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ON THE INVERSE OF SOME COVARIANCE MATRICES OF TOEPLITZ TYPE

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RAUL PEDRO MENTZ

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THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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Contents

1. Introduction*

A matrix $\underset{\sim}{A}$ with components a_{ij}, $i,j = 1,2,\ldots,h$, [written $\underset{\sim}{A} = (a_{ij})$] is called a Toeplitz matrix if $a_{ij} = a_{i-j}$. A particular case is when $a_{ij} = a_{|i-j|}$.

In mathematical statistics Toeplitz matrices arise in several contexts; see, for example, Grenander and Szegö [8]. Consider the following frequently occurring case. Let $\{x_t: t = \ldots,-1,0,1,\ldots\}$ be a wide-sense stationary stochastic process with $\mathcal{E} x_t = 0$ for all t. Its covariance sequence satisfies

$$\mathcal{E} x_t x_s = \mathrm{Cov}(x_t, x_s) = \sigma_{|s-t|}, \qquad s,t = \ldots,-1,0,1,\ldots, \tag{1.1}$$

*The author acknowledges financial assistance from the Ford Foundation and leave from the University of Tucumán, Argentina.

that is, a function of $|s-t|$ only. If $\underset{\sim}{x} = (x_1,\ldots,x_T)'$ is a finite segment of $\{x_t\}$, then its covariance matrix is the $T \times T$ matrix

$$(1.2) \qquad \underset{\sim}{\Sigma}_T = \begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{T-1} \\ \sigma_1 & \sigma_0 & \cdots & \sigma_{T-2} \\ \vdots & \vdots & & \vdots \\ \sigma_{T-1} & \sigma_{T-2} & \cdots & \sigma_0 \end{pmatrix};$$

$\underset{\sim}{\Sigma}_T$ is a matrix of Toeplitz type.

It is important to note that $\underset{\sim}{\Sigma}_T$ can also be viewed as

$$(1.3) \qquad \underset{\sim}{\Sigma}_T = \sum_{k=0}^{T-1} \sigma_k \underset{\sim}{G}_k ,$$

where $\underset{\sim}{G}_k = (g_{ij}^{(k)})$ and

$$(1.4) \qquad \begin{aligned} g_{ij}^{(k)} &= 1, & |i-j| &= k, \\ &= 0, & |i-j| &\neq k, \end{aligned}$$

$\underset{\sim}{\Sigma}_T$ is a linear combination of simple known matrices $\underset{\sim}{G}_0 = \underset{\sim}{I}$, $\underset{\sim}{G}_1,\ldots,\underset{\sim}{G}_{T-1}$, the coefficients being the parameters σ_k. This structure has been exploited to find the maximum likelihood estimators of the σ_k's under normality; see, for example, Anderson [1], [2], [4], and references therein. Here we shall be concerned with finding the inverse of $\underset{\sim}{\Sigma}_T$, and it turns out that the linear structure can be used to devise a practical procedure for such purpose. For a solution of a similar problem, with different $\underset{\sim}{G}_j$'s, see Mustafi [11].

It must be pointed out at the outset that there exists wide interest in finding either exact or approximate values for the components of such inverse matrices, since that knowledge can be used to derive the statistical theory for procedures defined in terms of them. For example, the author's interest in the inverse analyzed in Sections 2, 4, 5 and 6 stems from the study of Walker's [16] estimation procedure for the moving-average time-series model.

In many cases the underlying assumptions imply that

$$\sigma_{i-j} = 0, \qquad |i-j| > m, \tag{1.5}$$

where m is a nonnegative integer. We may call these processes "finitely correlated of order m". The case of lack of correlation corresponds to $m = 0$. If $m < T-1$, then $\underset{\sim}{\Sigma}_T$ has m diagonals above and m below the main diagonal with (possibly) nonzero components, and all other components are zero. For some of these matrices the inverse is known. When $m = 1$, Shaman [12], [13] gave several forms of the exact inverse, and several approximations. One of the methods in his paper is extended in Section 2 to the case $m = 2$.

In Section 3 a different approach is used for the particular case $\underset{\sim}{\Sigma}_T = \underset{\sim}{I} + \rho \underset{\sim}{G}_1$, and a new expression for the exact inverse, and some approximations, are presented. The method is then applied in

Section 4 to invert $\underset{\sim}{I} + \rho_1\underset{\sim}{G}_1 + \rho_2\underset{\sim}{G}_2$, and in Section 5 to the general matrix with such a structure. The proposed method is used at some points in conjunction with a condition that is stated and proved in an Appendix; it is related to "diagonal matrices of type r" (cf. Greenberg and Sarhan [7] and others) defined by the fact their components satisfy $a_{ij} = 0$ whenever $|i-j| \geq r$.

Finally Section 6 discusses an approximation based on a well-known relation between autoregressive and moving-average time series.

Since for $\sigma_0 \neq 0$

$$(1.6) \qquad \sum_{j=0}^{m} \sigma_j \underset{\sim}{G}_j = \sigma_0 \sum_{j=0}^{m} \rho_j \underset{\sim}{G}_j,$$

where $\rho_0 = 1$; $\rho_j = \sigma_j/\sigma_0$, $j = 1,2,\ldots,m$, we see that there is no loss of generality in taking the coefficient of $\underset{\sim}{G}_0$ in (1.3) to be one, as will be done below whenever it is convenient.

2. The inverse of $\underset{\sim}{I} + \rho_1 \underset{\sim}{G}_1 + \rho_2 \underset{\sim}{G}_2$ by evaluation of cofactors.

Let $\underset{\sim}{\Sigma}_T = (\sigma_{ij}) = \underset{\sim}{I} + \rho_1 \underset{\sim}{G}_1 + \rho_2 \underset{\sim}{G}_2$, $\rho_2 \neq 0$, and $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T = (w_{ij}^{(T)})$. The components of $\underset{\sim}{W}_T$ can be computed from

$$w_{ij}^{(T)} = \frac{\text{cofactor of } \sigma_{ji}}{|\underset{\sim}{\Sigma}_T|} . \tag{2.1}$$

The method presented in this section consists in expressing all the determinants that may appear in (2.1), in terms of some determinants, like $|\underset{\sim}{\Sigma}_T|$, each one of which satisfies a certain difference equation.

In this section we use the following notation, where a subscript denotes the order of the corresponding matrix or determinant, and we omit the superscripts in the components to simplify the writing. We also use the notation of partitioned matrices:

$$\Sigma_s = |\underset{\sim}{\Sigma}_s| \; ; \tag{2.2}$$

$$\underset{\sim}{L}_s = \begin{pmatrix} \rho_1 & 1 & \rho_1 & \rho_2 & 0 & \cdots & 0 & 0 & 0 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & 0 & 0 & 0 \\ 0 & \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_2 & \rho_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \rho_2 & \rho_1 \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} \rho_1 \\ \rho_2 \\ 0 \\ \vdots \\ 0 \end{matrix} & \underset{\sim}{\Sigma}_{s-1} \\ \hline 0 & 0 \; \cdots \; 0 \; \rho_2 \; \rho_1 \end{array}\right),$$

(2.3)

$$L_s = |\underset{\sim}{L}_s| \; ;$$

$$
\underset{\sim}{K}_s = \begin{pmatrix} \rho_1 & \rho_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \rho_1 & 1 & \rho_1 & \rho_2 & 0 & \cdots & 0 & 0 & 0 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_1 & 1 & \rho_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_2 & \rho_1 & 1 \end{pmatrix} = \left(\begin{array}{c|c} \rho_1 & \rho_2 \; 0 \; \cdots \; 0 \\ \hline \rho_1 & \\ \rho_2 & \\ 0 & \underset{\sim}{\Sigma}_{s-1} \\ \vdots & \\ 0 & \end{array}\right), \tag{2.4}
$$

$$
K_s = |\underset{\sim}{K}_s| \; .
$$

By expanding Σ_T in terms of the components in its first row, Durbin ([5], p. 315) found that the determinants satisfy the linear, homogeneous, fifth-order difference equation

$$
-\Sigma_n+(1-\rho_2)\Sigma_{n-1}+(\rho_2-\rho_1^2)\Sigma_{n-2}+\rho_2(\rho_1^2-\rho_2)\Sigma_{n-3}+\rho_2^3(\rho_2-1)\Sigma_{n-4}+\rho_2^5\Sigma_{n-5} = 0. \tag{2.5}
$$

The associated polynomial equation

$$
-z^5+(1-\rho_2)z^4+(\rho_2-\rho_1^2)z^3+\rho_2(\rho_1^2-\rho_2)z^2+\rho_2^3(\rho_2-1)z+\rho_2^5 = 0 \tag{2.6}
$$

can be written in a symmetric way using the substitution $-\rho_2 x = z$; after division by ρ_2^5 we obtain

$$
x^5 + \frac{1-\rho_2}{\rho_2}x^4 + \frac{\rho_1^2-\rho_2}{\rho_2^2}x^3 + \frac{\rho_1^2-\rho_2}{\rho_2^2}x^2 + \frac{1-\rho_2}{\rho_2}x + 1 = 0. \tag{2.7}
$$

Zero is not a root of (2.7), and if x^* is a root so is $1/x^*$. Since there must be five roots, $+1$ or -1 must be one of them. (They are the only "self inverses".) By inspection we see it is -1. Then (2.7) factors as

$$
\begin{aligned}
0 &= (x+1)(x-x_1)(x-\frac{1}{x_1})(x-x_2)(x-\frac{1}{x_2}) \\
&= (x+1)(x^2-d_1x+1)(x^2-d_2x+1) \\
&= (x+1)[x^4-(d_1+d_2)x^3+(2+d_1d_2)x^2-(d_1+d_2)x+1] \\
&= x^5+(1-d_1-d_2)(x^4+x)+(2+d_1d_2-d_1-d_2)(x^3+x^2)+1,
\end{aligned}
\tag{2.8}
$$

where $d_i \equiv x_i + 1/x_i$, $i = 1,2$. Equating coefficients in (2.7) and the last line of (2.8) we obtain

$$
1-d_1-d_2 = \frac{1-\rho_2}{\rho_2}, \qquad 2+d_1d_2-d_1-d_2 = \frac{\rho_1^2-\rho_2}{\rho_2^2}. \tag{2.9}
$$

If we define

$$
u = \frac{2\rho_2-1}{\rho_2}, \qquad v = \frac{\rho_1^2-2\rho_2+2\rho_2^2}{\rho_2^2}, \tag{2.10}
$$

then (2.9) can be written as

$$
d_1+d_2 = u, \qquad 2+d_1d_2 = v. \tag{2.11}
$$

This system has solutions

$$(2.12)\qquad d_1,d_2 = \frac{1}{2}\left[u \pm \sqrt{u^2+4(2-v)}\right] = \frac{1}{2\rho_2}\left[2\rho_2-1 \pm \sqrt{(2\rho_2+1)^2-4\rho_1^2}\right],$$

and another pair with the roles of d_1 and d_2 interchanged. Since we want to determine the roots x_j from $x^2-d_ix+1 = 0$, we see that (2.12) gives rise to all possible different roots. They are given by

$$(2.13)\qquad x_1,x_3 = \frac{1}{2}\left(d_1 \pm \sqrt{d_1^2-4}\right), \quad x_2,x_4 = \frac{1}{2}\left(d_2 \pm \sqrt{d_2^2-4}\right).$$

Note that $x_1x_3 = x_2x_4 = 1$. We conclude that the roots of (2.7) can be labeled

$$(2.14)\qquad x_1 = -1, \quad x_2 = \frac{1}{2}\left(d_1+\sqrt{d_1^2-4}\right), \quad x_3 = \frac{1}{x_2}, \quad x_4 = \frac{1}{2}\left(d_2+\sqrt{d_2^2-4}\right), \quad x_5 = \frac{1}{x_4}.$$

Substituting back $z = -\rho_2x$, (2.6) has the following roots:

$$(2.15)\qquad z_1 = \rho_2, \quad z_2 = \frac{-\rho_2}{2}\left(d_1+\sqrt{d_1^2-4}\right), \quad z_3 = \frac{-2\rho_2}{d_1+\sqrt{d_1^2-4}}, \quad z_4 = \frac{-\rho_2}{2}\left(d_2+\sqrt{d_2^2-4}\right),$$

$$z_5 = \frac{-2\rho_2}{d_2+\sqrt{d_2^2-4}}.$$

In general the roots (2.15) can be real or complex, and some or all can be identical. Hence the solution of (2.5) will take different forms depending on this fact. As an example, which will be also used as illustration in subsequent derivations, if all roots are distinct then (2.5) has solution

$$\Sigma_n = \sum_{i=1}^{5} C_i z_i^n , \tag{2.16}$$

where the z_i are the roots given in (2.15).

Since Σ_n is defined only for $n \geq 1$, (2.5) holds for $n \geq 6$, and the sequence satisfying the difference equation and for which (2.16) is the general solution is $\Sigma_1, \Sigma_2, \ldots$. The boundary conditions to determine C_i, $i = 1,\ldots,5$, can be taken to be (2.16) for $n = 1,\ldots,5$, with the left-hand sides evaluated explicitly as

$$\begin{aligned}
\Sigma_1 &= 1, \\
\Sigma_2 &= 1-\rho_1^2, \\
\Sigma_3 &= (1-\rho_2)(1+\rho_2-2\rho_1^2), \\
\Sigma_4 &= \Sigma_3-(\rho_1^2+\rho_2^2)+(\rho_1^4+\rho_2^4)+2\rho_1^2\rho_2-2\rho_1^2\rho_2^2, \\
\Sigma_5 &= \Sigma_4-\rho_1^2\Sigma_3+2\rho_1^2\rho_2(1-\rho_1^2-\rho_2+\rho_2^2)-\rho_2^2(1-\rho_1^2-\rho_2^2).
\end{aligned} \tag{2.17}$$

Following the same approach we expand L_T in terms of the components in its first row and find that

$$L_n - \rho_1 L_{n-1} + \rho_2 L_{n-2} - \rho_1\rho_2^2 L_{n-3} + \rho_2^4 L_{n-4} = 0. \tag{2.18}$$

The polynomial equation is

$$y^4 - \rho_1 y^3 + \rho_2 y^2 - \rho_1\rho_2^2 y + \rho_2^4 = 0, \tag{2.19}$$

and after replacing $\rho_2 x = y$ it becomes

$$(2.20) \qquad \rho_2^4 x^4 - \rho_1 \rho_2^3 x^3 + \rho_2^3 x^2 - \rho_1 \rho_2^3 x + \rho_2^4 = 0,$$

which has symmetric coefficients and can be studied in the same way as equation (2.7). The roots y_s, $s = 1,2,3,4$, of (2.19) are obtained from

$$(2.21) \qquad d_1 = \frac{1}{2\rho_2}\left(\rho_1 + \sqrt{\rho_1^2 - 4\rho_2 + 8\rho_2^2}\right), \qquad d_2 = \frac{1}{2\rho_2}\left(\rho_1 - \sqrt{\rho_1^2 - 4\rho_2 + 8\rho_2^2}\right);$$

$$y_1 = \frac{\rho_2}{2}\left(d_1 + \sqrt{d_1^2 - 4}\right), \quad y_2 = \frac{2\rho_2}{d_1 + \sqrt{d_1^2 - 4}}, \quad y_3 = \frac{\rho_2}{2}\left(d_2 + \sqrt{d_2^2 - 4}\right), \quad y_4 = \frac{2\rho_2}{d_2 + \sqrt{d_2^2 - 4}} .$$

The particular case of all roots distinct leads to solving (2.18) by the sequence

$$(2.22) \qquad L_n = \sum_{i=1}^{4} C_i^* y_i^n, \qquad n = 1,2,\ldots .$$

The four boundary conditions needed to determine the C_i^*'s can be taken to be (2.22) for $n = 1,2,3,4$ with the left-hand sides evaluated explicitly as

$$(2.23) \qquad \begin{aligned} L_1 &= \rho_1, \\ L_2 &= \rho_1^2 - \rho_2, \\ L_3 &= \rho_1^3 + \rho_2^2 \rho_1 - 2\rho_1 \rho_2, \\ L_4 &= \rho_1 L_3 - \rho_2[(\rho_1^2 - \rho_2) - \rho_2(\rho_1^2 - \rho_2^2)]. \end{aligned}$$

Expanding K_T by the components in its first row we have

$$K_n = \rho_1 \Sigma_{n-1} - \rho_2 K_{n-1}, \qquad n = 2,3,\ldots \tag{2.24}$$

In the special case that Σ_n is given by (2.16),

$$K_n + \rho_2 K_{n-1} = \rho_1 \sum_{i=1}^{5} c_i z_i^{n-1}, \tag{2.25}$$

which is a first-order, inhomogeneous, linear difference equation. The complete solution is the sum of the general solution of the homogeneous case $[\tilde{C}(-\rho_2)^n]$ and a particular solution of the inhomogeneous case. Provided that only one root (for Σ_T) equals ρ_2,

$$K_n = \tilde{C}(-\rho_2)^n + \frac{1}{2}\rho_1 c_1 \frac{1}{\rho_2}\rho_2^n + \rho_1 \sum_{j=2}^{5} c_j \frac{1}{z_j+\rho_2} z_j^n . \tag{2.26}$$

The second summand corresponds to the root $z_1 = \rho_2$; no other z_j can be equal to ρ_2 in (2.26); if more than one root equals ρ_2, instead of the factor $1/(z_j+\rho_2)$ we have to use $1/2\rho_2$.

The new constant $\tilde{C}$ in (2.26) will be evaluated from (2.26) for $n = 2$, with $K_1 = \rho_1$. Note that

$$\begin{aligned} K_2 &= \rho_1(1-\rho_2), \\ K_3 &= \rho_1(1-\rho_1^2) - \rho_1\rho_2(1-\rho_2), \\ K_4 &= \rho_1\Sigma_3 - \rho_2 K_3 = \rho_1(1-\rho_2)(1+\rho_2-2\rho_1^2) - \rho_2 K_3. \end{aligned} \tag{2.27}$$

With this background we now find expressions for the components w_{ij} of $\underset{\sim}{W}_T = \underset{\sim}{\Sigma}_T^{-1}$. Since $\underset{\sim}{W}_T$ is symmetric we restrict attention to the components on and above the main diagonal.

<u>1st case: $i = j$.</u> Then $w_{ii} = B_{ii}/\Sigma_T$, where B_{ii} is the cofactor of σ_{ii}. In terms of submatrices

$$(2.28) \qquad B_{ii} = \begin{vmatrix} \underset{\sim}{\Sigma}_{i-1} & \rho_2 \underset{\sim}{E}^{*'} \\ \rho_2 \underset{\sim}{E}^{*} & \underset{\sim}{\Sigma}_{T-i} \end{vmatrix},$$

where $\underset{\sim}{E}^*$ has its upper right-hand element equal to 1 and all other elements equal to zero. We use Laplace's expansion in terms of minors of the first $i - 1$ columns; then

$$(2.29) \qquad B_{ii} = \Sigma_{i-1}\Sigma_{T-i} - \rho_2^2 \Sigma_{i-2}\Sigma_{T-i-1}.$$

To make (2.29) valid for all i, we define $\Sigma_0 = 1$, $\Sigma_{-1} = 0$.

<u>2nd case: $i < j$.</u>

$$(2.30) \qquad (-1)^{i+j} B_{ij} = \begin{vmatrix} \underset{\sim}{\Sigma}_{i-1} & \underset{\sim}{F} & \underset{\sim}{0} \\ \rho_2 \underset{\sim}{E}^{*} & \underset{\sim}{L}_{j-i} & \underset{\sim}{F} \\ \underset{\sim}{0} & \rho_2 \underset{\sim}{E}^{*} & \underset{\sim}{\Sigma}_{T-j} \end{vmatrix},$$

where $\underset{\sim}{F}$ has its lower left-hand element equal to ρ_1, the two adjacent elements equal to ρ_2, and all other elements equal to 0. We expand (2.30) by Laplace's formula in terms of minors of the first $i-1$ columns. In these columns there are three non-vanishing minors with non-zero complementary minors, namely

$$(2.31) \qquad |\underset{\sim}{\Sigma}_{i-1}| = \Sigma_{i-1},$$

$$(2.32) \qquad \left|\begin{array}{c|c} \underset{\sim}{\Sigma}_{i-2} & \begin{matrix} 0 \\ \vdots \\ \rho_2 \\ \rho_1 \end{matrix} \\ \hline 0 \ \cdots\cdots \ 0 & \rho_2 \end{array}\right| = \rho_2 \Sigma_{i-2},$$

and

$$(2.33) \qquad \left|\begin{array}{c|c} \underset{\sim}{K}^*_{i-2} & \begin{matrix} 0 \\ \vdots \\ \rho_2 \\ 1 \end{matrix} \\ \hline 0 \ \cdots\cdots \ 0 & \rho_2 \end{array}\right| = \rho_2 K_{i-2},$$

where $\underset{\sim}{K}^*_s$ is $\underset{\sim}{K}_s$ flipped about its secondary diagonal, so that $|\underset{\sim}{K}^*_s| = K_s$. If we denote by $A_s(i,j)$, $s = 1,2,3$, the corresponding cofactors, then

$$(2.34) \qquad (-1)^{i+j} B_{ij} = \Sigma_{i-1} A_1(i,j) - \rho_2 \Sigma_{i-2} A_2(i,j) + \rho_2 K_{i-2} A_3(i,j).$$

The $A_s(i,j)$'s are computed using Laplace's expansion in terms of the last $T-j$ columns. Then

$$A_1(i,j) = \Sigma_{T-j}L_{j-i}-\rho_2K_{T-j}L_{j-i-1}+\rho_2^3\Sigma_{T-j-1}L_{j-i-2},$$

(2.35)
$$A_2(i,j) = \Sigma_{T-j}(\rho_1L_{j-i-1}-\rho_2^2L_{j-i-2})-\rho_2K_{T-j}(\rho_1L_{j-i-2}-\rho_2^2L_{j-i-3})$$
$$+\rho_2^3\Sigma_{T-j-1}(\rho_1L_{j-i-3}-\rho_2^2L_{j-i-4}),$$

$$A_3(i,j) = \rho_2\Sigma_{T-j}L_{j-i-1}-\rho_2^2K_{T-j}L_{j-i-2}+\rho_2^4\Sigma_{T-j-1}L_{j-i-3}.$$

For $j = [T/2]+1,\ldots,T$, say, these formulas are valid for all $i < j$, provided we define $\Sigma_0 = L_0 = 1$, $K_0 = 0$, $\Sigma_{-s} = L_{-s} = K_{-s} = 0$ for $s > 0$. For $j < [T/2]+1$ similar arrangements could be made. In fact, due to the structure of $\underset{\sim}{W}_T$ we only need to compute those components of the last $[(T+1)/2]$ columns on and between the principal and secondary diagonals, and then deduce the remaining components using the symmetry of W_T and its <u>persymmetry</u> (symmetry with respect to its secondary diagonal). They lead for example to

(2.36) $$w_{ij} = w_{T-i+1,T-j+1}, \qquad i = j,\ldots,T-j; \quad j = 1,2,\ldots,[T/2].$$

We summarize these results as follows:

<u>Proposition 2.1. Let</u> $\underset{\sim}{\Sigma}_T = \underset{\sim}{I}+\rho_1\underset{\sim}{G}_1+\rho_2\underset{\sim}{G}_2$, <u>with</u> $\rho_2 \neq 0$, <u>and</u> $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T = (w_{ij}^{(T)})$. <u>Then</u>

(2.37) $$w_{ij}^{(T)} = (-1)^{i+j}\frac{B_{ij}}{\Sigma_T}, \quad i = j,\ldots,T-j+1; \quad j = [T/2]+1,\ldots,T,$$

where the B_{ij} are given in (2.29) when $i = j$ and in (2.34) and (2.35) when $i < j$, in terms of the determinants Σ_s, L_s, K_s, which are defined in (2.2)-(2.4) and satisfy the difference equations (2.5), (2.18) and (2.24), respectively. The remaining elements of $\underset{\sim}{W}_T$ are obtained using $w_{ij}^{(T)} = w_{ji}^{(T)}$ and (2.36).

If $\rho_2 = 0$ but $\rho_1 \neq 0$, then $L_s = \rho_1^s$, $K_s = \rho_1 \Sigma_{s-1}$, $(-1)^{i+j} B_{ij} = \Sigma_{i-1} A_1(i,j)$, $A_1(i,j) = \rho_1^{j-i} \Sigma_{T-j}$, and the solution reduces to

$$w_{ij}^{(T)} = (-\rho_1)^{j-i} \frac{\Sigma_{i-1}\Sigma_{T-j}}{\Sigma_T}, \tag{2.38}$$

which is of form given, for example, by Shaman [12].

Hence, in principle at least, this method gives a complete solution to the problem of finding $\underset{\sim}{\Sigma}_T^{-1}$. For large T the computations involved may be quite laborious. As we saw in (2.29) the components along the main diagonal are functions of the Σ determinants only; but if $i \neq j$, even in row $i = 1$ we already have all types of determinants. In effect

$$\Sigma_T w_{1j} (-1)^{j+1} = \Sigma_{T-j} L_{j-1} - \rho_2 K_{T-j} L_{j-2} + \rho_2^3 \Sigma_{T-j-1} L_{j-3}, \tag{2.39}$$

$$j = 2,3,\ldots,T.$$

3. The inverse of $\underset{\sim}{I} + \rho \underset{\sim}{G}_1$ by solving difference equations.

A different approach will now be used to find $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T$ in the case of $m = 1$, that is, $\underset{\sim}{\Sigma}_T = \underset{\sim}{I} + \rho \underset{\sim}{G}_1$. Of course, we assume $\rho \neq 0$. By the definition of an inverse

$$(3.1) \qquad \underset{\sim}{I} = \underset{\sim}{\Sigma}_T \underset{\sim}{W}_T = (\underset{\sim}{I} + \rho \underset{\sim}{G}_1)\underset{\sim}{W}_T = \underset{\sim}{W}_T + \rho \underset{\sim}{G}_1 \underset{\sim}{W}_T.$$

In terms of components (3.1) is

$$(3.2) \qquad \delta_{ij} = \rho w_{i-1,j} + w_{ij} + \rho w_{i+1,j}, \qquad j = 1,2,\ldots,T; \quad i = 2,3,\ldots,T-1,$$

$$(3.3) \qquad \delta_{1j} = w_{1j} + \rho w_{2j}, \qquad j = 1,2,\ldots,T \ (i=1),$$

$$(3.4) \qquad \delta_{Tj} = \rho w_{T-1,j} + w_{Tj}, \qquad j = 1,2,\ldots,T \ (i=T),$$

where δ_{ij} is Kronecker's delta function. We consider the solution of (3.2) as a second-order, linear difference equation in i, for each fixed j; that is, we proceed column by column. The associated polynomial equation is

$$(3.5) \qquad \rho x^2 + x + \rho = 0,$$

which has roots

$$(3.6) \qquad x_1 = \frac{-1+\sqrt{1-4\rho^2}}{2\rho}, \qquad x_2 = \frac{-1-\sqrt{1-4\rho^2}}{2\rho}.$$

For any $\rho \neq 0$, $x_1 x_2 = 1$. For covariance matrices we further restrict our attention to the case $|\rho| < \frac{1}{2}$. Hence $|x_1| < 1$ and $|x_2| > 1$, or $|x_1| > 1$ and $|x_2| < 1$. We choose to present all results as functions of x_1, $|x_1| < 1$.

The general solution of (3.2) in the homogeneous case is then

$$w_{ij} = A(j)x_1^i + B(j)x_1^{-i}. \tag{3.7}$$

To find the complete solution we must take δ_{ij} into account. This can be done for example by expressing δ_{ij} as a linear combination of sines and cosines, since it is true that

$$(\delta_{ij}) = \underset{\sim}{I} = \underset{\sim}{M}\underset{\sim}{M}', \tag{3.8}$$

where $\underset{\sim}{M}$ is orthonormal. See Anderson [3], Section 4.2.2. Instead of this direct approach, we shall use (3.2) only when $\delta_{ij} = 0$; in particular we shall restrict attention to the w_{ij}'s above and on the main diagonal.

Let us consider the T-th column first. Its components satisfy

$$\begin{array}{lll} 0 = & w_{1T} + \rho w_{2T} & \text{(row 1)}, \\ 0 = & \rho w_{1T} + w_{2T} + \rho w_{3T} & \text{(row 2)}, \\ \vdots & \vdots & \vdots \\ 0 = & \rho w_{T-2,T} + w_{T-1,T} + \rho w_{TT} & \text{(row T-1)}, \\ 1 = & \rho w_{T-1,T} + w_{TT} & \text{(row T)}. \end{array} \tag{3.9}$$

The equations from rows 2 through T-1 have as solution the sequence

$$(3.10) \qquad w_{iT} = A(T)x_1^{i}+B(T)x_1^{-i}, \qquad i = 1,2,\ldots,T.$$

The equations for rows 1 and T will be used as boundary conditions to find $A = A(T)$ and $B = B(T)$:

$$\begin{aligned}
0 &= (Ax_1+Bx_1^{-1}) + \rho(Ax_1^2+Bx_1^{-2}) \\
&= Ax_1(1+\rho x_1) + Bx_1^{-2}(x_1+\rho) \\
&= Ax_1\left(-\frac{\rho}{x_1}\right) + Bx_1^{-2}(-\rho x_1^2) = -\rho(A+B), \\
1 &= \rho(Ax_1^{T-1}+Bx_1^{-T+1}) + (Ax_1^{T}+Bx_1^{-T}) \\
&= Ax_1^{T-1}(\rho+x_1) + Bx_1^{-T}(\rho x_1+1) \\
&= -\rho[Ax_1^{T+1}+Bx_1^{-(T+1)}].
\end{aligned}$$

Hence

$$(3.11) \qquad \begin{pmatrix} A(T) \\ B(T) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_1^{T+1} & x_1^{-(T+1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -\frac{1}{\rho} \end{pmatrix} = \frac{x_1^{T+1}}{\rho(1-x_1^{2T+2})} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and

$$(3.12) \qquad w_{iT} = \frac{1}{\rho(1-x_1^{2T+2})}(x_1^{T+1+i}-x_1^{T+1-i}), \qquad i = 1,2,\ldots,T.$$

We next consider the (T-1)-st column, for which the set of equations is

$$
\begin{array}{llr}
 & 0 = \qquad\qquad w_{1,T-1} + \rho w_{2,T-1} & \text{(row 1)}, \\
 & 0 = \rho w_{1,T-1} + w_{2,T-1} + \rho w_{3,T-1} & \text{(row 2)}, \\
 & \vdots \qquad\qquad \vdots & \vdots \\
(3.13) & & \\
 & 0 = \rho w_{T-3,T-1} + w_{T-2,T-1} + \rho w_{T-1,T-1} & \text{(row T-2)}, \\
 & 1 = \rho w_{T-2,T-1} + w_{T-1,T-1} + \rho w_{T,T-1} & \text{(row T-1)}, \\
 & 0 = \rho w_{T-1,T-1} + w_{T,T-1} & \text{(row T)}.
\end{array}
$$

The equations in rows 2 through T-2 have solution $w_{i,T-1} = A(T-1)x_1^i + B(T-1)x_1^{-i}$. Row 1 provides one boundary condition and row T-1 the other, provided we make the value of $w_{T,T-1}$ explicit (Rows 2 through T-2 involve $w_{i,T-1}$ only up to $i = T-1$.) Since $\underset{\sim}{W}_T$ is symmetric, this is achieved by letting $w_{T,T-1} = w_{T-1,T}$, where $w_{T-1,T}$ was already evaluated in column T.

In this manner we proceed column by column to derive a general expression for w_{ij}, $i \leq j$. We now prove that for $s = 0,1,\ldots,T-3$, say,

$$(3.14) \quad w_{i,T-s} = \frac{1+x_1^2+\ldots+x_1^{2s}}{\rho(1-x_1^{2T+2})}\left(x_1^{T+1-s+i} - x_1^{T+1-s-i}\right), \quad i \leq T-s.$$

We already proved that (3.14) holds for $s = 0$ ($j = T$). Suppose it holds for $s (0 \le s \le T-5$, say); it suffices to show that from this assumption we can show it holds for $s + 1$.

Column $T-s-1$ gives rise to the equations

$$
\begin{aligned}
0 &= w_{1,T-s-1} + \rho w_{2,T-s-1}, \\
0 &= \rho w_{i-1,T-s-1} + w_{i,T-s-1} + \rho w_{i+1,T-s-1}, \qquad i = 2,3,\ldots,T-s-2, \\
1 &= \rho w_{T-s-2,T-s-1} + w_{T-s-1,T-s-1} + \rho w_{T-s,T-s-1},
\end{aligned}
\tag{3.15}
$$

and other equations for components below the main diagonal. We use the first and third lines of (3.15) as boundary conditions to determine $A(T-s-1)$ and $B(T-s-1)$, taking $w_{T-s,T-s-1} = w_{T-s-1,T-s} \equiv w^*$ as given from the known solution of column $T-s$. The equations are

$$
\begin{aligned}
A(T-s-1) + B(T-s-1) &= 0, \\
A(T-s-1)x_1^{T-s} + B(T-s-1)x_1^{-T+s} &= w^* - \frac{1}{\rho},
\end{aligned}
\tag{3.16}
$$

and the solution is

$$
\begin{pmatrix} A(T-s-1) \\ B(T-s-1) \end{pmatrix} = \frac{x_1^{T-s}}{1-x_1^{2T-2s}} \left(w^* - \frac{1}{\rho}\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\tag{3.17}
$$

$$= \frac{x_1^{T-s}}{1-x_1^{2T-2s}} \left[- \frac{1}{\rho} \frac{(1-x_1^{2s+4})(1-x_1^{2T-2s})}{(1-x_1^2)(1-x_1^{2T+2})} \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{1+x_1^2+\dots+x_1^{2s+2}}{\rho(1-x_1^{2T+2})} \; x_1^{T-s} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Hence

$$(3.18) \quad w_{i,T-s-1} = \frac{1+x_1^2+\dots+x_1^{2s+2}}{\rho(1-x_1^{2T+2})} \left(x_1^{T-s+i} - x_1^{T-s-i}\right), \qquad i = 1,2,\dots,T-s-1,$$

which is what we wanted to show.

When s equals $T-1$ or $T-2$, i.e. when we deal with columns $j = 1$ and 2, the system (3.15) does not hold because then most of the components in the column are below the main diagonal. Since $\underset{\sim}{W}_T$ is symmetric with respect to its two main diagonals, from the components in columns $T, T-1$, etc. we can deduce those in columns $1, 2$, etc., respectively, by means of the relations

$$(3.19) \qquad w_{ij} = w_{ji} = w_{T-i+1,T-j+1}, \qquad i,j = 1,2,\dots,T.$$

Substituting $j = T-s$ in (3.14), and $(1-x_1^{2s+2})/(1-x_1^2) = 1 + x_1^2 + \dots + x_1^{2s}$, we have

Proposition 3.1. Let $\Sigma_T = I_T + \rho G_1$, with $\rho \neq 0$, and $\Sigma_T^{-1} = W_T = (w_{ij}^{(T)})$. Let $x_1 = (1/2\rho)(-1+\sqrt{1-4\rho^2})$ and assume that $1-4\rho^2 > 0$. Then

$$(3.20)\quad w_{ij}^{(T)} = \frac{1-x_1^{2T-2j+2}}{\rho(1-x_1^2)(1-x_1^{2T+2})}(x_1^{j+1+i} - x_1^{j+1-i}), \quad j = 1,2,\ldots,T; \quad i \leq j.$$

Two different forms of this formula are derived in the Appendix using a different argument.

Shaman [12] gave this result as

$$w_{ij}^{(T)} = \frac{(-1)^{j-i}[(-x_1)^{-i}-(-x_1)^{i}][(-x_1)^{-T+j-1}-(-x_1)^{T-j+1}]}{\sqrt{1-4\rho^2}\,[(-x_1)^{-T-1}-(-x_1)^{T+1}]}$$

$$(3.21)\quad = \frac{x_1^{-T+j-1}(1-x_1^{2T-2j+2})(x_1^{-i}-x_1^{i})}{x_1^{-T-1}\sqrt{1-4\rho^2}\,(1-x_1^{2T+2})}$$

$$= \left[-\frac{\rho(1-x_1^2)}{x_1\sqrt{1-4\rho^2}}\right]\frac{1-x_1^{2T-2j+2}}{\rho(1-x_1^2)(1-x_1^{2T+2})}(x_1^{j+1+i} - x_1^{j+1-i}),$$

which agrees with (3.20) because the quantity inside the brackets equals 1.

From (3.20) we can derive some approximations, provided $|x_1| < 1$ and T is large. Taking $i \leq j$ as in (3.20) we have

$$(3.22) \qquad w_{ij}^{(T)} \cong \frac{(1-x_1^{2T-2j+2})(x_1^{j+1+i}-x_1^{j+1-i})}{\rho(1-x_1^2)}, \qquad \text{for any } j;$$

$$(3.23) \qquad w_{ij}^{(T)} \cong \frac{x_1^{j+1+i}-x_1^{j+1-i}}{\rho(1-x_1^2)}, \qquad \text{for } j \text{ small};$$

$$(3.24) \qquad w_{ij}^{(T)} \cong -\frac{(1-x_1^{2T-2j+2})x_1^{j+1-i}}{\rho(1-x_1^2)}, \qquad \text{for } j \text{ large.}$$

In particular, under the same assumptions, we have that the following approximations are very good for columns 1 and T:

$$(3.25) \qquad w_{i1}^{(T)} \cong -\frac{x_1^i}{\rho}, \qquad w_{iT}^{(T)} \cong -\frac{x_1^{T+1-i}}{\rho}.$$

The present author [10] found use for approximations like those in (3.22)-(3.25) in developing some asymptotic statistical theory, since the omitted parts behave like x_1^T or x_1^{2T}, and when $|x_1| < 1$ they could be safely neglected for limit purposes.

Shaman [12] gave the approximation

$$(3.26) \qquad w_{ij}^{(T)} \cong \frac{(-1)^{j-i}(-x_1)^{j-i}}{\sqrt{1-4\rho^2}} = \frac{x_1^{j+1-i}}{x_1(1+2\rho x_1)}, \qquad \text{all } j, \quad i \leq j,$$

based on a different argument.

4. The inverse of $\underset{\sim}{I} + \rho_1\underset{\sim}{G}_1 + \rho_2\underset{\sim}{G}_2$ by solving difference equations.

We now apply the basic idea of Section 3 to the case $m = 2$, that is, $\underset{\sim}{\Sigma}_T = \underset{\sim}{I} + \rho_1\underset{\sim}{G}_1 + \rho_2\underset{\sim}{G}_2$, $\rho_2 \neq 0$. $\underset{\sim}{\Sigma}_T$ is assumed positive definite, but otherwise no further restrictions are placed on ρ_1 and ρ_2. Then

$$\underset{\sim}{I} = \underset{\sim}{\Sigma}_T\underset{\sim}{W}_T = \underset{\sim}{W}_T+\rho_1\underset{\sim}{G}_1\underset{\sim}{W}_T+\rho_2\underset{\sim}{G}_2\underset{\sim}{W}_T \ . \tag{4.1}$$

In terms of components (4.1) is, for all j,

$$\delta_{1j} = w_{1j}+\rho_1 w_{2j} +\rho_2 w_{3j} \qquad (i=1), \tag{4.2}$$

$$\delta_{2j} = \rho_1 w_{1j} +w_{2j}+\rho_1 w_{3j} +\rho_2 w_{4j} \qquad (i=2), \tag{4.3}$$

$$\delta_{ij} = \rho_2 w_{i-2,j}+\rho_1 w_{i-1,j}+w_{ij}+\rho_1 w_{i+1,j}+\rho_2 w_{i+2,j} \ , \quad i=3,4,\ldots,T-2, \tag{4.4}$$

$$\delta_{T-1,j} = \rho_2 w_{T-3,j}+\rho_1 w_{T-2,j}+w_{T-1,j}+\rho_1 w_{Tj} \qquad (i=T-1), \tag{4.5}$$

$$\delta_{Tj} = \rho_2 w_{T-2,j}+\rho_1 w_{T-1,j}+w_{Tj} \qquad (i=T). \tag{4.6}$$

We plan to solve the equations in (4.2)-(4.6) for each column (j fixed) to find the components of $\underset{\sim}{W}_T$ above the main diagonal. The characteristic equation associated with the fourth-order linear difference equation (4.4) is

$$\rho_2 x^4+\rho_1 x^3+x^2+\rho_1 x+\rho_2 = 0, \tag{4.7}$$

or dividing through by $\rho_2 (\rho_2 \neq 0)$,

(4.8)
$$x^4 + \frac{\rho_1}{\rho_2} x^3 + \frac{1}{\rho_2} x^2 + \frac{\rho_1}{\rho_2} x + 1 = 0.$$

Since (4.8) is symmetric in its coefficients it can be treated as equations (2.7) or (2.20). We are thus led to find the roots of

(4.9)
$$x^2 - d_i x + 1 = 0, \qquad i = 1,2,$$

where the d_i satisfy the system

(4.10)
$$d_1 + d_2 = -\frac{\rho_1}{\rho_2}, \qquad 2 + d_1 d_2 = \frac{1}{\rho_2}.$$

This system is like (2.11) and hence has solutions

(4.11)
$$d_1 = \frac{1}{2}\left[-\frac{\rho_1}{\rho_2} + \sqrt{\left(\frac{\rho_1}{\rho_2}\right)^2 + 4\left(2 - \frac{1}{\rho_2}\right)}\right], \quad d_2 = \frac{1}{2}\left[-\frac{\rho_1}{\rho_2} - \sqrt{\left(\frac{\rho_1}{\rho_2}\right)^2 + 4\left(2 - \frac{1}{\rho_2}\right)}\right],$$

and the four roots of (4.7) can be labeled x_1, $1/x_1$, x_2, $1/x_2$.

If the roots are distinct, the solution of (4.4) in the homogeneous case is given by the sequence

(4.12)
$$w_{ij} = C_1^*(j)x_1^i + C_2^*(j)x_1^{-i} + C_3^*(j)x_2^i + C_4^*(j)x_2^{-i}, \quad j=1,2,\ldots,T; \quad i=1,2,\ldots,T.$$

If the roots are real and either $x_1 = x_2$ or $x_1 = 1/x_2$, then the solution can be written as

$$(4.13)\quad w_{ij} = [C_1(j)+iC_2(j)]x_1^i+[C_3(j)+iC_4(j)]x_1^{-i}, \quad i,j = 1,2,\ldots,T.$$

For the sake of illustration we now consider the evaluation of the constants in (4.13), where we take $|x_1| < 1$. This case arose in the study of Walker's paper [16], with $\rho_1 = \frac{2\rho}{1+2\rho^2}$, $\rho_2 = \frac{\rho^2}{1+2\rho^2}$. See Mentz [10].

The components in column T of $\underset{\sim}{W}_T$ satisfy the equations

$$
\begin{aligned}
0 &= \phantom{\rho_2 w_{i-2,T} + \rho_1 w_{i-1,T} +{}} w_{1T} + \rho_1 w_{2T} + \rho_2 w_{3T} && \text{(row 1)},\\
0 &= \phantom{\rho_2 w_{i-2,T} +{}} \rho_1 w_{1T} + w_{2T} + \rho_1 w_{3T} + \rho_2 w_{4T} && \text{(row 2)},\\
(4.14)\quad 0 &= \rho_2 w_{i-2,T} + \rho_1 w_{i-1,T} + w_{iT} + \rho_1 w_{i+1,T} + \rho_2 w_{i+2,T}, && i = 3,\ldots,T-2,\\
0 &= \rho_2 w_{T-3,T} + \rho_1 w_{T-2,T} + w_{T-1,T} + \rho_1 w_{TT} && \text{(row } T-1),\\
1 &= \rho_2 w_{T-2,T} + \rho_1 w_{T-1,T} + w_{TT} && \text{(row } T).
\end{aligned}
$$

The solution of the difference equation for rows $3,\ldots,T-2$ is (4.13). We can use the equations for rows $1,2,T-1$ and T as boundary conditions to specify the $C_h = C_h(T)$, $h = 1,2,3,4$. For example the first equation gives

$$
\begin{aligned}
0 &= (C_1+C_2)x_1+(C_3+C_4)x_1^{-1}+\rho_1(C_1+2C_2)x_1^2+\rho_1(C_3+2C_4)x_1^{-2}\\
&\qquad +\rho_2(C_1+3C_2)x_1^3+\rho_2(C_3+3C_4)x_1^{-3}\\
(4.15)\qquad &= C_1(x_1+\rho_1x_1^2+\rho_2x_1^3)+C_2(x_1+2\rho_1x_1^2+3\rho_2x_1^3)+C_3(x_1^{-1}+\rho_1x_1^{-2}+\rho_2x_1^{-3})\\
&\qquad +C_4(x_1^{-1}+2\rho_1x_1^{-2}+3\rho_2x_1^{-3})\\
&\equiv C_1a_{11}+C_2a_{12}+C_3a_{13}+C_4a_{14},
\end{aligned}
$$

say. The remaining conditions lead to

$$0 = C_1(\rho_1x_1+x_1^2+\rho_1x_1^3+\rho_2x_1^4)+C_2(\rho_1x_1+2x_1^2+3\rho_1x_1^3+4\rho_2x_1^4)$$

$$+C_3(\rho_1x_1^{-1}+x_1^{-2}+\rho_1x_1^{-3}+\rho_2x_1^{-4})+C_4(\rho_1x_1^{-1}+2x_1^{-2}+3\rho_1x_1^{-3}+4\rho_2x_1^{-4})$$

$$\equiv C_1a_{21}+C_2a_{22}+C_3a_{23}+C_4a_{24} \;;$$

$$(4.16)\quad 0 = C_1x_1^T(\rho_2x_1^{-3}+\rho_1x_1^{-2}+x_1^{-1}+\rho_1)+C_2Tx_1^T(\tfrac{T-3}{T}\rho_2x_1^{-3}+\tfrac{T-2}{T}\rho_1x_1^{-2}+\tfrac{T-1}{T}x_1^{-1}+\rho_1)$$

$$+C_3x_1^{-T}(\rho_2x_1^3+\rho_1x_1^2+x_1+\rho_1)+C_4Tx_1^{-T}(\tfrac{T-3}{T}\rho_2x_1^3+\tfrac{T-2}{T}\rho_1x_1^2+\tfrac{T-1}{T}x_1+\rho_1)$$

$$\equiv C_1x_1^Ta_{31}+C_2Tx_1^Ta_{32}^{(0)}+C_3x_1^{-T}a_{33}+C_4Tx_1^{-T}a_{34}^{(0)} \;;$$

$$1 = C_1x_1^T(\rho_2x_1^{-2}+\rho_1x_1^{-1}+1)+C_2Tx_1^T(\tfrac{T-2}{T}\rho_2x_1^{-2}+\tfrac{T-1}{T}\rho_1x_1^{-1}+1)$$

$$+C_3x_1^{-T}(\rho_2x_1^2+\rho_1x_1+1)+C_4Tx_1^{-T}(\tfrac{T-2}{T}\rho_2x_1^2+\tfrac{T-1}{T}\rho_1x_1+1)$$

$$\equiv C_1x_1^Ta_{41}+C_2Tx_1^Ta_{42}^{(0)}+C_3x_1^{-T}a_{43}+C_4Tx_1^{-T}a_{44}^{(0)} .$$

The use of the superscript "(0)" will be clarified below. If $\underset{\sim}{C}(T) = (C_1(T),C_2(T),C_3(T),C_4(T))'$ is the vector of constants for column T, then

$$(4.17)\quad \underset{\sim}{C}(T) = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \\ x_1^T\begin{pmatrix} a_{31} & Ta_{32}^{(0)} \\ a_{41} & Ta_{42}^{(0)} \end{pmatrix} & x_1^{-T}\begin{pmatrix} a_{33} & Ta_{34}^{(0)} \\ a_{43} & Ta_{44}^{(0)} \end{pmatrix} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} \\ x_1^T \underset{\sim}{A}_{21}^{(0)} & x_1^{-T} \underset{\sim}{A}_{22}^{(0)} \end{pmatrix}^{-1} \begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{u} \end{pmatrix} \equiv \begin{pmatrix} \underset{\sim}{B}_{11} & \underset{\sim}{B}_{12} \\ \underset{\sim}{B}_{21} & \underset{\sim}{B}_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{u} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{B}_{12} \underset{\sim}{u} \\ \underset{\sim}{B}_{22} \underset{\sim}{u} \end{pmatrix},$$

where $\underset{\sim}{u} = (0,1)'$ and the introduction of the $\underset{\sim}{A}_{ij}$'s and $\underset{\sim}{B}_{ij}$'s is self-explanatory. By partitioned inversion

$$\underset{\sim}{B}_{22} = \left(x_1^{-T} \underset{\sim}{A}_{22}^{(0)} - x_1^T \underset{\sim}{A}_{21}^{(0)} \underset{\sim}{A}_{11}^{-1} \underset{\sim}{A}_{12}\right)^{-1} = x_1^T \left(\underset{\sim}{A}_{22}^{(0)} - x_1^{2T} \underset{\sim}{A}_{21}^{(0)} \underset{\sim}{A}_{11}^{-1} \underset{\sim}{A}_{12}\right)^{-1}$$

$$(4.18) \qquad = x_1^T \begin{pmatrix} a_{33} + x_1^{2T} q_{33}^{(0)} & Ta_{34}^{(0)} + x_1^{2T} q_{34}^{(0)} \\ a_{43} + x_1^{2T} q_{43}^{(0)} & Ta_{44}^{(0)} + x_1^{2T} q_{44}^{(0)} \end{pmatrix}^{-1}$$

$$= \frac{x_1^T}{\Delta_0} \begin{pmatrix} Ta_{44}^{(0)} + x_1^{2T} q_{44}^{(0)} & -Ta_{34}^{(0)} - x_1^{2T} q_{34}^{(0)} \\ -a_{43} - x_1^{2T} q_{43}^{(0)} & a_{33} + x_1^{2T} q_{33}^{(0)} \end{pmatrix},$$

where

$$(4.19) \qquad \Delta_0 = T\left(a_{33} a_{44}^{(0)} - a_{34}^{(0)} a_{43}\right) + x_1^{2T}\left(a_{33} q_{44}^{(0)} + Ta_{44}^{(0)} q_{33}^{(0)} - Ta_{34}^{(0)} q_{43}^{(0)} - a_{43} q_{34}^{(0)}\right) + x_1^{4T}\left(q_{33}^{(0)} q_{44}^{(0)} - q_{43}^{(0)} q_{34}^{(0)}\right),$$

and $(q_{ij}^{(0)}) = -\underset{\sim}{A}_{21}^{(0)} \underset{\sim}{A}_{11}^{-1} \underset{\sim}{A}_{12}$. Note that $q_{ij}^{(0)} = q_{ij}^{'(0)} + q_{ij}^{''(0)}$, for some coefficients $q_{ij}^{'(0)}$ and $q_{ij}^{''(0)}$ that depend on T only through expressions like $(T-t)/T$, $t = 1,2,3$. Note also that

$$(4.20) \qquad \underset{\sim}{B}_{22} = \frac{x_1^T}{\Delta_0}\left[\begin{pmatrix} Ta_{44}^{(0)} & -Ta_{34}^{(0)} \\ -a_{43} & a_{33} \end{pmatrix} + x_1^{2T}\begin{pmatrix} q_{44}^{(0)} & -q_{34}^{(0)} \\ -q_{43}^{(0)} & q_{33}^{(0)} \end{pmatrix}\right].$$

Now

$$\underset{\sim}{B}_{12} = -\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\underset{\sim}{B}_{22}$$

$$(4.21) \qquad = \frac{x_1^T}{\Delta_0}\left[-\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\begin{pmatrix} Ta_{44}^{(0)} & -Ta_{34}^{(0)} \\ -a_{43} & a_{33} \end{pmatrix} - x_1^{2T}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\begin{pmatrix} q_{44}^{(0)} & -q_{34}^{(0)} \\ -q_{43}^{(0)} & q_{33}^{(0)} \end{pmatrix}\right]$$

$$= \frac{x_1^T}{\Delta_0}\left[\begin{pmatrix} b_{13}^{'(0)} & b_{14}^{'(0)} \\ b_{23}^{'(0)} & b_{24}^{'(0)} \end{pmatrix} + x_1^{2T}\begin{pmatrix} b_{13}^{''(0)} & b_{14}^{''(0)} \\ b_{23}^{''(0)} & b_{24}^{''(0)} \end{pmatrix}\right],$$

say. Hence $\underset{\sim}{B}_{12}$ and $\underset{\sim}{B}_{22}$ have the same kind of structure. Finally we have that

$$(4.22) \qquad \underset{\sim}{C}(T) = \frac{x_1^T}{\Delta_0} \begin{pmatrix} b_{14}^{'(0)} + b_{14}^{''(0)} x_1^{2T} \\ b_{24}^{'(0)} + b_{24}^{''(0)} x_1^{2T} \\ -Ta_{34}^{(0)} - q_{34}^{(0)} x_1^{2T} \\ a_{33} + q_{33}^{(0)} x_1^{2T} \end{pmatrix},$$

and this completes the solution of (4.13) for column T.

For column $T-1$ the set of equations is obtained from (4.14) by replacing T by $T-1$ in the column index of w_{ij}, and interchanging 0 with 1 in the left-hand sides of rows $T-1$ and T. All components $w_{i,T-1}$, $i = 1,2,\ldots,T$, satisfy the homogeneous difference equation and hence the last two equations in the set can still be used as boundary conditions. Hence

$$(4.23) \qquad \underset{\sim}{C}(T-1) = \frac{x_1^T}{\Delta_0} \begin{pmatrix} b_{13}^{'(0)} + b_{13}^{''(0)} x_1^{2T} \\ b_{23}^{'(0)} + b_{23}^{''(0)} x_1^{2T} \\ Ta_{44}^{(0)} + q_{44}^{(0)} x_1^{2T} \\ -a_{43} - q_{43}^{(0)} x_1^{2T} \end{pmatrix}.$$

The remaining components on and above the main diagonal can be found using Proposition A1 of the Appendix. It is argued there that in the present case condition (A.1) can be replaced by an equivalent one relating the first two rows to each of the remaining ones. The conditions we use here are

$$\theta_{1i}w_{1j}+\theta_{2i}w_{2j} = w_{ij}, \qquad i = 3,4,\ldots,T; \quad j \geq i, \tag{4.24}$$

for columns $j = [T/2]+1,\ldots,T$; cf. Greenberg and Sarhan, [7]. Then the following steps will provide the needed components:

Step 1. Find columns T and $T-1$ of $\underset{\sim}{W}_T$, from (4.12), (4.13), or similar, and (4.22), (4.23), or similar, depending upon the nature of the roots of (4.7);

Step 2. Using (3.19) deduce rows 1 and 2 from columns T and $T-1$.

Step 3. Determine the proportionality constants θ_{st} in (4.24), from columns T and $T-1$;

Step 4. Using repeatedly (4.24), find those components w_{ij} satisfying $i = j,\ldots,T-j+1$; $j = [T/2]+1,\ldots,T-2$.

Proposition 4.1. Let $\underset{\sim}{\Sigma}_T$ and $\underset{\sim}{W}_T$ be as in Proposition 2.1. Then the $w_{ij}^{(T)}$'s are determined by means of the procedure consisting of Steps 1 through 4 described above, together with (3.19).

Proposition 4.1 gives a constructive way to obtain $\underset{\sim}{\Sigma}^{-1}$. We now proceed recursively as in Section 3, trying to gain some insight into the final form of the $w_{ij}^{(T)}$'s, and to suggest some approximations. It must be anticipated that for this case, when $m = 2$, it was not possible to obtain a closed, explicit expression like that of (3.20) when $m = 1$.

From the solution of columns T and $T-1$, namely (4.13) together with (4.22) and (4.23), we can derive sequentially all columns of $\underset{\sim}{W}_T$. Consider column $T-s$, for $s = 2,3,\ldots,T-5$, say. Its components satisfy the equations

$$
\begin{aligned}
0 &= w_{1,T-s} + \rho_1 w_{2,T-s} + \rho_2 w_{3,T-s} && (i=1),\\
0 &= \rho_1 w_{1,T-s} + w_{2,T-s} + \rho_1 w_{3,T-s} + \rho_2 w_{4,T-s} && (i=2),\\
0 &= \rho_2 w_{i-2,T-s} + \rho_1 w_{i-1,T-s} + w_{i,T-s} + \rho_1 w_{i+1,T-s} + \rho_2 w_{i+2,T-s}, && i=3,\ldots,T-s-2,\\
0 &= \rho_2 w_{T-s-3,T-s} + \rho_1 w_{T-s-2,T-s} + w_{T-s-1,T-s} + \rho_1 w_{T-s,T-s} + \rho_2 w_{T-s+1,T-s} && (i=T-s-1),\\
1 &= \rho_2 w_{T-s-2,T-s} + \rho_1 w_{T-s-1,T-s} + w_{T-s,T-s} + \rho_1 w_{T-s+1,T-s} + \rho_2 w_{T-s+2,T-s} && (i=T-s),
\end{aligned}
\tag{4.25}
$$

plus other homogeneous equations for rows $i > T-s$.

The solution of the equations in rows 3 through $T-s-2$ is given by the sequence (4.13) for $j = T-s$ and $i = 1,2,\ldots,T-s$. The equations for $i = 1,2$ in (4.25) provide two boundary conditions. Using the symmetry of $\underset{\sim}{W}_T$ we can link the equations for rows $T-s-1$ and $T-s$ with the components in columns $T-s+1$ and $T-s+2$, which we take as known, by letting $w_{T-s+1,T-s} = w_{T-s,T-s+1} \equiv w_s^*$, and $w_{T-s+2,T-s} =$

$= w_{T-s,T-s+2} \equiv w_s^{**}$. Hence the boundary conditions are

$$
(4.26)\quad
\begin{aligned}
&w_{1,T-s} + \rho_1 w_{2,T-s} + \rho_2 w_{3,T-s} = 0,\\
&\rho_1 w_{1,T-s} + w_{2,T-s} + \rho_1 w_{3,T-s} + \rho_2 w_{4,T-s} = 0,\\
&\rho_2 w_{T-s-3,T-s} + \rho_1 w_{T-s-2,T-s} + w_{T-s-1,T-s} + \rho_1 w_{T-s,T-s} = -\rho_2 w_s^{*},\\
&\rho_2 w_{T-s-2,T-s} + \rho_1 w_{T-s-1,T-s} + w_{T-s,T-s} = 1-\rho_1 w_s^{*} - \rho_2 w_s^{**}.
\end{aligned}
$$

These equations can be solved as we did in (4.15)-(4.23) when $s = 0$ and 1. The system for $s = 2,3,\ldots,T-5$ is

$$
(4.27)\quad
\underset{\sim}{C}(T-s) = \begin{pmatrix}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \\
x_1^{T-s}\begin{pmatrix} a_{31} & (T-s)a_{32}^{(s)} \\ a_{41} & (T-s)a_{42}^{(s)} \end{pmatrix} & x_1^{-T+s}\begin{pmatrix} a_{33} & (T-s)a_{34}^{(s)} \\ a_{43} & (T-s)a_{44}^{(s)} \end{pmatrix}
\end{pmatrix}^{-1}
\begin{pmatrix} 0 \\ 0 \\ -\rho_2 w_s^{*} \\ 1-\rho_1 w_s^{*}-\rho_2 w_s^{**} \end{pmatrix},
$$

where for $s = 0,1,\ldots,T-5$,

$$
(4.28)\quad
\begin{aligned}
a_{32}^{(s)} &= \frac{T-s-3}{T-s}\rho_2 x_1^{-3} + \frac{T-s-2}{T-s}\rho_1 x_1^{-2} + \frac{T-s-1}{T-s} x_1^{-1} + \rho_1 ,\\
a_{34}^{(s)} &= \frac{T-s-3}{T-s}\rho_2 x_1^{3} + \frac{T-s-2}{T-s}\rho_1 x_1^{2} + \frac{T-s-1}{T-s} x_1 + \rho_1 ,\\
a_{42}^{(s)} &= \frac{T-s-2}{T-s}\rho_2 x_1^{-2} + \frac{T-s-1}{T-s}\rho_1 x_1^{-1} + 1 ,\\
a_{44}^{(s)} &= \frac{T-s-2}{T-s}\rho_2 x_1^{2} + \frac{T-s-1}{T-s}\rho_1 x_1 + 1 .
\end{aligned}
$$

By an argument parallel to that of (4.17)-(4.23), it follows that

(4.29)
$$\underset{\sim}{C}(T-s) = \frac{x_1^{T-s}}{\Delta_s}\begin{pmatrix} \left[\begin{pmatrix} b_{13}^{'(s)} & b_{14}^{'(s)} \\ b_{23}^{'(s)} & b_{24}^{'(s)} \end{pmatrix} + x_1^{2(T-s)}\begin{pmatrix} b_{13}^{''(s)} & b_{14}^{''(s)} \\ b_{23}^{''(s)} & b_{24}^{''(s)} \end{pmatrix}\right] \begin{pmatrix} -\rho_2 w_s^{*} \\ 1-\rho_1 w_s^{*}-\rho_2 w_s^{**} \end{pmatrix} \\ \left[\begin{pmatrix} (T-s)a_{44}^{(s)} & -(T-s)a_{34}^{(s)} \\ -a_{43} & a_{33} \end{pmatrix} + x_1^{2(T-s)}\begin{pmatrix} q_{44}^{(s)} & -q_{34}^{(s)} \\ -q_{43}^{(s)} & q_{33}^{(s)} \end{pmatrix}\right] \begin{pmatrix} -\rho_2 w_s^{*} \\ 1-\rho_1 w_s^{*}-\rho_2 w_s^{**} \end{pmatrix} \end{pmatrix},$$

where

$$\Delta_s = (T-s)(a_{33}a_{44}^{(s)}-a_{34}^{(s)}a_{43})-x_1^{2(T-s)}\Big[(T-s)(a_{34}^{(s)}q_{43}^{(s)}-a_{44}^{(s)}q_{33}^{(s)})$$
$$+(a_{43}q_{34}^{(s)}-a_{33}q_{44}^{(s)})\Big]+x_1^{4(T-s)}(q_{44}^{(s)}q_{33}^{(s)}-q_{43}^{(s)}q_{34}^{(s)}),$$

$$(q_{ij}^{(s)}) = -\underset{\sim}{A}_{21}^{(s)}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}, \qquad i,j = 3,4,$$

(4.30)
$$\begin{pmatrix} b_{13}^{'(s)} & b_{14}^{'(s)} \\ b_{23}^{'(s)} & b_{24}^{'(s)} \end{pmatrix} = -\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\begin{pmatrix} (T-s)a_{44}^{(s)} & -(T-s)a_{34}^{(s)} \\ -a_{43} & a_{33} \end{pmatrix},$$

$$\begin{pmatrix} b_{13}^{''(s)} & b_{14}^{''(s)} \\ b_{23}^{''(s)} & b_{24}^{''(s)} \end{pmatrix} = -\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\begin{pmatrix} q_{44}^{(s)} & -q_{34}^{(s)} \\ -q_{43}^{(s)} & q_{33}^{(s)} \end{pmatrix},$$

and the $\underset{\sim}{A}_{ij}$ arise from (4.27).

From (4.29) it is clear that we can justify the following:

Proposition 4.2. Let $\underset{\sim}{\Sigma}_T$ and $\underset{\sim}{W}_T$ be as in Proposition 2.1. Then the $w_{ij}^{(T)}$'s have the form

$$(4.31)\quad w_{i,T-s}^{(T)} = (F_1+iF_2)x_1^{T-s+i}+(F_3+iF_4)x_1^{T-s-i}+(F_5+iF_6)x_1^{3(T-s)+i}$$
$$+(F_7+iF_8)x_1^{3(T-s)-i}, \quad s = 2,\dots,T-5; \quad i \leq T-s,$$

where the F_h's depend on s (column index), provided equation (4.7) has roots $x_1 = x_2$, $x_3 = x_4 = 1/x_1$.

Analogous expressions can be derived for other patterns of roots. We omit those details here.

For many purposes it appears that in (4.22) and (4.23) one can discard the part with x_1^{2T} as a factor, and further approximate Δ_0 in (4.19) by its first term. Then

$$(4.32)\quad \underset{\sim}{C}(T) \cong \frac{x_1^T}{T(a_{33}a_{44}^{(0)}-a_{34}^{(0)}a_{43})}\begin{pmatrix} b_{14}^{'(0)} \\ b_{24}^{'(0)} \\ -Ta_{34}^{(0)} \\ a_{33} \end{pmatrix}, \quad \underset{\sim}{C}(T-1) \cong \frac{x_1^T}{T(a_{33}a_{44}^{(0)}-a_{34}^{(0)}a_{43})}\begin{pmatrix} b_{13}^{'(0)} \\ b_{23}^{'(0)} \\ Ta_{44}^{(0)} \\ -a_{43} \end{pmatrix}.$$

Further for T-s reasonably large, $a_{32}^{(s)}$, $a_{34}^{(s)}$, $a_{42}^{(s)}$ and $a_{44}^{(s)}$ can be taken as equal for all relevant s, and respectively equal to

$$
(4.33)\qquad
\begin{aligned}
a_{32} &= \rho_2 x_1^{-3} + \rho_1 x_1^{-2} + x_1^{-1} + \rho_1 = a_{31}, \\
a_{34} &= \rho_2 x_1^{3} + \rho_1 x_1^{2} + x_1 + \rho_1 = a_{33}, \\
a_{42} &= \rho_2 x_1^{-2} + \rho_1 x_1^{-1} + 1 = a_{41}, \\
a_{44} &= \rho_2 x_1^{2} + \rho_1 x_1 + 1 = a_{43} .
\end{aligned}
$$

Then there would exist more similarity between (4.17) and (4.29) for $s > 0$, and the computations in the recursive procedure will be simplified.

5. <u>A general case.</u>

In this section we analyze the general case

$$\underset{\sim}{\Sigma}_T = \sum_{j=0}^{m} \rho_j \underset{\sim}{G}_j, \tag{5.1}$$

where $\underset{\sim}{G}_0 = \underset{\sim}{I}$, $\rho_0 = 1$, $\rho_m \neq 0$, and $1 \leq m < T$.

The approach of Section 2 can in principle be attempted to find the components $w_{ij}^{(T)}$ of $\underset{\sim}{W}_T = \underset{\sim}{\Sigma}_T^{-1}$. By a careful examination one can find out which of the non-zero minors in the first $i-1$ columns of the cofactor of σ_{ij} have non-zero complementary cofactors.

We observed in the case $m = 2$ that we needed three classes of determinants satisfying linear difference equations of orders 5, 4 and 2, respectively; of course if these classes could all be reduced to a single one (say the Σ_T), this would be achieved at the cost of augmenting the order of the difference equations involved. Hence in general we expect complicated expressions for the $w_{ij}^{(T)}$.

The approaches of Sections 3 and 4 seem more promising to find $\underset{\sim}{W}_T$, or at least some of its columns (or rows). By the definition of inverse

$$\underset{\sim}{I} = \underset{\sim}{\Sigma}_T \underset{\sim}{W}_T = \sum_{j=0}^{m} \rho_j \underset{\sim}{G}_j \underset{\sim}{W}_T = \underset{\sim}{W}_T + \sum_{j=1}^{m} \rho_j \underset{\sim}{G}_j \underset{\sim}{W}_T, \tag{5.2}$$

which in terms of components is

$$
(5.3)\qquad
\begin{array}{lcl}
\delta_{1j} & = & w_{1j} + \rho_1 w_{2j} + \cdots + \rho_m w_{m+1,j} \quad (i=1), \\
\delta_{2j} & = & \rho_1 w_{1j} + w_{2j} + \rho_1 w_{3j} + \cdots + \rho_m w_{m+2,j} \quad (i=2), \\
\vdots & & \qquad \vdots \\
\delta_{mj} & = & \rho_{m-1} w_{1j} + \cdots + \rho_1 w_{m-1,j} + w_{mj} + \rho_1 w_{m+1,j} + \cdots + \rho_m w_{2m,j} \quad (i=m), \\
\delta_{ij} & = & \rho_m w_{i-m,j} + \cdots + \rho_1 w_{i-1,j} + w_{ij} + \rho_1 w_{i+1,j} + \cdots + \rho_m w_{i+m,j}, \quad i=m+1,\ldots,T-m, \\
\delta_{T-m+1,j} & = & \rho_m w_{T-2m+1,j} + \cdots + \rho_1 w_{T-m,j} + w_{T-m+1,j} + \rho_1 w_{T-m+2,j} + \cdots + \rho_{m-1} w_{Tj} \quad (i=T-m+1), \\
\vdots & & \qquad \vdots \\
\delta_{Tj} & = & \rho_m w_{T-m,j} + \cdots + \rho_1 w_{T-1,j} + w_{Tj} \quad (i=T).
\end{array}
$$

This system will be solved for each j (fixed column) and $i \leq j$, that is, for components above and on the main diagonal.

The linear difference equation of order $2m$ corresponding to rows $m+1,\ldots,T-m$, in the homogeneous case, has the associated characteristic equation

$$
(5.4)\qquad
\begin{aligned}
0 &= \rho_m z^{2m} + \rho_{m-1} z^{2m-1} + \cdots + \rho_m \\
&= \rho_m (z^{2m}+1) + \rho_{m-1}(z^{2m-1}+z) + \cdots + \rho_1 (z^{m+1}+z^{m-1}) + z^m,
\end{aligned}
$$

which is symmetric in its coefficients. Dividing through by $\rho_m (\rho_m \neq 0)$ we have

$$
(5.5)\qquad 0 = (z^{2m}+1) + \frac{\rho_{m-1}}{\rho_m}(z^{2m-1}+z) + \cdots + \frac{\rho_1}{\rho_m}(z^{m-1}+z^{m+1}) + \frac{1}{\rho_m} z^m.
$$

The roots of (5.5) occur in pairs, z_j and $1/z_j (z_j \neq 0)$. Hence (5.5) equals

$$(5.6) \qquad 0 = \prod_{j=1}^{2m} (z-z_j) = \prod_{j=1}^{m} (z-z_j)(z - \frac{1}{z_j}) = \prod_{j=1}^{m} (z^2-zd_j+1),$$

where

$$(5.7) \qquad d_j = z_j + \frac{1}{z_j}, \qquad z_j^2 - d_j z_j + 1 = 0, \qquad j = 1,2,\ldots,m,$$

or

$$(5.8) \qquad z_j = \frac{1}{2}(d_j \pm \sqrt{d_j^2-4}), \qquad j = 1,2,\ldots,m.$$

Equating coefficients between (5.5) and (5.6) we are left with a system of nonlinear equations to be solved for the d_j's. We already found they are given by (4.10) when $m = 2$; when $m = 3$ they are

$$(5.9) \qquad \begin{aligned} -\frac{\rho_2}{\rho_3} &= d_1+d_2+d_3, \\ \frac{\rho_1}{\rho_3} &= 3+d_1d_2+d_1d_3+d_2d_3, \\ -\frac{1}{\rho_3} &= 2d_1+2d_2+2d_3+d_1d_2d_3. \end{aligned}$$

In general we will have m nonlinear equations to be solved for $d_1,\dots,d_m$. Even when the order of the difference equation is $2m$, the system is of order m. The system and (5.8) will furnish the $2m$ roots of (5.4), say, $z_1, 1/z_1, \dots, z_m, 1/z_m$.

The procedure can be viewed in another equivalent formulation. (5.5) is equal to

$$
\begin{aligned}
0 &= \frac{1}{\rho_m} z_m + \sum_{h=1}^{m} \frac{\rho_h}{\rho_m} (z^{m+h}+z^{m-h}) \\
&= \frac{1}{\rho_m} z_m + z^m \sum_{h=1}^{m} \frac{\rho_h}{\rho_m} z^{-h}(z^{2h}+1).
\end{aligned}
\tag{5.10}
$$

Let us substitute $s = z+1/z = (z^2+1)/z$, i.e., $zs = z^2+1$. Then

$$
\begin{aligned}
z^n s^n = (z^2+1)^n &= z^{2n}+1+nz^2[z^{2(n-2)}+1]+\cdots+\binom{n}{r}z^n, && \text{if } n = 2r, \\
&= z^{2n}+1+nz^2[z^{2(n-2)}+1]+\cdots+\binom{n}{r}z^{n-1}(z^2+1), && \text{if } n = 2r+1,
\end{aligned}
\tag{5.11}
$$

and after successive substitutions

$$
z^{2n}+1 = z^n(s^n+C_{n2}s^{n-2}+\cdots), \tag{5.12}
$$

for some coefficients C_{nj}. Using (5.12) we see that (5.10) becomes

$$0 = z^m\left[\frac{1}{\rho_m} + \sum_{h=1}^{m} \frac{\rho_h}{\rho_m}\,\frac{z^{2h}+1}{z^h}\right]$$

$$(5.13)\qquad = z^m\left[\frac{1}{\rho_m} + \sum_{h=1}^{m} \frac{\rho_h}{\rho_m}(s^h + C_{h2}s^{h-2} + \cdots)\right]$$

$$= z^m\left[\frac{\rho_h}{\rho_m}\, s^m + \cdots + \left(\frac{1}{\rho_m} + \cdots\right)\right],$$

where $z \neq 0$. For full details see Anderson [4].

Hence we have reduced the problem of solving (5.4), which is a polynomial equation of order $2m$ (in z), to that of solving (5.13), which has also a polynomial equation (in s), but of order m. Once $s_1,\ldots,s_m$ are available, the roots of (5.4) are given by

$$(5.14)\qquad z_i = \frac{s_i}{2} - \sqrt{\left(\frac{s_i}{2}\right)^2 - 1}, \qquad \frac{1}{z_i} = \frac{s_i}{2} + \sqrt{\left(\frac{s_i}{2}\right)^2 - 1}, \qquad i = 1,2,\ldots,m.$$

If $z_1, z_2, \ldots, z_m$ are distinct, then

$$(5.15)\qquad w_{ij}^{(T)} = \sum_{h=1}^{m}\left[C_h(j)\,z_h^{i} + C_h'(j)\,z_h^{-i}\right], \qquad i \leq j.$$

The form (5.15) will have to be altered if some of the z_h's coincide.

The $2m$ constants in the final expression for $w_{ij}^{(T)}$ can be evaluated from boundary conditions extracted from (5.3), for each j. In particular for columns $1,2,\ldots,m$ and $T-m+1,\ldots,T$, the boundary conditions are just the first m and last m equations in (5.3); in fact one of the resulting linear systems will determine the constants for all columns.

To determine the rest of the components, say, above and on the main diagonal, we can use the two approaches introduced in Section 4: the procedure involving the result stated in the Appendix, and the recursive procedure that works column by column.

Application of the result in the Appendix leads to:

Proposition 5.1. Let $\underset{\sim}{\Sigma}_T$ be given by (5.1), with $\underset{\sim}{G}_0 = \underset{\sim}{I}$, $\rho_0 = 1$, and $\rho_m \neq 0$, $1 \leq m < T$, and let $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T = (w_{ij}^{(T)})$. Then the $w_{ij}^{(T)}$'s are determined by means of a procedure consisting of the following four steps:

Step 1. Find columns $T-m+1,\ldots,T$ of $\underset{\sim}{W}_T$ from (5.15) or similar expressions, depending upon the nature of the roots of (5.4), where the $2m$ constants are evaluated from boundary conditions provided by the first m and last m equations in (5.3);

Step 2. Using (3.19) deduce rows $1,2,\ldots,m$, from columns $T-m+1,\ldots,T$;

Step 3. Determine the proportionality constants θ_{st} in

(5.16) $$\theta_{1i}w_{1j}^{(T)}+\theta_{2i}w_{2j}^{(T)}+\cdots+\theta_{mi}w_{mj}^{(T)} = w_{ij}, \qquad i = m+1,\ldots,T; \qquad j \geq i,$$

(for columns $j=[T/2]+1,\ldots,T-m$) from columns $T-m+1,\ldots,T$;

Step 4. Using repeatedly (5.16) find those components w_{ij} satisfying $i = j,\ldots,T-j+1$; $j = [T/2]+1,\ldots,T-m$, and all others using (3.19).

The application of the "column by column" procedure can be summarized as follows:

Proposition 5.2. Under the same hypotheses of Proposition 5.1, the $w_{ij}^{(T)}$ for $i \leq j$ are given by (5.15) or similar expressions, depending upon the nature of the roots of (5.4). The $2m$ constants are evaluated from boundary conditions extracted from the system (5.3), as follows: For columns $T,T-1,\ldots,T-m+1$, the conditions are the first m and last m equations in (5.3); for columns $j = T-m,T-m-1,\ldots,[T/2]+1$, say, we assume that columns $j+1,j+2,\ldots,j+m$ are already available, and then the conditions are the first m equations, plus equations for rows $j-m+1,\ldots,j$ with the substitutions

$$
\begin{aligned}
w_{j+1,j}^{(T)} &= w_{j,j+1}^{(T)} \equiv w_{1j}^{*}, \\
w_{j+2,j}^{(T)} &= w_{j,j+2}^{(T)} \equiv w_{2j}^{*}, \\
&\vdots \\
w_{j+m,j}^{(T)} &= w_{j,j+m}^{(T)} \equiv w_{mj}^{*};
\end{aligned}
\tag{5.17}
$$

the remaining columns can be obtained using (3.19), by symmetry of $\underset{\sim}{W}_T$.

To prove Proposition 5.1 one only notes that (5.16) is equivalent to the condition of Proposition A.1 in the Appendix; see comments there.

To prove Proposition 5.2 we note that for column j, $[T/2]+1 \leq j \leq T-m$, the complete system of equations is composed of the first m in (5.3), plus the homogeneous equations

$$(5.18)\qquad w_{ij} + \sum_{h=1}^{m} \rho_h (w_{i-h,j} + w_{i+h,j}) = 0, \qquad i = m+1, m+2, \ldots, j-m,$$

plus equations

$$(5.19)\qquad \begin{array}{lllllll} 0 = & \rho_m w_{j-2m+1,j} + \cdots + \rho_1 w_{j-m,j} + w_{j-m+1,j} + \rho_1 w_{j-m+2,j} + \cdots + \rho_m w_{j+1,j} & (i=j-m+1), \\ \vdots & \vdots & \vdots \\ 0 = & \rho_m w_{j-m-1,j} + \cdots + \rho_1 w_{j-2,j} + w_{j-1,j} + \rho_1 w_{jj} + \cdots + \rho_m w_{j+m-1,j} & (i=j-1), \\ 1 = & \rho_m w_{j-m,j} + \cdots + \rho_1 w_{j-1,j} + w_{jj} + \rho_1 w_{j+1,j} + \cdots + \rho_m w_{j+m,j} & (i=j), \end{array}$$

plus other (homogeneous) equations for rows $i > j$. The sequence satisfying (5.18) is w_{ij}, $i = 1,2,\ldots,j$, so that the components in the left-hand sides of (5.17) have to be obtained from the columns already determined.

<u>Note 1</u>. To avoid vacuous statements, T is taken to be large enough compared with m.

<u>Note 2</u>. In Proposition 5.2 the equation for row j is inhomogeneous, and all other are homogeneous (i.e., $\delta_{ij} = 0$).

Note 3. In some particular instances the procedures described in Propositions 5.1 or 5.2 or both will disclose an explicit general expression for $w_{ij}^{(T)}$, $i \leq j$, as was exemplified in Section 3.

6. An approximation.

Suppose $\underset{\sim}{\Sigma}$ is $T \times T$ of Toeplitz type with components $\sigma_{|i-j|}$ that satisfy (1.5) for some m, $1 \leq m < T-1$. Further suppose that $\sigma_0, \sigma_1, \ldots, \sigma_m, 0, 0, \ldots$ are the covariances of a stationary process (that is, a covariance matrix of any order with these elements is positive semi definite). Then one can always find coefficients $\alpha_0, \alpha_1, \ldots, \alpha_m$ such that $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_{MA}$, the covariance matrix of a vector $(x_1, \ldots, x_T)$ of random variables generated by the "moving-average model"

$$(6.1) \qquad x_t = \alpha_0 \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \cdots + \alpha_m \varepsilon_{t-m}, \qquad t = \ldots, -1, 0, 1, \ldots,$$

where the ε_t's are independent, $\mathscr{E}\varepsilon_t = 0$, $\mathscr{E}\varepsilon_t^2 = 1$, for all t. See, for example, Anderson [3], Chapter 5.

This way of looking at $\underset{\sim}{\Sigma}$ gives rise to a different approximation for $\underset{\sim}{\Sigma}^{-1}$ that is frequently used, at least for small values of m, and that we now discuss in the general case. It consists in

$$(6.2) \qquad \underset{\sim}{\Sigma}_{MA}^{-1} \cong \underset{\sim}{\Sigma}_{AR},$$

where $\underset{\sim}{\Sigma}_{AR}$ is the covariance matrix of $(y_1, \ldots, y_T)$ generated by the "autoregression"

$$(6.3) \qquad \alpha_0 y_t + \alpha_1 y_{t-1} + \cdots + \alpha_m y_{t-m} = \varepsilon_t, \qquad t = \ldots, -1, 0, 1, \ldots,$$

where the ε_t's have the properties of those of (6.1).

Let $\Sigma_{MA} = [\sigma_{MA}(i,j)] = [\sigma_{MA}(i-j)]$, $\Sigma_{AR} = [\sigma_{AR}(i,j)] = [\sigma_{AR}(i-j)]$. The autoregressive covariances satisfy the Yule-Walker equations

$$(6.4) \qquad \sum_{r=0}^{m} \alpha_r \sigma_{AR}(-r) = 1, \qquad \sum_{r=0}^{m} \alpha_r \sigma_{AR}(s-r) = 0, \qquad s = 1,2,\ldots;$$

if the roots $x_1,\ldots,x_m$ of the polynomial equation

$$(6.5) \qquad M(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m = 0$$

are different, then

$$(6.6) \qquad \sigma_{AR}(h) = \sum_{s=1}^{m} K_s^* x_s^h, \qquad h = 1,2,3,\ldots,$$

and the constants K_s^* are determined from the first m equations in (6.4), namely

$$(6.7) \qquad \begin{aligned} 1 &= \alpha_0 \sigma_{AR}(0) + \cdots + \alpha_{m-1}\sigma_{AR}(-m+1) + \alpha_m \sigma_{AR}(-m), \\ 0 &= \alpha_0 \sigma_{AR}(1) + \cdots + \alpha_{m-1}\sigma_{AR}(-m+2) + \alpha_m \sigma_{AR}(-m+1), \\ &\ \vdots \\ 0 &= \alpha_0 \sigma_{AR}(m-1) + \cdots + \alpha_{m-1}\sigma_{AR}(0) + \alpha_m \sigma_{AR}(-1), \end{aligned}$$

together with

$$(6.8) \qquad \sigma_{AR}(-h) = \sigma_{AR}(h), \qquad h = 1,2,\ldots,m;$$

see Anderson [3], Chapter 5. Hence if $\underset{\sim}{\Sigma}_{MA}^{-1} = (w_{ij}^{(T)})$, then

$$(6.9) \qquad w_{ij}^{(T)} \cong \sigma_{AR}(i,j) = \sum_{s=1}^{m} K_s^* x_s^{j-i} = \sum_{s=1}^{m} K_s(j) x_s^{-i}, \qquad j \geq i,$$

where $K_s(j) = K_s^* x_s^j$ is constant along columns of $\underset{\sim}{\Sigma}_{AR}$. We want to compare (6.9) with the exact form (5.15), that is, with

$$(6.10) \qquad w_{ij}^{(T)} = \frac{1}{\sigma_{MA}(0)} \sum_{s=1}^{m} [C_s(j) z_s^i + C_s'(j) z_s^{-i}], \qquad j \geq i;$$

here the z_s are taken to be distinct, and they, together with their reciprocals, are the $2m$ roots of (5.4), that we now write as

$$(6.11) \qquad \begin{aligned} 0 &= \sigma_{MA}(m) z^{2m} + \cdots + \sigma_{MA}(m) \\ &= z^m [\sigma_{MA}(m) z^m + \cdots + \sigma_{MA}(m) z^{-m}] \\ &= z^m \sum_{h=-m}^{m} \sigma_{MA}(h) z^h, \end{aligned}$$

because $\sigma_{MA}(h) = \sigma_{MA}(-h)$; since no root equals zero, this shows that $z_1,\ldots,z_m,1/z_1,\ldots,1/z_m$ are also the $2m$ roots of

$$(6.12) \qquad 0 = \sum_{h=-m}^{m} \sigma_{MA}(h) z^h.$$

The key fact to interpret the approximation is the relation

$$\sum_{h=-m}^{m} \sigma_{MA}(h)z^h = M(z)M(z^{-1}), \tag{6.13}$$

taken from Anderson [3], Section 5.7. It asserts that the expression in z in the left-hand side of (6.13), formed by the covariances of the moving-average model, admits a factorization in terms of polynomials in z and z^{-1}, each formed by the coefficients of the model.

If in the approximation (6.9) we choose all roots x_s to be less than one in absolute value, which without loss of generality we can always do when dealing with second-order moments, it then appears that the approximation (6.9) consists in omitting from the exact expression (6.10) the part involving positive powers of those roots.

The constants in (6.9) and (6.10) are related in a similar fashion: the m equations in (6.7) bear to those in (5.19) to be used for columns not near the end points, the same type of relation that $M(z)$ bears to $M(z)M(z^{-1})$; further the m boundary conditions provided by rows $i = 1,2,\ldots,m$ of (5.3) to determine the C_s and C_s', constitute an "end effect" that is neglected as part of the approximating process. The first assertion follows because (5.19) is also expressible as

$$(6.14)\qquad \begin{aligned} \sum_{r=-m}^{m} \sigma_{MA}(r)w_{j-s+r,j} &= \sigma_{MA}(0), && s = 0, \\ &= 0, && s = 1,2,\ldots,m-1, \end{aligned}$$

while (6.7) is

$$\text{(6.15)}\qquad \sum_{r=0}^{m} \alpha_r \sigma_{AR}(s-r) = 1, \qquad s = 0,$$
$$= 0, \qquad s = 1,2,\ldots,m-1;$$

writing (6.13) as

$$\text{(6.16)}\qquad \sum_{r=-m}^{m} \sigma_{MA}(r) z^r = \left(\sum_{r=0}^{m} \alpha_r z^{s-r}\right)\left(\sum_{r'=0}^{m} \alpha_{r'} z^{-s+r'}\right),$$

the relation follows by identifying $w_{j-s+r,j}$ with z^r in the left-hand side, and $\sigma_{AR}(s-r)$ with z^{s-r} in the right-hand side, for each s. Further (6.8) can be taken to correspond to (5.17)

As an illustration, when $m = 1$, $\alpha_0 = 1$ and $\alpha_1 = \alpha$, (6.10) becomes

$$\text{(6.17)}\qquad w_{ij}^{(T)} = \frac{A(j)}{1+\alpha^2} x_1^i + \frac{B(j)}{1+\alpha^2} x_1^{-i},$$

where $x_1 = (1/2\rho)(-1+\sqrt{1-4\rho^2})$, $\rho = \alpha/(1+\alpha^2)$, and hence $x_1 = -\alpha$. From (3.20)

$$\text{(6.18)}\qquad A(j) = -B(j) = \frac{(1-x_1^{2T+2-2j})x_1^{j+1}}{\rho(1-x_1^2)(1-x_1^{2T+2})} = -x_1^j \frac{(1+x_1^2)(1-x_1^{2T+2-2j})}{(1-x_1^2)(1-x_1^{2T+2})},$$

so that

$$(6.19) \qquad w_{ij}^{(T)} = -\frac{x_1^j}{1-x_1^2}\,\frac{1-x_1^{2T+2-2j}}{1-x_1^{2T+2}}\,(x_1^i - x_1^{-i}), \qquad j \geq i.$$

For the first-order autoregression $y_t + \alpha y_{t-1} = \varepsilon_t$, the covariance sequence is $\sigma_{AR}(r) = (-\alpha)^r/(1-\alpha^2)$, so that

$$(6.20) \qquad w_{ij}^{(T)} \cong \frac{(-\alpha)^{j-i}}{1-\alpha^2} = \frac{x_1^j}{1-x_1^2}\,x_1^{-i} \equiv K(j)x_1^{-i}, \qquad j \geq i.$$

Hence in this case it is verified that the term containing x_1^i is neglected, and the "end effect" consists in taking $(1-x_1^{2T+2-2j})/(1-x_1^{2T+2})$ as approximately equal to one in the coefficient of x_1^{-i}.

When $m = 2$, if (6.5) has roots $x_1 \neq x_2$, then $\sigma_{AR}(r)$ is given by Anderson [3], page 174, and

$$(6.21) \qquad w_{ij}^{(T)} \cong \sigma_{AR}(j-i) = \frac{1}{(x_1-x_2)(1-x_1x_2)}\left(\frac{x_1^{j-i+1}}{1-x_1^2} - \frac{x_2^{j-i+1}}{1-x_2^2}\right), \qquad j \geq i,$$

which is of the form (6.9).

Another approach that is often used to analyze the approximations, is to relate $\underset{\sim}{\Sigma}_{MA}$ to $\underset{\sim}{\Sigma}_{AR}^{-1}$; see e.g. Durbin [5]. For $m = 1$, $\alpha_0 = 1$, $\alpha_1 = \alpha$,

$$(6.22)\qquad \underset{\sim}{\Sigma}_{AR}^{-1} = \begin{pmatrix} 1 & \alpha & 0 & \cdots & 0 & 0 \\ \alpha & 1+\alpha^2 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\alpha^2 & \alpha \\ 0 & 0 & 0 & \cdots & \alpha & 1 \end{pmatrix},$$

which differs from $\underset{\sim}{\Sigma}_{MA}$ in that the components in places $i = j = 1$ and $i = j = T$ are 1 instead of $1+\alpha^2$. For $m = 2$, $\alpha_0 = 1$,

$$(6.23)\qquad \underset{\sim}{\Sigma}_{AR}^{-1} = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_1 & 1+\alpha_1^2 & \alpha_1+\alpha_1\alpha_2 & \alpha_2 & \cdots & 0 & 0 & 0 \\ \alpha_2 & \alpha_1+\alpha_1\alpha_2 & 1+\alpha_1^2+\alpha_2^2 & \alpha_1+\alpha_1\alpha_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+\alpha_1^2+\alpha_2^2 & \alpha_1+\alpha_1\alpha_2 & \alpha_2 \\ 0 & 0 & 0 & 0 & \cdots & \alpha_1+\alpha_1\alpha_2 & 1+\alpha_1^2 & \alpha_1 \\ 0 & 0 & 0 & 0 & \cdots & \alpha_2 & \alpha_1 & 1 \end{pmatrix},$$

which differs from $\underset{\sim}{\Sigma}_{MA}$ in that the components along the main diagonal should all be $1 + \alpha_1^2 + \alpha_2^2$, and along the two adjacent diagonals should be $\alpha_1 + \alpha_1\alpha_2$. The forms of $\underset{\sim}{\Sigma}_{AR}^{-1}$ for different values of m were given by Siddiqui [14] and Wise [17].

For any m, there are differences between those components in two submatrices of order $m \times m$, located at both end points of the main diagonal.

APPENDIX

On a necessary and sufficient condition for a covariance matrix to have an inverse of diagonal type. In this appendix we state and prove in detail a result that appears to have originated with Guttman [9] and Ukita [15]. The conditions were used by Greenberg and Sarhan [7], who call them sufficient conditions.

A matrix $\underset{\sim}{A} = (a_{ij})$ is said to be "diagonal of type r" if $a_{ij} = 0$ whenever $|i-j| \geq r$. From (1.5) it follows that a stochastic process which is finitely correlated of order m, gives rise to covariance matrices which are diagonal of type $r = m+1$.

Proposition A.1. Let $\underset{\sim}{\Sigma} = (\sigma_{ij})$ be a $T \times T$ symmetric and positive definite matrix. A necessary and sufficient condition that $\underset{\sim}{\Sigma}^{-1} = (w_{ij})$ be diagonal of type r is that there exist constants b_{ts} such that for $t = 1,2,\ldots,T-r+1$

$$\text{(A.1)} \qquad \sigma_{tt'} + b_{t1}\sigma_{t+1,t'} + \cdots + b_{t,r-1}\sigma_{t+r-1,t'} = 0, \qquad t'=t+1,\ldots,T.$$

Proof. For the necessity of the condition, suppose that $\underset{\sim}{y} = (y_1,\ldots,y_T)'$ is a vector of jointly distributed random variables with $\mathscr{E}\underset{\sim}{y} = \underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}$ such that $\underset{\sim}{\Sigma}^{-1}$ is diagonal of type r. There exists an upper triangular matrix $\underset{\sim}{B} = (b_{ij})$ with $b_{ii} = 1$, and a diagonal matrix $\underset{\sim}{D} = (d_{ij})$ with $d_{ii} > 0$, such that

(A.2) $$\Sigma^{-1} = B'DB;$$

see e.g. Graybill [6], Section 8.6; B is "upper triangular of type r", i.e. $b_{ij} = 0$ if $j-i \geq r$.* Let us introduce the new random vector $u = (u_1,\ldots,u_T)'$, defined by

(A.3) $$u = By.$$

Then $\mathcal{E}u = 0$, $\mathcal{E}uu' = B\Sigma B' = BB^{-1}D^{-1}B'^{-1}B' = D^{-1}$, and the u_t's are uncorrelated. From (A.3) we deduce that

(A.4) $$u_t = y_t + b_{t1}y_{t+1} + \cdots + b_{t,r-1}y_{t+r-1}, \qquad t = 1,2,\ldots,T-r+1.$$

Solving (A.3) for y, we note that $B^{-1} = (b^{ij})$ is also upper triangular and hence

(A.5) $$y_t = \sum_{s=t}^{T} b^{ts}u_s, \qquad t = 1,2,\ldots,T.$$

---

* From Graybill [6] it follows that there exists an upper triangular matrix T such that $\Sigma^{-1} = T'T$. Since Σ^{-1} is positive definite, $t_{ii} \neq 0$. Since Σ^{-1} is diagonal of type r, T is upper triangular of type r. By choosing the diagonal matrix Δ conveniently, $\Delta T = B$ can be left upper triangular of type r and with $b_{ii} = 1$. Then $D^{-1} = \Delta\Delta' = \Delta^2$, and $d_{ii} > 0$.

Multiplying (A.4) by (A.5) with t replaced by t', $t' = t+1,\ldots,T$, and taking expected values we have that

$$\text{(A.6)} \qquad 0 = \sigma_{tt'} + b_{t1}\sigma_{t+1,t'} + \cdots + b_{t,r-1}\sigma_{t+r-1,t'}, \qquad t' = t+1,\ldots,T,$$

which is (A.1).

Sufficiency. We have to show that if (A.1) holds, $w_{ij} = 0$ for $|i-j| \geq r$. Consider $i > j$, that is, w_{ij} is below the main diagonal. The $(T-1) \times (T-1)$ cofactor of σ_{ji} will be evaluated by Laplace's expansion using minors, formed by its last $T-r-j$ columns. For any one of these minors, if its complementary minor does not include row 1 of $\underset{\sim}{\Sigma}$, then the first r columns of the latter are linearly dependent. If row 1 of $\underset{\sim}{\Sigma}$ is included, by using the relation (A.1) successively and at most $j-1$ times the complementary minor can be brought into the equivalent form

$$\text{(A.7)} \qquad \begin{vmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} \\ \underset{\sim}{0} & \underset{\sim}{M}_{22} \end{vmatrix},$$

where $\underset{\sim}{M}_{11}$ is upper triangular and the first r columns of $\underset{\sim}{M}_{22}$ are linearly dependent. In either case the complementary minor is 0 and hence $w_{ij} = 0$. Q.E.D.

Condition (A.1) states the existence of a linear relation between successive sets of r adjacent rows of $\underset{\sim}{\Sigma}$; an equivalent and often useful formulation is to relate the first $r-1$ rows to each of the remaining ones. For $r = 2$ and 3 the conditions were written explicitly by Greenberg and Sarhan [7]; when $r = 2$ these reduce to

$$\text{(A.8)} \qquad \frac{\sigma_{ij}}{\sigma_{1j}} = \lambda_i, \qquad i = 2,3,\ldots,T; \quad j = i,i+1,\ldots,T,$$

where of course we require the relevant components in the first row to be non-zero.

As an application of (A.8), let us rederive (3.20). In Section 3 the given matrix, $\underset{\sim}{\Sigma}$, is known to be diagonal of type $r = 2$.

From (3.12) we know that

$$\text{(A.9)} \qquad w_{iT} = \frac{x_1^{T+1}}{\rho(1-x_1^{2T+2})}(x_1^i - x_1^{-i}), \qquad i = 1,2,\ldots,T;$$

using (3.19)

$$\text{(A.10)} \qquad w_{i1} = w_{T-i+1,T} = \frac{x_1^{T+1}}{\rho(1-x_1^{2T+2})}(x_1^{T-i+1} - x_1^{-T+i-1}), \quad i = 1,2,\ldots,T.$$

By symmetry of $\underset{\sim}{\Sigma}^{-1}$, the components in its first row are

$$\text{(A.11)} \qquad w_{1j} = \frac{x_1^{T+1}}{\rho(1-x_1^{2T+2})}(x_1^{T-j+1} - x_1^{-T+j-1}), \qquad j = 1,2,\ldots,T.$$

The proportionality constants are obtained from (A.9):

$$\text{(A.12)} \qquad \lambda_i = \frac{w_{iT}}{w_{1T}} = \frac{x_1^i - x_1^{-i}}{x_1 - x_1^{-1}}\,, \qquad i = 2,3,\ldots,T.$$

Applying these to (A.11) we obtain

$$\text{(A.13)} \qquad \begin{aligned} w_{ij} &= \frac{x_1^{T+1}(x_1^i - x_1^{-i})(x_1^{T-j+1} - x_1^{-T+j-1})}{\rho(x_1 - x_1^{-1})(1 - x_1^{2T+2})} \\ &= -\frac{(x_1^i - x_1^{-i})(x_1^{T-j+1} - x_1^{-(T-j+1)})}{\rho(x_1 - x_1^{-1})(x_1^{T+1} - x_1^{-(T+1)})}\,, \qquad i \le j, \end{aligned}$$

which are two new versions of (3.20).

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The covariance matrix $\underset{\sim}{\Sigma}_T = (\sigma_{ts})$ of $x_1,\ldots,x_T$, where $\{x_t: t = \ldots,-1,0,1,\ldots\}$ is a wide-sense stationary stochastic process, is of Toeplitz type; that is, $\sigma_{ts} = \sigma_{|t-s|}$, the covariance of x_t and x_s. If $\{x_t\}$ is a moving average process of order m, then $\sigma_j = 0$ for $j > m$. For $m = 2$ $\underset{\sim}{\Sigma}_T^{-1}$ is obtained by evaluation of cofactors. For $m = 1$ and 2 difference equations are set up for σ^{ts} for fixed s, where $\underset{\sim}{\Sigma}_T^{-1} = (\sigma^{ts})$. These two methods are extended to the case of arbitrary m. Approximations to $\underset{\sim}{\Sigma}_T^{-1}$ are also given.

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