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A MICROSTRUCTURE THEORY FOR THE BUCKLING AND VIBRATION OF A LAMINATED BEAM

Gary L. Anderson

Watervliet Arsenal
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BENÉT WEAPONS LABORATORY

WATERVLIET ARSENAL
Watervliet, New York

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Beam Theory

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TABLE OF CONTENTS

	Page
Abstract	1
Introduction	4
Statement of the General Problem	5
The Variational Formulation	6
Some Preliminary Considerations from Beam Theory	8
Energies for a Laminated Beam	11
The Equations of Motion	17
Solution of the Field Equations	22
The Hinged-Hinged Beam	22
The Clamped-Clamped Beam	23
Numerical Results	34
The Hinged-Hinged Beam	35
The Clamped-Clamped Beam	48
Conclusions	54
References	56
Appendix: Effective Modulus Theory	57
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FIGURES

1. Geometry and dimensions of the composite beam. 12

2. Variation of the critical load coefficient λ_{cr} with depth-to-length ratio ζ for a hinged-hinged beam. 37

Page

3. Variation of the first flexural frequency $\omega(n=1)$ with the applied load parameter λ and the depth-to-length ratio ζ for a hinged-hinged beam. 39

4. Variation of the second flexural frequency $\omega(n=2)$ with λ and ζ for a hinged-hinged beam. 40

5. Variation of the third flexural frequency $\omega(n=3)$ with λ and ζ for a hinged-hinged beam. 41

6. Variation of the first flexural frequency $\omega(n=1)$ with ζ and the number of layer pairs for a hinged-hinged beam. 42

7. Variation of the second flexural frequency $\omega(n=2)$ with ζ and the number of layer pairs for a hinged-hinged beam. 43

8. Variation of the third flexural frequency $\omega(n=3)$ with ζ and the number of layer pairs for a hinged-hinged beam. 44

9. Variation of the first thickness-shear frequency $\omega(n=1)$ versus ζ for a hinged-hinged beam. 46

10. Variation of the first three thickness-shear frequencies with ζ as determined from the microstructure theory for a hinged-hinged beam. 47

11. Variation of the critical load coefficient λ_{cr} with the depth-to-length ratio ζ for a clamped-clamped beam. 49

12. Variation of ω_1 with ζ and λ for a clamped-clamped beam. 50

13. Variation of ω_2 with ζ and λ for a clamped-clamped beam. 52

14. Variation of ω_3 with ζ and λ for a clamped-clamped beam. 53

INTRODUCTION

Recently C.-T. Sun [1] developed a mathematical theory for a continuum model with microstructure for laminated beams. These laminated beams were assumed to consist of many parallel alternating layers of two homogeneous, isotropic elastic materials. Each constituent layer was treated as a Timoshenko beam, and expressions for the kinetic and strain energies, as well as the work done by external forces, were derived. The smoothing technique, which had previously been used so effectively in deriving a general field theory for laminated composites [2-5], was applied, and the equations of motion and associated boundary conditions were derived from Hamilton's principle. To test the accuracy of the resulting theory, the dispersion curve for flexural waves in an infinite laminated beam were determined from (i) the exact theory, (ii) the new theory, and (iii) the effective modulus theory for a composite beam consisting of five stiff layers and four soft layers. It was found that the curves obtained from the exact theory and the microstructure theory were in excellent agreement over the entire range of wave-number considered. By contrast, the effective modulus theory exhibited agreement only in the very low frequency range.

The effective modulus theory treats the laminated composite as a homogeneous but transversely isotropic medium. Based on this point of view, Brunelle [6-9] has investigated the stability and vibration characteristics of transversely isotropic Timoshenko beams under initial stress. He showed that the values of buckling loads and natural frequencies of vibration are strongly dependent upon the nature of the boundary restraint and the relative degree of anisotropy, i.e., the ratio of the longitudinal Young's modulus to the transverse shear modulus. Decreases in the values of the buckling coefficients and natural frequencies of vibration relative to the classical values, computed on the basis of the Euler-Bernoulli theory for beams, were observed to be particularly pronounced in the case of the clamped-clamped beam.

The objective of the present investigation is to extend Sun's microstructure theory for laminated beams [1] so as to include the effect of initial stresses and to solve a pair of stability and free vibration problems within the framework of the resulting theory. The numerical results for buckling coefficients and frequency parameters are compared with Brunelle's results [6-9] derived from Timoshenko beam theory for transversely isotropic materials.

STATEMENT OF THE GENERAL PROBLEM

We consider an elastic solid of volume V bounded by a finite surface S. Displacements are prescribed on the portion S_u of the surface S and tractions are applied on the remaining portion S_τ. Initially the body is assumed to be at rest and is subjected to a state of initial stress σ_{ij}, i, j = 1,2,3, which arises from conservative forces applied on the surface S_τ of V. We assume that the perturbed and linearized field equations [10,11] are

$$\tau_{ij,j} + (\sigma_{jk} u_{i,k})_{,j} + F_i = \rho\ddot{u}_i, \quad \text{in } V, \tag{1}$$

$$(\tau_{ij} + \sigma_{jk}\, u_{i,k})n_j \;=\; \beta p_i, \quad \text{on } S_\tau, \tag{2}$$

$$u_i \text{ prescribed}, \quad \text{on } S_u \tag{3}$$

$$\tau_{ij} \;=\; c_{ijk\ell}\, \varepsilon_{k\ell}, \qquad \varepsilon_{ij} \;=\; \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

where τ_{ij} denotes the perturbed state of stress, u_i is the displacement vector measured from the undisturbed state. F_i is the body force field per unit volume, ρ is the mass density, and n_j is the unit exterior normal to the surface S. The quantity β is a parameter associated with the magnitude of the externally applied surface tractions, and the p_j are the components of perturbations of the applied surface tractions on S_τ. In (4), the general stress-strain relationship is given. The familiar summation convention for Cartesian tensors is employed in (1)-(4), commas denote spatial derivatives, and dots time derivatives.

THE VARIATIONAL FORMULATION

In order to derive a form of Hamilton's principle that will generate (1)-(3), we multiply (1) by δu_i and form the integral of the volume V of the result:

$$\int_V \tau_{ij,j}\delta u_i dV + \beta\int_V (\sigma_{jk}u_{i,k})_{,j}\delta u_i dV + \int_V F_i \delta u_i dV = \int_V \rho \ddot{u}_i \delta u_i dV. \tag{5}$$

Application of the divergence theorem to (5) yields

$$\oint_S (\tau_{ij} + \beta\sigma_{jk}u_{i,k})n_j\delta u_i dS - \int_V \delta(W + \frac{1}{2}\beta\sigma_{jk}u_{i,k}u_{i,j})dV +$$

$$+ \int_V F_i \delta u_i dV \quad = \quad \int_V \rho \ddot{u}_i \delta u_i dV, \tag{6}$$

where we have exploited the symmetry properties of τ_{ij} and σ_{ij} to write

$$\delta W \quad = \quad (\partial W/\partial \varepsilon_{ij})\delta\varepsilon_{ij} \quad = \quad \tau_{ij}\delta\varepsilon_{ij} \quad = \quad \tau_{ij}\delta u_{i,j}$$

$$\sigma_{jk}u_{i,k}\delta u_{i,j} \quad = \quad \frac{1}{2}\,\delta(\sigma_{jk}u_{i,k}u_{i,j}),$$

with W denoting the strain energy density. Furthermore, if we assume that the body force vector can be represented as

$$F_i \quad = \quad f_i + a_{ij}u_j, \qquad f_i \quad = \quad f_i(x,t), \qquad a_{ij}(x) \quad = \quad a_{ji}(x),$$

then

$$F_i\delta u_i \quad = \quad \delta(f_iu_i + \frac{1}{2}\,a_{ij}u_iu_j).$$

Hence, in view of (2) and (3), (6) can be expressed as

$$\delta \int_V [W + \frac{1}{2}\,\beta\sigma_{jk}u_{i,k}u_{i,j} - f_iu_i - \frac{1}{2}\,a_{ij}u_iu_j]dV +$$

$$+ \int_V \rho\ddot{u}_i\delta u_i dV \quad = \quad \delta\int_{S_\tau} \beta p_i u_i dS. \tag{7}$$

Forming the time integral of (7) and integrating by parts in the term involving $\ddot{u}_i\ \delta u_i$, we obtain

$$\delta \int_{t_0}^{t_1} \mathcal{L}\, dt \quad = \quad 0, \tag{8}$$

where

$$\mathcal{L} = \int_V [\frac{1}{2}\rho\dot{u}_i\dot{u}_i - W - \frac{1}{2}\beta\sigma_{jk}u_{i,k}u_{i,j} - f_iu_i - \frac{1}{2}a_{ij}u_iu_j]dV + \int_{S_\tau} \beta p_iu_i dS. \tag{9}$$

Eq. (8) represents Hamilton's principle associated with the problem (1)-(3). In conjunction with (9), (8) can be used effectively for the purpose of deriving plate and beam theories when initial stresses are present.

SOME PRELIMINARY CONSIDERATIONS FROM BEAM THEORY

To derive an elementary theory for extensional deformations and a Timoshenko theory for flexural deformations in beams, we approximate the displacement field as follows:

$$u_1(x,t) = u(x_1,t) - x_2\phi(x_1,t), \quad u_2(x,t) = w(x_1,t)$$
$$u_3(x,t) = 0, \tag{10}$$

where the x_1 - axis coincides with the axis of the beam which passes through the centroid of every cross-section and the x_2 - axis denotes the transverse direction. Here u is the longitudinal deflection of the axis of the beam, w the transverse deflection, and ϕ the rotation of a section in the x_1x_2 - plane. The strains computed from (10) are easily shown to be

$$\varepsilon_{11} = u_{,1} - x_2\phi_{,1} \quad \varepsilon_{12} = \frac{1}{2}(w_{,1} - \phi), \quad \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{22} = \varepsilon_{33} = 0. \tag{11}$$

In the usual manner, we shall ignore the relations $\varepsilon_{22} = \varepsilon_{33} = 0$ in favor of setting, for an isotropic material,

$$\tau_{22} = (2\mu + \lambda)\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{33}) = 0,$$

$$\tau_{33} = (2\mu + \lambda)\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22}) = 0.$$

Thus, it follows that

$$\varepsilon_{22} = \varepsilon_{33} = -\lambda\varepsilon_{11}/2(\lambda + \mu), \qquad \varepsilon_{ii} = \mu\varepsilon_{11}/(\mu + \lambda). \tag{12}$$

The expression for the strain energy is

$$\begin{aligned} W &= \frac{1}{2}\tau_{ij}\varepsilon_{ij} = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} \\ &= \mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2) + 2\mu(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2) + \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}. \end{aligned} \tag{13}$$

Substitution of (11) and (12) into (13) leads to

$$W = \frac{1}{2}E\varepsilon_{11}^2 + 2\mu\varepsilon_{12}^2 \tag{14}$$

since $E = \mu(2\mu+3\lambda)/(\mu+\lambda)$, E being Young's modulus. In the development of the Timoshenko beam theory, it is customary to introduce a shear correction factor κ into (14). Hence, we replace ε_{12}^2 by $\kappa\varepsilon_{12}^2$, so that (14) becomes

$$W = \frac{1}{2}E\varepsilon_{11}^2 + 2\mu\kappa\varepsilon_{12}^2,$$

or, by virtue of (11)

$$W = \frac{1}{2}E(u_{,1}^2 - 2x_2u_{,1}\phi_{,1} + x_2^2\phi_{,1}^2) + \frac{1}{2}\kappa\mu(w_{,1} - \phi)^2. \tag{15}$$

An integration of (15) over the cross-sectional area A of the beam yields

$$U = \int_A W dA = \frac{1}{2} EAu_{,1}^2 + \frac{1}{2} EI\phi_{,1}^2 + \frac{1}{2} \kappa\mu A(w_{,1} - \phi)^2, \tag{16}$$

where $I = \int_A x_2^2 dA$ and $\int_A x_2 dA = 0$ since the x_1 - axis passes through the centroid of the cross-section.

In a similar fashion, for the kinetic energy density, we have, in view of (10),

$$\mathcal{J} = \frac{1}{2} \rho\dot{u}_i\dot{u}_i = \frac{1}{2} \rho(\dot{u}^2 - 2x_2\dot{u}\dot{\phi} + x_2^2\dot{\phi}^2) + \frac{1}{2} \rho\dot{w}^2$$

and thus

$$T = \int_A \frac{1}{2} \rho\dot{u}_i\dot{u}_i dA = \frac{1}{2} \rho(A\dot{u}^2 + I\dot{\phi}^2 + A\dot{w}^2). \tag{17}$$

Furthermore, we can also show that

$$\begin{aligned} \sigma_{jk}u_{i,k}u_{i,j} = {} & \sigma_{11}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) + \sigma_{22}(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2) + \\ & + \sigma_{33}(u_{1,3}^2 + u_{2,3}^2 + u_{3,3}^2) + 2\sigma_{12}(u_{1,2}u_{1,1} + \\ & + u_{2,2}u_{2,1} + u_{3,2}u_{3,1}) + 2\sigma_{13}(u_{1,3}u_{1,1} + u_{2,3}u_{2,1} + \\ & + u_{3,3}u_{3,1}) + 2\sigma_{23}(u_{1,3}u_{1,2} + u_{2,3}u_{2,2} + u_{3,3}u_{3,2}). \end{aligned}$$

If we assume that all $\sigma_{ij} = 0$ except σ_{11}, then, in view of (10), we have

$$\sigma_{jk}u_{i,k}u_{i,j} = \sigma_{11}(u_{1,1}^2 + u_{2,1}^2) = \sigma_{11}(u_{,1}^2 - 2x_2u_{,1}\phi_{,1} + x_2^2\phi_{,1}^2 + w_{,1}^2)$$

and

$$U^* = \frac{1}{2}\int_A \sigma_{jk}u_{i,k}u_{i,j}dA = \frac{1}{2}\sigma_{11}(Au^2_{,1} + I\phi^2_{,1} + Aw^2_{,1}). \tag{18}$$

If $f_i = a_{ij} = p_i = 0$, then, using (16)-(18), we can reduce (9) to

$$\mathcal{L} = \int_0^\ell (T - U - U^*)dx_1. \tag{19}$$

ENERGIES FOR A LAMINATED BEAM

In this section we intend to extend the theory developed by Sun [1] for laminated composite beams to include the effect of the initial stress σ_{11}. From this theory we should be able to compute buckling loads for such beams. We consider a composite beam consisting of a large number of parallel alternating layers of two homogeneous, isotropic elastic materials (See Figure 1). We assume that the cross section of each layer is rectangular and of width b. The depth of the cross-section shall be denoted by h. We shall designate the properties of the stiff layer by the subscript 1 and of the soft layer by the subscript 2. The thicknesses of the layers are d_1 and d_2, and the Young's moduli, shear moduli, and the mass densities for the two types of constituents are denoted by E_1, μ_1, ρ_1 and E_2, μ_2, ρ_2, respectively. In addition, we shall use a superscript k to refer to the properties of the k-th stiff and soft layers.

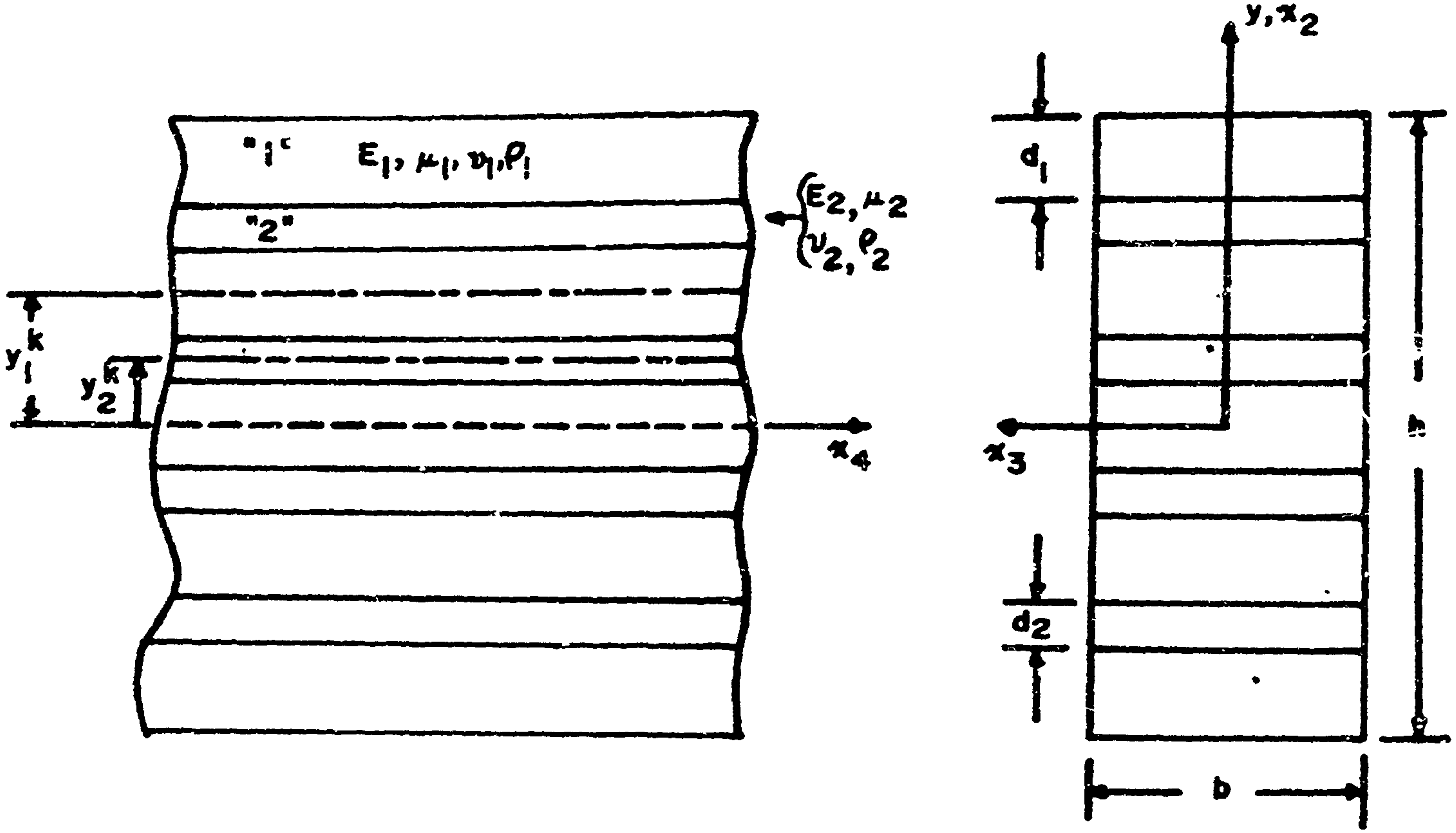

Figure 1. Geometry and dimensions of the composite beam.

According to (16), the strain energies per unit length in the k-th stiff and soft layers due to bending, shear deformation, and extension are

$$U_1^k = \frac{1}{2} E_1 A_1 (u_{1,1}^k)^2 + \frac{1}{2} E_1 I_1 (\phi_{1,1}^k)^2 + \frac{1}{2} \kappa_1 \mu_1 A_1 (w_{1,1}^k - \phi_1^k)^2, \tag{20}$$

$$U_2^k = \frac{1}{2} E_2 A_2 (u_{2,1}^k)^2 + \frac{1}{2} E_2 I_2 (\phi_{2,1}^k)^2 + \frac{1}{2} \kappa_2 \mu_2 A_2 (w_{2,1}^k - \phi_2^k)^2, \tag{21}$$

respectively, where, as is evident from Figure 1,

$$A_\alpha = bd_\alpha, \qquad I_\alpha = bd_\alpha^3/12, \qquad \alpha = 1,2. \tag{22}$$

By virtue of (18), the potential energies per unit length due to the initial stress are

$$U_1^{*k} = \frac{1}{2} \sigma_{11} [A_1 (u_{1,1}^k)^2 + I_1 (\phi_{1,1}^k)^2 + A_1 (w_{1,1}^k)^2], \tag{23}$$

$$U_2^{*k} = \frac{1}{2} \sigma_{11} [A_2 (u_{2,1}^k)^2 + I_2 (\phi_{2,1}^k)^2 + A_2 (w_{2,1}^k)^2]. \tag{24}$$

Similarly, the kinetic energies per unit length for the k-th stiff and soft layers are

$$T_1^k = \frac{1}{2} \rho_1 [A_1 (\dot{u}_1^k)^2 + I_1 (\dot{\phi}_1^k)^2 + A_1 (\dot{w}_1^k)^2], \tag{25}$$

$$T_2^k = \frac{1}{2} \rho_2 [A_2 (\dot{u}_2^k)^2 + I_2 (\dot{\phi}_2^k)^2 + A_2 (\dot{w}_2^k)^2], \tag{26}$$

respectively, in view of (17).

Sun [1] introduced an additional approximation on the extensional components u_α^k, namely,

$$u_\alpha^k = - y_\alpha^k \psi(x_1,t), \qquad \alpha = 1,2, \tag{27}$$

where $\psi(x_1,t)$ denotes the gross rotation of the composite beam and y_α^k the coordinates of the centroidal axes of the k-th stiff and soft layers. Insertion of (27) into (20)-(26) leads to

$$U_\alpha^k = \frac{1}{2} E_\alpha A_\alpha (y_\alpha^k)^2 \psi_{,1}^2 + \frac{1}{2} E_\alpha I_\alpha (\phi_{\alpha,1}^k)^2 + \frac{1}{2} \kappa_\alpha \mu_\alpha A_\alpha (w_{\alpha,1}^k - \phi_\alpha^k)^2, \tag{28}$$

$$U_\alpha^{*k} = \frac{1}{2} \sigma_{11} [A_\alpha (y_\alpha^k)^2 \psi_{,1}^2 + I_\alpha (\phi_{\alpha,1}^k)^2 + A_\alpha (w_{\alpha,1}^k)^2], \tag{29}$$

$$T_\alpha = \frac{1}{2} \rho_\alpha [A_\alpha (y_\alpha^k)^2 \dot{\psi}^2 + I_\alpha (\dot{\phi}_\alpha^k)^2 + A_\alpha (\dot{w}_\alpha^k)^2]. \tag{30}$$

Under the assumption that there are n pairs of stiff and soft layers in the composite beam, the total strain, kinetic, and potential energies per unit length can be expressed as

$$(U, T, U^*) = \sum_{k=1}^{n} \sum_{\alpha=1}^{2} (U_\alpha^k, T_\alpha^k, U_\alpha^{*k}). \tag{31}$$

Still following Sun [1], we introduce a smoothing operation through which the summation over k in (31) may be approximated by the following weighted integrations over y:

$$(U, T, U^*) = \int_{-h/2}^{h/2} \frac{1}{d_1+d_2} \sum_{\alpha=1}^{2} (U_\alpha, T_\alpha, U_\alpha^*) dy, \tag{32}$$

where y_1^k and y_2^k are to be replaced by y and the superscript k is deleted. Eq. (32) is also valid for the case in which the number of stiff layers is not equal to the number of soft layers.

Inserting (28-(30) into (32), we obtain

$$U = \frac{1}{2}\xi\sum_{\alpha=1}^{2}[E_\alpha I_\alpha(\phi_{\alpha,1})^2 + \kappa_\alpha\mu_\alpha A_\alpha(w_{\alpha,1}-\phi_\alpha)^2 + (h^2/12)E_\alpha A_\alpha\psi_{,1}^2], \tag{33}$$

$$U^* = \frac{1}{2}\xi\sum_{\alpha=1}^{2}[(h^2/12)A_\alpha\psi_{,1}^2 + I_\alpha(\phi_{\alpha,1})^2 + A_\alpha(w_{\alpha,1})^2], \tag{34}$$

$$T = \frac{1}{2}\xi\sum_{\alpha=1}^{2}[\rho_\alpha A_\alpha\dot{w}_\alpha^2 + \rho_\alpha I_\alpha\dot{\phi}_\alpha^2 + (h^2/12)\rho_\alpha A_\alpha\dot{\psi}^2], \tag{35}$$

where $\xi = h/(d_1+d_2)$.

The number of dependent variables appearing in (33)-(35) can be reduced because the continuity conditions at the interface between stiff and soft layers introduces two relationships involving w_α^k, ϕ_α^k, and ψ. In terms of the discrete variables, the continuity of displacements at the interface of the k-th pair of layers leads to

$$w_1^k = w_2^k \text{ and } u_1^k + \frac{1}{2}d_1\phi_1^k = u_2^k - \frac{1}{2}d_2\phi_2^k$$

or

$$u_2^k - u_1^k = \frac{1}{2}\sum_{\alpha=1}^{2} d_\alpha\phi_\alpha^k, \tag{36}$$

But in view of (27), (36) becomes

$$(y_1^k - y_2^k)\psi = \frac{1}{2}\sum_{\alpha=1}^{2} d_\alpha \phi_\alpha^k,$$

and since $y_1^k = y_2^k + \frac{1}{2}(d_1+d_2)$ this last result reduces to

$$\psi = \eta\phi_1^k + (1-\eta)\phi_2^k, \tag{37}$$

where

$$\eta = d_1/(d_1+d_2). \tag{38}$$

Applying the smoothing operation to (37), we have

$$\psi = \eta\phi_1 + (1-\eta)\phi_2.$$

Solving for ϕ_2, we find

$$\phi_2 = (\psi-\eta\phi)/(1-\eta), \tag{39}$$

where we now set $\phi_1 \equiv \phi$. Therefore, substitution of (39) and $w_1 = w_2 \equiv w$ into (33)-(35) yields

$$U = \frac{1}{2}\xi E_1 I_1 \phi_{,1}^2 + \frac{1}{2}\xi\kappa_1\mu_1 A_1 (w_{,1}-\phi)^2 + \frac{1}{2} I_b[\eta E_1 + (1-\eta)E_2]\psi_{,1}^2 + \frac{1}{2}\xi\frac{E_2 I_2}{(1-\eta)^2}(\psi_{,1}-\eta\phi_{,1})^2 + \frac{1}{2}\xi\kappa_2\mu_2 A_2\left[w_{,1} - \frac{(\psi-\eta\phi)}{1-\eta}\right]^2, \tag{40}$$

$$U^* = \frac{1}{2}\sigma_{11} I_b \psi_{,1}^2 + \frac{1}{2}\sigma_{11} A w_{,1}^2 + \frac{1}{2}\xi\sigma_{11} I_1 \phi_{,1}^2 + \frac{\xi\,\sigma_{11} I_2}{2(1-\eta)^2}(\psi_{,1}-\eta\phi_{,1})^2, \tag{41}$$

$$T = \frac{1}{2}\,\xi(\rho_1 A_1+\rho_2 A_2)\dot{w}^2 + \frac{1}{2}\,\xi\rho_1 I_1\dot{\phi}^2 + \frac{\xi\,\rho_2\,I_2}{2(1-\eta)^2}\,(\dot{\psi}-\eta\dot{\phi})^2 +$$

$$+\frac{1}{2}\,I_b\,[\eta\rho_1 + (1-\eta)\rho_2]\dot{\psi}^2, \tag{42}$$

where

$$A = bh, \qquad I_b = bh^3/12. \tag{43}$$

Hence, we have now completely prepared the integrand in (19) for use with (8).

THE EQUATIONS OF MOTION

If we now substitute (40)-(42) into (19), we obtain

$$\mathcal{L} = \int_0^{\ell} [\,\frac{1}{2}\,\xi\,a_4\dot{w}^2 + \frac{1}{2}\,\xi\,a_{13}\dot{\phi}^2 + \frac{1}{2}\,\xi\,a_9\dot{\psi}^2 - \xi\,a_{10}\dot{\psi}\dot{\phi} -$$

$$-(\xi\,a_1 + \sigma_{11}A)w^2_{,1} + \xi\,a_3 w_{,1}\phi + \xi\,a_2\psi w_{,1} - \frac{1}{2}\,\xi(a_5+a_{14}\sigma_{11})\psi^2_{,1} -$$

$$-\frac{1}{2}\,\xi\,a_6\psi^2 + \xi\,a_8\psi\phi + \xi\,a_7\,(1+\sigma_{11}/E_2)\psi_{,1}\phi_{,1} - \frac{1}{2}\,\xi\,a_{12}\phi^2\,\cdot$$

$$-\frac{1}{2}\,\xi(a_{11}+a_{15}\sigma_{11})\phi^2_{,1}]dx_1, \tag{44}$$

where

$$a_1 = \kappa_1\mu_1 A_1 + \kappa_2\mu_2 A_2 = \kappa b\,(\mu_1 d_1 + \mu_2 d_2),$$

$$a_2 = \kappa_2\mu_2A_2/(1-\eta) = \kappa b\mu_2(d_1 + d_2),$$

$$a_3 = \kappa_1\mu_1A_1 - \kappa_2\eta\mu_2A_2/(1-\eta) = \kappa bd_1(\mu_1 - \mu_2)$$

$$a_4 = \rho_1A_1 + \rho_2A_2 = b(\rho_1d_1 + \rho_2d_2),$$

$$a_5 = E_2I_2/(1-\eta)^2 + \eta E_1I_b/\xi + (1-\eta)E_2I_b/\xi$$

$$= (b/12)\ [E_2d_2(d_1 + d_2)^2 + h^2(E_1d_1 + E_2d_2)],$$

$$a_6 = a_2/(1-\eta) = (b/d_2)\kappa\mu_2\ (d_1 + d_2)^2,$$

$$a_7 = \eta E_2I_2/(1-\eta)^2 = (b/12)E_2d_1d_2(d_1 + d_2),$$

$$a_8 = \eta a_6 = (b/d_2)\kappa\mu_2d_1\ (d_1 + d_2),$$

$$a_9 = \rho_2I_2/(1-\eta)^2 + \eta\rho_1I_b/\xi + (1-\eta)\rho_2I_b/\xi$$

$$= (b/12)\ [\rho_2d_2\ (d_1 + d_2)^2 + h^2(\rho_1d_1 + \rho_2d_2)],$$

$$a_{10} = \eta\rho_2I_2/(1-\eta)^2 = (b/12)\rho_2d_1d_2\ (d_1 + d_2),$$

$$a_{11} = E_1I_1 + \eta^2E_2I_2/(1-\eta)^2 = (b/12)d_1^2(E_1d_1 + E_2d_2),$$

$$a_{12} = \kappa_1\mu_1A_1 + \kappa_2\eta^2\mu_2A_2/(1-\eta)^2 = (b/d_2)\kappa d_1(\mu_1d_2 + \mu_2d_1),$$

$$a_{13} = \rho_1I_1 + \eta^2\rho_2I_2/(1-\eta)^2 = (b/12)d_1^2\ (\rho_1d_1 + \rho_2d_2),$$

$$a_{14} = I_b/\xi + I_2/(1-\eta)^2 = (b/12)(d_1 + d_2)[h^2 + d_2(d_1 + d_2)],$$

$$a_{15} = I_1 + \eta^2I_2/(1-\eta)^2 = (b/12)d_1^2(d_1 + d_2), \tag{45}$$

where $\kappa_1 = \kappa_2 \equiv \kappa = 0.822$ for a rectangular cross-section.

Insertion of (44) into Hamilton's principle (8) leads to the following system of displacement equations of motion in $0<x_1<\ell$:

$$(a_1 + \sigma_{11}A/\xi)w_{,11} - a_2\psi_{,1} - a_3\phi_{,1} = a_4\ddot{w},$$

(46)

$$a_2w_{,1} + (a_5 + a_{14}\sigma_{11})\psi_{,11} - a_6\psi - a_7(1 + \sigma_{11}/E_2)\phi_{,11} +$$
$$+ a_8\phi = a_9\ddot{\psi} - a_{10}\ddot{\phi},$$

(47)

$$a_3w_{,1} - a_7(1 + \sigma_{11}/E_2)\psi_{,11} + a_3\psi + (a_{11} + a_{15}\sigma_{11})\phi_{,11} -$$
$$- a_{12}\phi = a_{13}\ddot{\phi} - a_{10}\ddot{\psi},$$

(48)

with the boundary conditions

(i) either $(a_1 + \sigma_{11}A/\xi)w_{,1} - a_2\psi - a_3\phi = 0$ or $\delta w = 0,$

(ii) either $(A_5 + a_{14}\sigma_{11})\psi_{,1} - a_7(1 + \sigma_{11}/E_2)\phi_{,1} = 0$ or $\delta\psi = 0,$

(iii) either $(a_{11} + a_{15}\sigma_{11})\phi_{,1} - a_7(1 + \sigma_{11}/E_2)\psi_{,1} = 0$ or $\delta\phi = 0,$

(49)

on $x_1 = 0,\ell$. If we set $\sigma_{11} = 0$ in (46)-(49), then these equations become identical to Sun's equations (45)-(50) in [1] in the absence of surface and end loads.

Let us now suppose that a compressive load P is applied along the axis of the beam, so that $\sigma_{11}A = - P$. Then (46)-(48) become

$$(b_1-\lambda b_2)w''(x,\tau)-\bar{\Psi}'(x,\tau)-b_3\bar{\Phi}'(x,\tau) = b_{13}\ddot{w}(x,\tau),$$

(50)

$$w'(x,\tau) + (b_4-\lambda b_5)\overline{\psi}''(x,\tau) - b_6\overline{\psi}(x,\tau) - b_7(1-\lambda b_8)\overline{\phi}''(x,\tau) +$$

$$+ b_9\overline{\phi}(x,\tau) = b_{14}\ddot{\overline{\psi}}(x,\tau) - b_{15}\ddot{\overline{\phi}}(x,\tau), \tag{51}$$

$$b_3 w'(x,\tau) - b_7(1-\lambda b_8)\overline{\psi}''(x,\tau) + b_9\overline{\psi}(x,\tau) + (b_{10}-\lambda b_{11})\overline{\phi}''(x,\tau) -$$

$$- b_{12}\overline{\phi}(x,\tau) = b_{16}\ddot{\overline{\phi}}(x,\tau) - b_{15}\ddot{\overline{\psi}}(x,\tau), \tag{52}$$

where $x = x_1/\ell$, $\tau = (\mu_2/\rho_2\ell^2)^{1/2}t$, $\overline{\phi} = \ell\phi$, $\overline{\psi} = \ell\psi$, primes and dots now refer to derivatives with respect to x and τ, respectively, and

$$\lambda = \frac{P(\ell/\pi)^2}{E_2\ I_b}, \qquad b_1 = \frac{\mu_1 d_1+\mu_2 d_2}{\mu_2(d_1+d_2)} = \frac{1+\gamma d}{1+d},$$

$$b_2 = \frac{E_2(\pi h/\ell)^2}{12\kappa\mu_2} = \frac{\delta(\pi\zeta)^2}{12\kappa}, \quad b_3 = \frac{d_1(\mu_1-\mu_2)}{\mu_2(d_1+d_2)} = \frac{d(\gamma-1)}{1+d},$$

$$b_4 = \frac{E_2 d_2(d_1+d_2)^2+h^2(E_1 d_1+E_2 d_2)}{12\kappa\mu_2\ell^2(d_1+d_2)} = \frac{\delta\zeta^2[1+\varepsilon d+(\beta_1+\beta_2)^2]}{12\kappa(1+d)},$$

$$b_5 = \frac{E_2(\pi h/\ell)^2}{144\kappa\mu_2\ell^2}[h^2+d_2(d_1+d_2)] = \frac{\delta\pi^2\zeta^4}{144\kappa}[1+\beta_2(\beta_1+\beta_2)],$$

$$b_6 = (d_1+d_2)/d_2 = 1+d, \quad b_7 = E_2 d_1 d_2/12\kappa\mu_2\ell^2 = \delta\beta_1\beta_2\zeta^2/12\kappa,$$

$$b_8 = (\pi h/\ell)^2/12 = (\pi\zeta)^2/12, \qquad b_9 = d_1/d_2 = d,$$

$$b_{10} = \frac{(d_1/\ell)^2(E_1d_1+E_2d_2)}{12\kappa\mu_2(d_1+d_2)} = \frac{\delta(\zeta\beta_1)^2(1+\epsilon d)}{12\kappa(1+d)},$$

$$b_{11} = \frac{E_2(d_1/\ell)^2(\pi h/\ell)^2}{144\kappa\mu_2} = \frac{\pi^2\delta\beta_1^2\zeta^4}{144\kappa}$$

$$b_{12} = \frac{d_1(\mu_1d_2+\mu_2d_1)}{\mu_2d_2(d_1+d_2)} = \frac{d(\gamma+d)}{1+d}$$

$$b_{13} = \frac{\rho_1d_1+\rho_2d_2}{\kappa\rho_2(d_1+d_2)} = \frac{1+\theta d}{\kappa(1+d)},$$

$$b_{14} = \frac{\rho_2d_2(d_1+d_2)^2+h^2(\rho_1d_1+\rho_2d_2)}{12\kappa\ell^2\rho_2(d_1+d_2)} = \frac{\zeta^2[\beta_2^2(1+d)^2+1+\theta d]}{12\kappa(1+d)},$$

$$b_{15} = \frac{d_1d_2}{12\kappa\ell^2} = \frac{\zeta^2\beta_1\beta_2}{12\kappa},$$

$$b_{16} = \frac{d_1^2(\rho_1d_1+\rho_2d_2)}{12\kappa\ell^2\rho_2(d_1+d_2)} = \frac{\beta_1^2\zeta^2(1+\theta d)}{12\kappa(1+d)},$$

with

$$\epsilon = E_1/E_2, \quad \gamma = \mu_1/\mu_2, \quad d = d_1/d_2, \quad \zeta = h/\ell,$$

$$\delta = E_2/\mu_2, \quad \theta = \rho_1/\rho_2, \quad \beta_\alpha = d_\alpha/h, \quad \alpha = 1,2.$$

SOLUTION OF THE FIELD EQUATIONS

The Hinged-Hinged Beam

We now consider the motion of a beam that is hinged at the ends $x = 0,1$. According to (49), the boundary conditions are

$$w = (b_4-\lambda b_5)\overline{\psi}' - b_7(1-\lambda b_8)\overline{\phi}' = b_7(1-\lambda b_8)\overline{\psi}' - (b_{10}-\lambda b_{11})\overline{\phi}' = 0 \tag{53}$$

at $x = 0,1$. It is a simple matter to verify that the functions

$$w(x,\tau) = A_1 \sin(n\pi x)\cos\omega\tau, \quad \overline{\psi}(x,\tau) = A_2 \cos(n\pi x)\cos\omega\tau,$$

$$\overline{\phi}(x,\tau) = A_3 \cos(n\pi x)\cos\omega\tau, \qquad n = 1,2,3,\ldots, \tag{54}$$

satisfy the boundary conditions (53) identically. Substitution of (54) into the equations of motion (50)-(52) yields the homogeneous algebraic system

$$\sum_{j=1}^{3} \alpha_{ij} A_j = 0, \tag{55}$$

where

$$\alpha_{11} = b_{13}\omega^2 - (n\pi)^2(b_1-\lambda b_2), \quad \alpha_{22} = b_{14}\omega^2 - b_6 - (n\pi)^2(b_4-\lambda b_5),$$

$$\alpha_{33} = b_{16}\omega^2 - b_{12} - (n\pi)^2(b_{10}-\lambda b_{11}), \quad \alpha_{12} = \alpha_{21} = n\pi$$

$$\alpha_{23} = \alpha_{32} = b_9 + (n\pi)^2 b_7(1-\lambda b_8) - b_{15}\omega^2, \quad \alpha_{13} = \alpha_{31} = n\pi b_3. \tag{56}$$

If (55) is to have a nontrivial solution, we must require that

$$\det(\alpha_{ij}) = 0. \tag{57}$$

The values of the natural frequencies ω_n and of the critical load parameter λ_{cr} (ω=0) will be computed from (57).

The Clamped-Clamped Beam

For combinations of boundary conditions other than those of the hinged-hinged case described above, simple expressions of the form (54) for the solution of (50)-(52) cannot be found. Nonetheless, we can obtain the values of the natural frequencies and critical loads for these more difficult cases by following a procedure described by Jones [12]. If we insert

$$w(x,\tau) = w(x)\cos\omega\tau, \quad \overline{\Psi}(x,\tau) = \psi(x)\cos\omega\tau,$$

$$\overline{\Phi}(x,\tau) = \phi(x)\cos\omega\tau,$$

into (50)-(52), we find the following set of ordinary differential equations:

$$c_1 w''(x) + c_2 w(x) - \psi'(x) - c_4\phi'(x) = 0,$$

$$w'(x) + c_5\psi''(x) + c_6\psi(x) - c_7\phi''(x) + c_8\phi(x) = 0,$$

$$c_4 w'(x) - c_7\psi''(x) + c_8\psi(x) + c_9\phi''(x) + c_{10}\phi(x) = 0, \tag{58}$$

where

$$c_1 = b_1-\lambda b_2, \quad c_2 = b_{13}\omega^2, \quad c_4 = b_3, \quad c_5 = b_4-\lambda b_5,$$

$$c_6 = b_{14}\omega^2-b_6, \quad c_7 = b_7(1-\lambda b_8), \quad c_8 = b_9-b_{15}\omega^2,$$

$$c_9 = b_{10}-\lambda b_{11}, \; c_{10} = b_{16}\omega^2 - b_{12}. \tag{59}$$

For the case of buckling, we set $\omega=0$, so that (58) reduces to

$$c_1 w''(x) - \psi'(x) - c_4\phi'(x) = 0,$$

$$w'(x) + c_5\psi''(x) + c_6\psi(x) - c_7\phi''(x) + c_8\phi(x) = 0,$$

$$c_4 w'(x) - c_7\psi''(x) + c_8\psi(x) + c_9\phi''(x) + c_{10}\phi(x) = 0, \tag{60}$$

where now

$$c_2 = 0, \quad c_6 = - b_6, \quad c_8 = b_9, \quad c_{10} = - b_{12}.$$

Since the equations in (58) and (60) possess constant coefficients, we seek a solution in the form

$$w(x) = Ae^{\bar{\lambda}x}, \quad \psi(x) = Be^{\bar{\lambda}x}, \quad \phi(x) = Ce^{\bar{\lambda}x}. \tag{61}$$

Substitution of (61) into (58) yields the following homogeneous system of algebraic equations in A, B, C:

$$(c_1\bar{\lambda}^2 + c_2)A - \bar{\lambda}B - \bar{\lambda}c_4C = 0,$$

$$\bar{\lambda}A + (c_5\bar{\lambda}^2 + c_6)B + (c_8 - c_7\bar{\lambda}^2)C = 0,$$

$$c_4\bar{\lambda}A + (c_8 - c_7\bar{\lambda}^2)B + (c_9\bar{\lambda}^2 + c_{10})C = 0. \tag{62}$$

This system of equations will have a nontrivial solution provided that the determinant of the coefficient matrix vanishes. Expansion of the determinant leads to the following polynomial of degree six in $\bar{\lambda}$ (bicubic in $\bar{\lambda}^2$):

$$c_1(c_5c_9 - c_7^2)\bar{\lambda}^6 + [c_1c_5c_{10} + c_9\ (c_2c_5 + c_1c_6) + 2c_4c_7 +$$

$$+ c_4^2c_5 - c_2c_7^2 + 2c_1c_7c_8 + c_9]\bar{\lambda}^4 + [c_2c_6c_9 + c_{10}\ (c_2c_5 + c_1c_6) -$$

$$- 2c_4c_8 + c_4^2c_6 - c_1c_8^2 + 2c_2c_7c_8 + c_{10}]\bar{\lambda}^2 + c_2(c_6c_{10} - c_8^2) = 0. \tag{63}$$

But with ω=0, (63) reduces to

$$\bar{\lambda}^2\{c_1(c_5c_9 - c_7^2)\bar{\lambda}^4 + [c_1(c_5c_{10} + c_6c_9 + 2c_7c_8) + 2c_4c_7 +$$

$$+ c_4^2c_5 + c_9]\bar{\lambda}^2 + c_1(c_6c_{10} - c_8^2) - 2c_4c_8 + c_4^2c_6 + c_{10}\} = 0. \tag{64}$$

A numerical study of (64) has revealed that, for the range of interest here (numerical values for the laminate's parameters are given in the next section), the roots of (64) may be expressed as

$$\bar{\lambda} = 0,\ 0,\ \pm\ i\eta_1,\ \pm\ \eta_2,$$

$$\eta_1,\ \eta_2 > 0, \qquad i = (-1)^{1/2}.$$

Consequently, the functions stated in (61) now assume the following forms:

$$w(x) = A_1 + A_2x + A_3\cos\eta_1x + A_4\sin\eta_1x + A_5\cosh\eta_2x + A_6\sinh\eta_2x,$$

$$\psi(x) = B_1 + B_2x + B_3\cos\eta_1x + B_4\sin\eta_1x + B_5\cosh\eta_2x + B_6\sinh\eta_2x,$$

$$\phi(x) = C_1 + C_2x + C_3\cos\eta_1x + C_4\sin\eta_1x + C_5\cosh\eta_2x + C_6\sinh\eta_2x. \tag{65}$$

The eighteen constants A_j, B_j, C_j, $j = 1,2,3,4,5,6$, in (65) are interrelated. Substitution of (65) into (60) yields a set of a dozen relationships among these coefficients. It is a straightforward algebraic exercise to demonstrate then that

$$A_2 = R_3C_1, \quad A_3 = R_1C_4, \quad A_4 = -R_1C_3, \quad A_5 = R_2C_6,$$

$$A_6 = R_2C_5, \quad B_1 = S_3C_1, \quad B_2 = 0, \quad B_3 = -S_1C_3,$$

$$B_4 = -S_1C_4, \quad B_5 = -S_2C_5, \quad B_6 = -S_2C_6, \quad C_2 = 0,$$

where

$$R_1 = \Delta_1^{-1}[c_8-c_4c_6+(c_7+c_4c_5)\eta_1^2], \quad R_2 = \Delta_2^{-1}[c_4c_6-c_8+(c_7+c_4c_5)\eta_2^2],$$

$$S_1 = (\eta_1/\Delta_1)[c_4+c_1c_8+c_1c_7\eta_1^2], \quad S_2 = (\eta_2/\Delta_2)[c_4+c_1c_8-c_1c_7\eta_2^2],$$

$$\Delta_1 = \eta_1[1 + c_1(c_6 - c_5\eta_1^2)], \quad \Delta_2 = \eta_2[1 + c_1(c_6 + c_5\eta_2^2)],$$

$$R_3 = (c_3^2 - c_6c_{10})/(c_4c_6 - c_8), \quad S_3 = (c_{10} - c_4c_8)/(c_4c_6 - c_8).$$

Consequently, (65) may now be expressed as

$$w(x) = A_1 + R_3C_1x + R_1C_4\cos\eta_1x - R_1C_3\sin\eta_1x + R_2C_6\cosh\eta_2x + $$
$$+ R_2C_5\sinh\eta_2x,$$

$$\psi(x) = S_3C_1 - S_1C_3\cos\eta_1x - S_1C_4\sin\eta_1x - S_2C_5\cosh\eta_2x - S_2C_6\sinh\eta_2x,$$

$$\phi(x) = C_1 + C_3\cos\eta_1x + C_4\sin\eta_1x + C_5\cosh\eta_2x + C_6\sinh\eta_2x. \tag{66}$$

If both ends of the beam are clamped, then, in view of (49) the boundary conditions are

$$w(0) = \psi(0) = \phi(0) = w(1) = \psi(1) = \phi(1) = 0. \tag{67}$$

Inserting (66) into the first three boundary conditions in (67), we obtain

$$A_1 + R_1C_4 + R_2C_6 = 0,$$

$$S_3C_1 - S_1C_3 - S_2C_5 = 0,$$

$$C_1 + C_3 + C_5 = 0.$$

With the help of these relationships, we can write (66) as

$$w(x) = R_1(\cos\eta_1x - 1)C_4 + (R_3\xi_{21}x + R_1\xi_{32}\sin\eta_1x +$$
$$+ R_2\sinh\eta_2x)C_5 + R_2(\cosh\eta_2x - 1)C_6,$$

$$\psi(x) = - S_1C_4\sin\eta_1x + (S_3\xi_{21} + S_1\xi_{32}\cos\eta_1x - S_2\cosh\eta_2x)C_5 -$$

$$- S_2C_6\sinh\eta_2x,$$

$$\phi(x) = C_4\sin\eta_1x + (\xi_{21} - \xi_{32}\cos\eta_1x + \cosh\eta_2x)C_5 + C_6\sinh\eta_2x,$$

with

$$\xi_{21} = (S_2 - S_1)/(S_3 + S_1), \quad \xi_{32} = (S_3 + S_2)/(S_3 + S_1).$$

Substitution of (68) into the last three boundary conditions in (67) leads to the following homogeneous system of algebraic equations in C_4, C_5, and C_6:

$$A_{ij}\, C_{j+3} = 0, \qquad i,j = 1,2,3, \tag{69}$$

where

$$A_{11} = R_1(\cos\eta_1 - 1), \quad A_{12} = R_3\xi_{21} + R_1\xi_{32}\sin\eta_1 + R_2\sinh\eta_2,$$

$$A_{13} = R_2(\cosh\eta_2 - 1), \quad A_{21} = - S_1\sin\eta_1,$$

$$A_{22} = S_3\xi_{21} + S_1\xi_{32}\cos\eta_1 - S_2\cosh\eta_2, \quad A_{23} = - S_2\sinh\eta_2,$$

$$A_{31} = \sin\eta_1, \quad A_{32} = \xi_{21} - \xi_{32}\cos\eta_1 + \cosh\eta_2,$$

$$A_{33} = \sinh\eta_2.$$

The system of equations (69) will have a nontrivial solution if and only if

$$\mathrm{Det}\,(A_{ij}) = 0. \tag{70}$$

Expansion of the determinant in (70) yields the relatively simple relationship

$$2R_2(\cosh\eta_2 - 1)\sin\eta_1 + 2R_1\xi_{32}(1-\cos\eta_1)\sinh\eta_2 + $$
$$+ R_3\xi_{21}\sin\eta_1 \sinh\eta_2 = 0. \tag{71}$$

However, if we introduce the identities

$$\sin\eta_1 = 2\sin(\eta_1/2)\cos(\eta_1/2), \quad \sinh\eta_2 = 2\sinh(\eta_2/2)\cosh(\eta_2/2),$$

$$1-\cos\eta_1 = 2\sin^2(\eta_1/2), \qquad \cosh\eta_2 - 1 = \sinh^2(\eta_2/2),$$

into (71), we obtain, after some rearrangement,

$$\sin(\eta_1/2)\ \sinh(\eta_2/2)\ [2R_2\tanh(\eta_2/2) + 2R_1\xi_{32}\tan(\eta_1/2) +$$
$$+ R_3\xi_{21}] = 0.$$

As in the case of the Timoshenko theory (see the Appendix) the smallest positive value of λ_{cr} can be shown to arise from

$$\sin\ (\eta_1/2) = 0.$$

Thus, $\eta_1 = 2\pi$, and since in (64)$\bar{\lambda}^2 = - \eta_1^2 = - (2\pi)^2$, we find that the value of λ_{cr} is to be computed from

$$c_1(c_5c_9-c_7^2)(2\pi)^4 - [c_1(c_5c_{10} + c_6c_9 + 2c_7c_8) + 2c_4c_7 + c_4^2c_5 +$$
$$+ c_9](2\pi)^2 + c_1(c_6c_{10} - c_8^2) - 2c_4c_8 + c_4^2c_6 + c_{10} = 0, \tag{72}$$

or the expanded form of this result which can be shown to be a polynomial of degree three in λ. A discussion of the numerical results obtained from (72) will be given in the next section.

Let us next address the vibration problem for a clamped-clamped laminated beam. For $\omega \neq 0$, we must use (63) to compute the characteristic exponents $\bar{\lambda}$. Considering (63) as a cubic in $\bar{\lambda}^2$, we have been able to show that, for the range of interest here, (63) has either one negative root and two positive roots or two negative roots and one positive root. For the present we confine our attention to the case

$$\bar{\lambda}_1^2 < 0, \qquad \bar{\lambda}_3^2 > \bar{\lambda}_2^2 > 0.$$

Now let

$$\eta_1 = |\bar{\lambda}_1^2|^{1/2}, \qquad \eta_2 = \bar{\lambda}_2, \qquad \eta_3 = \bar{\lambda}_3,$$

it being understood that $\eta_j > 0$, $j = 1,2,3$. Therefore, (61) leads to

$$w(x) = A_1 \cos\eta_1 x + A_2 \sin\eta_1 x + A_3 \cosh\eta_2 x + A_4 \sinh\eta_2 x + $$
$$+ A_5 \cosh\eta_3 x + A_6 \sinh\eta_3 x,$$

$$\psi(x) = B_1 \cos\eta_1 x + B_2 \sin\eta_1 x + B_3 \cosh\eta_2 + B_4 \sinh\eta_2 x + $$
$$+ B_5 \cosh\eta_3 x + B_6 \sinh\eta_3 x,$$

$$\phi(x) = C_1 \cos\eta_1 x + C_2 \sin\eta_1 x + C_3 \cosh\eta_2 x + C_4 \sinh\eta_2 x + $$
$$+ C_5 \cosh\eta_3 x + C_6 \sinh\eta_3 x. \tag{73}$$

Proceeding as we did with (65), we can show that

$$A_1 = R_1 C_2, \qquad A_2 = -R_1 C_1, \qquad A_3 = R_2 C_4, \qquad A_4 = R_2 C_3,$$

$$A_5 = R_3 C_6, \qquad A_6 = R_3 C_5, \qquad B_1 = S_1 C_1, \qquad B_2 = S_1 C_2,$$

$$B_3 = S_2 C_3, \qquad B_4 = S_2 C_4, \qquad B_5 = S_3 C_5, \qquad B_6 = S_3 C_6,$$

where

$$R_1 = (\eta_1/\Delta_1)\ [c_4c_6 - c_8 - (c_4c_5 + c_7)\eta_1^2],$$

$$R_2 = (\eta_2/\Delta_2)\ [c_4c_6 - c_8 + (c_4c_5 + c_7)\eta_2^2],$$

$$R_3 = (\eta_3/\Delta_3)\ [c_4c_6 - c_8 + (c_4c_5 + c_7)\eta_3^2],$$

$$S_1 = \Delta_1^{-1}\ [c_1c_7\eta_1^4 + (c_4 + c_1c_8 - c_2c_7)\eta_1^2 - c_2c_8],$$

$$S_2 = \Delta_2^{-1}\ [c_1c_7\eta_2^4 - (c_4 + c_1c_8 - c_2c_7)\eta_2^2 - c_2c_8],$$

$$S_3 = \Delta_3^{-1}\ [c_1c_7\eta_3^4 - (c_4 + c_1c_8 - c_2c_7)\eta_3^2 - c_2c_8],$$

$$\Delta_1 = c_1c_5\eta_1^4 - (1 + c_1c_6 + c_2c_5)\eta_1^2 + c_2c_6,$$

$$\Delta_2 = c_1c_5\eta_2^4 + (1 + c_1c_6 + c_2c_5)\eta_2^2 + c_2c_6,$$

$$\Delta_3 = c_1c_5\eta_3^4 + (1 + c_1c_6 + c_2c_5\eta_3^2 + c_2c_6.$$

Therefore, (73) becomes

$$w(x) = R_1[C_2\cos\eta_1 x - c_1\sin\eta_1 x] + R_2[c_4\cosh\eta_2 x + c_3\sinh\eta_2 x] + \\ + R_3[C_6\cosh\eta_3 x + c_5\sinh\eta_3 x],$$

$$\psi(x) = S_1[C_1\cos\eta_1 x + C_2\sin\eta_1 x] + S_2[C_3\cosh\eta_2 x + C_4\sinh\eta_2 x] + \\ + S_3[C_5\cosh\eta_3 x + C_6\sinh\eta_3 x],$$

$$\phi(x) = C_1\cos\eta_1 x + C_2\sin\eta_2 x + C_3\cosh\eta_2 + C_4\sinh\eta_2 x + \\ + C_5\cosh\eta_3 x + C_6\sinh\eta_3 x. \tag{74}$$

Invoking the first three boundary conditions in (67), we can reduce the number of constants in (74) to three:

$$w(x) = R_1R_2(\cosh\eta_2x - \cos\eta_1x)C_4' + [R_1(S_3 - S_1)\sin\eta_1x + R_2(S_3 - S_1)\sinh\eta_2x + (S_1 - S_2)R_3\sinh\eta_3x]C_5' + R_1R_3(\cosh\eta_3x - \cos\eta_1x)C_6',$$

$$\psi(x) = (S_2R_1\sinh\eta_2x - S_1R_2\sin\eta_1x)C_4' + [S_1(S_2 - S_3)\cos\eta_1x + S_2(S_3 - S_1)\cosh\eta_2x + S_3(S_1 - S_2)\cosh\eta_3x]C_5' + (S_3R_1\sinh\eta_3x - S_1R_3\sin\eta_1x)C_6',$$

$$\phi(x) = (R_1\sinh\eta_2x - R_2\sin\eta_1x)C_4' + [(S_2 - S_3)\cos\eta_1x + (S_3 - S_1)\cosh\eta_2x + (S_1 - S_2)\cosh\eta_3x]C_5' + (R_1\sinh\eta_3x - R_3\sin\eta_1x)C_6',$$

where we have set

$$C_4 = R_1C_4', \qquad C_6 = R_1C_6', \qquad C_5 = (S_1 - S_2)C_5'.$$

The frequency equation is now derived by inserting (75) into the last three boundary conditions in (67). In the usual manner, we again obtain (70), where now

$$A_{11} = R_1R_2(\cosh\eta_2 - \cos\eta_1),$$

$$A_{12} = R_1(S_3-S_2)\sin\eta_1 + R_2(S_3-S_1)\sinh\eta_2 + (S_1-S_2)R_3\sinh\eta_3,$$

$$A_{13} = R_1R_3(\cosh\eta_3 - \cos\eta_1), \quad A_{21} = S_2R_1\sinh\eta_2 - S_1R_2\sin\eta_1,$$

$$A_{22} = S_1(S_2-S_3)\cos\eta_1 + S_2(S_3-S_1)\cosh\eta_2 + S_3(S_1-S_2)\cosh\eta_3,$$

$$A_{23} = S_3R_1\sinh\eta_3 - S_1R_3\sin\eta_1, \quad A_{31} = R_1\sinh\eta_2 - R_2\sin\eta_1,$$

$$A_{32} = (S_2-S_3)\cos\eta_1 + (S_3-S_1)\cosh\eta_2 + (S_1-S_2)\cosh\eta_3,$$

$$A_{33} = R_1\sinh\eta_3 - R_3\sin\eta_1.$$

Expansion of the determinant in (70) leads to the frequency equation

$$2R_1R_2\gamma_{23}\gamma_{31}(1-\cos\eta_1 \cosh\eta_2)\sinh\eta_3 + 2R_2R_3\gamma_{12}\gamma_{31}(1-\cosh\eta_2 \cosh\eta_3)\sin\eta_1 + 2R_1R_3\gamma_{12}\gamma_{23}(1-\cos\eta_1 \cosh\eta_3)\sinh\eta_2 + (R_1^2\gamma_{23}^2 - R_3^2\gamma_{12}^2 - R_2^2\gamma_{31}^2)\sin\eta_1 \sinh\eta_2 \sinh\eta_3 = 0, \tag{76}$$

where

$$\gamma_{23} = S_2 - S_3, \qquad \gamma_{31} = S_3 - S_1, \qquad \gamma_{12} = S_1 - S_2.$$

If the value of ω^2 is such that η_2^2 assumes a negative value, then we set $\eta_2^2 = -\bar{\eta}_2^2$. With the identities

$$\sinh(i\bar{\eta}_2) = i\sin\bar{\eta}_2, \qquad \cosh(i\bar{\eta}_2) = \cos\bar{\eta}_2,$$

we can now express (76) in the form

$$2R_1\bar{R}_2\bar{\gamma}_{23}\gamma_{31}(1-\cos\eta_1 \cos\bar{\eta}_2)\sinh\eta_3 + 2\bar{R}_2R_3\bar{\gamma}_{12}\gamma_{31}(1-\cos\bar{\eta}_2 \cosh\eta_3)\sin\eta_1 + 2R_1R_3\bar{\gamma}_{12}\bar{\gamma}(1-\cos\eta_1 \cosh\eta_3)\sin\bar{\eta}_2 + (R_1^2\bar{\gamma}_{23}^2 - R_3^2\bar{\gamma}_{12}^2 + \bar{R}_2^2\gamma_{31}^2)\sin\eta_1 \sin\bar{\eta}_2 \sinh\eta_3 = 0 \tag{77}$$

where

$$\bar{\Delta}_2 = c_1 c_5 \bar{\eta}_2^4 - (1 + c_1 c_6 + c_2 c_5)\bar{\eta}_2^2 + c_2 c_6,$$

$$\bar{R}_2 = (\bar{\eta}_2/\bar{\Delta}_2)\,[c_4 c_6 - c_8 - (c_4 c_5 + c_7)\bar{\eta}_2^2],$$

$$\bar{S}_2 = \bar{\Delta}_2^{-1}\,[c_1 c_7 \bar{\eta}_2^4 + (c_4 + c_1 c_8 - c_2 c_7)\bar{\eta}_2^2 - c_2 c_8],$$

$$\bar{\gamma}_{23} = \bar{S}_2 - S_3, \qquad \bar{\gamma}_{12} = S_1 - \bar{S}_2.$$

For a certain set of values of ω^2, say ω_n^2, $n = 1,2,3,\ldots$, the left side of (76 or 77) will vanish. In the numerical work we assign a value to ω^2 and compute the η_j's (i.e., the $\bar{\lambda}_j$'s) from (63). The value of ω^2 and η_j are then inserted into (76 or 77). We evaluate the left side of (76 or 77), and if its value is different from zero a new value is assigned to ω^2 and the process is repeated until a value of ω^2 that causes the left side of (76 or 77) to vanish is found.

NUMERICAL RESULTS

Sun [1] compared the dispersion curves obtained from the effective modulus theory and the microstructure theory for laminated beams, using the following numerical values for the parameters:

$$\gamma = 100, \qquad \theta = 2, \qquad \eta = 0.8,$$

$$\xi = 4.8, \qquad \nu_1 = 0.2, \qquad \nu_2 = 0.35. \tag{78}$$

The values of η and ξ given in (78) correspond to a composite beam consisting of five stiff layers and four soft layers. We shall also use these values here as well as other values of ξ, i.e., the number of layer pairs in the composite will be varied.

The numerical results obtained here from the microstructure theory are compared with the corresponding results obtained from the effective modulus theory (see the Appendix), which is comprised of Timoshenko beam theory, with effective moduli and density, including the effect of a uniform axial compressive load. For a two-phase composite medium of the type under consideration in this investigation, Sun [1] assumed the effective Young's modulus, effective shear modulus, and effective mass density to be

$$E = \eta E_1 + (1-\eta)E_2, \quad \mu = \eta\mu_1 + (1-\eta)\mu_2,$$

$$\rho = \eta\rho_1 + (1-\eta)\rho_2, \tag{79}$$

respectively.

The Hinged-Hinged Beam

The critical value of the buckling parameter λ and the natural frequencies of vibration are computed from (57) for a hinged-hinged beam. For a compressive load P applied in a conservative manner, we may calculate the value of λ_{cr} from (57) by setting the frequency parameter ω equal to zero and n=1 in (56).

In Figure 2, the variation of the critical load coefficient, λ_{cr}, is plotted against the depth-to-length ratio, ζ, over the range $0<\zeta<0.2$. The corresponding results obtained from the effective modulus theory (see the Appendix) are also shown. The microstructure and the effective modulus theories are in rather close agreement for relatively long beams. However, as the values of ζ increases, the values of λ_{cr} as calculated from the effective modulus theory decrease relatively slowly, whereas the values as computed from the microstructure theory decrease very rapidly. At $\zeta = 0.2$, the microstructure value is approximately 38% of the effective modulus value. The effect of increasing the number of layer pairs is also shown in Figure 2. Increasing the number of layer pairs beyond twenty has a negligible effect on the value of λ_{cr}. The values of λ_{cr} for the beam consisting of only five stiff layers and four soft layers are only a few percent greater than those for a beam consisting of a large number of layer pairs.

Eq. (57) may be considered as a polynomial of degree three in ω^2. Hence, for each choice of the integer n, (57) will yield three branches on a frequency plot, and these branches may be labelled the (i) flexural, (ii) thickness-shear, and (iii) microstructure rotational branches. Since this third branch is associated with relatively high frequencies, we shall concentrate our attention in this study on the more important flexural and thickness-shear branches. In Figures 3-5, the variation of ω_n with ζ for the first three flexural modes (n=1,2,3) with $\lambda = -5,0,5$, is plotted for a beam made of five stiff layers and

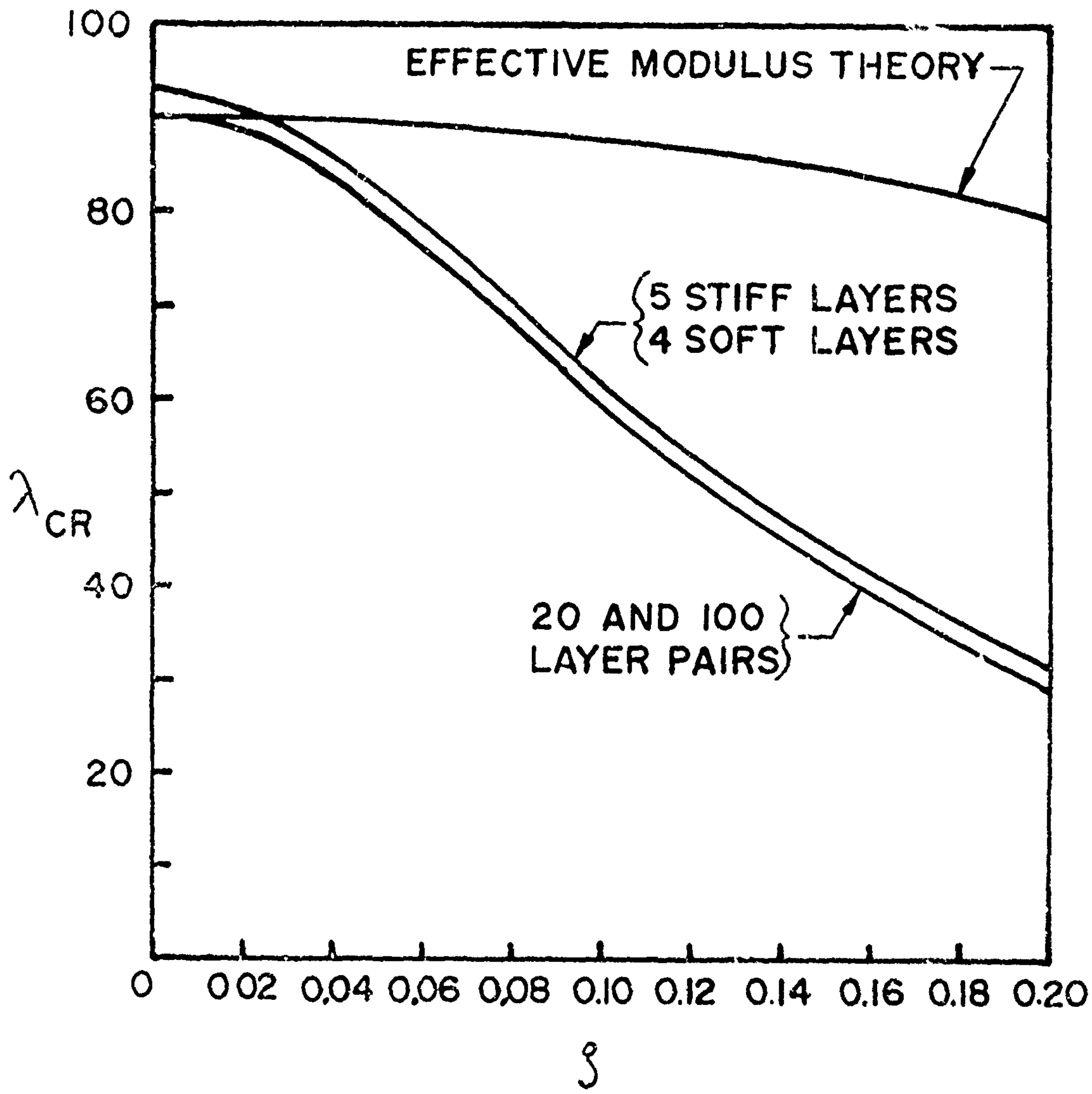

Figure 2. Variation of the critical load coefficient λ_{cr} with depth-to-length ratio ζ for a hinged-hinged beam.

four soft layers. For small values of ζ, i.e., relatively long beams, the effective stiffness and microstructure results are in excellent agreement, however as the value of ζ increases the microstructure frequencies are considerably less than the effective modulus frequencies, the magnitude of the effect increasing with the mode number n. 's is to be expected, the presence of a tensile load (negative λ) tends ι increase the natural frequency, whereas a compressive load (positive λ) tends to decrease it relative to its value in the absence of an axial load.

In Figures 6-8, the first three flexural frequencies for $\lambda = 0$ are again plotted, however now the effect of the number of layer pairs on the frequency is demonstrated. An increase in the number of layer pairs tends to decrease the value of ω_n, the difference becoming more pronounced as ζ and the mode number n increase. For sufficiently small values of ζ the effect of increasing the number of layer pairs is virtually negligible. In addition, increasing the number of layer pairs beyond twenty has very little effect upon the values of the ω_n's.

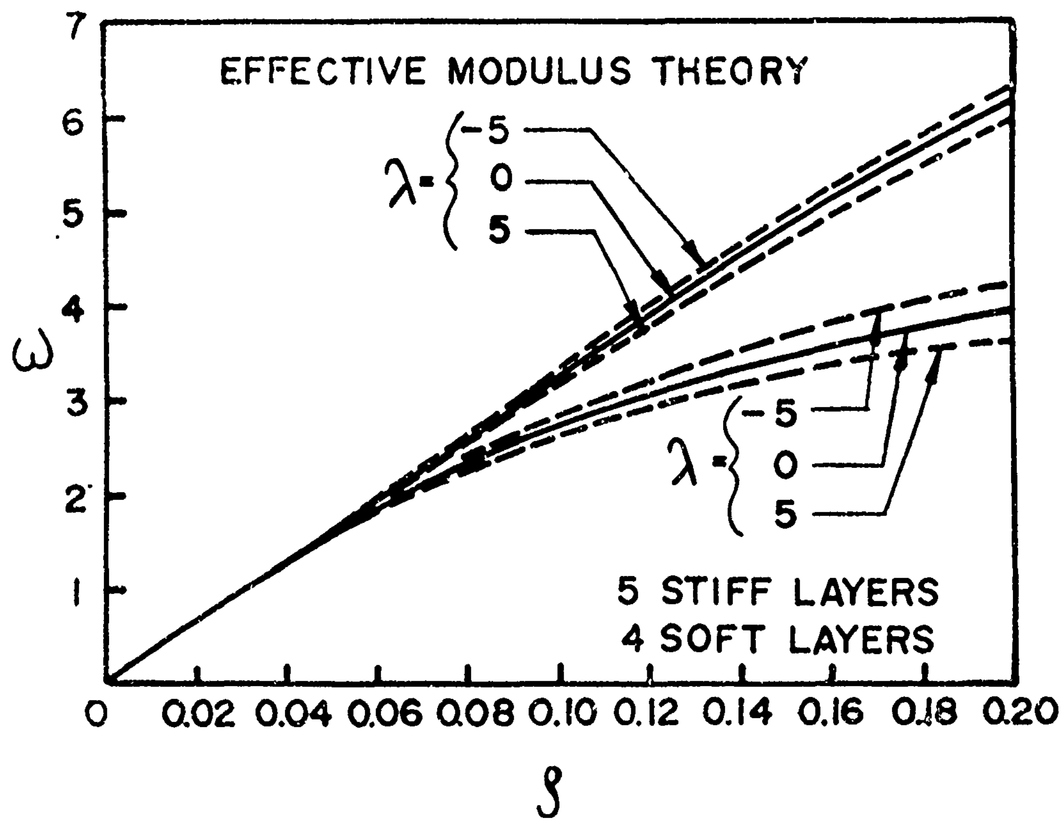

Figure 3. Variation of the first flexural frequency $\omega(n=1)$ with the applied load parameter λ and the depth-to-length ratio ζ for a hinged-hinged beam.

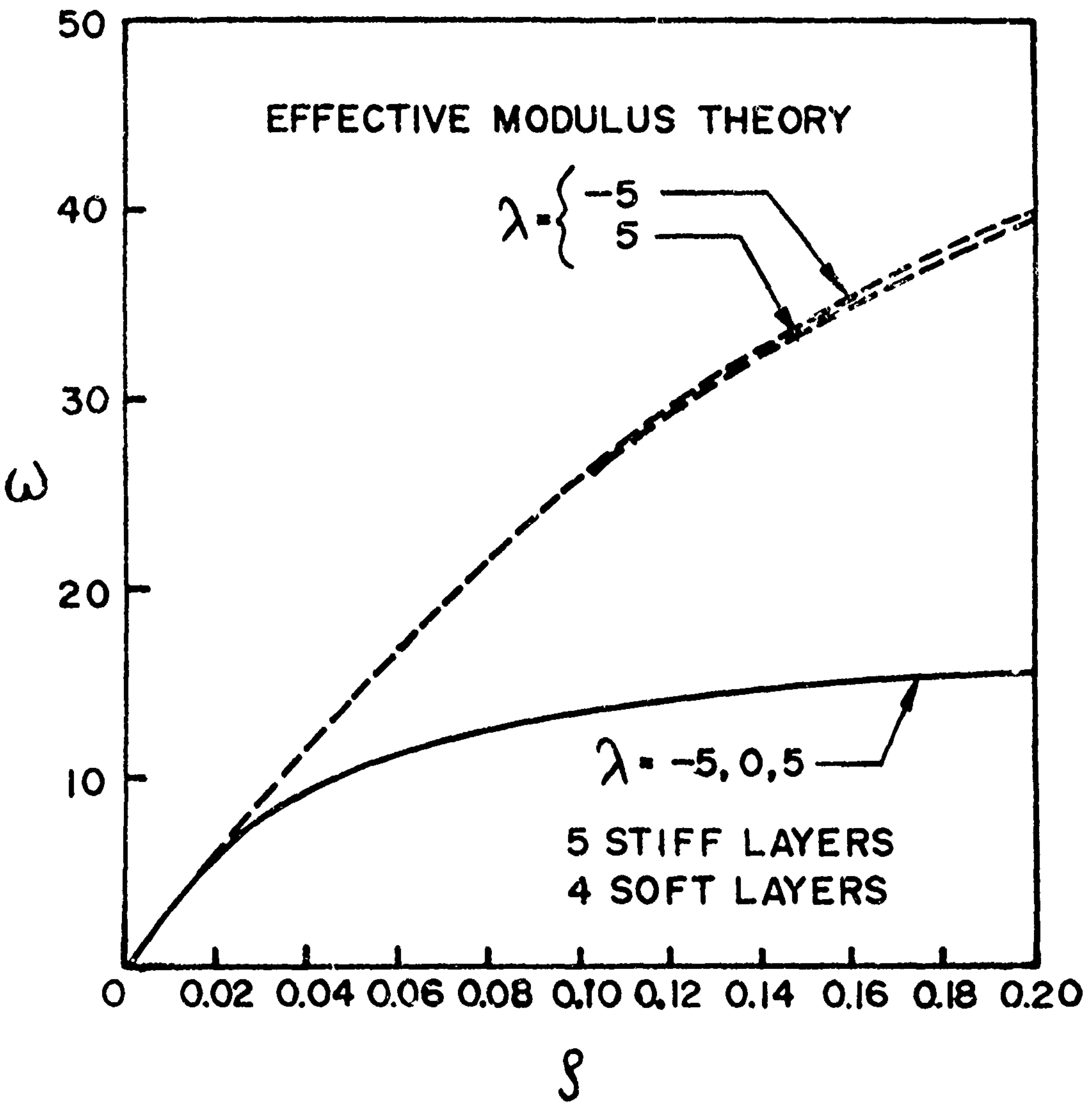

Figure 4. Variation of the second flexural frequency $\omega(n=2)$ with λ and ζ for a hinged-hinged beam.

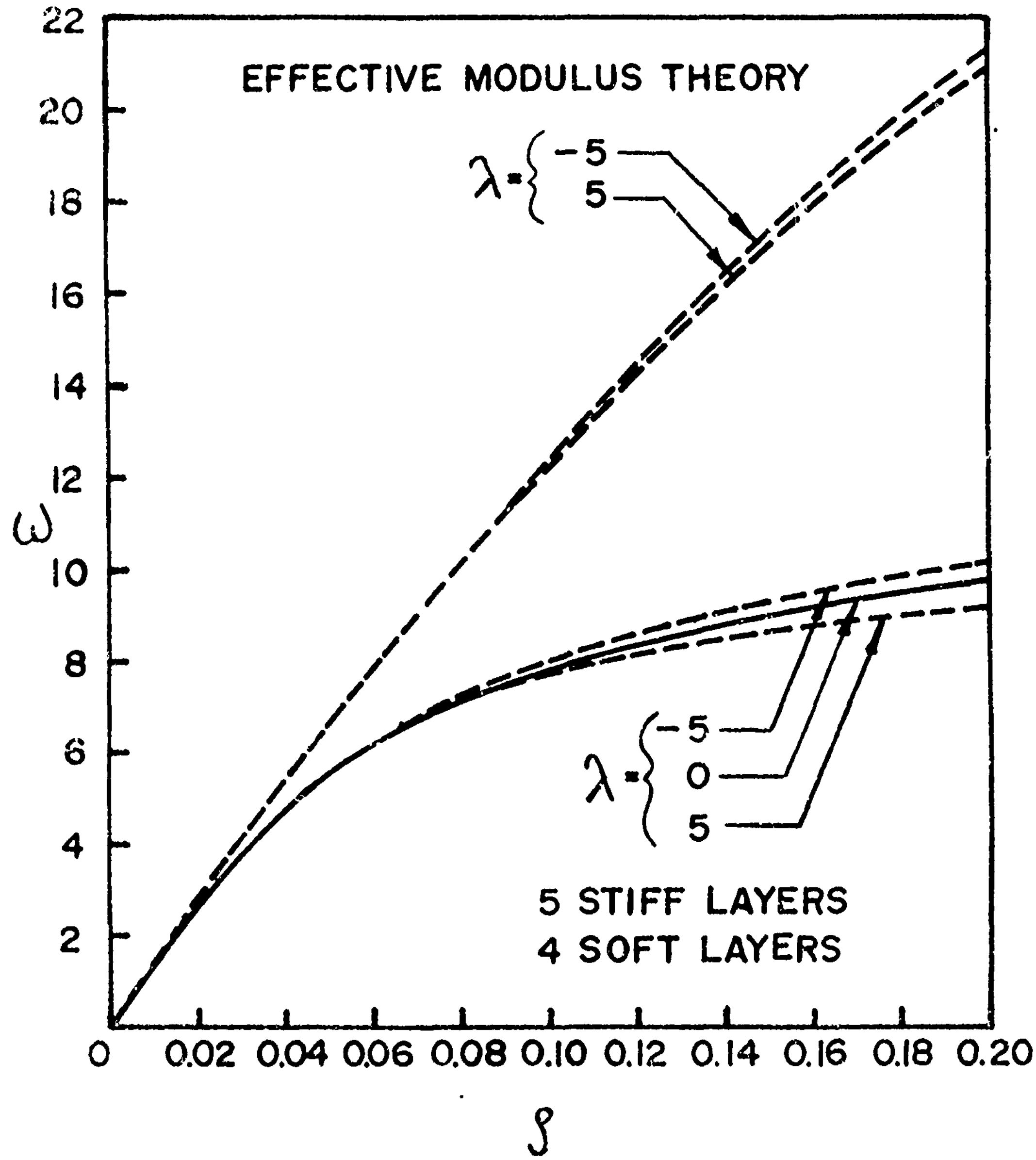

Figure 5. Variation of the third flexural frequency $\omega(n=3)$ with λ and ζ for a hinged-hinged beam.

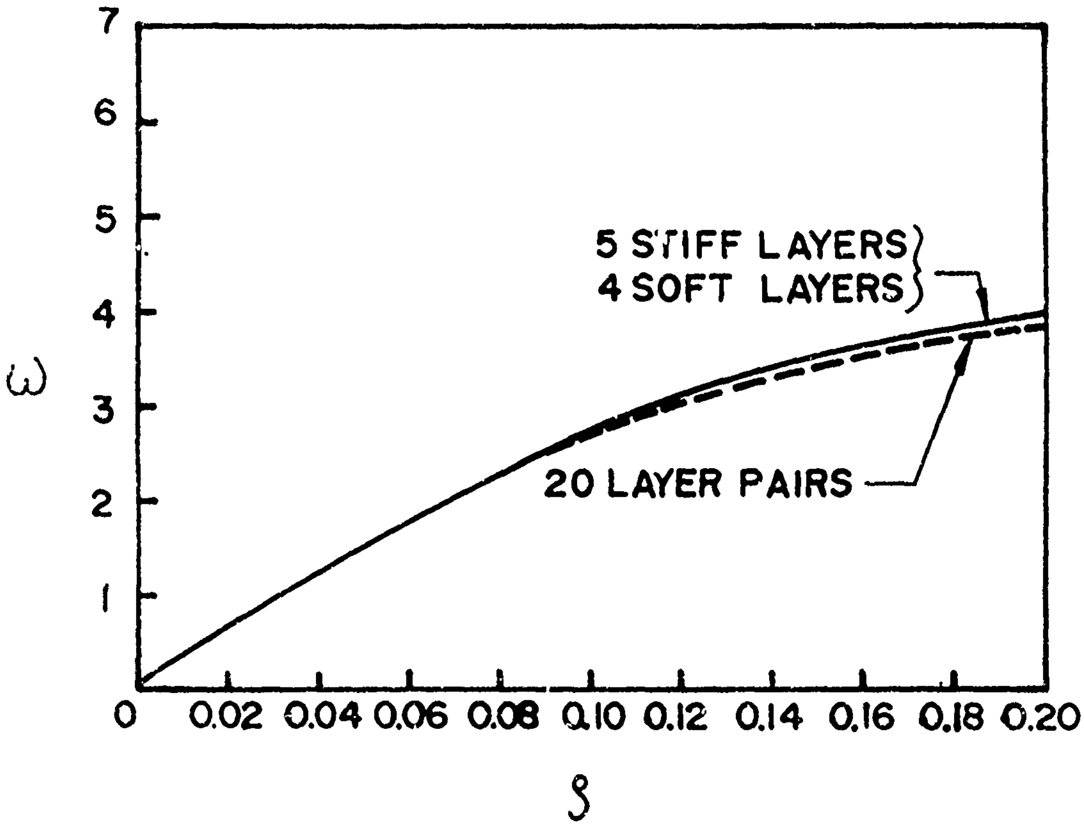

Figure 6. Variation of the first flexural frequency $\omega(n=1)$ with ζ and the number of layer pairs for a hinged-hinged beam.

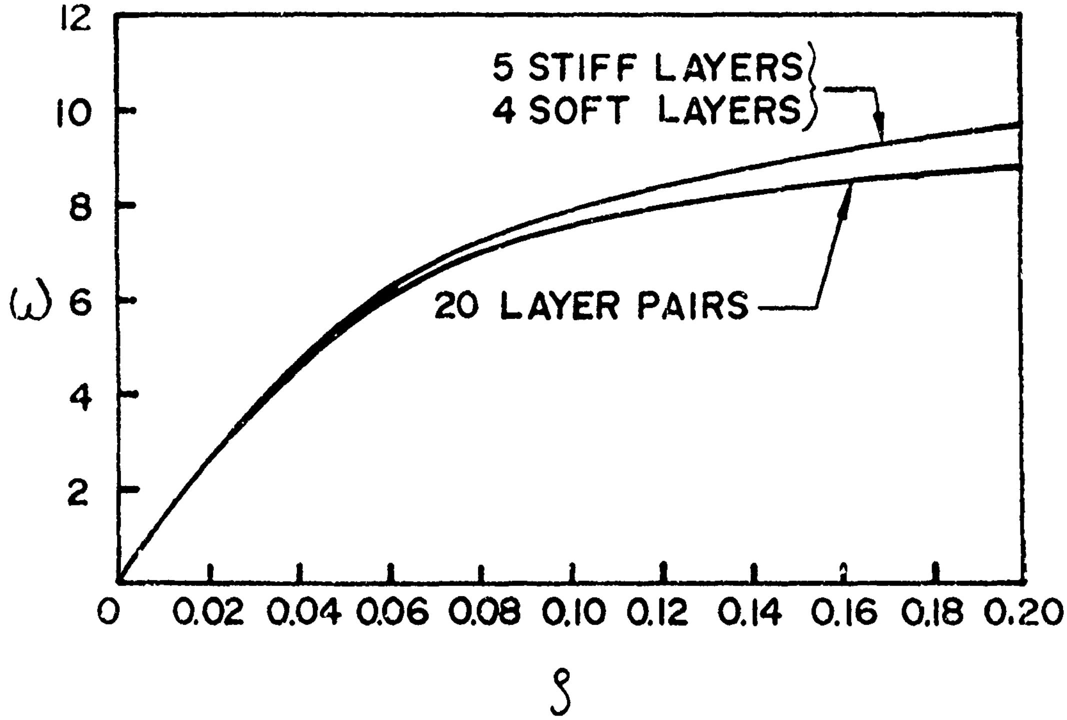

Figure 7. Variation of the second flexural frequency $\omega(n=2)$ with ζ and the number of layer pairs for a hinged-hinged beam.

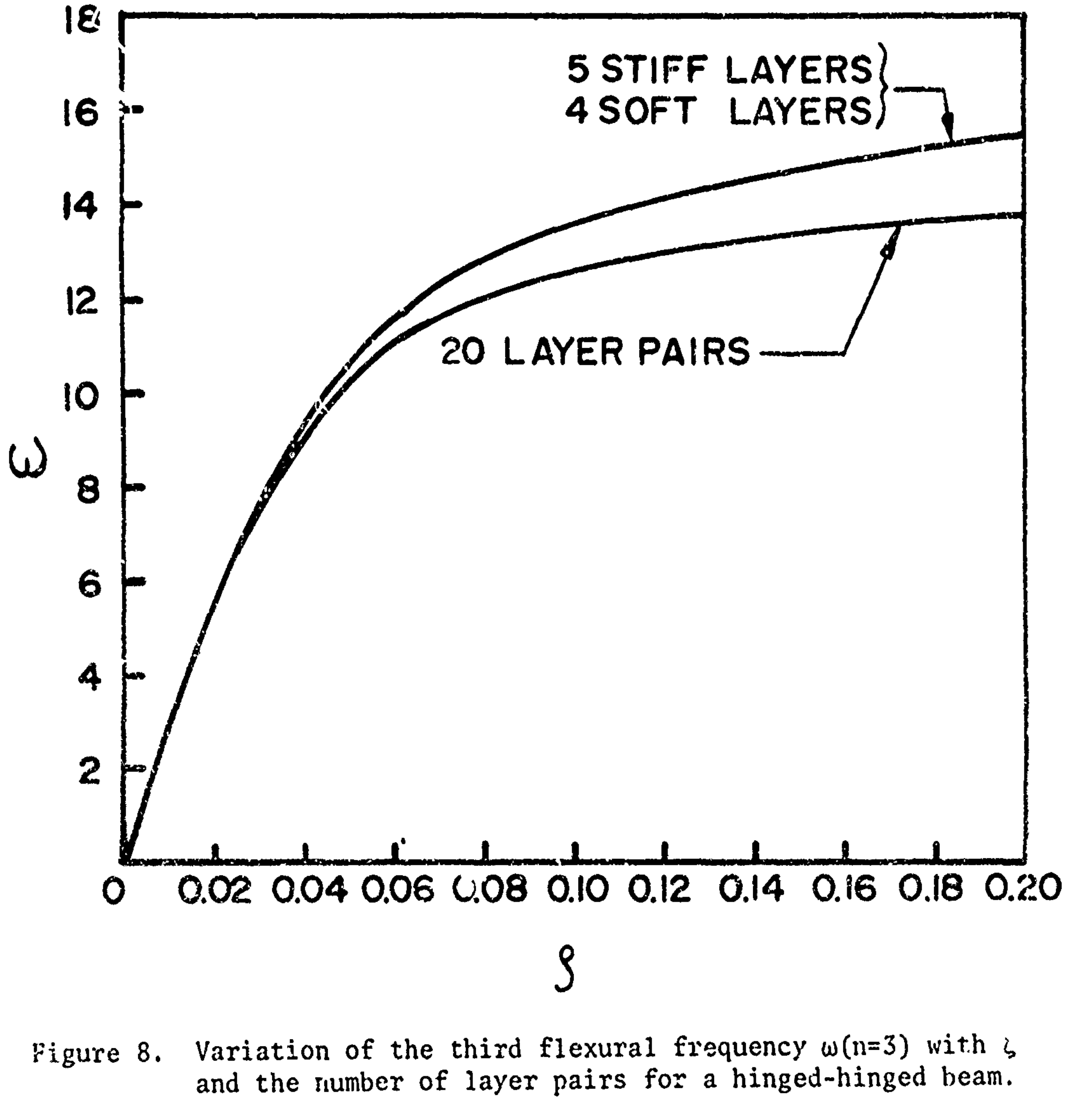

Figure 8. Variation of the third flexural frequency $\omega(n=3)$ with ζ and the number of layer pairs for a hinged-hinged beam.

In Figure 9, the lowest thickness-shear frequency is plotted as a function of ζ. In the range $|\lambda| \leq 5$, the calculations performed here were virtually insensitive to the values of λ. The effective modulus theory leads to values for ω_n that are again greater than the thickness-shear frequencies computed from the microstructure theory. Based upon the microstructure theory, the frequencies for the first three thickness-shear modes were determined, and their variation with ζ is shown in Figure 10.

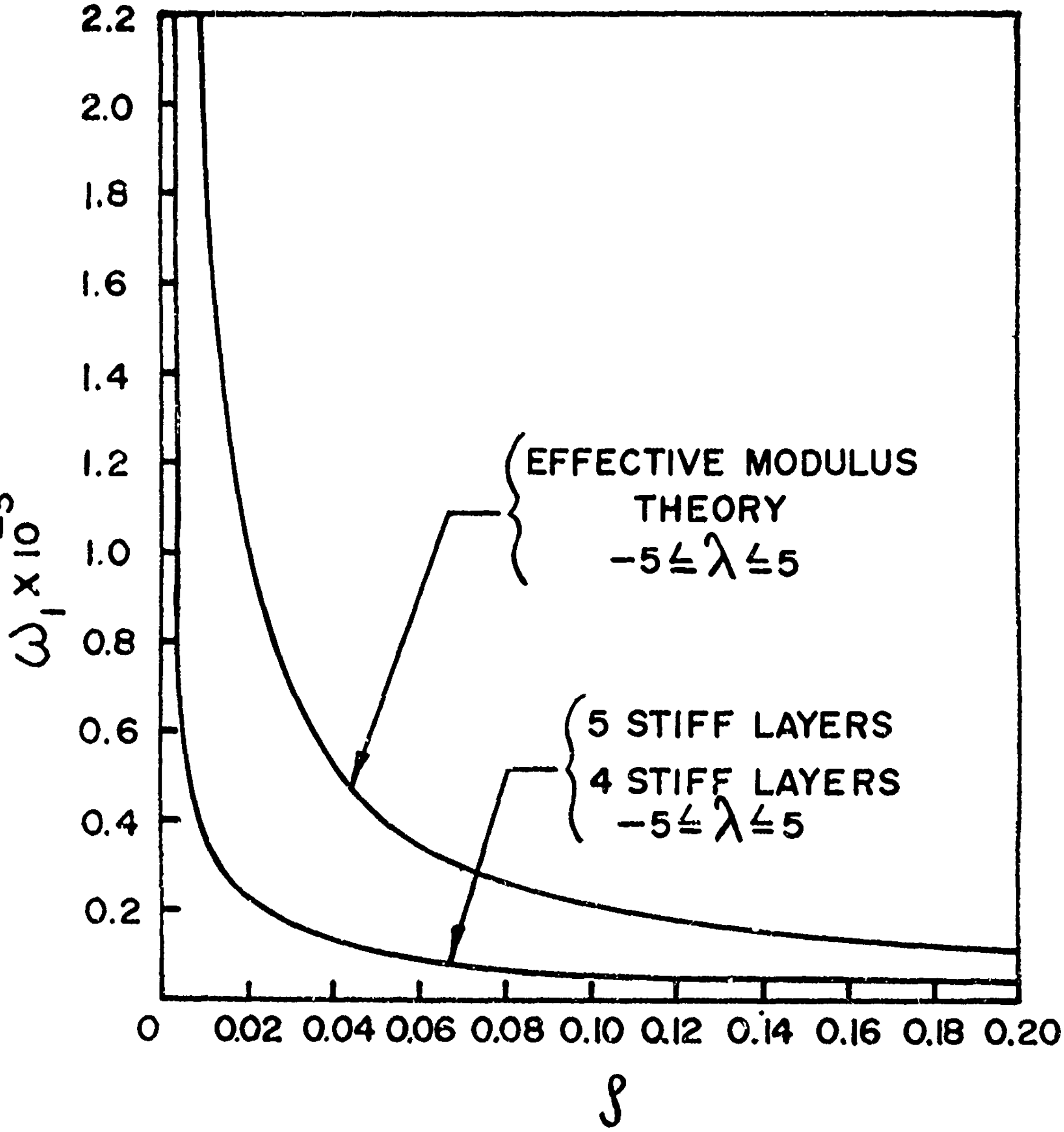

Figure 9. Variation of the first thickness-shear frequency $\omega(n=1)$ versus ζ for a hinged-hinged beam.

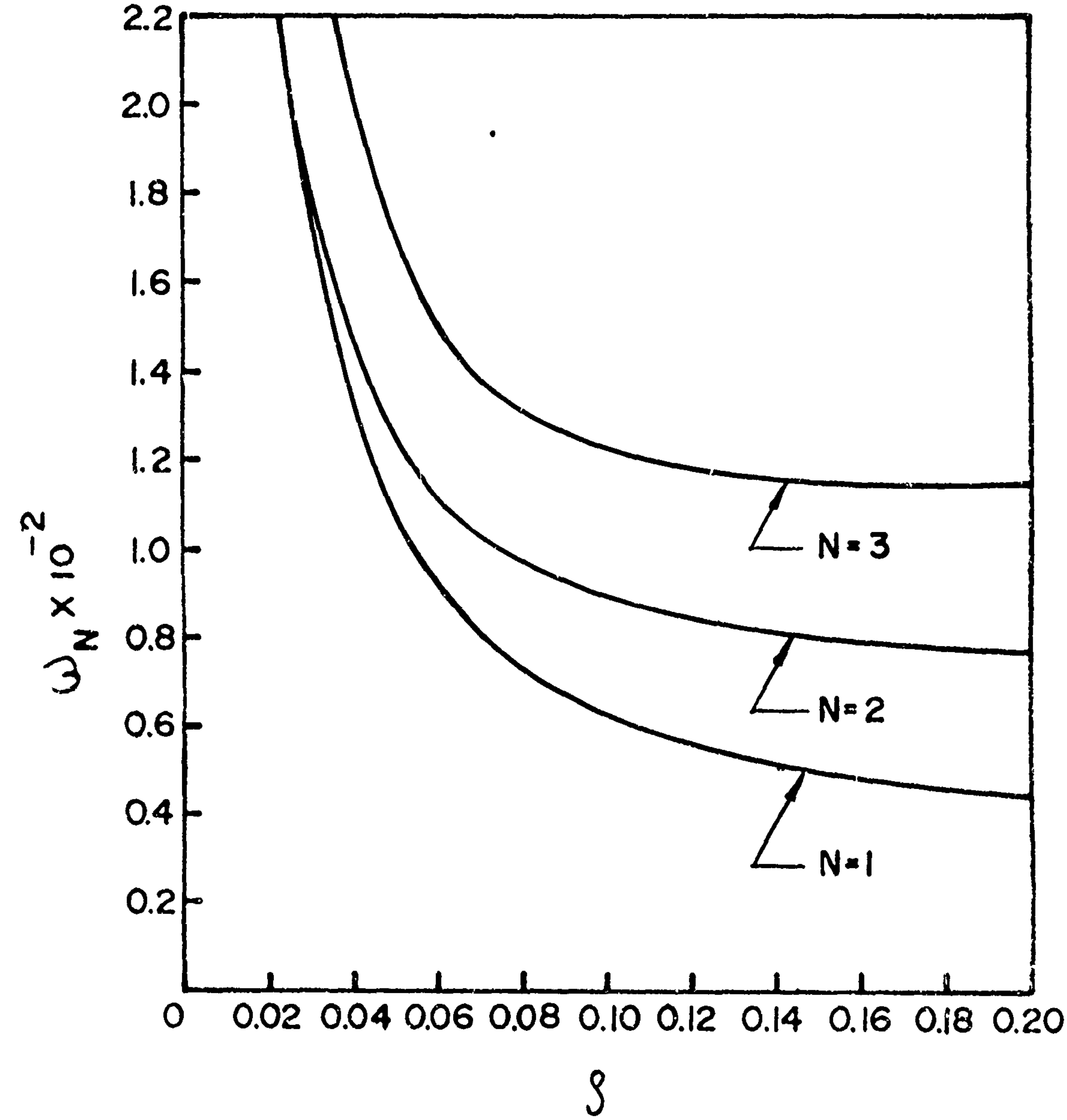

Figure 10. Variation of the first three thickness-shear frequencies with ζ as determined from the microstructure theory for a hinged-hinged beam.

The Clamped-Clamped Beam

The critical values of the buckling coefficient λ are calculated from (72). In Figure 11, the variation of λ_{cr} is plotted as a function of ζ over the range $0<\zeta\leq 0.2$. Results of the microstructure calculations, (72), and the effective modulus calculations (see the Appendix) are shown. For very long beams both theories predict essentially the same values for λ_{cr}, but as ζ increases, i.e., as shorter and shorter beams are considered, the microstructure theory predicts much lower values for the critical load coefficient than does the effective modulus theory. In fact at $\zeta = 0.2$, the microstructure value for λ_{cr} is approximately one-sixth of the effective modulus value.

Using the frequency equations (76) and (77) in conjunction with the characteristic polynomial (63), we have computed the frequency parameters ω_n, n=1,2,3, for the lowest three modes in a clamped-clamped beam. These results are shown in Figures 12, 13, and 14. It is evident that the values obtained from the effective modulus and microstructure theories agree only for small values of ζ. For $\zeta>0.02$ the respective curves diverge, and the values of ω_n predicted by the microstructure theory are considerably less than the corresponding values computed from the effective modulus theory. In order to show the effect of initial tensile or compressive axial loads, we have plotted the values of ω_n as functions of ζ for λ = - 150,0,150 for the effective modulus theory and for λ = - 50,0,50 for the microstructure

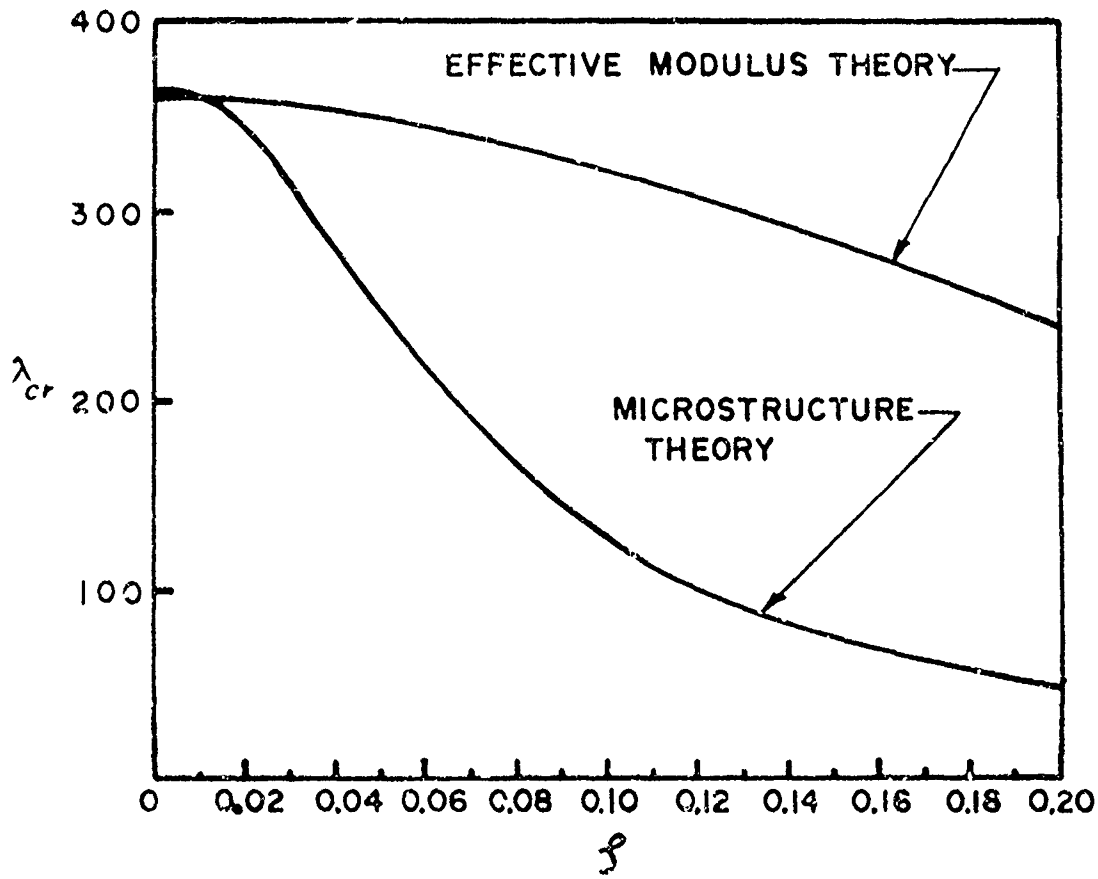

Figure 11. Variation of the critical load coefficient λ_{cr} with the depth-to-length ratio ζ for a clamped-clamped beam.

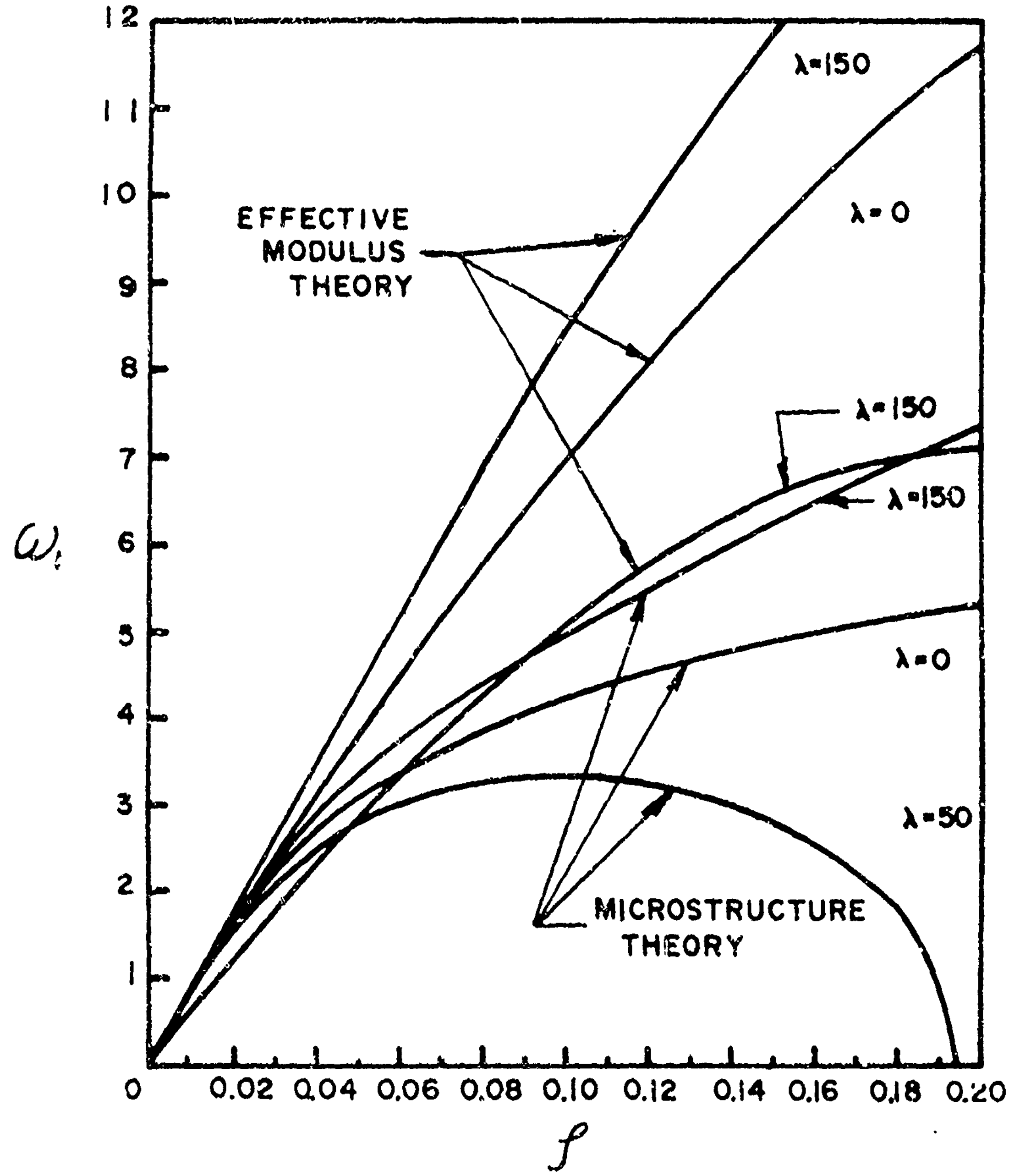

Figure 12. Variation of ω_1 with ζ and λ for a clamped-clamped beam.

theory. Indeed, it should be observed in Figure 12 that for $\lambda = 50$ the value of the frequency parameter ω_1 starts at zero for $\zeta = 0$ and increases to a maximum as ζ increases and then decreases to zero as ζ increases to $\zeta = 0.1945$. Since $\omega_1 = 0$ for $\lambda = 50$ and $\zeta = 0.1945$, this value of ζ corresponds to the value of the critical buckling coefficient for a clamped-clamped beam, the length-to-depth ratio of which is 0.1945 (compare Figure 11). The curves for ω_2 and ω_3 with $\lambda = 50$ show similar characteristics but approach zero at higher values of ζ.

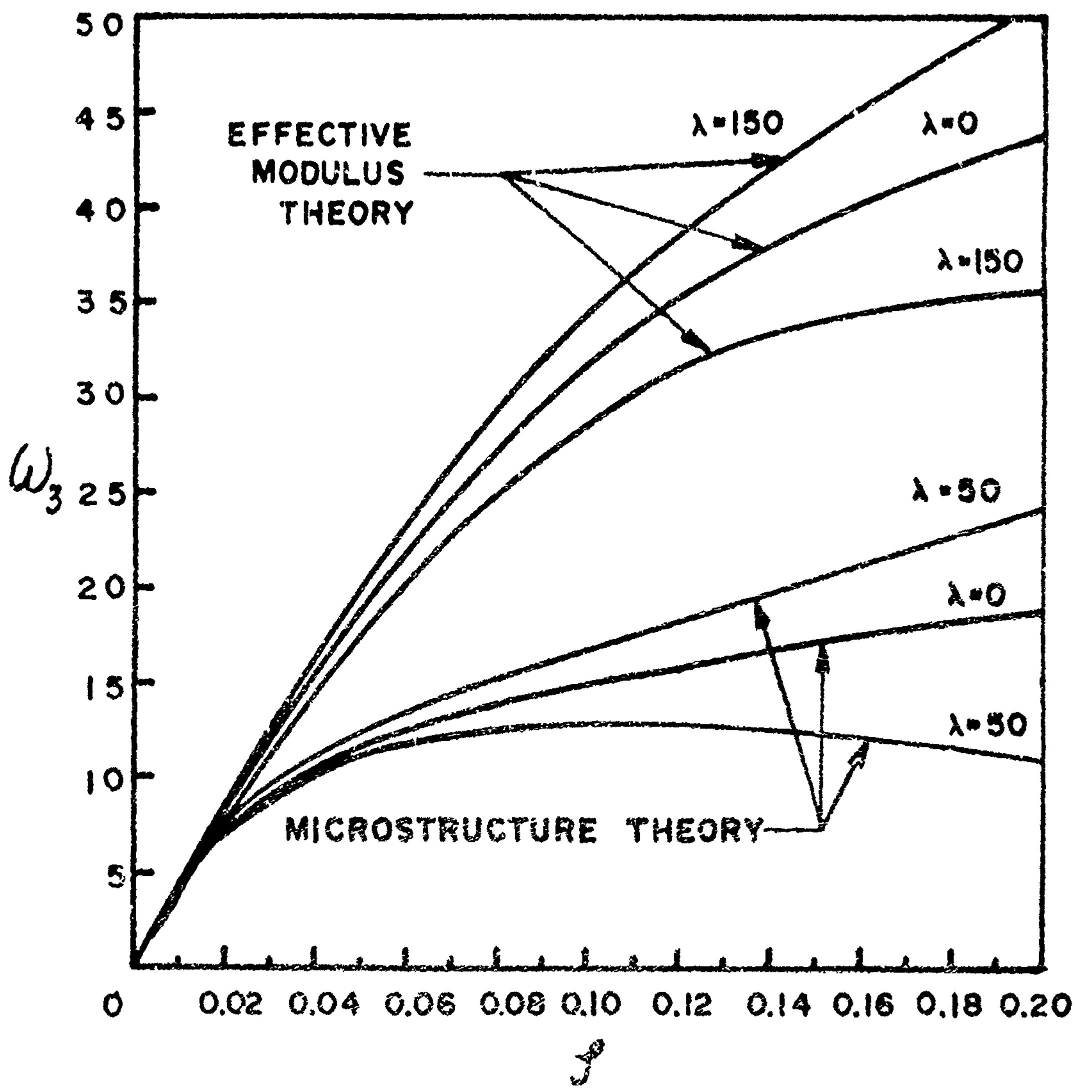

Figure 13. Variation of ω_2 with ζ and λ for a clamped-clamped beam.

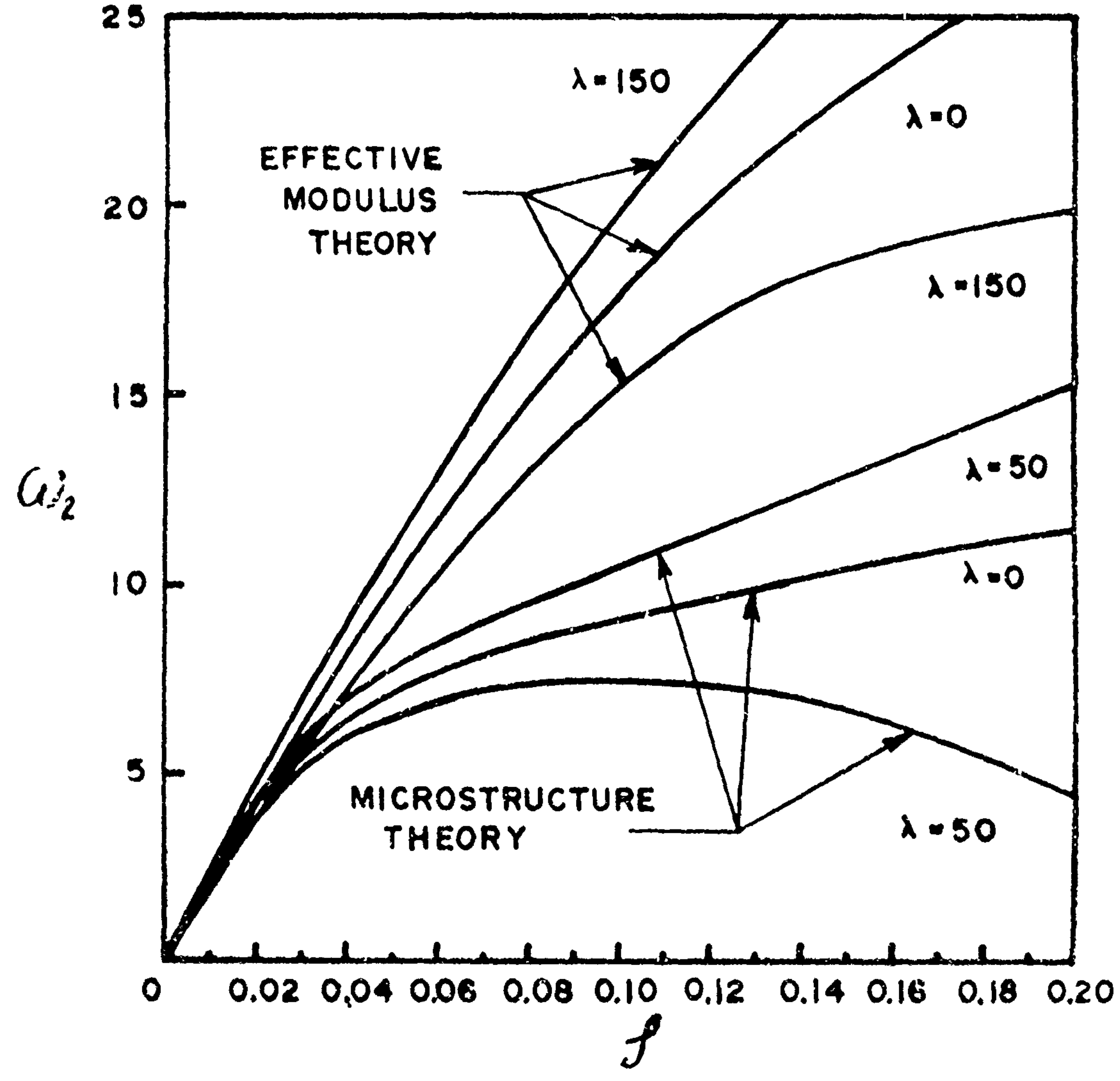

Figure 14. Variation of ω_3 with ζ and λ for a clamped-clamped beam.

CONCLUSIONS

Based upon the numerical results reported herein for the buckling coefficients and natural frequencies of vibration of a laminated beam subject to either hinged-hinged or clamped-clamped end conditions, it is evident that the effective modulus theory, i.e., Timoshenko beam theory for a transversely isotropic bar subject to initial axial stress, appears to be reliable for engineering design purposes only in the limit of fairly long beams. The extended version of the microstructure theory that was developed and investigated in the present study predicts much lower buckling coefficients and natural frequencies in laminated beams with depth-to-length ratios of moderate size, the magnitude of the effect being more pronounced in the case of the clamped-clamped beam. In view of the dispersion curves reported by Sun [1], this was to be expected.

Based upon the effective modulus theory, the results reported in References [6]-[9] for buckling coefficients and natural frequencies indicated much lower values than those predicted by the classical Euler-Bernoulli beam theory, which does not account for the disparity in the longitudinal and transverse elastic moduli. Indeed, the results become more and more divergent as the ratio of the transverse Young's modulus to the longitudinal shear modulus, E/G, increases, the effects being particularly pronounced when E/G exceeds, circa, 50. The numerical results presented in this report indicate still lower values for the buckling coefficients and the natural frequencies, even though the

E/G ratio was only slightly greater than 3. Consequently, the validity of adopting a transversely isotropic beam to model a laminated beam must be seriously questioned and challenged.

It is currently being proposed at the Watervliet Arsenal that an experimental study of the buckling and vibrational response of laminated beams and plates be initiated. Such a study will provide a basis for rendering further judgment on the ranges of applicability of both the microstructure and effective modulus theories.

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APPENDIX

EFFECTIVE MODULUS THEORY

With the aid of (16)-(19), the fundamental differential equations for the effective modulus theory may be derived in a very simple fashion. In view of the studies reported in [6]-[9], it is known that the extensional deflection $u(x_1,t)$ in the beam is uncoupled from the transverse deflection $w(x_1,t)$ and the rotation $\phi(x_1,t)$. Hence, the function $u(x_1,t)$ may be neglected. Thus, we replace (16)-(18) by

$$U = \frac{1}{2}[EI\phi_{,1}^2 + \kappa\mu A(w_{,1} - \phi)^2], \tag{A-1}$$

$$T = \frac{1}{2}\rho[I\dot{\phi}^2 + A\dot{w}^2], \tag{A-2}$$

$$U^* = \frac{1}{2}\sigma_{11}[I\phi_{,1}^2 + Aw_{,1}^2], \tag{A-3}$$

respectively.

Therefore, application of (19) in the familiar manner leads to the following equations of motion:

$$\kappa\mu bh(w_{,11} - \phi_{,1}) - Pw_{,11} - \rho bh\ddot{w} = 0, \tag{A-4}$$

$$(E-P/A)I_b\phi_{,11} + \kappa\mu A(w_{,1} - \phi) - \rho I_b\ddot{\phi} = 0, \tag{A-5}$$

with the boundary conditions

(i) either $\kappa\mu A(w_{,1} - \phi) + Pw_{,1} = 0$ or $\delta w = 0$

(ii) either $(E-P/A)I_b\phi_{,1} = 0$ or $\delta\phi = 0$

$$\tag{A-6}$$

on $x_1 = 0,\ell$. We have set $\sigma_{11} = P/A$.

Setting $x_1 = \ell x$, $\phi = \overline{\phi}/\ell$, and $t = (\rho_2\ell^2/\mu_2)^{1/2}\tau$, we can express (A-4) and (A-5) as follows:

$$(b_1 - \lambda b_2)w''(x,\tau) - b_1\overline{\phi}'(x,\tau) = b_{13}\ddot{w}(x,\tau), \tag{A-7}$$

$$(b^*_{10} - \lambda b^*_{11})\overline{\phi}''(x,\tau) + b_1(w' - \overline{\phi}) = b^*_{16}\ddot{\overline{\phi}}(x,\tau), \tag{A-8}$$

where b_1, b_2, and b_{13} are given in the text and

$$b^*_{10} = \frac{\delta\zeta^2(1+\epsilon d)}{12\kappa(1+d)}, \quad b^*_{11} = \frac{\delta\pi^2\zeta^4}{144\kappa}, \quad b^*_{16} = \frac{\zeta^2}{12\kappa}\ \frac{1+\theta d}{1+d} .$$

For simply-supported ends, the boundary conditions for the beam are $w = \overline{\phi}' = 0$ at $x = 0,1$. If we insert

$$w(x,\tau) = A_1\sin n\pi x \cos\omega\tau, \quad \overline{\phi}(x,\tau) = A_2\cos n\pi x \cos\omega\tau$$

into (A-7) and (A-8), we obtain, after some manipulation, the frequency equation

$$b_{13}b^*_{16}\omega^4 - \{b_1b_{13}+(n\pi)^2\,[b_{13}b^*_{10}+ b_1b^*_{16}- \lambda(b_{13}b^*_{11}+b_2b^*_{16})]\}\omega^2 + $$
$$+ (n\pi)^2[(n\pi)^2(b^*_{10}- \lambda b^*_{11})(b_1- \lambda b_2) - \lambda b_1b_2] = 0 \tag{A-9}$$

The frequencies for the flexural and thickness-shear modes may be obtained from (A-9) with the aid of the quadratic formula. Numerical results calculated in this manner have been plotted in Figures 3-5 and 9.

In the case of buckling, the value of the critical load coefficient is obtained by setting $\omega=0$ and $n=1$ in (A-9):

$$\pi^2(b_{10}^* - \lambda b_{11}^*)(b_1 - \lambda b_2) - \lambda b_1 b_2 = 0.$$

It is easily shown that

$$2\pi^2 b_{11}^* b_2 \lambda_{cr} = \pi^2(b_2 b_{10}^* + b_1 b_{11}^*) + b_1 b_2 - \{[\pi^2(b_2 b_{10}^* + b_1 b_{11}^*) + b_1 b_2]^2 - 4\pi^4 b_1 b_2 b_{10}^* b_{11}^*\}^{1/2}. \tag{A-10}$$

Numerical results derived from (A-10) have been plotted in Figure 2.

Let us consider the buckling and vibration of a clamped-clamped beam separately. In the former we may assume that w and $\bar{\phi}$ are independent of time τ, so that (A-7) and (A-8) reduce to the following pair of ordinary differential equations:

$$(b_1 - \lambda b_2)w''(x) - b_1\bar{\phi}'(x) = 0, \tag{A-11}$$

$$(b_{10}^* - \lambda b_{11}^*)\bar{\phi}''(x) + b_1 [w'(x) - \bar{\phi}(x)] = 0. \tag{A-12}$$

The solutions of (A-11) and (A-12) are easily found to be

$$\bar{\phi}(x) = A_1 \cos\Lambda x + A_2 \sin\Lambda x + A_3, \tag{A-13}$$

$$w(x) = A_4 + A_3 x + (c/\Lambda)A_1 \sin\Lambda x - (c/\Lambda)A_2 \cos\Lambda x, \tag{A-14}$$

where

$$c = b_1/(b_1 - \lambda b_2), \qquad \Lambda^2 = \lambda b_1 b_2/(b_1 - \lambda b_2)(b_{10}^* - \lambda b_{11}^*).$$

The boundary conditions for a clamped-clamped beam are

$$\bar{\phi} = w = 0 \quad \text{at} \quad x = 0,1. \tag{A-15}$$

Substitution of (A-13) and (A-14) into (A-15) leads to a system of homogeneous algebraic equations in the A_i, $i = 1,2,3,4$, which, in the usual fashion, leads to the equation

$$\sin\Lambda = 2(c/\Lambda)\ (1-\cos\Lambda), \tag{A-16}$$

from which we can calculate the value of the critical load coefficient. Using the trigonometric identities

$$\sin\Lambda = 2 \sin(\Lambda/2) \cos(\Lambda/2), \qquad \cos\Lambda = 1 - 2 \sin^2(\Lambda/2),$$

we can verify that (A-16) leads to

$$\sin(\Lambda/2) = 0 \quad \text{and} \quad \tan(\Lambda/2) = \Lambda/2c.$$

The smallest value for λ_{cr} is found to originate from sin $(\Lambda/2) = 0$, i.e., $\Lambda = 2\pi$. Now in view of the definition of Λ^2 above, it is a straightforward matter to demonstrate that λ_{cr} is to be computed from

$$b_2 b_{11}^* \lambda_{cr}^2 - (b_1 b_{11}^* + b_2 b_{10}^* + b_1 b_2/4\pi^2)\lambda_{cr} + b_1 b_{10}^* = 0,$$

Numerical results obtained from this expression have been plotted in Figure 11.

To determine the frequencies of vibration of a clamped-clamped beam, we consider (A-7) and (A-8) subject to (A-15). Inserting

$$w(x,\tau) = w(x) \cos \omega\tau, \qquad \overline{\Phi}(x,\tau) = \phi(x) \cos \omega\tau$$

into (A-7), (A-8), we have

$$(b_1 - \lambda b_2)w''(x) - b_1\phi'(x) = - b_{13}\omega^2 w(x), \tag{A-17}$$

$$(b_{10}^* - \lambda b_{11}^*)\phi''(x) + b_1 [w'(x) - \phi(x)] = - b_{16}^*\omega^2\phi(x). \tag{A-18}$$

If we introduce a new dependent variable $\psi(x)$, according to

$$\phi(x) = \psi'(x),$$

then, without loss of generality, (A-18) can be replaced by

$$(b_{10}^* - \lambda b_{11}^*)\psi''(x) + b_1 [w(x) - \psi(x)] = - b_{16}^*\omega^2\psi(x),$$

whence

$$b_1 w(x) = (b_1 - b_{16}^*\omega^2)\psi(x) - (b_{10}^* - \lambda b_{11})\psi''(x). \tag{A-19}$$

Substitution of (A-19) into (A-17) leads to

$$\psi^{IV}(x) + p_2\psi''(x) - p_o\psi(x) = 0, \tag{A-20}$$

where

$$p_2 = \bar{p}_2/\bar{p}_4 \quad \text{and} \quad p_o = \bar{p}_o/\bar{p}_4,$$

$$\bar{p}_o = \frac{b_1^2 b_{13}}{b_{16}^*}\,\Omega^2(1-\Omega^2), \quad \bar{p}_2 = \frac{b_1 b_{13}}{b_{16}^*}(b_{10}^*-\lambda b_{11}^*)+b_1^2 \quad \Omega^2+\lambda b_1 b_2(1-\Omega^2),$$

$$\bar{p}_4 = (b_1 - \lambda b_2)(b_{10}^* - \lambda b_{11}^*), \qquad \Omega^2 = b_{16}^*\omega^2/b_1.$$

However, it can be shown that (A-20) may be represented as

$$(D^2 + \delta_1^2)(D^2 + \delta_2^2)\psi(x) = 0, \qquad D \quad d/dx, \tag{A-21}$$

where, with $\alpha = 1,2$,

$$2c_4\delta_\alpha^2 = c_1\Omega^2 + c_2(1-\Omega^2) - (-1)^\alpha\,[c_1^2\Omega^4 + 2c_5\Omega^2(1-\Omega^2) + c_2^2\,(1-\Omega^2)^2]^{1/2}, \tag{A-22}$$

with

$$c_1 = \frac{b_1 b_{13}}{b_{16}^*}(b_{10}^*-\lambda b_{11}^*) + b_1^2, \quad c_2 = \lambda b_1 b_2, \quad c_3 = \frac{b_1^2 b_{13}}{b_{16}^*},$$

$$c_4 = (b_1 - \lambda b_2)\,(b_{10}^* - \lambda b_{11}^*),$$

$$c_5 = \lambda b_1^3 b_2 + \frac{b_1^2 b_{13}}{b_{16}^*}(b_{10}^* - \lambda b_{11}^*)(2\,b_1 - \lambda b_2).$$

It is now important to observe that

$$\delta_1^2 > 0 \quad \text{for} \quad \Omega > 0$$

and that

$$\delta_2^2 \begin{cases} < 0 & \text{if} \quad 0 < \Omega < 1 \\ = 0 & \text{if} \quad \Omega = 1 \\ > 0 & \text{if} \quad 1 < \Omega. \end{cases} \tag{A-23}$$

The significance of the sign of δ_2^2 in the two frequency ranges rests in the form of the solution of (A-21). By the principle of superposition in the theory of differential equations, we can replace (A-21) by

$$(D^2 + \delta_1^2)\psi_1(x) = 0, \qquad (D^2 + \delta_2^2)\psi_2(x) = 0, \tag{A-24}$$

with

$$\psi(x) = \psi_1(x) + \psi_2(x). \tag{A-25}$$

It is evident from (A-23) and (A-24) that $\psi_1(x)$ will always be given by

$$\psi_1(x) = A_1 \cos \delta_1 x + B_1 \sin \delta_1 x, \tag{A-26}$$

whereas $\psi_2(x)$ will have the form

$$\psi_2(x) = A_2 \cosh \bar{\delta}_2 x + B_2 \sinh \bar{\delta}_2 x, \qquad 0 < \Omega < 1, \tag{A-27}$$

with $\bar{\delta}_2^2 = - \delta_2^2$, or the form

$$\psi_2(x) = A_2 \cos \delta_2 x + B_2 \sin \delta_2 x, \qquad 1 < \Omega. \tag{A-28}$$

In terms of our present notation, the boundary conditions (A-15) can now be expressed as

$$w = \psi' = 0 \quad \text{at} \quad x = 0,1. \tag{A-29}$$

The second condition in (A-29), in view of (A-25), leads to

$$\psi_1' + \psi_2' = 0 \quad \text{at} \quad x = 0,1. \tag{A-30}$$

By virtue of (A-24) and (A-25), (A-19) becomes

$$w(x) = 1 - \Omega^2 + \frac{\delta_1^2}{b_1}(b_{10}^* - \lambda b_{11}^*) \quad \psi_1(x) +$$

$$+ \quad 1 - \Omega^2 + \frac{\delta_2^2}{b_1}(b_{10}^* - \lambda b_{11}^*) \quad \psi_2(x). \tag{A-31}$$

Thus, the first boundary conditions in (A-29) become

$$F_1\psi_1 + F_2\psi_2 = 0 \quad \text{at} \quad x = 0,1, \tag{A-32}$$

where

$$F_\alpha = 1 - \Omega^2 + (\delta_\alpha^2/b_1)(b_{10}^* - \lambda b_{11}^*), \quad \alpha = 1,2.$$

Consequently, we have established that the boundary conditions for the clamped-clamped beam may be expressed as (A-30) and (A-32).

Case I. $0 < \Omega < 1$. In this case $\delta_2^2 < 0$, so we set $\overline{\delta}_2^2 = -\delta_2^2$. Then

$$\overline{F}_2 = 1 - \Omega^2 - (\overline{\delta}_2^2/b_1)(b_{10}^* - \lambda b_{11}^*).$$

Substitution of (A-26) and (A-27) into (A-30) and (A-32) leads to a system of homogeneous algebraic equations in the coefficients A_α, B_α, $\alpha = 1,2$, from which we can derive the frequency equation

$$2F_1\overline{F}_2\delta_1\overline{\delta}_2(1-\cos\delta_1\cosh\overline{\delta}_2) + (F_1^2\overline{\delta}_2^2 - \overline{F}_2^2\delta_1^2)\sin\delta_1\sinh\overline{\delta}_2 = 0. \tag{A-33}$$

Case II. $1 < \Omega$. In this situation, (A-27) is replaced by (A-28), and the analysis of the previous paragraph is repeated. Because $\overline{\delta}_2^2 = -\delta_2^2$ $(\delta_2^2 > 0)$, we have $\overline{\delta}_2 = is\delta_2$, $i = (-1)^{1/2}$ and $s = \pm 1$. Now

$$F_2 = 1 - \Omega^2 + (\delta_2^2/b_1)(b_{10}^* - \lambda b_{11}^*)$$

and

$$\cosh\overline{\delta}_2 = \cos\delta_2, \qquad \sinh\overline{\delta}_2 = is\sin\delta_2.$$

It is next easy to verify that (A-33) becomes

$$2F_1F_2\delta_1\delta_2(1-\cos\delta_1\cos\delta_2) - (F_1^2\delta_2^2 + F_2^2\delta_1^2)\sin\delta_1\sin\delta_2 = 0, \tag{A-34}$$

for $1 < \Omega$. Upon computing the value of Ω from (A-33) or (A-34), we can determine ω from

$$\omega = \Omega\,(b_1/b_{16}^*)^{1/2}.$$