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INTRODUCTION

In the current literature on stopping of fast heavy particles the stopping power S is given by (see for example reference [1-5])

$$S = \frac{2\pi e_i^2 \cdot e^2 \cdot N}{mv^2} \left[\ln \frac{(2mv_o^2\psi^2)}{I^2} - 2\beta^2 - 2\Delta_1 - 2\Delta_2 \right] \qquad (1)$$

where e_i and v are the charge and velocity of the heavy particle, m the rest mass of the electron, N the total number of electrons per unit volume in the target, $\beta = v/c$ and c the velocity of light, $\gamma^2 = (1-\beta^2)^{-1}$. Δ_1 accounts for the shell effect, i.e. it is a correction term extending the range of validity of equation (1) down to incident particle velocities comparable to that of the electrons in the target, and Δ_2 is a correction term accounting for the polarization in condensed matter. I is the average ionization potential and defined by (see e.g. Eq. 34 in ref 1).

$$\ln I = \sum f_q \cdot \ln I_q \qquad (2)$$

where f_q is the oscillator strength of the excitation energy I_q of the electronic state in the target material.

One of the main difficulties in the theoretical estimates of the stopping power according to Eq. (1) is our inadequate knowledge of the oscillator strength f_q and the corresponding excitation energies I_q. The problem has been solved by Bethe [6-7] and Moller [8] in case of hydrogen.

Bethe [6] then sought to solve the problem for more complex atoms, but computation difficulties have prevented good estimates of the stopping power except in case of simple atoms e.g. helium. Bloch [9-10] sought to solve the problem of complex atoms by use of the Thomas-Fermi Statistical model of the atom and derived that the average ionization potential was given approximately by

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$$I = K_o Z \tag{2a}$$

where K_o is a constant and Z the atomic number. Jensen [11] using the Thomas-Fermi-Dirac model predicted that

$$I = K_o \left(1 + k_o Z^{-2/3}\right) Z \tag{2b}$$

where k_o is another constant

The theoretical estimates of K_o and k_o are rather poor. These constants have therefore been determined experimentally. Following Fermi's [12-13] explanation of the effect of the polarization of ponderable matter by the field of the incident particle, the interest in determining I from the oscillator strengths and the excitation energies was renewed making use of these quantities. Sternheimer [14-18] in particular has used this approach and given procedures to determine the average ionization potential from our knowledge of oscillator strengths and excitation energies. For the excitation energies he used the eigen values for the energy or the measured ionization potentials. Sternheimer found it necessary to multiply the calculated logarithmic average of the ionization potentials of the inner states by a factor ρ to make the calculation agree with the experiment. This factor ρ was found by Sternheimer to be of the order of 1.2 to 2. The theoretical determination of I was thus rather inaccurate.

A different approach for determining the average ionization potential is outlined by Lindhard and Scharff [19] who, guided by the theory for stopping power in electronic plasma, give the following equation for the average ionization potential I

$$I = \frac{1}{2} \int d^3r \cdot \rho(r) \ln \left[\chi \left(4\pi e^2 \rho(r)/m\right)^{1/2}\right] \tag{2c}$$

where $\rho(r)$ is the electronic density function in the atom in matter, the integration is performed over the volume of the atom. The quantity χ is a correction constant for estimating the effect of the binding of the electrons to the nucleus as compared with the inter-electronic forces determining the plasma frequency. They estimate that in the statistical model of the atom the factor χ is of the order of $\sqrt{2}$, while in light atoms χ is close to one. This theory appears to be in fair agreement with experiment (see e.g. Table 2 of ref. [19]); However a more exact determination requires better estimates of χ which are rather difficult to obtain.

Lindhard and Scharff [19], Lindhard and Winther [20] and Bonderup [4] also give an estimate of the shell correction Δ_1 which, for velocity v of the incident particle a few times greater than the velocity u of the electrons in the matter, is given by

$$\Delta_1 = \frac{< u^2 >}{v^2} \tag{3}$$

where $< u^2 >$ is the average velocity of the electrons under consideration in the matter [i.e. in case of Lindhard and Winthers article $< u^2 >$ is proportional to the kinetic energy of the electrons in the plasma].

Shell corrections have also been calculated by Bethe and Livingstone [21], Brown [22] and Walske [23-24], Bethe et al. [25] and by Fano [1].

The polarization effect Δ_2 according to Sternheimer, Eq. (1) of reference [14], is given by

$$\Delta_2 = \sum f_q \ln \frac{(\hbar\omega_{q_1})^2+a^2}{(\hbar\omega_{q_1})^2} - \frac{(1-\beta^2)\cdot a^2 m}{4\pi e^2 N} \tag{4}$$

where f_q is the oscillator strength of an oscillator with eigenfrequency (cyclic frequency) ω_q and where

$$\omega_{q_1}^2 = \omega_q^2 - \frac{1}{3} \cdot \frac{4\pi e^2 f_q \cdot N}{m} \tag{4a}$$

and a^2 is given by

$$\frac{1}{\beta^2} - 1 = \sum \frac{4\pi e^2 f_q N/m}{(\hbar\omega_{q_1})^2+a^2} \tag{4b}$$

THEORY

The close connection between the classical treatment and the quantum mechanical treatment is often obscure in the current literature. The following treatment to a large extent unifies the formulations by Bohr, Bethe for individual oscillators as well as Fermi's formulation of the interaction phenomena in ponderable matter containing single type of oscillators and its extension to matter containing a multiplicity of oscillator types.

Individual Oscillator. We may Fourier analyze the field of the incident particle and calculate the energy absorbed by an oscillator with eigenfrequency ω_q and located at a distance p from the assymptote of the hyperbolic track of the incident particle. p is the usual impact parameter in Bohr's formulation. The Fourier analyzed field may be written in the form

$$F(\omega,p) = \frac{e_i}{\pi v p} \cdot \left[\frac{p\omega}{\gamma v} \cdot K_1 \left(\frac{p\omega}{\gamma v} \right) + i \cdot \frac{1}{\gamma} \cdot \frac{p\omega}{\gamma v} \cdot K_0 \left(\frac{p\omega}{\gamma v} \right) \right] \quad (5)$$

where e_i, v are the charge and the velocity of the incident particle, p the impact parameter, ω the cyclic frequency of the Fourier harmonic, $K_0(x)$ and $K_1(x)$ are the modified Bessel functions, $\gamma = (1-v^2/c^2)^{-1/2}$ and $i = \sqrt{-1}$.

We may then let this field act on an oscillator q and the dynamical equation is then

$$m_q \ddot{r}_q + \alpha_q m_q \dot{r}_q + m_q \omega_q^2 r_q = e_q \cdot F(\omega,p)\, e^{i\omega t} \quad (6)$$

where m_q, is the mass associated with the oscillation, e_q the charge following that mass, r_q the complex distance vector of the charge from the "equilibrium" position and $\dot{r}_q$ and $\ddot{r}_q$ the first and the second time derivatives of r_q : α_q is the damping constant. We assume that $r_q \ll c$ so that we may disregard the magnetic field forces. We also assume that the oscillations are around fixed position.

The energy S_q absorbed by the oscillator may be derived by multiplying the right side of Eq. (6) by $\frac{1}{2} \dot{r}_q(\omega)$ and adding to that result its complex conjugate. An integration over t, ω', ω and p may be performed exactly and we get:

$$S_q = \frac{2\pi e_i^2 e_q^2 N_q}{m_q v^2} \left[(zK_1K_0 + z^*K_1^*K_0) - \frac{v^2}{c^2} z\, z^* (K_1K_1^* - K_0K_0^*) \right]_{z_2}^{z_1} \quad (7)$$

where N_q is the number of oscillators per cm^3. $K_0 = K_0(z)$ and $K_1 = K_1(z)$ are the modified Bessel functions of the complex variable z and where z, K_0 and K_1 are the complex-conjugates of z, K_0 and K_1. z is given by

$$z = \frac{\bar{\omega}_q \cdot p}{\gamma v} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

and $\bar{\omega}_q = \sqrt{\omega_q^2 - \alpha_q^2/4} + i\alpha_q/2$

and where ω_q and α_q are the eigenfrequency and the damping constant respectively of the oscillator. We have also that

$$z_1 = \frac{\bar{\omega}_q \cdot p_1}{\gamma v} \quad \text{and} \quad z_2 = \frac{\bar{\omega}_q \cdot p_2}{\gamma v}$$

In the approximation of small arguments z_1 and for $p_2 \to \infty$ we have that Eq. (7) is equivalent to

$$S_q = \frac{2 e_i^2 e_q^2 N_q}{m_q v^2} \left[\ln \frac{k^2 \cdot \gamma^2 v^2}{\omega_q^2 p_1^2} - \frac{v^2}{c^2} \right] \tag{7a}$$

and where $k = 1.122919 \simeq 1.123$

This equation is identical to that deduced by N. Bohr [26] in a different manner. We have assumed in the derivation of Eq. (5) that the incident particle's track was a straight line. Bohr was able to correct for this and showed that classically we have

$$p = \frac{|e_i \cdot e_q| (m_q + M)}{m_q \cdot M v^2} \tag{8}$$

which is valid in non relativistic limit.

On the other hand, if we introduce in Eq. (7) the quantum mechanical limitation that the angular momentum changes such that $p \cdot \Delta(m\gamma v) = \hbar$, where $\Delta(m\gamma v)$ is the change in momentum of the incident particle, then we may derive Bethe's equation in the usual form, which assumes that the average energy transfer ΔE is uniquely defined by $\Delta E = [\Delta(m\gamma v)]^2 / 2m$.

We have then

$$p_1 = \frac{\hbar}{\sqrt{2 m_q T_{max}}} \quad \text{or} \quad z_1 = \frac{\hbar \omega_q}{2 m \gamma^2 v^2} \tag{9}$$

$$p_2 = \frac{\hbar}{\sqrt{2 m_q T_{min}}} \quad \text{or} \quad z_2 = \frac{\hbar \omega_q}{E_q} \tag{10}$$

where T_{max} is the maximum energy transferred. For heavy incident particles we have

$$T_{max} = 2 m_q \gamma^2 v^2 \tag{11}$$

The minimum energy T_{min} transferred is

$$T_{min} = \frac{(E_m - E_o)^2}{2 m_q \gamma^2 v^2} = \frac{E_q^2}{2 m_q \gamma^2 v^2} \tag{12}$$

If we impose the quantum mechanical limitations given by Eqs. (9), (10), (11) and (12) on the impact parameter in Eq. (7) we get

$$S_q = \frac{2\pi e_i^2 e^2 N_q}{m_q v^2} \left[\ln \frac{(2m\gamma^2 v^2)}{\hbar^2 \omega_q^2 \cdot \delta_q^2} - \frac{v^2}{c^2} \cdot b_q \right] \qquad (7b)$$

where

$$\delta_q = \frac{\exp(z_2 \cdot K_0(z_2) \cdot K_1(z_2))}{k}$$

$$b = 1 - (|K_1(z_2)|^2 - |K_0(z_2)|^2)$$

and $z_2 = \hbar \bar{\omega}_q / E_q$

For $E_q = \hbar\omega_q$ we have $\delta_q = 1.1474$ and $b_q = 0.815$. For hydrogen Bethe found an average δ_q to be 1.103. Eq. (7a) was deduced by disregarding the magnetic field, and is valid for spinless particles. Bethe [7] showed that for protons with spin 1/2 an additional term $-v^2/c^2$ must be added inside the brackets of Eq. (7a). For the interaction of incident electrons we must also make adjustment for spin, exchange effects and Paulis exclusions principle as shown by Moller [8].

The quantum mechanical probabilities for transition from state n to state m may be derived using similar methods and by replacing T_{max} by E_{mn} and E_q by E_{mn}, where E_{mn} is the energy difference between the two states.

We have assumed in the formulation above that the incident particles velocity is large compared to velocity of the atomic electrons. When this assumption is not valid an adjustment must be made. This adjustment is the shell correction. In Bethe's formulation this correction results from the fact that the momentum of the oscillating electron or its energy after the collision is not uniquely defined by the momentum transfer. In the parametric formulation above, the shell correction arises because the single parameter p does not define well the Fourier harmonics at the place of the electrons, since during the collision the atomic oscillators are moving on their own. On similar grounds we may argue that the largest impact parameter corresponding to $z_2 = \hbar\omega_q / E_q$ is also not well defined, and the value $z_2 = \hbar\omega_q / \hbar\omega_q = 1$, an upper limit usually used by Bohr, may be a better average upper limit corresponding to the average oscillator frequency ω_q.

We may compare the use of these two z_2 values with experiments and more detailed calculations. In the hydrogen molecule the binding energy for the two atoms is approximately 4.52 eV and the binding energy of the two electrons then $2 \cdot 13.6 + 4.52 = 2 \cdot$

15.86 or 15.86 eV per electron which is close to the ionization threshold 15.6. We have then that $E_q \simeq \hbar\omega_q$ and therefore the average excitation potential $I_{av} = 13.86 \cdot {}^{q}1.147 = 18.2$. The preferred experimental value [5] is $I_{av} = 18.7$ corresponding to $\delta_q = 1.179$. In helium the average rotational frequency is approximately given by $\hbar\omega_q = 13.6\ (2 - 5/16)^2 = 38.73$ eV while the ionization potential is $E_q = 24.46$. We then have $\hbar\omega_q/E_q = 1.583$ and $\delta_q = 0.96$ and $I_{av} = 38.73 \cdot 0.96 = 37.18$ while $\delta_q = {}^{q}1.1474$ result in ${}^{q}I_{av}$ $38.73 \cdot 1.1474 = 44.45$. The preferred experimental value for helium is 42 eV [5] corresponding to $\delta_q = 1.084$. This adjustment of δ_q however does not depend (or only very slightly depends) on the energy of the incident particle. For sparse matter, we will therefore consider the adjusted δ_q as constant for each electron and independent of the velocity of the incident electron.

PONDERABLE MATTER WITH MULTIPLICITY OF OSCILLATOR TYPES.

Interaction of incident charged particles with ponderable matter containing a multiplicity of oscillators is of greatest importance. While the theoretical results derived above for sparse matter are in general agreement with current theory, the following results for stopping in ponderable matter deviate considerably from the current theory.

Using procedures similar to those used in deriving Eq. (7a), we find that the energy absorption by the oscillator q is given by

$$S_q = \frac{2\pi e^2 e^1 N}{m_q \cdot v^2}\, {}^{i}_{q}\, {}^{q}_{q}\, C_q \left[(z \cdot K_1 \cdot K_0 + z^* K_1^* K_0)\right]_{z_2}^{z_1} \qquad (15)$$

where $K = K(z)$ and $K = K(z)$ are the modified Bessel functions and where

$$z_1 = \frac{\hbar\bar{\omega}_{q2}}{2m\gamma v^2} \qquad (16)$$

$$z_2 = \frac{\hbar\omega_{q2} \cdot \gamma}{\hbar\bar{\omega}_{q2}} = \gamma \qquad (17)$$

in Eq. (17), as argued before, we have replaced E_q in Eq. (10) by $\hbar\omega_q$. In ponderable matter, however, we get $z_2 = {}^{q}\gamma$ instead of $z_2 = 1$ in sparse matter.

This simple form of Eq. (15) results from the fact that at the resonance frequency we have:

$$\Gamma_q = 1 - \frac{v^2}{c^2} \cdot \epsilon_{q_2} = 1 \qquad (18)$$

where ϵ_{q_2} is the dielectric constant at the resonance frequency $\bar{\omega}_{q_2}$.

The dielectric constant is given by

$$\varepsilon = 1 + \frac{\sum \frac{A_p}{\omega_p^2-\omega^2+i\alpha_p\omega}}{1 - \sum \frac{b_p \cdot A_p}{\omega_p^2-\omega^2+i\alpha_p\omega}} \tag{19}$$

where $A_p = 4\pi N_p \cdot e_p^2/m_p$

and the summations are made over all the p oscillators in the medium. N_p is the number of oscillators of type p per cm^3. The shielding constants b_p, in the derivation of Eq. (15), are assumed to be real and we equate them to 1/3 for all inner electrons and for the outer electrons in amorphic matter and matter with cubic lattices, while $b_p = 0$ for the conduction electrons.
The resonance frequency $\bar{\omega}_q$ for energy absorption is given by

$$\bar{\omega}_{q_2} = \sqrt{\omega_{q_2}^2 - \alpha_{q_2}^2/4} + i\alpha_{q_2}/2 \tag{20}$$

where

$$\omega_{q_2}^2 = \omega_q^2 + \frac{(1-b_q)\cdot A_q}{1 + \sum \frac{(1-b_\ell)\, A_\ell}{\omega_\ell^2 - \omega_{q_2}^2}} \tag{21}$$

and

$$\alpha_{q_2} = \alpha_q + \frac{(1-b_q)A_q \sum \frac{(1-b_\ell)A_\ell(\alpha_\ell-\alpha_{q_2})}{(\omega_\ell^2 - \omega_{q_2}^2)^2}}{[1 + \sum \frac{(1-b_\ell)A_\ell}{(\omega_\ell^2 - \omega_{q_2}^2)}]^2} \tag{22}$$

The value of C_q in Eq. (15) is

$$C_q = \frac{\alpha_q(\omega_{q_2})}{\alpha_{q_2}(\omega_{q_2})} \frac{[1 + \sum \frac{(b_q-b_\ell)A_\ell}{\omega_\ell^2 - \omega_{q_2}^2 - i\alpha_{q_2}\omega_{q_2}}][1 + \sum \frac{(b_q-b_\ell)\, A_\ell}{\omega_\ell^2 - \omega_{q_2}^2 + i\alpha_{q_2}\omega_{q_2}}]}{[1 + \sum \frac{(1-b_\ell)A_\ell}{\omega_\ell^2 - \omega_{q_2}^2 - i\alpha_{q_2}\omega_{q_2}}][1 + \sum \frac{(1-b_\ell)\, A_\ell}{\omega_\ell^2 - \omega_{q_2}^2 + i\alpha_{q_2}\omega_{q_2}}]} \tag{23}$$

The summations in Eqs. (21) (22) and (23) are made over all the oscillators $\ell \neq q$.

For fast incident particles we may use the logarthimic approximations and we derive

$$S_q = \frac{2\pi e_i^2 e_q^2 N_q}{m_q v^2} \cdot C_q \cdot \ln \frac{(2m\gamma v^2)^2}{\hbar^2 \omega_{q_2}^2 \delta_{q_2}^2} \tag{24}$$

where

$$\delta_{q_2} = \frac{\exp(z_2 \cdot K_1(z_2) \cdot K_0(z_2))}{k} \tag{25}$$

and $k = 1.123$. The total energy absorption by all of the oscillators is derived by summation of expressions of the form of Eq. (15) or its approximation, Eq. (24).

There are actually more absorptions frequencies, i.e. mathematical singularities or poles, besides ω_{q_2}. These frequencies are close to ω_{q_2}, but the correspondent C_q are so small that we have found it reasonable to disregard them in ordinary matter.

The Čerenkov Radiation

The Čerenkov radiation consists of those Fourier components of the incident particles field that are too slow to keep up with the particle. The energy contained in this field may be integrated and we get

$$S_{ce} = \frac{e_i^2 \cdot \sum A_p}{2v^2} \left[\frac{A_p \ln \frac{\omega_{p_2}^2}{\omega_{p_2}^2 + a^2}}{\sum A_p} - \ln(1-\beta^2) - \beta^2 + \frac{(1-\beta^2)a^2}{\sum A_p} \right] \tag{26}$$

where

$$A_p = 4\pi e_p^2 N_p / m_p \tag{27}$$

and where a^2 is determined by

$$\frac{1+\beta^2}{\beta^2} = \frac{\sum \frac{A_p (\omega_p^2 + a^2)}{(\omega_p^2 + a^2)^2 - \alpha_p^2 a^2}}{1 - \sum \frac{b_p A_p (\omega_p^2 + a^2)}{(\omega_p^2 + a^2)^2 - \alpha_p^2 a^2}} \tag{28}$$

where $\beta^2 = v^2/c^2$

The summation in Eqs. (27) and (28) are made over all the p oscillators. The first and the last term in the bracket of Eq. (26) correspond to the polarization terms given by Eq. (4) according to reference (14). The difference is due partly to the more accurate present treatment of Lorentz' correction of the dielectric constant and partly to a slight error in reference [14]. (Eq. (52) of this reference has the correct value 1, which corresponds to our definition of $\omega_{p_{2_\ell}}$ in our Eq. (22), while Eq. (1) of reference [14] uses $\bar{\nu}_j = {}^{p_2}{}_{\ell_j} - f_j$, which we believe is incorrect.) The difference between Eq. (28) and (4b) is due to better treatment of Lorentz' correction, in the present calculations.

The Total Energy Loss

The total energy loss consists of the energy absorptions by the different oscillators and the Čerenkov radiation emitted by the incident particle. We derive

$$S = \frac{2\pi e_1^2 e^2 N}{mv^2}\left[\ln \frac{(2m\gamma^2v^2)^2}{I_2^2} - (2)\,\beta^2 - 2\Delta_1 - 2\Delta_2 - 2\Delta_3 - 2\Delta_4\right] \tag{29}$$

where

$$\ln I_2^2 = \sum \frac{f_q}{N} \ln \hbar^2 \omega_{q_2}^2 \tag{30}$$

Δ_1 = the usual shell correction

$$\Delta_2 = \frac{1}{2}\left[\sum \frac{f_q}{N} \ln \frac{\omega_{q_2}^2 + a^2}{\omega_{q_2}^2} - \frac{(1-\beta^2)a^2 m}{4\pi e^2 N}\right] \tag{31}$$

$$\Delta_3 = \frac{1}{2}\sum \frac{f_q(1-C_q)}{N} \ln \frac{(2m\gamma v^2)^2}{\hbar^2 \omega_{q_2}^2 \delta_{q_2}^2} \tag{32}$$

$$\Delta_4 = z_2 \cdot K_1(z_2) \cdot K_0(z_2) - \ln k \tag{33}$$

and $z_2 = \gamma$ and $k = 1.123$ (34)

$$\omega_{q_2}^2 = \omega_q^2 + \frac{(1-b_q)\,A_q}{1 + \sum_{\ell \neq q} \frac{(1-b_\ell)A_\ell}{\omega_\ell^2 - \omega_{q_2}^2}} \tag{35}$$

$A_q = 4\pi e^2 f_q/m$ and f_q is the effective number of electrons. f_q

may differ from N_q, the actual number of electrons q, because of transfer of oscillator strength. b_q and b_ℓ are equal to 1/3 for the inner electrons and for the valance electrons in amorphic dielectrics and matter forming cubic lattices but $b_q = 0$ for the conduction electrons. Further we replace Eq. (28) by the approximation:

$$\frac{1-\beta^2}{\beta^2} = \frac{\sum \frac{A_p}{\omega_p^2+a^2}}{1 - \sum \frac{b_p A_p}{\omega_p^2+a^2}} \tag{36}$$

The summation is made over all p oscillators. C_q is given by Eqs. (22) and (23). But according to Eq. (22) we expect $\alpha_{q_2} \approx \alpha_q$ and we will approximate C_q by

$$C_q = \frac{[1 + \sum \frac{(b_q-b_\ell)A_\ell}{\omega_\ell^2-\omega_{q_2}^2}]^2}{[1 + \sum \frac{(1-b_\ell)A_\ell}{\omega_\ell^2-\omega_{q_2}^2}]^2} \tag{37}$$

The Energy Loss in Aluminum

For theoretical calculations of the stopping power in aluminum we have used Eqs. (29) - (37).

We have used the tabulation provided by Herman and Skilman [27] to derive the field potentials and then used these to calculate the "rotational frequencies" ω_q. To improve the accuracy we have used the measured ionization potentials, E_q, to calculate the field potentials from the tables rather than the theoretical ones calculated by Herman and Skillman. In Table I we list the measured ionization potential E_q from reference [28], the calculated rotational frequencies $\hbar\omega_q$ in units of eV, and the plasma related frequencies $\hbar^2 A_q$ in $(eV)^2$ where A_q is given by Eq. (27). We show also the frequencies $\hbar\omega_{q_2}$ in eV where $\omega_{q_2}^2$ is given by Eq. (21), and in the last column we show the value of $\omega_{q_2}^2(1-C_q)$ where C_q is

Table I

q	E_q	$\hbar\omega_q$	h^2A_q	$\hbar\omega_{q_2}$	$1 - C_q$	
$1S^2$	1559.6	1895	166.153	1895	-3.41	10^{-4}
$2S^2$	117.7	210	166.153	210.7^{26}	-3.29	10^{-2}
$2P^6$	73.1	131	498.46	132.3^{36}	-1.07	10^{-2}
$3S^2p^1$	(4.08)	---	249.23	15.61	+6.41	10^{-2}

calculated according to Eq. (37) The last row of Table I is calculated assuming that there are 3 conduction electrons in aluminum. We may then use Table I and Eq. (30) to calculate the average ionization potential, and we get $I_2 = 130.65$.

Table II

Proton Energy in MeV	$\Delta_1 \cdot 100$	$\Delta_2 \cdot 100$	$\Delta_3 \cdot 100$	$\Delta_4 \cdot 100$	lnI' theory	lnI' exp
1	23.4	0.00	4.67	13.74	5.290	
2	20.0	0.01	4.99	13.64	5.259	
3	16.6	0.01	5.18	13.59	5.226	5.202[a]
4	14.0	0.01	5.32	13.55	5.201	5.186[a]
5	12.4	0.02	5.43	13.49	5.185	5.172[a]
6	11.2	0.02	5.51	13.44	5.174	5.161[a]
8	9.14	0.03	5.65	13.34	5.154	5.147[a]
10	7.74	0.03	5.75	13.23	5.140	5.140[a]
12	6.74	0.04	5.84	13.13	5.130	5.135[a]
15	(5.42)	0.05	5.94	12.98	5.116	
20	(4.09)	0.08	6.07	12.72	5.102	(5.100)[b]
30	(2.77)	0.13	6.27	12.20	5.086	(5.081)[b]
100	(0.92)	0.71	6.82	8.59	5.042	
300	(0.40)	3.77	7.30	+ 1.39	5.001	(4.997)[c]
1000	(0.17)	18.80	7.79	- 8.70	5.053	
3000	(0.17)	59.31	8.22	-11.43	5.434	
10000	0.17	177.5	8.73	-11.59	6.621	

a. See reference [29]
b. See reference [30] "Average" I'_{exp} at 35-50 MeV and 50-75 MeV respectively
c. See reference [31] "Average" $\mathbb{C}'_{exp}$ at 338 MeV

In Table II we list the values Δ_1, Δ_2, Δ_3, and Δ_4 as functions of the proton energy in MeV. Δ_1 is from the calculations of Bonderup [4] for $I_{av} = 158$. We extrapolated his calculations beyond 12

MeV by using the equation

$$\Delta_1 = \frac{\text{constant}}{v^2}$$

where v is the velocity of the incident particle. To calculate Δ_2 from Eq. (31) we determine first a^2 by successive approximations inserting the values from Table I in Eq. (36). Δ_3 is calculated by inserting the values from Table I in Eq. (32). Δ_4 is determined from Eqs. (33) and (34). In the last two columns we list the value of $\ln I'_{th}$ and $\ln I'_{exp}$ where

$$\ln I'_{th} = \ln I_2 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$$

The corresponding experimental value $\ln I'_{exp}$ as determined by different authors, is listed in the last column. The last three experimental values in Table II are in parentheses because they are derived from the experiments in accordance with the old theory and should for exact comparison be reevaluated in light of the present theory. However, because the range is mainly determined by the high energy portion of the incident particles track, the comparison is still fairly good. It is interesting to note that the experimental results by Mather and Segrè [50] are now in good agreement with the theory and with the many other experiments. The experimental results by Vasilevskii et al. [32] appear also to be in general agreement with the above theory. It is of interest to note that the shell corrections which are based on the work of Lindhard and Winther [20] and Bonderup [4], at least for aluminum, agree well with the present theory and experimental results.

References

[1] Fano, U.; Ann. Rev. Nucl. Sci. 13, 1 (1964)
[2] Bichel, H.; 2nd ed. McGraw-Hill, New York, sec. 8c American Institute of Physics Handbook (1963)
[3] Brandt, W.; Phys. Rev. 104, 691, (1956)
[4] Bonderup, E.; Kgl. Danske Videnskab Selskab, Mat.-Fys.-Medd. 35, No. 17 (1967)
[5] Barkas, W.H. and Berger, M.J.: Natl. Acad. Sci. Natl. Res. Council Publ. 1133 (1964)
[6] Bethe, H., A.: Ann. d. Phys. (5), 325 (1930).
[7] Bethe, H.A.: Ch. 3 of Handbuch der Physik, Vol 24, 1st part (Ed. A. Smekal, Julius Springer Verlag, Berlin, 1933) see in particular pp. 491-523.
[8] Møller, C. Ann. d. Phys. 14, 531 (1932)
[9] Bloch, F.; Ann. d. Phys., 16, 285 (1933)
[10] Bloch, F.; Zs. f. Phys., 81, 363 (1933)
[11] Jensen, H.; Zs. f. Phys., 106, 620, (1937)
[12] Fermi, E.; Phys. Rev., 56, 1242, (1939)
[13] Fermi, E.; Phys. Rev., 57, 485 (1940)
[14] Sternheimer, R.M.: Phys. Rev., 88, 851 (1952)
[15] Sternheimer, R.M.: Phys. Rev. 89, 1148 (1953) ibid, 91, 256 (1953); and errata ibid 93, 1434 (1954)
[16] Sternheimer, R.M.: Phys. Rev., 103, 511, (1956)
[17] Sternheimer, R.M.: Phys. Rev., 145, 247 (1966)
[18] Sternheimer, R.M.: Phys. Rev.; 164, 349. (1967)
[19] Lindhard, J.; and Scharff, J.; Kgl. Danske Videnskab. Selskab Mat. - Fys. Medd., 27, No. 15 (1953)
[20] Lindhard, J. and Winther, A.; Kgl. Danske Videnskab. Selskab Mat. - Fys. Medd. 34, No. 4 (1964)
[21] Livingston, M.S.: Bethe, H.A: Rev. Mod. Phys., 9, 245 (1937)
[22] Brown, L.M. Phys. Rev. 79, 297 (1950)
[23] Walske, M.C.: Phys. Rev. 88, 1283 (1952)
[24] Walske, M.C.: Phys. Rev., 101, 940 (1956)
[25] Bethe, H.A., Brown, L.M.and Walske M.C. Phys. Rev. 79, 413 (1950)
[26] Bohr, N.; Phil. Mag. 26, 1 (1913) ibid 30, 581 (1915)
[27] Herman, F. and Skillman, S.;
[28] Bearden, J.A. and Burr, A.F. Revs. Mod. Phys., 39, 125 (1967)
[29] Andersen, H.H., Garfinkel, Hanke, C.C. and Sorensen H.; Kg. Danske Videnskab. Selskab Mat. - Fys. Medd. 35, No. 4 (1966)
[30] Bloemberger and van Heerden, P.J.: Phys. Rev. 83, 561 (1951)
[31] Mather, R. and Segrè, E. Phys. Rev. 34, 191 (1951)
[32] Vasilevskii I.M., and Prokoshkin; Sov. J. Nuc. Phys. 4, 390, 1967

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