ASYMPTOTIC EVALUATION OF THE PROBABILITIES OF MISCLASSIFICATION BY LINEAR DISCRIMINANT FUNCTIONS

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T. W. ANDERSON

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1. Introduction

The problem of classifying an observation into one of two multivariate normal populations with a common covariance matrix might be called the classical classification problem. Fisher's linear discriminant function [Fisher (1936)] serves as a criterion when samples are used to estimate the parameters of the two distributions. The exact probabilities of misclassifications when using this criterion are difficult to compute because the distribution of the criterion is virtually intractable. Wald (1944) made considerable progress towards finding the distribution, but only managed to express the criterion as a function of three angles whose distribution he gave. T. W. Anderson (1951) and Rosedith Sitgreaves (1952) continued with the problem. For further references see T. W. Anderson, Das Gupta, and Styan (1972), Subject Matter Code 6.2.

If the parameters are known, the Neyman-Pearson Fundamental Lemma can be applied to the classical classification problem [as done by Wald (1944)] to obtain a discriminant function that is linear in the components of the vector to be classified. The distribution of this statistic is normal; the mean and variance depends only on the Mahalanobis distance between the two populations. Since the procedure for classification is to classify into one population or the other depending on whether this statistic is greater or less than a constant, the probabilities of misclassification are found directly from the normal distribution. If the constant is 0, the probabilities are equal and the procedure is minimax.

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When the parameters are unknown and there is available a sample from each population, the mean of each population is estimated by the mean of the respective sample and the common covariance matrix of the populations is estimated by using deviations from the respective means in the two samples. The classification function W, proposed by T. W. Anderson (1951), is obtained by replacing the parameters in the linear function resulting from the Neyman-Pearson Fundamental Lemma by the estimates; the substitution for parameters has been called "plugging in" estimates. This criterion differs from Fisher's discriminant function by subtraction of the average of the Fisher discriminant function at the two sample means. Then the distribution depends only on the population distance, and this fact makes the distribution problem simpler [T. W. Anderson (1951) and Sitgreaves (1952)], though it is still rather intractable.

When the sizes of the two samples increase, the limiting distribution of W approaches a normal distribution, whose mean and variance depend on the Mahalanobis distance; if the limiting mean is subtracted from W and the difference is divided by the limiting standard deviation, the statistic has the standard normal distribution as its limiting distribution. Bowker and Sitgreaves (1961) and Okamoto (1963) with correction (1968) have given asymptotic expansions of the distributions to the order of the reciprocal of the square of the sample sizes. The approximate probability depends on the unknown parameter (the distance).

The "Studentized" W statistic is W less the estimate of its limiting mean divided by the estimate of its limiting standard deviation. It, too, has the standard normal distribution as its limiting distribution. If a statistician wants to set his cut-off point to achieve a specified probability of misclassification, he can use this Studentized W. An asymptotic expansion of the distribution of this statistic has been given by T. W. Anderson (1972).

In this paper we compare these two approximations to the probabilities of misclassification and their uses. For further discussion of the classification problem see Anderson (1958), Chapter 6.

2. The asymptotic expansion of the distribution of the classification statistic W

Let the two populations be $N(\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\Sigma})$ and $N(\underset{\sim}{\mu}^{(2)}, \underset{\sim}{\Sigma})$, and let the two samples be $\underset{\sim}{x}_1^{(1)}, \ldots, \underset{\sim}{x}_{N_1}^{(1)}$ and $\underset{\sim}{x}_1^{(2)}, \ldots, \underset{\sim}{x}_{N_2}^{(2)}$, respectively. The observation to be classified is $\underset{\sim}{x}$, which has the distribution $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, where $\underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}$ or $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1^{(2)}$. The classification statistic W is

(1) $$W = (\underset{\sim}{\bar{x}}^{(1)} - \underset{\sim}{\bar{x}}^{(2)})' \underset{\sim}{S}^{-1} [\underset{\sim}{x} - \tfrac{1}{2} (\underset{\sim}{\bar{x}}^{(1)} + \underset{\sim}{\bar{x}}^{(2)})] ,$$

where

(2) $$\underset{\sim}{\bar{x}}^{(1)} = \frac{1}{N_1} \sum_{j=1}^{N_1} \underset{\sim}{x}_j^{(1)} , \quad \underset{\sim}{\bar{x}}^{(2)} = \frac{1}{N_2} \sum_{j=1}^{N_2} \underset{\sim}{x}_j^{(2)} ,$$

(3) $$n\underset{\sim}{S} = \sum_{j=1}^{N_1} (\underset{\sim}{x}_j^{(1)} - \underset{\sim}{\bar{x}}^{(1)})(\underset{\sim}{x}_j^{(1)} - \underset{\sim}{\bar{x}}^{(1)})' + \sum_{j=1}^{N_2} (\underset{\sim}{x}_j^{(2)} - \underset{\sim}{\bar{x}}^{(2)})(\underset{\sim}{x}_j^{(2)} - \underset{\sim}{\bar{x}}^{(2)})' ,$$

and $n = N_1 + N_2 - 2$. The rule is to classify $\underset{\sim}{x}$ as coming from $N(\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\Sigma})$ if $W > c$ and from $N(\underset{\sim}{\mu}^{(2)}, \underset{\sim}{\Sigma})$ if $W \leq c$, where c may be a constant, particularly 0, or a function of $\underset{\sim}{\bar{x}}^{(1)}$, $\underset{\sim}{\bar{x}}^{(2)}$, and $\underset{\sim}{S}$.

The squared Mahalanobis distance is

$$\alpha = (\underset{\sim}{\mu}^{(1)} - \underset{\sim}{\mu}^{(2)})' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{\mu}^{(1)} - \underset{\sim}{\mu}^{(2)}) , \tag{4}$$

which can be estimated by

$$a = (\underset{\sim}{\bar{x}}^{(1)} - \underset{\sim}{\bar{x}}^{(2)})' \underset{\sim}{S}^{-1} (\underset{\sim}{\bar{x}}^{(1)} - \underset{\sim}{\bar{x}}^{(2)}) . \tag{5}$$

The limiting distribution of W as $N_1 \to \infty$ and $N_2 \to \infty$ is normal with variance α and mean $\frac{1}{2}\alpha$ if $\underset{\sim}{x}$ is from $N(\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\Sigma})$ and mean $-\frac{1}{2}\alpha$ if $\underset{\sim}{x}$ is from $N(\underset{\sim}{\mu}^{(2)}, \underset{\sim}{\Sigma})$; that is, the standard normal distribution $N(0, 1)$ is the limiting distribution of $(W - \frac{1}{2}\alpha)/\sqrt{\alpha}$ for $\underset{\sim}{x}$ coming from $N(\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\Sigma})$ and of $(W + \frac{1}{2}\alpha)/\sqrt{\alpha}$ for $\underset{\sim}{x}$ coming from $N(\underset{\sim}{\mu}^{(2)}, \underset{\sim}{\Sigma})$.

Okamoto's expansion of the probability distribution [(1963), Corollary 1] to terms of order n^{-1} is

$$\begin{aligned}(6)\quad \Pr\left\{\frac{W - \frac{1}{2}\Delta^2}{\Delta} \le u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}\right\} &= \Phi(u) + \frac{1}{n}\phi(u)\Bigg\{\Delta\left[1 + \frac{p}{2}k - \frac{p-2}{2}\frac{1}{k}\right)\frac{1}{\Delta^2} - \frac{p-1}{2}\Bigg] \\ &\quad - \left[(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k})\frac{p-3}{\Delta^2} + \frac{3p-2}{2} + \frac{1}{2}\frac{1}{k} + \frac{\Delta^2}{4}\right] u \\ &\quad - \Delta\left[(1 + \frac{1}{k})\frac{1}{\Delta^2} + 1\right] u^2 - \left[(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k})\frac{1}{\Delta^2} + 1\right] u^3\Bigg\} \\ &\quad + O(n^{-2}) ,\end{aligned}$$

where $k = \lim_{n\to\infty} N_1/N_2$ as $N_1 \to \infty$ and $N_2 \to \infty$, $\Delta^2 = \alpha$, and $\Phi(\ )$ and $\phi(\ )$ are the cumulative distribution function and density of $N(0, 1)$, respectively. If $\lim_{n\to 1} N_1/N_2 = 1$, then

$$\begin{aligned}(7)\quad \Pr\left\{\frac{W - \frac{1}{2}\Delta^2}{\Delta} \le u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}, \lim_{n\to\infty}\frac{N_1}{N_2} = 1\right\} &= \Phi(u) + \frac{1}{n}\phi(u) \quad \Delta\left[\frac{2}{\Delta^2} - \frac{p-1}{2}\right] \\ &\quad - 2\left[\frac{p-3}{\Delta^2} + \frac{3p-1}{2} + \frac{\Delta^2}{4}\right] u - \Delta\left[\frac{2}{\Delta^2} + 1\right] u^2 - \left[\frac{2}{\Delta^2} + 1\right] u^3\Bigg\} + O(n^{-2})\end{aligned}$$

$$= \Phi(u) + \frac{1}{n}\phi(u)\left\{\left[\frac{2}{\Delta^2} + 1\right](\Delta + u)(1-u^2) - \frac{p+1}{2}\Delta\right.$$

$$\left. - \left[2\,\frac{p-2}{\Delta^2} + \frac{3p+1}{2} + \frac{\Delta^2}{4}\right]u\right\} + O(n^{-2}) \ .$$

The relation between the cut-off point c and the argument u is

$$c = u\Delta + \frac{1}{2}\Delta^2 \ , \quad u = \frac{c - \frac{1}{2}\Delta^2}{\Delta} \ . \tag{8}$$

The probability of misclassification when $\underset{\sim}{x}$ is from $N(\underset{\sim}{\mu}^{(1)}, \underset{\sim}{\Sigma})$ is (6) [or (7)] with u given by (8); the probability depends importantly on the parameter .

A cut-off point of particular interest is $c = 0$, which corresponds to $u = -\frac{1}{2}\Delta$. If $N_1 = N_2$, this defines a minimax procedure. In this case the probability of misclassification is

$$\Pr\left\{W \le 0 \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}, \lim_{n\to\infty}\frac{N_1}{N_2} = 1\right\} = \Phi\left(-\frac{\Delta}{2}\right) + \frac{1}{n}\,\phi\left(\frac{\Delta}{2}\right)\left\{\frac{p-1}{\Delta} + \frac{p}{4}\Delta\right\} + O(n^{-2}) \ . \tag{9}$$

As far as this approximation goes, the correction term is positive; that is, the probability of a misclassification error is greater than the value of the normal approximation. For a given value of Δ the correction term and hence the probability (to order n^{-1}) increases with p. For a given value of p the probability (to order n^{-1}) decreases with Δ.

Okamoto (as well as Bowker and Sitgreaves) expanded the characteristic function. The method of Anderson (1972) could be used to obtain the result.

3. The asymptotic expansion of the distribution of the Studentized W

To use the approximate probability given by (6) one must know the parameter $\alpha = \Delta^2$, but this is generally unknown; then the statistician cannot achieve, even approximately, a desired probability. However, he can use the fact that a is a consistant estimate of α and therefore $(W - \frac{1}{2} a)/\sqrt{a}$ and $(W + \frac{1}{2} a)/\sqrt{a}$ have $N(0, 1)$ as the limiting distribution in cases $\underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}$ and $\underset{\sim}{\mu} = \underset{\sim}{\mu}^{(2)}$, respectively.

We can write

$$W - \frac{1}{2} a = (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\underset{\sim}{x} - \bar{\underset{\sim}{x}}^{(1)}) . \tag{10}$$

Then

$$\Pr\left\{\frac{W - \frac{1}{2} a}{\sqrt{a}} \le u\right\} = \Pr\left\{(\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \le u \sqrt{(\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})} + (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\bar{\underset{\sim}{x}}^{(1)} - \underset{\sim}{\mu})\right\} . \tag{11}$$

Since $\underset{\sim}{x}$ has the distribution $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ independently of $\bar{\underset{\sim}{x}}^{(1)}$, $\bar{\underset{\sim}{x}}^{(2)}$, and $\underset{\sim}{S}$, the conditional distribution of $(\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})$ is $N[0, (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} \underset{\sim}{\Sigma} \underset{\sim}{S}^{-1} (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})]$, and

$$r = \frac{(\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})}{\sqrt{(\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})' \underset{\sim}{S}^{-1} \underset{\sim}{\Sigma} \underset{\sim}{S}^{-1} (\bar{\underset{\sim}{x}}^{(1)} - \bar{\underset{\sim}{x}}^{(2)})}} \tag{12}$$

has the distribution $N(0, 1)$. Then (11) is

$$(13)\quad \Pr\left\{\frac{W-\frac{1}{2}a}{\sqrt{a}} \le u\right\} = \Pr\left\{r \le \frac{u\sqrt{(\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}(\bar{x}^{(1)}-\bar{x}^{(2)})} + (\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}(\bar{x}^{(1)}-\mu)}{\sqrt{(\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}\Sigma S^{-1}(\bar{x}^{(1)}-\bar{x}^{(2)})}}\right\}$$

$$= \mathcal{E}\,\Phi\left[\frac{u\sqrt{(\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}(\bar{x}^{(1)}-\bar{s}^{(2)})} + (\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}(\bar{x}^{(1)}-\mu)}{(\bar{x}^{(1)}-\bar{x}^{(2)})' S^{-1}\Sigma S^{-1}(\bar{x}^{(1)}-\bar{x}^{(2)})}\right],$$

where the expectation is with respect to $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, and S.

When $\mu = \mu^{(1)}$, $\bar{x}^{(1)} - \bar{x}^{(2)}$, $\bar{x}^{(1)} - \mu$, and S converge in probability to $\mu^{(1)} - \mu^{(2)}$, 0, and Σ, respectively. We can expand the argument of $\Phi(\ )$ in a Taylor's series in terms of $\sqrt{n}$ times the differences between the estimates and their probability limits. When the expansion includes third degree terms and the expecations computed, the result is

$$(14)\quad \Pr\left\{\frac{W-a}{\sqrt{a}} \le u \,\middle|\, \mu=\mu^{(1)}\right\} = \Phi(u) + \frac{1}{n}\phi(u)\left[\frac{(p-1)}{\Delta}(1+k) - (p - \frac{1}{4} + \frac{1}{2}k)u - \frac{1}{4}u^3\right] + O(n^{-2})$$

Interchanging N_1 and N_2 gives

$$(15)\quad \Pr\left\{\frac{W+\frac{1}{2}a}{\sqrt{a}} \le v \,\middle|\, \mu=\mu^{(2)}\right\} = \Phi(v) - \frac{1}{n}\phi(v)\left[\frac{p-1}{\Delta}(1+\frac{1}{k}) + (p - \frac{1}{4} + \frac{1}{2k})v + \frac{1}{4}v^3\right] + O(n^{-2}) .$$

The proof of these results was given by T. W. Anderson (1972). If $\lim_{n\to\infty} N_1/N_2 = k = 1$,

$$(16)\quad \Pr\left\{\frac{W-a}{\sqrt{a}} \le u \,\middle|\, \mu=\mu^{(1)},\ \lim_{n\to\infty}\frac{N_1}{N_2} = 1\right\} = \Phi(u) + \frac{1}{n}\phi(u)\left\{2\,\frac{p-1}{\Delta} - (p+\frac{1}{4})u - \frac{1}{4}u^3\right\} + O(n^{-2}) .$$

The correction term in (14) [(15) or (16)] is positive for $u < 0$. If $p = 1$, the correction term does not depend on Δ; if $p > 1$, the correction term decreases with Δ. For $u < 0$, the correction term increases with p.

For $u = -\frac{1}{2}\Delta$ (which is not $c = 0$)

$$(17) \qquad \Pr\left\{\frac{W-a}{\sqrt{a}} \leq -\frac{\Delta}{2} \Big| \underset{\sim}{\mu}=\underset{\sim}{\mu}^{(1)},\ \lim_{n\to\infty}\frac{N_1}{N_2}=1\right\} = \Phi\left(-\frac{\Delta}{2}\right) + \frac{1}{n}\phi\left(\frac{\Delta}{2}\right)\left\{2\,\frac{p-1}{\Delta} + \frac{4p+1}{8}\,\Delta + \frac{\Delta^3}{32}\right\} + O(n^{-2}) \ .$$

4. Numerical values of the correction term for the Studentized W when $N_1 = N_2$

We can obtain an idea of the importance of the term of order $1/n$ by studying numerical values of it. We consider the second term in (16), which is the error to order n^{-1} of using $\Phi(u)$ for the probability of misclassification. The correction relative to the nominal probability of misclassification is

$$(18) \qquad \frac{1}{n}\,\frac{\phi(u)}{\Phi(u)}\left[2\,\frac{p-1}{\Delta} - \left(p+\frac{1}{4}\right)u - \frac{u^3}{4}\right].$$

Table 1 gives values of the term in brackets for the five values of u corresponding to values of $\Phi(u)$ of .1, .05, .025, .01, and .005, and various values of p and Δ. It is 4.0893 for $u = -1.28155$ [$\Phi(u) = .1$], $p = 2$, and $\Delta = 2$. The correction relative to the nominal probability of misclassification is the value in the table multiplied by the ratio $\phi(u)/\Phi(u)$ divided by $n = N_1 + N_2 - 2$. In the example above it is $4.0893 \times 1.755 = 7.1767$ divided by n. If $N_1 = N_2 = 25$, then $n = 48$ and the correction relative to the nominal probability of misclassification is about .15. Here the correction would be rather small. For values of N_1 and N_2 somewhat larger, one might be willing to neglect the correction. One would hope that for these values of N_1 and N_2 the error when using this correction term would be rather small.

We might also be interested in the correction at $u = -\frac{1}{2}\Delta$. Table 2 gives the information. For example, for $\Delta = 4$ $\Phi(-\frac{1}{2}\Delta) = .022750$ (which would be the minimax probability if the parameters were known) and the correction is the appropriate number in the fourth column multiplied by .053991 divided by n. If $N_1 = N_2 = 25$ and $p = 2$, then $n = 48$ and the correction relative to the nominal probability is $7 \times 2.383/48$ $= .3475$.

5. Comparison of the expansions of the distributions of W and the Studentized W

It is striking that the asymptotic expansion of the distribution of the Studentized W is much simpler than that of W itself [the comparison of (6) with (14) and (7) with (16)], except for the particular case of $u = -\frac{1}{2}\Delta$ [(9) with (17)] which has special meaning for W ($c = 0$), but not for the Studentized W.

It is of interest to compare the correction terms of the two asymptotic expansions. The difference is

$$(19)\quad \Pr\left\{\frac{W - \frac{1}{2}a}{\sqrt{a}} \leq u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}\right\} - \Pr\left\{\frac{W - \frac{1}{2}\alpha}{\sqrt{\alpha}} \leq u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}\right\} = \frac{1}{n}\phi(u)\left\{\frac{p-2}{2}\,\frac{2+k+1/k}{\Delta} + \frac{p-1}{2}\Delta \right.$$
$$+ \left[\left(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k}\right)\frac{p-3}{\Delta^2} + \frac{2p-3}{4} - \frac{1}{2}k + \frac{1}{2}\frac{1}{k} + \frac{\Delta^2}{4}\right] u$$
$$\left. + \left[\left(1 + \frac{1}{k}\right)\frac{1}{\Delta} + \Delta\right] u^2 + \left[\frac{2+k+1/k}{2\Delta^2} + \frac{3}{4}\right] u^3 \right\} + O(n^{-2}) .$$

If $\lim_{n\to\infty} N_1/N_2 = k = 1$, the expression simplifies to

$$(20) \quad \Pr\left\{\frac{W - \frac{1}{2}a}{\sqrt{a}} \leq u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}, \lim_{n\to\infty} \frac{N_1}{N_2} = 1\right\} - \Pr\left\{\frac{W - \frac{1}{2}\alpha}{\sqrt{\alpha}} \leq u \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}_1, \lim_{n\to\infty} \frac{N_1}{N_2} = 1\right\}$$

$$= \frac{1}{n}\phi(v)\left\{2\,\frac{p-2}{\Delta} + \frac{p-1}{2}\Delta + 2\left[\frac{p-3}{\Delta^2} + \frac{2p-3}{4} + \frac{\Delta^2}{4}\right]u + \left[\frac{2}{\Delta} + \Delta\right]u^2 - \left[\frac{2}{\Delta^2} + \frac{3}{4}\right]u^3\right\} + O(n^{-2}) .$$

In particular, for $u = -\frac{1}{2}\Delta$ the difference is

$$(21) \quad \Pr\left\{\frac{W - \frac{1}{2}a}{\sqrt{a}} \leq -\frac{\Delta}{2} \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}, \lim_{n\to\infty} \frac{N_1}{N_2}\right\} - \Pr\left\{W \leq 0 \,\middle|\, \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}, \lim_{n\to\infty} \frac{N_1}{N_2} = 1\right\}$$

$$= \frac{1}{n}\phi\left(\frac{\Delta}{2}\right)\left\{\frac{p-1}{\Delta} + \left(\frac{p}{4} + \frac{1}{8}\right)\Delta + \frac{1}{32}\Delta^3\right\} + O(n^{-2}) .$$

Put another way, the correction term for $\Pr\{(W-a)/\sqrt{a} \leq -\frac{1}{2}\Delta\}$ is twice the correction term for $\Pr\{W \leq 0\}$ plus $\phi(\frac{1}{2}\Delta)\{\Delta/8 + \Delta^3/32\}/n$. The latter term, which does not depend on p, is usually small; values of $\Delta/8 + \Delta^3/32$ are given in Table 3. Comparison with Table 2 shows that for $p > 1$ this term is small except for large Δ. Thus, roughly speaking, the correction for the Studentized W is about that of W itself.

Okamoto (1963) has given numerical values of the term of order $1/n$ and the term of order $1/n^2$ in the expansion of $\Pr\{W \leq 0 | \underset{\sim}{\mu} = \underset{\sim}{\mu}^{(1)}\}$ for $N_1 = N_2 = 100$ $(n = 198)$ for various values of p and Δ. His values for $1/n$ are about twice the values we can compute from Table 2. In his table for small values of p and Δ the ratio of the term of order $1/n^2$ to the term of order $1/n$ is very roughly $1/n$. The maximum of the $1/n^2$ term over Δ increases with p. At $p = 7$, for example, it is about .0008. The table suggests that for small or moderate values of p the second correction term can be safely ignored for moderately large values of N_1 and N_2.

6. Comparison of approximate densities and moments

Corresponding to the approximate distributions of $(W-\alpha)/\sqrt{\alpha}$ and $(W-a)/\sqrt{a}$ (for $\underset{\sim}{\mu}=\underset{\sim}{\mu}^{(1)}$) are densities and moments. It is of some interest to compare these.

The approximate density of $(W - \frac{1}{2}\Delta^2)/\Delta$ is

$$(22) \qquad \phi(u)\Bigg\{1 - \frac{1}{n}\Bigg[(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k})\frac{p-3}{\Delta^2} + \frac{3p-2}{2} + \frac{1}{2}\frac{1}{k} + \frac{\Delta^2}{4}$$

$$+ \left(\frac{3 + \frac{1}{2}p\,k - \frac{1}{2}(p-6)/k}{\Delta} - \frac{p-5}{2}\Delta\right)u$$

$$- \left(\frac{p-6}{\Delta^2}(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k}) + \frac{3p-8}{2} + \frac{1}{2}\frac{1}{k} + \frac{\Delta^2}{4}\right)u^2$$

$$+ \left(\frac{1 + 1/k}{\Delta} + \Delta\right)u^3 + \left(\frac{1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k}}{\Delta^2} + 1\right)u^4\Bigg]\Bigg\} \quad ,$$

which for $k = 1$ is

$$(23) \qquad \phi(u)\Bigg\{1 - \frac{1}{n}\Bigg[2\,\frac{p-3}{\Delta^2} + \frac{3p-1}{2} + \frac{\Delta^2}{4} + \left(\frac{6}{\Delta} - \frac{p-5}{2}\Delta\right)u - \left(2\,\frac{p-6}{\Delta^2} + \frac{3p-7}{2} + \frac{\Delta^2}{4}\right)u^2$$

$$+ \left(\frac{2}{\Delta} + \Delta\right)u^3 + \left(\frac{2}{\Delta^2} + 1\right)u^4\Bigg]\Bigg\} \quad .$$

The approximate density of $(W - \frac{1}{2}a)/\sqrt{a}$ is

$$(24) \qquad \phi(u)\Bigg\{1 - \frac{1}{n}\Bigg[p - \frac{1}{4} + \frac{1}{2}k + \frac{(p-1)(1+k)}{\Delta}u - (p - 1 + \frac{1}{2}k)u^2 - \frac{u^4}{4}\Bigg]\Bigg\} ,$$

which for $k = 1$ is

$$(25) \qquad \phi(u)\Bigg\{1 - \frac{1}{n}\Bigg[p + \frac{1}{4} + 2\,\frac{p-1}{\Delta}u - (p - \frac{1}{2})u^2 - \frac{u^4}{4}\Bigg]\Bigg\} .$$

The approximate mean of $(W - \frac{1}{2}\Delta^2)/\Delta$ is

$$(26) \qquad -\frac{1}{n}\left[\frac{6 + \frac{1}{2}pk - \frac{1}{2}(p-12)/k}{\Delta} - \frac{p-11}{2}\Delta\right],$$

which for $k = 1$ is

$$(27) \qquad -\frac{1}{n}\left[\frac{12}{\Delta} - \frac{p-11}{2}\Delta\right];$$

the approximate second-order moment is

$$(28) \qquad 1 + \frac{1}{n}\left[\frac{(2p-30)\left(1 + \frac{1}{2}k + \frac{1}{2}\frac{1}{k}\right)}{\Delta^2} + 3p + 26 - \frac{1}{k} + \frac{1}{2}\Delta^2\right],$$

which for $k = 1$ is

$$(29) \qquad 1 + \frac{1}{n}\left[\frac{4p-60}{\Delta^2} + 3p - 25 + \frac{1}{2}\Delta^2\right].$$

The approximate mean of $(W - \frac{1}{2}a)/\sqrt{a}$ is

$$(30) \qquad -\frac{1}{n}\frac{(p-1)(1+k)}{\Delta},$$

which for $k = 1$ is

$$(31) \qquad -\frac{1}{n}\,2\,\frac{p-1}{\Delta};$$

the approximate second-order moment is

$$(32) \qquad 1 + \frac{1}{n}(2p + 1 + k),$$

which for $k = 1$ is

$$(33) \qquad 1 + \frac{1}{n}(2p + 2).$$

In each case the "approximate" moment is the moment of the approximate density. The approximate second-order moment is also the approximate variance. For $(W - \frac{1}{2} a)/\sqrt{a}$ the approximate mean is negative for $p > 1$ (while it is 0 for the standard normal distribution); its numerical value increases with p and decreases with Δ. The approximate variances are greater than 1 (the value for the standard normal distribution); it increases with p, but does not depend on Δ.

7. Achieving a given probability of misclassification

Suppose one wants to achieve a given probability p of misclassification when $\underset{\sim}{\mu}=\underset{\sim}{\mu}^{(1)}$, say. How should one choose the cut-off point $c = u\sqrt{a} + \frac{1}{2} a$ for W or equivalently u for $(W - \frac{1}{2} a)/\sqrt{a}$?

Let u_0 be the number such that $\Phi(u_0) = p$. Then the probability of misclassification is

$$(34) \quad p + \frac{1}{n} \phi(u_0) \left[\frac{(p-1)(1+k)}{\Delta} - (p - \frac{1}{4} + \frac{1}{2} k)u_0 - \frac{1}{4} u_0^3\right] + O(n^{-2}) \quad .$$

The correction term of order n^{-1} contains the unknown parameter Δ (if $p > 1$). However, Δ can be estimated by $\sqrt{a}$. These facts suggest taking

$$(35) \qquad u = u_0 - \frac{1}{n} \left[\frac{(p-1)(1+k)}{\sqrt{a}} - (p - \frac{1}{4} + \frac{1}{2} k) u_0 - \frac{1}{4} u_0^3\right] .$$

Then

$$(36) \quad \Pr\left\{\frac{W - \frac{1}{2} a}{\sqrt{a}} \leq u \,\middle|\, \underset{\sim}{\mu}=\underset{\sim}{\mu}^{(1)}\right\} = \Pr\left\{\frac{W - \frac{1}{2} a}{\sqrt{a}} + \frac{1}{n} \frac{(p-1)(1+k)}{\sqrt{a}} \leq u^*\right\} ,$$

where

$$(37) \qquad u^* = u_0 + \frac{1}{n}\left[\left(p - \frac{1}{4} + \frac{1}{2}k\right) u_0 + \frac{1}{4} u_0^3\right] .$$

If $p = 1$, this probability is (14) with $u = u^*$, which is $p+O(n^{-2})$. When $p > 1$, we calculate the probability of misclassification as

$$(38) \qquad \Pr\{W - \frac{1}{2} a \leq u^* \sqrt{a} - \frac{1}{n}(p-1)(1+k)\} = \Pr\{(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})$$

$$\leq u^* \sqrt{(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-1}(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})} + (\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-1}(\bar{\underset{\sim}{x}}^{(1)}-\underset{\sim}{\mu}) - \frac{1}{n}(p-1)(1+k)\}$$

$$= \Phi\left[\frac{u^* \sqrt{(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-1}(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})} + (\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-1}(\bar{\underset{\sim}{x}}^{(1)}-\underset{\sim}{\mu}) - \frac{1}{n}(p-1)(1+k)}{\sqrt{(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})'\underset{\sim}{S}^{-2}(\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)})}}\right],$$

where $\bar{\underset{\sim}{x}}^{(1)}-\bar{\underset{\sim}{x}}^{(2)}$, $\bar{\underset{\sim}{x}}^{(1)}-\underset{\sim}{\mu}$ and $\underset{\sim}{S}$ have the joint distribution given in Anderson (1972). Then the expansion of $\Phi(\ )$ is

$$(39) \qquad \Phi\{u^* + \frac{1}{\sqrt{n}} C^*(\underset{\sim}{Z},\underset{\sim}{V}) + \frac{1}{n} D^*(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V}) + r^*_{7n}(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$$

$$- \frac{1}{n}(p-1)(1+k)\left[\frac{1}{\Delta} - \frac{1}{\Delta^3\sqrt{n}}(\underset{\sim}{\delta}'v_1-\underset{\sim}{\delta}\underset{\sim}{V}\underset{\sim}{\delta}) + r^*(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})\right]$$

$$= \Phi(u^*) + \phi(u^*)\left\{\frac{1}{\sqrt{n}} C^*(\underset{\sim}{Z},\underset{\sim}{V}) + \frac{1}{n}\left[D^*(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})\right.\right.$$

$$\left. - \frac{1}{2} u^* C^{*2}(\underset{\sim}{Z},\underset{\sim}{V}) - \frac{1}{n}(p-1)(1+k)\frac{1}{\Delta}\right]$$

$$\left. + \frac{1}{\Delta^3 n^{3/2}}(p-1)(1+k)\ (\underset{\sim}{\delta}'\underset{\sim}{Y}-\underset{\sim}{\delta}'\underset{\sim}{V}\underset{\sim}{\delta})\right\} + \frac{1}{n^{3/2}} r^*_8(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V}) + \frac{1}{n^2} r^*_9(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$$

$$+ r^*_{10n}(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V}) ,$$

where $C^*(\underset{\sim}{Z},\underset{\sim}{V})$, $D^*(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$, and $r^*_{7n}(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$ are $C(\underset{\sim}{Z},\underset{\sim}{V})$, $D(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$ and $r_{7n}(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$ of Anderson (1972), with u replaced by u^* and $r^*(\underset{\sim}{Y},\underset{\sim}{Z},\underset{\sim}{V})$ in the remainder term in (19) of Anderson (1972). The expected value of $\Phi(\ )$ is

$$(40) \quad \Phi(u^*) + \frac{1}{n}\,\phi(u^*)\,[-\,(p - \frac{1}{4} + \frac{1}{2k})\,u^* - \frac{1}{4}\,u^{*3}] + O(n^{-2})$$

$$= \Phi(u_0) + O(n^{-2})$$

$$= p + O(n^{-2}) \ .$$

TABLE 1

$$2\,\frac{p-1}{\Delta} - (p + \frac{1}{4})u - \frac{u^3}{4}$$

	p \ Δ	1	2	3	4	6	∞
$u = -1.28155$	1	2.13	2.13	2.13	2.13	2.13	2.13
$\phi(u) = .100$	2	5.41	4.41	4.08	3.91	3.74	3.41
$\Phi(u) = .17550$	4	11.97	8.97	7.97	7.47	6.97	5.97
$\phi(u)/\Phi(u) = 1.755$	8	25.10	18.10	15.77	14.60	13.43	11.10

	p \ Δ	1	2	3	4	6	∞
$u = -1.64485$	1	3.17	3.17	3.17	3.17	3.17	3.17
$\Phi(u) = .05$	2	6.81	5.81	5.48	5.31	5.15	4.81
$\phi(u) = .10314$	4	14.10	11.10	10.10	9.50	9.10	8.10
$\phi(u)/\Phi(u) = 2.063$	8	28.68	21.68		18.18	17.06	14.68

	p \ Δ	1	2	3	4	6	∞
$u = -1.95996$	1	4.33	4.33	4.33	4.33	4.33	4.33
$\Phi(u) = .025$	2	8.29	7.29	6.96	6.79	6.63	6.29
$\phi(u) = .05844$	4	16.21	13.21	12.21	11.71	11.21	10.21
$\phi(u)/\Phi(u) = 2.338$	8	32.05	25.05	22.72	21.55	20.39	18.05

	$p \backslash \Delta$	1	2	3	4	6	∞
$u = -2.32635$	1	6.06	6.06	6.06	6.06	6.06	6.06
$\phi(u) = .01$	2	10.38	9.38	9.05	8.88	8.72	8.38
$\Phi(u) = .02665$	4	19.03	16.03	15.03	14.53	14.03	13.03
$\phi(u)/\Phi(u) = 2.665$	8	36.34	29.34	27.01	25.84	24.67	22.34

	$p \backslash \Delta$	1	2	3	4	6	∞
$u = -2.57583$	1	7.49	7.49	7.49	7.49	7.49	7.49
$\phi(u) = .005$	2	11.07	11.07	10.73	10.57	10.40	10.07
$\Phi(u) = .01446$	4	18.22	18.22	17.22	16.72	16.22	15.22
$\phi(u)/\Phi(u) = 2.892$	8	32.52	32.52	30.19	29.02	27.86	25.52

TABLE 2

$$2\,\frac{p-1}{\Delta} + \left(\frac{p}{2} + \frac{1}{8}\right) + \frac{\Delta^3}{32}$$

$p \backslash \Delta$	1	2	3	4	6
1	.65625	1.50000	2.71875	4.50000	10.50000
2	3.15625	3.50000	4.88542	7.00000	13.83333
4	8.15125	7.50000	9.21875	12.00000	20.00000
8	18.15625	15.50000	17.88542	22.00000	40.83333
$\phi(-\frac{1}{2}\Delta)$	.35206	.24197	.129518	.053991	.0044318
$\Phi(-\frac{1}{2}\Delta)$	.30854	.15866	.066807	.022750	.0013499
$\phi(-\frac{1}{2}\Delta)/\Phi(\frac{1}{2}\Delta)$	1.141	1.525	1.939	2.383	3.283

TABLE 3

$$\frac{\Delta}{8} + \frac{\Delta^3}{32}$$

Δ	1	2	3	4	6
$\frac{\Delta}{8} + \frac{\Delta^2}{32}$	.15626	.50000	1.21875	2.50000	7.50000

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Linear discriminant functions are used to classify an observation as coming from one of two normal populations with common covariance matrices and different means when samples are used to estimate the parameters of the distributions. Okamoto's asymptotic expansion of the distribution of the classification statistic W is compared with Anderson's expansion for the Studentized W (that is, W standardized by estimates of its mean and standard deviation). Some numerical evaluations of the term of order of the reciprocal of the sample sizes is given. The uses of the two approximate distributions are discussed.

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