1$. Thus, inequality (12) is satisfied for $x = \alpha_1/\alpha_2$, and the lemma is proved.

The converse of Lemma 4 is stated as Theorem 2.

Theorem 2:

If $|A_2| > |A_1|$ and $\|(\alpha_1/\alpha_2)|A_2| - (\alpha_2/\alpha_1)|A_1|\| < \| |A_2| - |A_1| \|$

then

$$|A_1| + |A_2| > (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|.$$

Proof: From the monotonicity of the function, F , defined in the proof of Lemma 4, and the values of $F(|A_1|/|A_2|)$ and $F(1)$ given in the proof of Lemma 4, it follows that (12) is satisfied for only those values of x such that $|A_1|/|A_2| < x < 1$. It is then clear that $|A_1|/|A_2| < \alpha_1/\alpha_2 < 1$ is implied by the hypothesis of the theorem. The conclusion of the theorem follows from Lemma 3.

Corollary 1:

For a given partition, if $|A_1|$ is the $\min|A_j|$, $j = 1, 2, \dots, J$, and $|A_2|$ is any other, $|A_2| > |A_1|$, and α_1, α_2 are such that Theorem 2 holds, then $\min\{(\alpha_2/\alpha_1)|A_1|, (\alpha_1/\alpha_2)|A_2|\} > |A_1|$. This

implies that the new partition results in a higher lower bound than that given in (5).

Proof:

From the conclusion of Theorem 2, $|A_1| + |A_2| \geq (\alpha_2/\alpha_1)|A_1| + (\alpha_1/\alpha_2)|A_2|$.

But from Lemma 3, this implies that

$$|A_1|/|A_2| < \alpha_1/\alpha_2 < 1.$$

Now, $\min\{(\alpha_2/\alpha_1)|A_1|, (\alpha_1/\alpha_2)|A_2|\} \geq |A_1|$ if and only if

$(\alpha_2/\alpha_1)|A_1| \geq |A_1|$ and $(\alpha_1/\alpha_2)|A_2| \geq |A_1|$. But these latter require-

ments are precisely the inequalities of Lemma 3, i.e., $|A_1|/|A_2| < \alpha_1/\alpha_2 < 1$.

Theorem 2 and Corollary 1 show that one iteration can sharpen both the upper and lower bounds given by (5). In fact, heuristically, one seeks partitions that tend to equalize the $|A|$'s, and this can be done systematically by interchanging units from one system with units from another system; these units satisfying the conditions of Theorem 2, e.g., interchanging units within the highest and lowest $|A|$'s. The algorithm would be continued until there are no pairwise interchanges satisfying the conditions of Theorem 2. When this occurs, the solution is "good" but not necessarily optimal. This can be seen by examining the following counterexample. Suppose there are 9 units with associated a 's to be divided into three systems as follows:

System 1:	.09,	.0501,	.03;	$ A_1 $	=	.00013527
System 2:	.06,	.20,	.02;	$ A_2 $	=	.000240
System 3:	.12,	.1001,	.015;	$ A_3 $	=	.00018018

For this partition $|A_1| + |A_2| + |A_3| = .00055545$, and cannot be improved by pairwise interchanges. However, consider the following partition:

System 1:	.12,	.0501,	.03;	$ A_1 $	=	.00018036
System 2:	.06,	.20,	.015;	$ A_2 $	=	.000180
System 3:	.09,	.1001,	.02;	$ A_3 $	=	.00018018

For this partition, which is, in fact, optimal, $|A_1| + |A_2| + |A_3| = .00054054$. Since the aforementioned algorithm only allows for pairwise interchanges, it need not lead to optimal solutions.

The problem considered to date has been in the context of taking M units and partitioning them into J disjoint systems, with units being interchangeable and the number required for each system not specified. Suppose the problem is now changed so that each system contains n different (non-interchangeable) types of components and J systems have to be assembled from the $M = nJ$ units, i.e. there are J units of each type available. Again, each system operates as a parallel system, and as before the objective is to find the partition that maximizes the expected number of systems that perform satisfactorily. How does this new problem compare with the problem previously treated? Fortunately, the admissible partitions for this new problem is a subset of the

partitions of the original problem, and furthermore, none of the results depended on the number of units assigned to each system. Hence, all the results previously obtained are applicable to the new problem. Of course, similar comments can be made for the case of interchangeable components, but where each system must contain $n = M/J$ units.

3. SINGLE SYSTEM PROBLEM

Another variation of the assembly problem is concerned with system design. Suppose a single system is to be constructed containing n units. Three cases will be considered, namely (1) the system will perform satisfactorily if all of the n units performs satisfactorily, i.e., it is a series system; (2) the system will perform satisfactorily if at least one of the n units performs satisfactorily, i.e., it is a parallel system; and (3) the system will perform satisfactorily if at least $k > 1$ units performs satisfactorily, i.e., it is a k out of n system. Attached to the m^{th} unit is a value p_m , $m = 1, 2, \dots, n$, which will denote the probability that the m^{th} unit will perform satisfactorily, $0 \leq p_m \leq 1$, and these probabilities are assumed to be independent. The n units are to be "manufactured", and the manufacturer has sufficient control of his process that he is able to produce at an aimed at "p" level, but his resources are limited so that there is a constraint imposed on the values of the p 's, namely $\sum_{m=1}^n p_m = A$, where A is a fixed positive number. The problem is to find the desired p 's so as to maximize the reliability of the system.

Case 1: Series System

For a series system, the problem is to find the p_1, p_2, \dots, p_n which maximizes the reliability, R , where

$$R = \prod_{m=1}^n p_m,$$

subject to

$$0 \leq p_m \leq 1, \quad m = 1, 2, \dots, n \text{ and } \sum_{m=1}^n p_m = A.$$

It can be assumed that $A \leq n$, otherwise the solution is to choose each $p_m = 1$. The problem is equivalent to maximizing $\sum_{m=1}^n \log p_m$ subject to the same conditions. Since $\log p_m$ is a concave function and ignoring the constraints $p_m \leq 1, m = 1, 2, \dots, n$, it is well known that the optimal values of p_1, p_2, \dots, p_n are $p_1^* = p_2^* = \dots = p_n^* = A/n$. Since $A \leq n$, the ignored constraints are satisfied. Hence $p_1^* = p_2^* = \dots = p_n^* = A/n$ is optimal.

Case 2: Parallel System

For a parallel system, the problem is to find the p_1, p_2, \dots, p_n which maximizes the reliability R , where

$$R = 1 - \prod_{m=1}^n (1 - p_m),$$

subject to

$$0 \leq p_m \leq 1, \quad m = 1, 2, \dots, n \text{ and } \sum_{m=1}^n p_m = A.$$

It can be assumed that $A < 1$, otherwise the optimal solution is to choose at least one of the p 's equal to 1 and the rest arbitrary (but subject to the constraint that $\sum_{m=1}^n p_m = A$). The problem is equivalent to minimizing

$$\sum_{m=1}^n \log(1 - p_m),$$

subject to

$$p_m \geq 0, m = 1, 2, \dots, n, \text{ and } \sum_{m=1}^n p_m = A.$$

The constraint $p_m \leq 1, m = 1, 2, \dots, n$ is superfluous since $A < 1$. The functions $\log(1 - p_m)$ are concave. Therefore $\sum_{m=1}^n \log(1 - p_m)$ is a concave function in (p_1, p_2, \dots, p_n) with the minimum at an extreme point of $\sum_{m=1}^n p_m = A, p_m \geq 0 (m = 1, 2, \dots, n)$. Each extreme point has one $p_m = A$ and the remaining equal to zero. Thus, in particular, $p_1^* = A, p_2^* = p_3^* = \dots = p_n^* = 0$ is optimal.

Case 3: k out of n System

For a k out of n system, the system will perform satisfactorily if at least k out of n units perform satisfactorily. Let $X_m, m = 1, 2, \dots, n,$ be independent Bernoulli random variables with parameter p_m . Let $Y = X_1 + X_2 + \dots + X_n$. Then the reliability R can be expressed as

$$R = P\{Y \geq k\}.$$

The problem is to maximize R subject to

$$0 \leq p_m \leq 1, \text{ and } \sum_{m=1}^n p_m \leq A.$$

It can be assumed that $A < k$, otherwise the optimal solution is to choose k of the p 's equal to 1 and the rest arbitrary. If the constraints $p_m \leq 1$ are ignored, the Kuhn-Tucker conditions indicate that if $p_1^*, p_2^*, \dots, p_n^*$ are optimal, then there exists a number $\lambda^* \geq 0$ satisfying the following:

$$\text{If } p_m^* = 0, \text{ then } \frac{\partial}{\partial p_m} P\{Y \geq k\} - \lambda^* \leq 0 \text{ at } p_m = p_m^*,$$

$$\text{for } m = 1, 2, \dots, n,$$

$$\text{If } p_m^* > 0, \text{ then } \frac{\partial}{\partial p_m} P\{Y \geq k\} - \lambda^* = 0 \text{ at } p_m = p_m^*,$$

$$\text{for } m = 1, 2, \dots, n,$$

$$\text{If } \lambda^* > 0, \text{ then } \sum_{m=1}^n p_m^* = A.$$

Suppose there are r variables which are greater than zero in the optimal solution and denote them by $p_1^*, p_2^*, \dots, p_r^*$. The $P\{Y \geq k\}$ can be expressed as

$$P\{Y \geq k\} = p_1 F(p_2, p_3, \dots, p_n) + (1 - p_1) G(p_2, p_3, \dots, p_n),$$

where the function F is $P\{X_2 + X_3 + \dots + X_n \geq k - 1\}$ and the function

G is $P\{X_2 + X_3 + \dots + X_n \geq k\}$.

Hence,

$$\frac{\partial}{\partial p_1} P\{Y \geq k\} = F(p_2, p_3, \dots, p_n) - G(p_2, p_3, \dots, p_n) = H(p_2, p_3, \dots, p_n).$$

Therefore, $H(p_2^*, p_3^*, \dots, p_n^*) = \lambda^*$ can be solved uniquely for p_2^* as a function of $p_3^*, p_4^*, \dots, p_n^*$ and λ^* .

Similarly,

$$\frac{\partial}{\partial p_2} P\{Y \geq k\} = H(p_1, p_3, \dots, p_n) = \lambda^*$$

can be solved uniquely for p_1^* as the same function of $p_3^*, p_4^*, \dots, p_n^*$ and λ^* so that $p_1^* = p_2^*$. In the same manner, it can be shown that $p_1^* = p_2^* = \dots = p_r^*$, and furthermore, it is clear that $p_i^* = A/r$, $i = 1, 2, \dots, r$. If $\frac{A}{r} \leq 1$, then the ignored constraints are satisfied. This is easily shown to be the case. It is clear that $r \geq k$ since if more than $(n - k)$ p 's are zero, then $P\{Y \geq k\} = 0$, which cannot be a maximum. Therefore $\frac{A}{r} \leq \frac{A}{k} < 1$, so that the ignored constraints are satisfied. The foregoing results do not indicate how to determine r . One way is to evaluate $P\{Y \geq k\}$ with $p_i^* = A/r$, $i = 1, 2, \dots, r$, choosing that value of $r = r^*$ which maximizes

$$P\{Y \geq k\} = \sum_{j=k}^r \binom{r}{j} \left(\frac{A}{r}\right)^j \left(1 - \frac{A}{r}\right)^{r-j} \quad \text{for } r=k, k+1, \dots, n. \quad (13)$$

Some insight is obtained by replacing the right hand side of (13) by the normal or Poisson approximation, depending upon which is appropriate. When r is large and A/r is near 0 or 1 the Poisson is appropriate; when A/r is "near" 1/2, the normal is appropriate. For either case, k (and consequently r) and n should be large. For the normal approximation $r = r^*$ is to be chosen which maximizes

$$\int_{\frac{k-r(A/r)}{\sqrt{r(A/r)(1-A/r)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{\frac{k-A}{\sqrt{A(1-A/r)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \quad (14)$$

For $r = k, k+1, \dots, n$. Since $k-A \geq 0$, clearly $r^* = n$. Thus,

$$p_1^* = p_2^* = \dots = p_n^* = A/n. \quad (15)$$

If the Poisson approximation is used for the smaller values of A , the right hand side of (13) yields

$$P\{Y \geq k\} = 1 - \sum_{j=0}^{k-1} \frac{e^{-A} A^j}{j!},$$

which is independent of r . That is, approximately speaking, the choice of r has very little influence on the left hand side of (13). Therefore, it can be said that if n and k are both large, an

approximately optimal solution is given by (15).

The results given in (15) are consistent with the n of n case (series system). It does not appear to be consistent with the 1 of n case (parallel system). However, this is not unexpected since k must be large in order for the approximation to be good.

The solution given in (15) is easily shown to be exact and optimal if the problem considered is to maximize the variance of the random variable Y , the number of units that perform satisfactorily, subject to the usual constraints. (Since $A \leq k$ and A is the expected number of components that function satisfactorily, maximization of the variance is desired.)

4. MULTIPLE SYSTEM PROBLEM

The problem considered in this section is the same as that considered in Section 3, except that nJ components are to be "manufactured" and assembled into J systems. Each system performs satisfactorily if at least k out of n units perform satisfactorily. It is desired to maximize the expected number of systems that perform satisfactorily, $E(N)$, subject to constraints on the p 's.

Motivated by the results obtained in Section 3 using the approximation, it will be assumed initially that the probability of each unit performing satisfactorily within the system will be equal, i.e., all units in the j^{th} system have probability p_j of functioning. The problem is to determine p_1, p_2, \dots, p_J to maximize

$$D(p_1, p_2, \dots, p_J) = \sum_{j=1}^J \sum_{i=k}^n \binom{n}{i} p_j^i (1 - p_j)^{n-i} \quad (16)$$

subject to

$$0 \leq p_j \leq 1, \quad j = 1, 2, \dots, J$$

and

$$\sum_{j=1}^J np_j \leq A, \quad \text{where } 0 \leq A < nJ.$$

The Kuhn-Tucker conditions indicate that if $p_1^*, p_2^*, \dots, p_J^*$ are optimal, then there must exist numbers $\lambda^* \geq 0$, $\mu_j^* \geq 0$, $j = 1, 2, \dots, n$, satisfying the following:

$$\text{If } p_j^* = 0, \text{ then } \frac{\partial}{\partial p_j} D(p_1, p_2, \dots, p_J) - n\lambda^* - \mu_j^* \leq 0 \text{ at } p_j = p_j^*,$$
$$\text{for } j = 1, 2, \dots, J. \quad (17a)$$

$$\text{If } p_j^* > 0, \text{ then } \frac{\partial}{\partial p_j} D(p_1, p_2, \dots, p_J) - n\lambda^* - \mu_j^* = 0 \text{ at } p_j = p_j^*,$$
$$\text{for } j = 1, 2, \dots, J. \quad (17b)$$

$$\text{If } \mu_j^* = 0, \text{ then } p_j^* \leq 1 \quad \text{for } j = 1, 2, \dots, J. \quad (17c)$$

$$\text{If } \mu_j^* > 0, \text{ then } p_j^* = 1 \quad \text{for } j = 1, 2, \dots, J. \quad (17d)$$

$$\text{If } \lambda^* = 0, \text{ then } n \sum_{j=1}^J p_j^* \leq A. \quad (17e)$$

$$\text{If } \lambda^* > 0, \text{ then } n \sum_{j=1}^J p_j^* = A. \quad (17f)$$

$$\text{Now, } \frac{\partial D}{\partial p_j} = n \binom{n-1}{n-k} p_j^{k-1} (1-p_j)^{n-k}$$

so that $\frac{\partial D}{\partial p_j} \geq 0$ for $0 \leq p_j \leq 1$ and equal to zero when $p_j = 0$ or 1 .
 If $\mu_j^* > 0$, then $p_j^* = 1$. When $p_j^* = 1$, $\frac{\partial D}{\partial p_j} = 0$ so that $\mu_j^* = -n\lambda^*$
 from (17b); this is a contradiction, and hence, $\mu_j^* = 0$ for all j ,
 $j = 1, 2, \dots, J$. Now if any $p_j^* = 1$, then from (17b) $\lambda^* = 0$, so
 that every p_j^* must be zero or one (from 17a or 17b) since $\frac{\partial D}{\partial p_j} = 0$
 only at these values of p . If $0 < p_j < 1$, then from (17b)

$$n \binom{n-1}{n-k} p_j^* (1-p_j^*)^{n-k} = n\lambda^* > 0.$$

For a given λ^* the equation has at most two possible roots in the
 range $0 < p_j < 1$. Therefore, an optimal solution is of the form
 $p_1^* = p_2^* = \dots = p_x^* = 0$, and $p_{x+1}^* = p_{x+2}^* = \dots = p_{x+y}^* = \bar{p}$,
 $p_{x+y+1}^*, p_{x+y+2}^*, \dots, p_J^* = \bar{p}$, for some x and y , where \bar{p}, \bar{p} are
 solutions to

$$n \binom{n-1}{n-k} p^{k-1} (1-p)^{n-k} = \lambda^*, \text{ for some } \lambda^* > 0 \quad (18a)$$

$$\text{and } y\bar{p} + (J-x-y)\bar{p} = \frac{A}{n}. \quad (18b)$$

An optimal solution may consist of $[s]$ p 's each having value
 one and $(J - [s])$ p 's each having value zero, where $[s]$ is the

greatest integer less than or equal to A/n . But this is consistent with the given form when $y = 0$, $\bar{p} = 0$, $\bar{p} = 1$.

The curve $\sum_{i=k}^n \binom{n}{i} p^i (1-p)^{n-i}$ being s shaped is concave over the range $\frac{k-1}{n-1} \leq p \leq 1$. It is evident that if A/nJ satisfies $\frac{k-1}{n-1} \leq \frac{A}{nJ} \leq 1$, then $p_1^* = p_2^* = \dots = p_J^* = \frac{A}{nJ}$ is optimal.

When $k = n$, the series case, the optimal solution is to have $p_j = 1$ for $j = 1, 2, \dots, [A/n]$, $p_{[A/n]+1} = A/n - [A/n]$, and $p_j = 0$ for $j = [A/n]+2, \dots, J$. This is immediate from convexity arguments, independent of the foregoing argument.

At the outset of this section it was assumed that the probability of each unit performing satisfactorily within a system was equal. This may not always be reasonable. In Section 3, it was evident that when $A \geq k$, the optimal solution was to choose k of the p 's equal to 1 and the rest 0. In the problem encountered in this section where nJ units are to be assembled into J systems, it may very well be that allocating k of the total resources to a system is appropriate even if $A \leq kJ$ and hence, the initial assumption about equal probabilities within a system may not be valid. Thus, any algorithm which seeks an optimal solution to this problem requires a consideration of such possible systems, and their contribution to the total of the expected number of systems that perform satisfactorily as well as allocating the remaining resources so as to maximize D in equation (16).

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