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MC COWAN'S SOLITARY WAVE EXPANSIONS

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Ballistic Research Laboratories

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by

Serge J. Zaroodny

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In 1894 McCowan proposed that the structure of solitary waves be represented by a certain series expansion; but he gave an estimate for only the first term of this expansion. Any truncation of such an expansion compromises the essential ideal of the steadiness of a solitary wave; also, the approximations which he adopted might not be the best ones for the diverse particular purposes for which this wave may be used. In this paper a more precise structure of this wave is constructed by implementing McCowan's expansion to higher orders. The coefficients in this expansion are determined by minimizing a certain reasonable measure of the imperfection of this expansion. This minimization is done by using the FNMIN subroutine.

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In 1894 McCowan proposed that the structure of solitary waves be represented by a certain series expansion; but he gave an estimate for only the first term of this expansion. Any truncation of such an expansion compromises the essential ideal of the steadiness of a solitary wave; also, the approximations which he adopted might not be the best ones for the diverse particular purposes for which this wave may be used. In this paper a more precise structure of this wave is constructed by implementing McCowan's expansion to higher orders. The coefficients in this expansion are determined by minimizing a certain reasonable measure of the imperfection of this expansion. This minimization is done by using the FNMIN subroutine.

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INTRODUCTION

In the Theory of Water Waves there often arises the concept of a solitary wave, often spoken of as a 'soliton'. It is a single, two-dimensional intumescence that propagates (over a level bottom, and with a series of traditonal simplifying assumptions) supposedly without any change in its profile and velocity; its velocity depends upon its intensity. No exact and concise expression for the structure of such a wave is known. Until recently, all theories of this wave have been in the form of expansions; if these theories may have appeared to be simple, this is so only because for various particular uses of this concept (and specifically, for solitary waves of low intensity) the first terms of such expansions have been considered sufficient. Any truncation of such expansions, of course, possesses shortcomings. Thus, the popular Boussinesq model of solitary waves[1*] compromises (as shown by Munk[2]) the generally presumed, Laplacian, governing equation; while the commonly quoted first-order variant of McCowan's theory compromises the dynamic boundary condition on the pressure at the surface. There have been a number of attempts to reduce such shortcomings by resorting to different expansions[3], or to higher-order terms of such expansions (with the coefficients of such terms determined analytically); but all such attempts drastically complicate these theories. Byat-Smith[4] and Strelkoff[5] have recently proposed a radical resolution of such shortcomings by obtaining the profile of this wave numerically; but this approach involves even more sophisticated and laborious procedures (involving the solution of an integro-differential equation), is not immune to 'numerical' (truncation-and-roundoff) errors, and requires much further work if the interior flow field is desired.

For certain practical purposes such as those of Reference 6, where in the representation of a solitary wave it is desired to select a good compromise between simplicity and precision (or at least to inspect, and to exhibit, the effectiveness of various tradeoffs between these desiderata) the most efficient course appeared to be that of extending

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McCowan's theory to higher orders, with the coefficients of the higher-order terms determined 'empirically' by computations.

GENERAL EXPANSION PROPOSED BY McCOWAN

McCowan originally proposed that the complex stream function of a solitary wave (in a stationary coordinate system, at the instant when the crest of the wave passes the plane of x = 0) be represented as

$$\psi + i\phi = \Sigma\, a_j \tan^j[\frac{m}{2}(z + ix)],\ j = 1, 3, 5... \qquad (1)$$

where x and z, respectively, are the horizontal and vertical distances to a water particle from the point at the bottom under the wave's crest; the real stream function ψ and the real velocity potential ϕ are here defined by the fact that the horizontal and vertical components of the water velocity are

$$u = \partial\phi/\partial x = \partial\psi/\partial z, \qquad v = \partial\phi/\partial z = - \partial\psi/\partial x, \qquad (2)$$

and the real constants a_j and m, as well as the wave's celerity C, remain to be determined. Incidentally, the parameter m can to some extent be specified arbitrarily, since it indicates the intensity of the wave (m is in the nature of a reciprocal of some "effective" length of the wave, and the more intense wave is the 'more concentrated', or the shorter, one). The intensity of the wave, however, may more conveniently be specified by the ratio γ of waveheight to undisturbed depth; thus a_j, m, and C ought to be expressed in terms of γ. It might be noticed that the notation of (1) and (2), particularly well suited to this problem, differs slightly from what would be more customary today.

McCowan, however, at once in effect confined his theory to the first term of this expansion - by using a couple of ingenious tricks for the choice of the coefficients a_1, m and C. The difficulties which he faced in his ground-breaking 1894 proposal were precisely of the type that are readily resolvable with today's computing technology.

REQUIREMENTS UPON THE COEFFICIENTS IN THE EXPANSION

An obvious reason for McCowan's choice of the basic function (the tangent of the complex coordinate in a certain scale) in this expansion is that the first-order term alone already yields a configuration that indeed is very similar to the ideal described by Russell empirically, and had in mind by Boussinesq and Rayleigh. The reason for McCowan's providing - at least in principle - the infinite number of coefficients a_j can be easily seen if we consider the requirements which a solitary wave must satisfy.

The resort to an analytic function of a complex coordinate at once satisfies the governing (Laplace's) equation. The choice of a function (the tangent) such that its real part vanishes for $z = 0$ satisfies the bottom boundary condition. The remaining conditions are best inspected in a coordinate system moving with the wave, in which the complex stream function becomes, say

$$\Psi + i\Phi = (\psi + i\phi) - C(z+ix) \tag{3}$$

so that transfer to this system reduces u by C and leaves v unchanged. In particular, the kinematic boundary condition on a wave's surface (the impenetrability of the free surface) then can be satisfied simply by definition: a streamline can be taken as the profile of the wave. The dynamic boundary condition (the continuity of pressure across the air-water interface) is further simplified because McCowan's theory is one of those which neglect the density, and hence the pressure, of air. This condition then reduces to $p \equiv 0$, and Bernoulli's equation for a water particle at the surface takes the form

$$\Phi_t + Q^2/2 + gZ = \text{constant}$$

$$= C^2/2 + gh \text{ as } x \to \pm\infty$$

where h is the undisturbed depth, Q^2 is $(U - C)^2 + V^2$, and Z, U, V are the surface values of z, u, v. For any instantaneous functions Z(x) and Q(x) this equation merely yields Φ_t; but the solitary wave is distinguished from other waves by still another, the fifth, condition, that of its steadiness: viz., by the fact that its Φ_t must vanish identically everywhere. It is in order to assure such vanishing that we have been given the freedom of choosing m, C and the a_j's.

A CONVENIENT CHOICE OF UNITS FOR ANALYSIS

If h and the unknown C are taken as units of distance and velocity (so that the unit of time is h/C; of acceleration, C^2/h; and of Ψ, Ch), this C drops out of our analysis, but can easily be recovered. In lieu of C there appears another unknown, the acceleration of gravity in these units, viz., the quantity $g^M = g/(C^2/h)$; but this g^M happens to enter the theory in such a manner that it is much more easily determinable than C. Another interpretation of g^M is convenient: writing its definition as

$$R \equiv (g^M)^{-1/2} = C/(gh)^{1/2} \tag{5}$$

and recollecting that $(gh)^{1/2}$ is the celerity of Airy low-and-long (or "shallow-water") waves of small amplitude, (5) is seen to correspond to the definition of Mach number in Compressible Fluid Mechanics - whereby the solitary wave corresponds to a shock wave. The term "Russell number", after the discoverer of this long-unrecognized wave, is suggested for R. It should be noted that in both the 1844 empirical concepts of Scott Russell, and in the 1871 Boussinesq theory it turns out that

$$R \approx (1+\gamma)^{1/2} \tag{6}$$

- a value which our extension of McCowan's theory modifies only very slightly (while in the original tricky version of his first-order theory R appeared to be much smaller).

The quantities m, a_1 and a_j for j = 3, 5... in these units will be denoted by M, N/M, and $A_j N/M$, where N turns out to be a convenient factor for all velocities. If the tangent in (1) is T, the complex stream function appears as $(NT/M)(1+A_3T^2+A_5T^4+\ldots)$. Our problem thus reduces to a determination of such M, N, A_3, A_5... and g^M as would best annihilate Φ_t.

CRITERION FOR THE IMPERFECTION OF A SOLITARY WAVE MODEL

Given a tentative set of parameters M, N, g^M, A_3, A_5... of a solitary wave, we may compute (in a manner that will presently be shown) for every x the quantity

$$D = 2\Phi_t = 1 - Q^2 - 2g^M(Z-1) \tag{7}$$

which we shall consider as a discrepancy, or unsteadiness. Clearly, in order to formulate a criterion that would characterize this tentative set of parameters without specifying x we need merely to compute D for several values of x and do some averaging. The point is that the manner of such averaging is essentially arbitrary. In particular, we find it both informative and practical to consider, and to minimize, the positive-definite quantity

$$F = \int_0^Y D^2 \, dY \tag{8}$$

where $Y \equiv Z - 1$. By viewing D as D(Y), and thus taking a set of points uniformly spaced in Y, we give weight mainly to points in the main body of the wave, and give only small weight to the points in the "far tails"; also, we take advantage of the fact that at Y = 0 (viz., at $x = \pm\infty$) D vanishes. In this connection it should be noted that one of McCowan's 'tricks' was the basically-arbitrary, and unnecessarily stringent, requirement as to the manner in which $D \to 0$ as $x \to \pm \infty$: he required that the curve of D(Y) be tangent to the Y-axis at the origin (at $Y \to 0$). He used this condition to determine the celerity C of the solitary wave. Unfortunately, this method amounted to determining C from the behavior of the "far tails" (which may be of little interest in such a context as that of Reference 6); and to disregarding the relatively large discrepancies D which thereby result in the main body of the wave. If his requirement on the manner of the vanishing of D at $Y \to 0$ is relaxed, much smaller F can be achieved, even when only the first term of (1) is used; and in particular, this reduction of F means that the dynamic boundary condition is more nearly satisfied in the main body of the wave.

COMPUTATION OF THE IMPERFECTION CRITERION

Computation of D for use in (8) requires a determination of x for a prescribed Z, viz., the determination of the wave's profile; for once x(Z) is known, U, V and Q can be gotten by use of (2). In the moving coordinate system the real stream function is $\Psi = \psi - z$; and at the surface, at $x \to \pm \infty$ (where $z \mp Z \to 1$ and $\psi \to 0$) this Ψ becomes - 1. Wave's profile thus is the streamline of $\Psi = -1$, whereby $\psi = \Psi + Z = Z - 1 = Y$. Thus by equating the real part of (1) to Z - 1, viz., by writing

$$\psi\ (x,Z) = Y \qquad (9)$$

where Z and Y are prescribed, we have an equation for determining x(Y). The two end points in the range $0 \leq Y \leq \gamma$ afford simplifications: $x(\gamma)$ is simply 0, while x(0) is of no interest since it yields D = 0. For other values of Y the equation (9) can readily be solved by the Newtonian iterations. In such iterations the first approximation for x(Z) can of course be taken as that obtained with some preceding tentative set of the parameters M, N, g^M, A_3, A_5...; incidentally, V appears as a byproduct of these iterations. With $x(\zeta)$ known, the determination of U by (2), D by (7) and F by (8) is straightforward.

The minimization of F - and hence, the determination of the best set of the wave's parameters for a given γ - is in practice made possible by the existence of the FORTRAN subroutine FNMIN ('function minimization without taking the derivatives'). The value of F computed for a tentative set of parameters is simply returned to that subroutine - which then decides what alternative combination of parameters is to be tried next, and the search continues until the subroutine is satisfied, viz., until all parameters are optimized to a prescribed precision.

STRUCTURE OF THE REFINED SOLITARY WAVE

Results of our minimization of F can be expressed by a statement of the values of parameters M, N/M, A_3, A_5... and R for a series of values of γ. Figure 1 shows the behavior of the function D(Y) for different models of a wave of γ = .5. Table I shows the wave parameters for this γ for a series of models of this wave, and also shows the precision criterion F, and the number, n, of intervals into which the range of Y has been divided in this computation. The expansion (1) is seen to be not of the 'orthogonal' type, in that the passage to the next-higher-order model does modify the coefficients that enter into the preceding model; but there is a modicum, so to say, of ideal orthogonality in that such modification is relatively minor. The inclusion of a jth term in (1) seems to contribute a ripple of (j+1)/2 humps that attempts, so to say, to fill in the imperfections in the profile of the wave that are left by the (j-2) th-order model. Table II shows the coefficients in (1) for the 1st, 3rd, and 5th order models. An inspection of this table[6] suggests that for the larger values of γ a greater number of terms in (1) ought to be considered; in particular, it has shown that for γ = .5 the expansion (1) ought to be extended at least to the 9th order. Table III, in particular, shows that the celerity of the solitary wave is much closer to its classical estimate by (6) than to McCowan's estimate.

The background, the theory, the limitations - and further numerical illustrations - of this extension of McCowan's theory of solitary waves are given in greater detail in Reference 6. In particular, we purposely refrain here from considering the much-discussed question of the estimate of the upper limit of γ for the solitary wave. For the higher γ the crest of this wave sharpens, and the approach along the lines of (1) becomes awkward; but there are other reasons for believing that such a limit is a phantom, for most of the classical assumptions postulated in all theories of solitary waves would cease to be applicable long before such a limit is approached.

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The author is indebted to Professor Michael D. Greenberg of the University of Delaware for the general direction of the work of Reference 6, of which this paper is a brief extract; to Mr. Glenn Beck of USAARDC, Aberdeen Proving Ground, Maryland, the author of the remarkable FNMIN routine; and to many of the latter's colleagues - and in particular, to Mr. Donald Taylor - for the instruction, and patient assistance, in the tricks of FORTRAN and BRLESC.

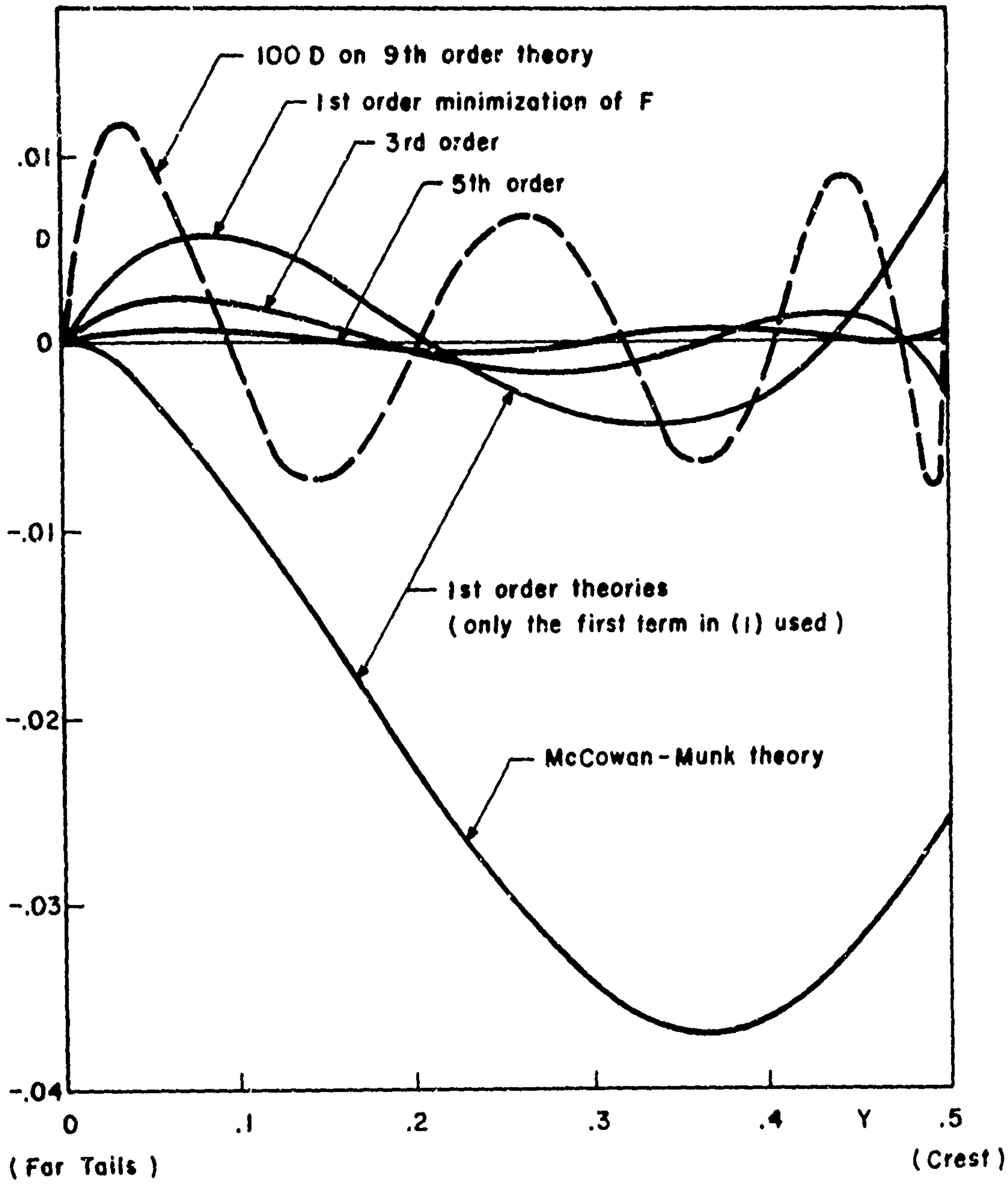

Figure 1. The variation of the discrepancy D, equation (7), for different models of solitary wave of height $\gamma = .5$.

Table 1. Parameters of Solitary Wave of Height $\gamma = .5$

Model	n	10^8F	R^2	M	N/M	A_3	A_5	A_7	A_9
Russell-Boussinesq			1.5						
McCowan-Munk*	10	159700	1.3746	.880	.570				
1**	10	2529	1.479509	.851975	.577811				
3	10	447[&]	1.480288	.894215	.651486	-.051773			
5	10	27	1.478435	.925042	.628340	-.068140	.008133		
7	20[&&]	3.05	1.478324	.940984	.617784	-.076766	.011944	-.002037	
9	20	.35	1.478160	.948329	.612894	-.080581	.013735	-.002885	.000586

The conventional theory, as in Reference 2.

***Using only the 1st order term in (1) with minimization of F.*

&*A misleading alternative ought to be mentioned. If g^M, M and N/M are kept as per 1st-order theory, and only A_3 is allowed to vary, the improvement turns out to be marginal: then $10^8F = 1431$.*

&&*Since for large n the precision criterion F is roughly proportional to n, the speed of the convergence of the expansion (1) would be even more pronounced if F/n were used as criterion.*

Table II. Parameters M, A_3, A_5, N/M of McCowan's Solitary Wave

Model	McC-M		1st Order		3rd Order			5th Order			
γ	M	N/M	M	N/M	M	A_3	N/M	M	A_3	A_5	N/M
.01	.1732	.34	.171467	.019753	.172120	-.003254	.114773	.172131	-.003284	.000026	.114753
.1	.510	.34	.497543	.177257	.513161	-.025921	.345606	.515882	-.028496	.001738	.343772
.2	.675	.32	.645511	.316497	.679097	-.041217	.466955	.689022	-.048028	.004781	.460069
.3	.770	.55	.731914	.426181	.779706	-.049469	.548736	.797934	-.060275	.007147	.535710
.4	.832	.604	.789382	.512050	.846762	-.052608	.608413	.872326	-.066705	.008242	.589647
.5	.880	.640	.831975	.577811	.894215	-.051773	.651486	.925042	-.068140	.008133	.628340
.6	.914	.664	.868274	.625314	.929811	-.047304	.679261	.963713	-.064968	.007239	.653427
.7	.940	.680	.905621	.654041	.958050	-.038060	.690096	.993021	-.056291	.005992	.663201

Table III. $R^2 = 1/g^M = C^2/gh$, the Square of "Russell Number" of Solitary Wave

Wave's Height γ	Russell-Boussinesq	McCowan-Munk	Wave's Model: Minimization of F, equation (8), to the order 1	3	5	7	9
0	1.	1.	1.	1.	1.		
.01	1.01	1.0101	1.0100	1.0100	1.0100		
.1	1.1	1.0968	1.0996	1.0995	1.0995		
.2	1.2	1.1858	1.1983	1.1977	1.1976		
.3	1.3	1.2593	1.2952	1.2945	1.2940		
.4	1.4	1.3195	1.3894	1.3890	1.3880		
.5	1.5	1.3746	1.4795	1.4803	1.4784	1.4783	1.4782
.6	1.6	1.4190	1.5627	1.5659	1.5630		
.7	1.7	1.4566	1.6337	1.6404	1.6361		

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