NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

A PROPOSED METHOD FOR DEFINING AND MEASURING WEAPONS DELIVERY SYSTEM ACCURACY

by

Richard Chester Macke

Thesis Advisor: D. R. Barr

March 1972

Approved for public release; distribution unlimited.

69

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Proposed Method for Defining and Measuring Weapons Delivery System Accuracy

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Master's Thesis; March 1972

5. AUTHOR(S) (First name, middle initial, last name)

Richard Chester Macke

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1972	69	24

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
The report was prepared	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

~~This paper~~ is in response to a growing concern about the adequacy of present measures and methodologies in depicting air-to-ground delivery system accuracy. An accuracy measure, measurement methodology and performance guarantee based on cost-effectiveness techniques are developed. The measure is based on the frequency of mission success. The methodology is based on Bayesian techniques using a multinomial distribution to represent the radial miss distance pattern. A technique for using an estimate of CEP to determine the prior parameters is developed. Actual and simulated impact data are used to compare the proposed methodology to historically accepted and other recently proposed techniques. The methods of cost-effectiveness are applied to guaranteeing an air-to-ground system in an effort toward making costs more controllable.

Security Classification

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Bayesian Analysis						
A-7E						
Cost-Effectiveness Analysis						
Contract Guarantee						
Delivery Accuracy						
Air-to-Ground Accuracy						
Weapon System Analysis						
Bomb Accuracy						

DD FORM 1 NOV 65 1473 (BACK)
101-807-6821

Security Classification

A-31409

A Proposed Method for Defining
and Measuring Weapons Delivery
System Accuracy

by

Richard Chester Macke
Lieutenant Commander, United States Navy
B.S., United States Naval Academy, 1960

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
March 1972

Author Richard C Macke

Approved by: Donald R Barr
Thesis Advisor

Jack R Borsting
Chairman, Department of Operations
Research and Administrative Sciences

Milton U. Clauser
Academic Dean

ABSTRACT

This paper is in response to a growing concern about the adequacy of present measures and methodologies in depicting air-to-ground delivery system accuracy. An accuracy measure, measurement methodology and performance guarantee based on cost-effectiveness techniques are developed. The measure is based on the frequency of mission success. The methodology is based on Bayesian techniques using a multinomial distribution to represent the radial miss distance pattern. A technique for using an estimate of CEP to determine the prior parameters is developed. Actual and simulated impact data are used to compare the proposed methodology to historically accepted and other recently proposed techniques. The methods of cost-effectiveness are applied to guaranteeing an air-to-ground system in an effort toward making costs more controllable.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

The author gratefully acknowledges the institutional aid of The Center For Naval Analysis (CNA), The Naval Postgraduate School (NPGS) and The Naval Air Test Center (NATC). Many individuals from all three organizations contributed to the concepts and development contained herein. Of special significance were the use of a multinomial distribution suggested by CDR. Warren ROGERS, CNA, the background in current analysis techniques provided by Mr. Frank PHILLIPS, NATC, and Dr. Helmut DOST, CNA, and the constructive comments of Prof. Alan WASHBURN, NPGS. The advisory comments, direction and editing of Prof. Donald BARR, NPGS, were essential to the successful completion of the thesis. The A-7E data provided by the Operations Evaluation Group of CNA were influential in testing the proposed technique. A debt of gratitude is owed to all those listed and all the others who aided the author.

I. INTRODUCTION

A. PURPOSE

The intent of this thesis is threefold; to propose a measure of effectiveness; to propose a method to determine the specific effectiveness; and to propose the concept of a cost-effectiveness guarantee. All three are proposed with respect to an air-to-ground (A/G) weapons delivery system in a dive maneuver. However, the techniques proposed, with the necessary modifications, appear to have wider applications. Some of the development is incomplete and requires further investigation.

B. NEED FOR ACCURACY MEASUREMENT

Prior to developing a method for measuring the accuracy of an A/G delivery system, it is important to understand the need for accuracy measurement. Several needs are discussed.

1. Specification Compliance

In the Test and Evaluation (T&E) community a determination of specification compliance is usually the prime motive for measuring accuracy. The accuracy measure and occasionally methodology are specified in the applicable contract guarantees. Confusion can be introduced by the use of different measurement units and methodologies in different specifications.

A typical, though fictitious, example of an accuracy specification is:

> "Exhibit a 15 mil circular error probable(CEP), corrected for aiming error, computed in the plane normal to the line of sight from the release point when delivering Mk-76 practice bombs at a release airspeed between 400 and 450 KIAS, a dive angle between 40 and 50 degrees and a release slant range to target less than 10,000 feet."

Testing to this specification would probably be expensive

Also inherent in the specification example is the singular usage of the measure of accuracy. This measure in itself has little value outside of the specification context. Therefore, it appears that a significant amount of money and time would be expended with negligible information gain beyond specification compliance.

2. <u>Sortie Predictions</u>

In the employment of an A/G system there is a need for a different measure of accuracy. Strike planners and weaponeers need to estimate the probability of a weapon impacting within a given distance of a target. (The methodology used for this purpose is described later.) This is an important need as it is required throughout the lifetime of the system.

3. <u>System Comparisons</u>

Often a need arises to compare a system with some other system or systems. One of the measures of effectiveness that should be used for comparison is accuracy. Thus, a need arises for a measure of accuracy, common to the various systems being compared. An identical need exists if a cost-effectiveness analysis is to be conducted.

Three general requirements for measurement of accuracy have been discussed. As presented, each of these requirements uses a different measure and the measures need not be compatibly defined. In theory, it could occur that several separate determinations of accuracy would be required for a single system. In practice, this is not usual, however, a common technique would be beneficial so that all requirements could use the same measure.

C. HISTORICAL MEASUREMENT METHODS

Historically the measurement of accuracy of nearly all types of weapons delivery systems has involved probability distributions and the parameters that define these distributions. Several terms that are commonly used in weapons accuracy analysis are shown in Figure 1.

1. Distributions

The normal family of probability distributions has long been accepted as the proper family for the distribution of weapon impacts. Recently this concept has come under closer scrutiny. It has been suggested that a normal distribution may not always accurately depict a parent distribution of impacts and other families such as the Cauchy have been proposed [1].

a. Normal and Variations

Many different variations of the normal distribution have been used or proposed but only the two most common are discussed.

(1) Bivariate Normal. The bivariate normal is the more flexible of the two variations discussed, permitting the errors in range and deflection to be correlated. Letting x denote range and y denote deflection, the density function is:

$$(1)\quad f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)}\exp\{\frac{-1}{2(1-\rho^2)}[(\frac{x-\mu_x}{\sigma_x})^2 - 2\rho\,\frac{x-\mu_x}{\sigma_x}\,\frac{y-\mu_y}{\sigma_y} + (\frac{y-\mu_y}{\sigma_y})^2]\}$$

Often the range and deflection errors are assumed to be nearly uncorrelated so that ρ is approximately zero. A rotation of the coordinate system can also be used to eliminate the correlation but the resulting variables do not represent true range and deflection errors. The uncorrelated density reduces to:

$$(2)\quad f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\{\frac{-1}{2}[(\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2]\}$$

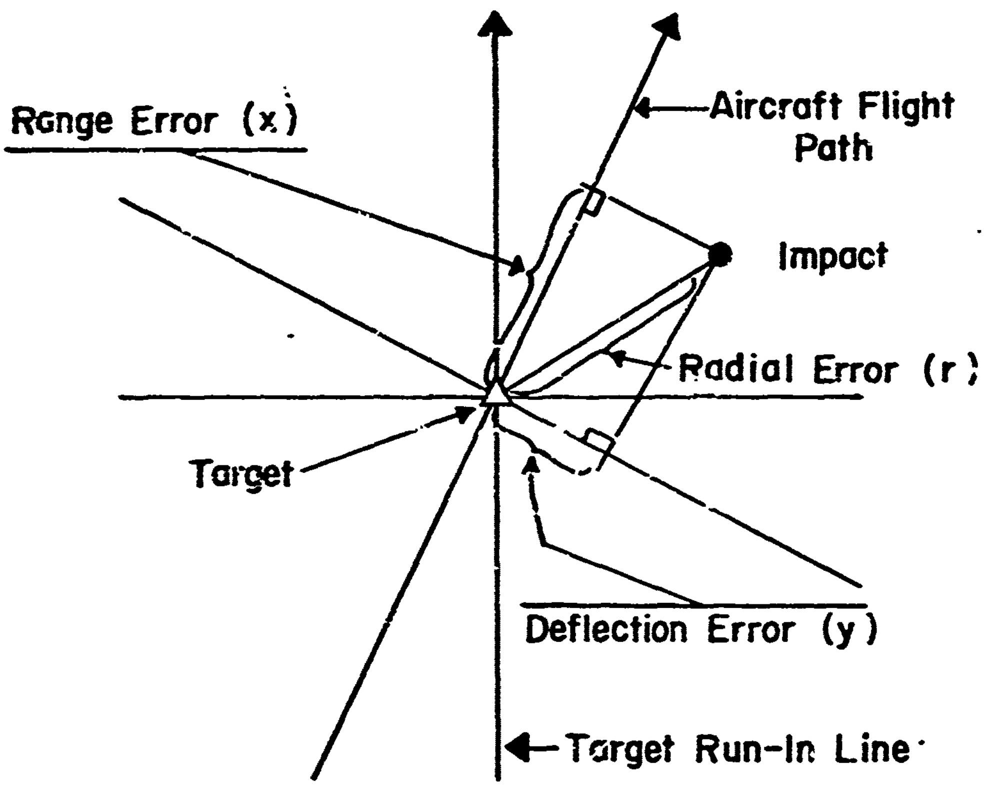


Figure I.

Depiction in the Ground Plane of terms commonly used in Weapons Accuracy Analysis.

The parameters are:

μ_x - mean of the range error distribution

μ_y - mean of the deflection error distribution

σ_x^2 - variance of the range error distribution

σ_y^2 - variance of the deflection error distribution

ρ - correlation between the range and deflection errors

Standard statistical techniques can be used to estimate these parameters from observed data.

(2) Circular Normal. The more commonly used normal distribution is the circular normal in which it is assumed that the range and deflection errors are independent (uncorrelated), have mean zero and have common variance. The circular normal cumulative function where r is the radial miss distance is:

$$(3) \quad F(r) = 1 - \exp\{-\frac{r^2}{2\sigma^2}\} \quad ; \quad r \geq 0$$

Procedures have been developed for use when $\sigma_x \neq \sigma_y$ [2, p. 3 and 4]; i.e., the distribution is an elliptical normal. An axis rotation can be used to eliminate the correlation between range and deflection.

An estimator for the σ^2 parameter is:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} r_i^2}{2N}$$

The circular normal has computational simplicity over the more general bivariate normal distribution, but the assumptions are quite restrictive.

b. Cauchy

An Air Force report [1] of an analysis of combat impacts observed in Southeast Asia found that the normal family did not yield a good distributional fit of the data. One of the distributions that was shown to closely approximate the data was the Cauchy.

The functional forms of the Cauchy were proposed;

(4) Rectangular Cauchy $F(x,y) = \frac{4}{\pi^2} \tan^{-1} \frac{x}{2\beta} \tan^{-1} \frac{y}{2\beta}$

(5) Circular Cauchy $F(r) = \frac{4}{\pi} \tan^{-1} \left(\frac{\beta^2}{\beta^2 + r^2} \right)^{1/2}$

The estimator for the parameter β was proposed as:

$\hat{\beta} = .455 \cdot$ (radial miss distance of the median impact)

The functional forms of the Cauchy proposed have the advantage of simplicity but lack mean values.

c. Direct Hits Plus Distribution

The Air Force study [1] pointed out that several impacts were observed to be directly on target contrary to the predictions of a continuous distribution function. This may be explained by considering the method of measuring the miss distances (photographic), the physical size of the target and some ego oriented, psychological attraction of the target itself. The method proposed to account for the positive mass at the target was to introduce a proportion of direct hits. The distribution function then becomes:

(6) $G(x,y) = \delta + (1-\delta) F(x,y)$ in the bivariate case, or

(7) $G(r) = \delta + (1-\delta) F(r)$ in the circular case.

The parameter δ could be estimated from prior experience or could possibly be some universal constant derived from many different systems.

d. Other Distributions

Literally hundreds of different distributions such as the Weibul, exponential or uniform or variations of distributions such as a mixture of two circular normals or a localized normal have been proposed. Two interesting facts emerge from these proposals. First, as the predictive ability of the distributions increases, the complexity of the functional form generally increases. This usually increases manipulation difficulty. Secondly, nearly all of the distributions or variations proposed are continuous.

2. Measures

There are several commonly accepted measures of accuracy which are functionally related to the parameters defining a unique distribution in an assumed family of impact distributions. These measures are graphically portrayed in Figure 2.

a. CEP

The circular error probable is defined as the radius of a circle centered at the target (or designated point) which contains 50 percent of the observed independent impacts, or, the radial distance to the median impact [4, p. 1-1, 1-2] The functional relationship of CEP to the distribution parameters is found from the following integral:

$$(8)\quad F(CEP) = \int_0^{CEP} f(r)\,dr = .5$$

As an example, the CEP using the circular normal distribution is derived as:

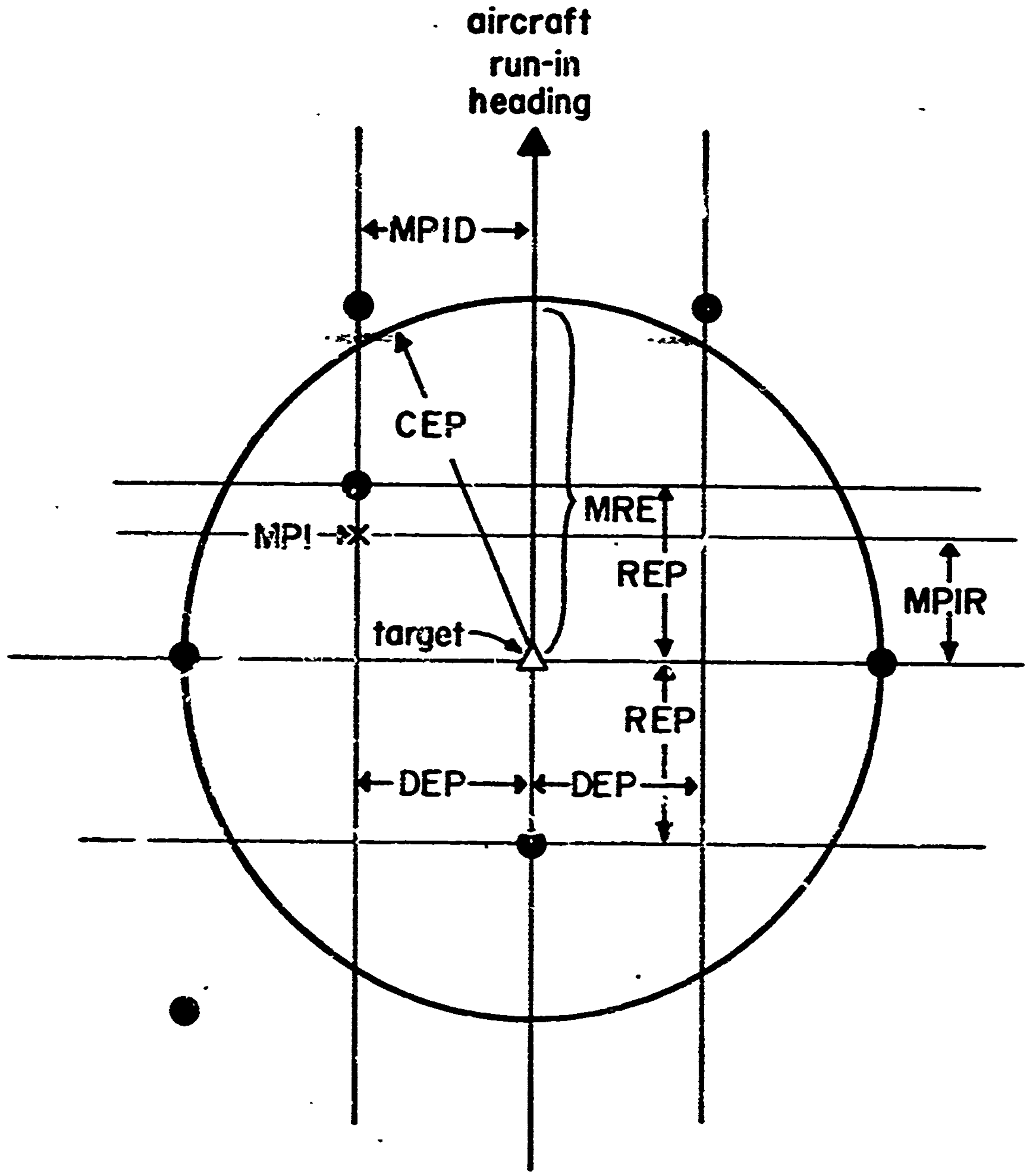


Figure 2.

Depiction of Accuracy Measures in the Ground Plane.

● Individual impact points
(7 randomly placed)

$$.5 = 1 - \exp\left(-\frac{CEP^2}{2\sigma^2}\right) = \exp\left(-\frac{CEP^2}{2\sigma^2}\right)$$

$$\ln(.5) = -\frac{CEP^2}{2\sigma^2}$$

$$\ln(2) = \frac{CEP^2}{2\sigma^2}$$

$$CEP = \sigma\sqrt{2\ln(2)} = 1.1774\sigma$$

b. REP/DEP

The range error probable and deflection error probable are similar to CEP. The REP is the distance from the target (or some designated point) to the median impact in range [4,p. 1-2,1-3]. The functional relationship of REP to the distributional parameters is:

$$(9)\quad F(REP) - F(-REP) = .5 = \int_{-REP}^{REP} f(x)\,dx$$ where $f(x)$ is the marginal distribution of x from $f(x,y)$

DEP is defined similarly. For a bivariate normal distribution, $REP = .674\sigma_x$ and $DEP = .674\sigma_y$.

It should be noted that if an axis rotation was used to uncorrelate the range errors and deflection errors, the functional form in equation (9) is no longer valid. This is a serious disadvantage of mathematically uncorrelating the data.

c. MRE

The mean radial error is defined as the mean of the radial miss distance distribution. It is derived below for the circular normal distribution:

$$f(r) = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}$$

$$MRE = E[r] = \int_0^{\infty} \frac{r^2}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} dr$$

this integral is solved [5,p. 458] as

$$E[r] = \frac{\sqrt{2\pi}}{2}\sigma = 1.253\sigma = MRE$$

Of the listed measures, the MRE is probably the least frequently used.

d. MPI

The mean point of impact is the point which has as its range and deflection coordinates the arithmetic means of the range and deflection coordinates of the individual impact points [4, p. 1-3]. The MPI is calculated from an observed sample of impacts rather than a hypothetical distribution. In many analyses (such as T&E) [Refs. 18, 20,23, and 24], the MPI is used as the center of the observed impact distribution instead of the target. The offset of the MPI from the target is termed a system bias and the CEP, REP, DEP and MRE are calculated with respect to the MPI. Obviously, this technique would not be useful in weaponeering applications.

Analogous to the MPI are the mean point of impact in range and deflection (MPIR, MPID) which are the arithmetic means of the impacts in range and deflection.

e. Other

Three other definitions given in the Joint Munitions Effectiveness Manual (JMEM) [4, p. 1-1,1-2] are of interest and are quoted.

(1) "Bombing Error. The combination of all errors which cause weapons to miss the target. Included are ballistic, aiming, release and aircraft system errors."

(2) "Ballistic Dispersion. The variation of the path of a weapon which is attributed to physical tolerances in the weapon dimensions and aerodynamic stability."

(3) "Delivery Accuracy. The measure of the ability of pilots[1] to put the weapon impact pattern center (usually MPI) on the target or aimpoint. The unit of measure of the variation in placement of the pattern center may be σ, CEP, or REP and DEP. It is these measures that are used in predicting the results of future weapon releases of the same type. Delivery accuracy is based on the errors in aiming, release and aircraft systems. It does not include ballistic errors."

3. Mils

In many applications the accuracy measures are expressed in mils (milliradians) perpendicular to the line of sight (LOS) at release or some other appropriate point along the aircraft flight path. The geometry involved in the computation of the mil is presented in Figure 3. A mil is usually defined as the angle subtended by a secant line of one foot length at a radius of 1000 feet [4, p. 4-2]. The deflection mil error (d_m) is related to the deflection foot error (d_f) below using the symbology of Figure 3:

(11) $$d_m = \frac{1000\ d_f}{s}$$

Due to the geometry, the deflection error in the ground plane is identical to the deflection error in the scoring plane, thus no correction is required. The range mil error is computed by the following equations:

(12) $$r_m = \frac{1000\ r_s}{s}$$ where r_s can be closely approximated (especially if s/r_s is large) by

(13) $$r_s = r_f \cos \theta$$

---

[1]It seems to this author that more is involved than just the ability of the pilot.

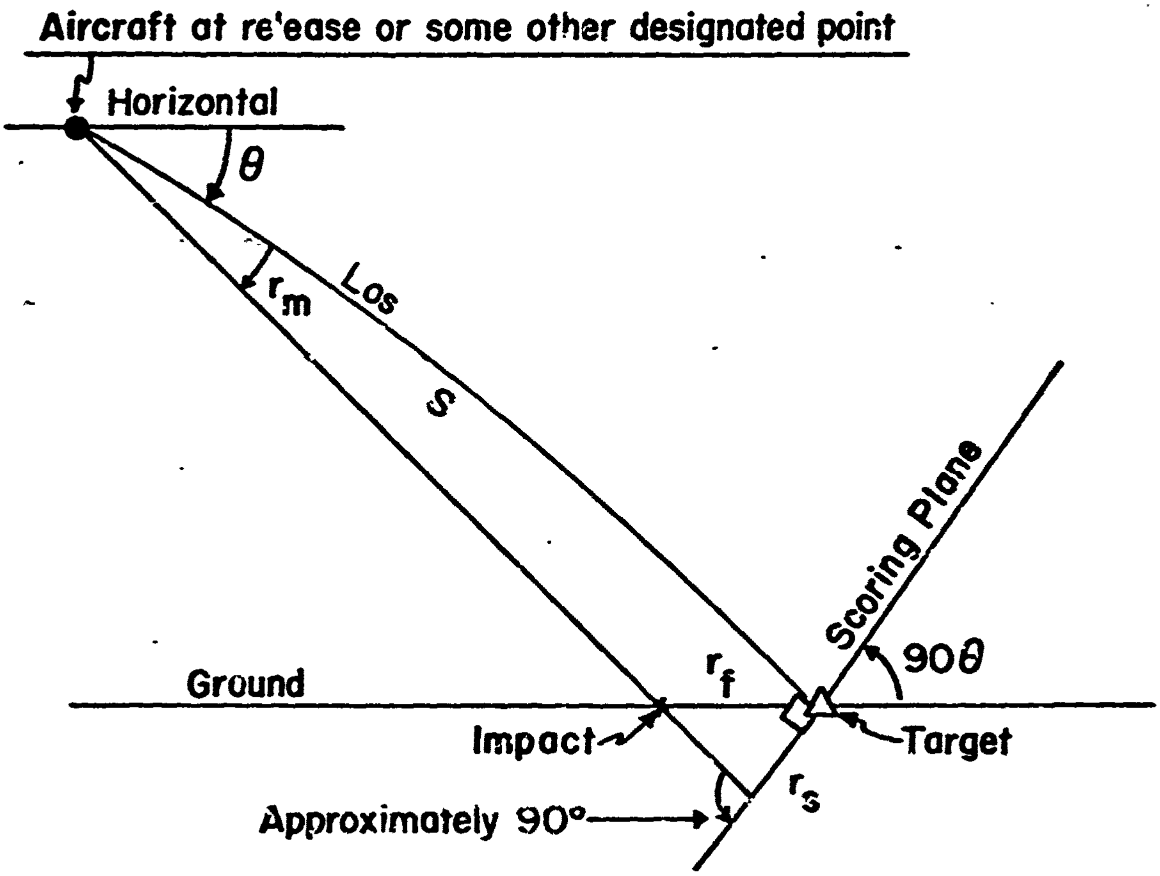


Figure 3.
Geometry of the Mil Definition

θ = Depression of the Los from the horizontal.

S = Slant range to the target in feet.

r_f = Range miss distance in the ground plane in feet.

r_s = Range miss distance in the scoring plane in feet.

r_m = Mil error.

Theoretically, the use of mils transforms impact data taken from various release slant ranges and dive angles to a common base. Traditionally, separate accuracy measures are specified for each given set of release conditions; i.e., slant range, flight path angle, airspeed and maneuver. The use of mils permits some aggregation of accuracy data over release conditions.

D. PROBLEM

From the material thus far presented it is concluded that a single, general accuracy measure and methodology could be advantageous. Some of the more important attributes of a good measure and methodology are:

a. suitability to each need for the measurement of accuracy.

b. exactness[1] of the estimation methodology.

c. minimal amount of testing required to achieve the desired estimation exactness.

d. freedom from distributional and other assumptions restricting the applicability.

e. independence from delivery conditions.

f. mathematical tractability and computational simplicity

1) for data analysis.

2) for weaponeering.

3) for maintenance of capability records.

g. adaptibility to new systems.

Even the few attributes listed above indicate the difficulty of deriving an "optimal" measure and measurement methodology. Additionally,

---

[1]Exactness is defined as the <u>minimum error</u> throughout this thesis.

the preferential ordering assigned the attributes by an individual will significantly influence his judgement of the "goodness" of a particular method.

By carefully defining the problem and attempting to give preference to all phases of the problem, a methodology has been developed. This is not proposed as the "optimal" method but it does have more of the attributes listed above than previous methodologies known to this author.

II. ACCURACY MEASURE

Prior to developing a methodology for determining accuracy, a specific measure of accuracy should be determined.

A. SYSTEM FUNCTION

The JMEM definition of delivery accuracy (quoted earlier) can be paraphrased as the placement of an impact pattern center on a designated point. The measure of accuracy being some description of the precision of that placement. However, accuracy of an A/G delivery system may also be thought of as the frequency with which a system performs its assigned function or functions. This definition differs from the JMEM in that it does not include assumptions about an impact pattern (it is distribution free) and it gives the accuracy measure in terms of the mission. An A/G delivery system has the singular function or mission in a combat role of target destruction.

Two important results can now be discerned. First, the singular function of target destruction greatly simplifies the development that follows. Second, a basis for the proposed definition of delivery accuracy has been reached. This basis can be built upon to derive a measure and methodology that are related directly to the system mission.

B. MEASURE OF EFFECTIVENESS

Based on a mission of target destruction, the accuracy measure definition is the frequency with which target destruction is achieved. An interpretation of the frequency could be the number of targets destroyed per sortie. Note that this is the inverse of a common weaponeering measure, the number of sorties required to destroy a target. The averaging over sorties is not essential. It could have been taken over weapons expended, attacks or other quantities. Sorties was chosen because it is commonly used as a normalizing quantity in aviation terminology.

It may seem that the number of targets destroyed per sortie is more a measure of effectiveness (MOE) than an accuracy measure. In many contexts, the MOE may be a function of an accuracy measure. For brevity, the term MOE will be used and should be interpreted as meaning both accuracy measure and measure of effectiveness.

The MOE chosen (targets destroyed per sortie) closely resembles that used by the Weapon System Effectiveness Industry Advisory Committee (WSEIAC) [6, p. 24,25] which presents some recent analyses of related problems.

C. WEAPONEERING EQUATION

The MOE chosen can be mathematically expressed by a probability statement; the probability that the target is destroyed in a certain number of sorties. This is symbolized in the familiar concept of the probability of failure equals the product of the probabilities of failure on each of several assumed independent trails as:

(14) $\alpha = 1 - (1 - \rho)^{n\ell}$ where;

- α is the probability that the target is destroyed.
- ρ is the probability of target destruction in a single attack.
- n is the number of sorties
- ℓ is the number of attacks per sortie.

Solving for the rate of target destruction per sortie ($1/n$) yields:

$$(15) \quad 1/n = \frac{\ell \cdot \ln(1-\rho)}{\ln(1-\alpha)}$$

Equation (15) will be called the "weaponeering equation".

For convenience in this thesis, the number of attacks per sortie (ℓ) will be assumed to be unity. In many applications, such as against heavily defended targets, ℓ is unity by policy. However, no loss of generality or mathematical inconsistencies are imposed due to taking ℓ to be unity.

Superficially, the probability of target destruction in a single attack (ρ) may appear to be identical to the historical measure of accuracy. A closer examination shows ρ to be more. Inherently, ρ depends on the destructive radius of the weapon/target combination (r_e). However, the results of the WSEIAC analysis [7,p. 22-33] give a deeper insight into ρ. The WSEIAC report concluded that ρ should be related to the availability, dependability and capability of the system by the following equation.

$$(16) \quad \rho = A^T DC \quad \text{where}$$

A^T is the transpose of the vector of probabilities that the system is in some state the start of the mission.

D is the matrix of probabilities that the system is in some state at the required mission time conditioned on the state of the system at the start of the mission.

C is the vector of capabilities conditioned on the system state.

Appendix C gives an example of the use of the above equation to enhance understanding of this important concept.

Now it can be seen that C, the capabilities, are what were commonly referred to historically as the measure of accuracy. An important distinction exists though that requires additional investigation.

D. CAPABILITY

As defined, the capability is related to r_e and conditioned on the system state. No mention is made concerning the delivery conditions. This is a radical departure from the historical method of tying the accuracy to a specific set of delivery conditions. In theory and in reality, the probability of placing a weapon within a given distance of the target is dependent on the delivery conditions. However, the experienced attack aviator who has flown in combat or attempted to achieve a specified set of delivery conditions over an unfamiliar target and terrain on the first attack will readily admit the difficulty involved. The presence of enemy defenses affects all delivery parameters and can cause large deviations from programmed dive angle, airspeed, release altitude[1] and run-in heading. Cloud conditions different from those predicted may dictate last minute changes in dive angle, release altitude and run-in heading. Winds not anticipated can cause dive angle and run-in heading to change during the attack. In close formation attacks each aircraft usually achieves a different set of delivery conditions.

In older, manual delivery systems, the sight setting used was based on a precise set of delivery conditions. Thus, a great importance was imposed on the achievement of these prescribed conditions. Now, sophisticated systems continually compute the predicted weapon impact point and automatically release the weapon when some designated point coincides with the predicted impact point. The pilot using this system is free to vary his delivery conditions "within reason" as he deems appropriate.

---

[1]Dive angle and release altitude define the slant range to the target which is an influential parameter in the accuracy.

It seems reasonable to insist that the capability measure also be free of delivery conditions.

The "within reason" phrase cited above gives qualitative bounds on the delivery conditions. These bounds should reflect the current operational tactics, and can be expected to change with time. For example, in the "Vietnam Era," the bounds might have been:

a. Dive angle — 25 to 60 degrees
b. Release altitude — 4000 to 7000 feet
c. Airspeed — 400 to 600 KIAS
d. Run-in heading — 0 to 360 degrees

III. ACCURACY ESTIMATION METHODOLOGY

The proposed MOE requires the estimation of $\rho = A^T DC$. The estimation of availability and dependability (A and D) has been well documented in numerous reports including the WSEIAC [7]. A method of estimating capability (C), as used to estimate ρ, is developed herein.

A. DERIVATION

The detailed derivation is presented in Appendix A. The derivation of the estimator for C follows a Bayesian approach. The techniques of the Bayesian approach are explained by DeGroot [8], Savage, Raiffa, Schlaifer and many others. The derivation is summarized below to provide continuity.

A squared error loss function was derived as:

(17) $L = \frac{a}{(\ln(1-\alpha))^2} [\ln(1-\hat{\rho})-\ln(1-\rho)]^2$ where $\hat{\rho}$ is the estimator of ρ.

The Bayes estimator[1] was found to be:

(18) $D^* = E[\ln(1-\rho)]$ (the symbol " * " indicates a Bayes estimator)

Members of the multinomial family of distributions were chosen as the sampling distributions because they can be used to approximate any distributional shape. The probability mass function of the multinomial is:

(19) $f(x_1,\ldots,x_k|N,p_1,\ldots,p_k) = \frac{N!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$

---

[1] A Bayes estimator is the one which minimizes the expected loss.

$$\text{where} \quad N = \sum_{i=1}^{k} x_i$$

$$1 = \sum_{i=1}^{k} p_i$$

$$p_i \geq 0 \qquad i=1,\ldots,k$$

and x_i is the number of impacts observed in the i^{th} interval.

The conjugate family for the multinomial is the Dirichlet. Its density function is:

$$(20) \quad f(p_1,\ldots,p_k|\alpha_1,\ldots,\alpha_k) = \frac{\Gamma(\alpha_1+\ldots+\alpha_k)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)} p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1}$$

$$\text{where} \quad 1 = \sum_{i=1}^{k} p_i$$

$$p_i > 0 \qquad i=1,\ldots,k$$

$$\alpha_i > 0 \qquad i=1,\ldots,k$$

Using the conjugate family, the parameters of the posterior Dirichlet distribution of the p_i are $\alpha_1 + x_1,\ldots, \alpha_k + x_k$.

With an assumption and an approximation, the Bayes estimator for C was found to be

$$(21) \quad \hat{C}^* = \sum_{i=1}^{m} \frac{\alpha_i + x_i}{\alpha_0} \qquad \text{where } \alpha_0 = \sum_{i=1}^{k} (\alpha_i + x_i)$$, and m is the smallest integer greater than or equal to r_e/interval length of the multinomial

It was found that the prior parameters could be assigned by assuming a prior value for CEP and making some assumptions.

Appendices B, C and D delineate the proposed procedure using an example problem.

B. ASSUMPTIONS

Several assumptions are made in the measure and measurement methodology derivation. This section will present a complete listing of all the assumptions and corresponding justifications.

1. Independent Trials

Independent trials were assumed in the weaponeering equation (section II.C). In testing, the independence can be achieved by using only the first weapon dropped in a series or by randomization of the delivery maneuvers and parameters so that no two consecutive deliveries are the same. For prediction, the independence assumption is conservative as it neglects the possibility of multipass improvement.

2. Accuracy Definition

The "frequency with which a system performs its assigned function(s)" definition is one of many possible definitions. The rationale behind the particular choice was the direct link to the mission as discussed in section II.A.

3. Accuracy Measure

The "rate of target destruction per sortie" is, again, one of many possible choices. The rationale was discussed in section II.B.

4. Single Attack per Sortie

The assumption that $\ell = 1$ was made for convenience and has no effect other than simplification. Any value may be assigned to ℓ without changing the derivation.

5. Independence from Delivery Conditions

The specific assumptions and an extensive discussion of the rationale was presented in section II.D.

6. Squared Error Loss Function

The rationale behind the choice of the quadratic loss function was the generality and tractibility of that form. The implicit assumption of the loss being equal for overestimates and underestimates is a simplification. If separate treatment is preferred, the methodology is still correct as shown in Ref. 17, p. 195-197.

7. Radial Miss Distance

The radial errors were analyzed vice separate treatment of the range and deflection errors. The rationale being that any emphasis placed on a particular heading, or more importantly, heading relative to some target axis may lead to erroneous conclusions. Ideally, use of the proper run-in heading can increase the accuracy but due to the reasons cited in section II.E, this will often result in an overestimate of the true accuracy.

The radial measure is expressed in the ground plane. The advantages of using mils in the scoring plane are obvious, especially in view of the varied release conditions. Unfortunately, targets are usually found in the ground plane and most weapons detonate on impact with the ground. The geometry of the problem will also show that weapons designed to detonate at a fixed altitude yield to a miss distance measured in a ground plane projected horizontally up to the burst height.

8. Perfect Reliability

The assumption that $A^T D = 1$ was made for simplicity. Assuming otherwise complicates the mathematics but does not alter the concept.

9. CEP as Assigned Prior

Assigning a CEP is one of several methods of assigning the α_i's required in the prior distribution. The rationale for choosing CEP is given in Appendix A but other schemes might suffice.

10. Uniform Distribution of p_i

The justification for setting all the $E[p_i]$ equal, up to the CEP, is based on the resulting mathematical simplicity. Other assignments might be acceptable but might also add to the difficulty of analysis.

11. Each p_i Distributed Symmetrically

Assuming each p_i is distributed symmetrically about its expected value is another simplifying step. Assignment of specific values to $V[p_i]$ is possible for $i=1,\ldots,j$ but adds another subjective decision to the analysis.

Deletion of assumptions 4 and 6 through 11 does not affect the methodology. The mathematics become more cumbersome and the estimator, D*, may differ, but the concept remains unchanged.

Assumptions 9 through 11 simplify the determination of the α_i's and as will be shown in the numerical analysis (section IV), yield good results. However, changing these assumptions does not alter the estimator.

C. APPLICABILITY

Both the accuracy measure and methodology have been derived. It is worthwhile now to reflect back to the listed attributes (section I.D) and comment on the compliance with them.

1. Suitability to Needs

The three needs cited were specification compliance, weaponeering, and system comparison. The suitability of the selected MOE to the latter two needs is apparent. The weaponeering measure was used as the MOE and an MOE is one of the essential elements of system comparison.

The sutiability to specification compliance is not obvious and in view of present specifications is even dubious. In section V it will be shown that specifications can be couched in terms of cost-effectiveness with many attendant advantages. The suitability should become apparent in that context.

2. Exactness

The methodology derivation results in a Bayes estimator of the accuracy measure. The Bayes estimator derived is a sufficient statistic [8, p. 159] which means that no more information relative to the estimate can be garnered from the data [8, p. 155]. The use of Bayes procedures also allows probability statements to be made about the parameter (p_i) of interest.

The numerical analysis (section IV) shows that the proposed methodology is the most exact of the several techniques compared.

3. Minimal Testing

The numerical analysis (section IV) shows that an average error in n[1] of approximately three sorties per target destroyed can be achieved with 100 data points. For r_e values in excess of 40 feet, the corresponding error is less than two. The 100 impacts is less than that required to achieve comparable exactness with the other techniques evaluated.

4. Distribution Free

The multinomial density permits the data to define its own distributional shape as shown in Appendix A.

5. Independent of Delivery Parameters

The independence from delivery conditions was discussed in section II.D.

---

[1] The reason for shifting to n vice 1/n is explained in section IV.

6. Tractibility and Simplicity

The use of a conjugate family simplifies the data analysis as shown in Appendix B. The weaponeering can be accomplished from one table and one graph as shown in Appendix D. As the number of data points increases, the influence of the prior decreases. Thus, operational units which usually collect extensive data, need only maintain the total number of impacts in each of the appropriate intervals.

7. Adaptibility

The adaptability is highly dependent on the nature of the adaption required. No quantitative assessment of the adaptability can be made without knowing the specific adaptation required.

IV. NUMERICAL ANALYSIS

Numerical analysis was used to compare the proposed method with other historical techniques. The Center for Naval Analysis graciously provided extensive A-7E weapons delivery data. There were 1244 impacts delivered from system dive maneuvers with reasonable release conditions. These data were used to make a comparison with real world data. Simulation data was used to make comparisons based on various known distributions with known parameters. A sensitivity analysis of the CEP value used for the prior was also conducted.

The quantity used for comparison was the inverse of the accuracy measure proposed. The magnitude of the actual minus predicted 1/n would be inversely related to the magnitude of n ($1/n\hat{n}|n-\hat{n}|$) and would lose meaning if n were not presented. In the interests of security, the value of n will not be presented. So, $\Delta = |n-\hat{n}|$ is the parameter compared. (This term, Δ, is referred to as the comparator in the figures.)

A. A-7E DATA

The accuracy parameters derived from the A-7E data are not given, again, in the interests of security.

The true distribution of impacts was assumed to be that defined by the data. The large number of impacts (1244) lend credibility to this assumption. The n used in computing Δ was derived from the percent of observed impacts within the appropriate r_e value.

The techniques compared to the proposed methodology were based on the commonly used circular normal distribution and on the recently proposed Cauchy distribution (see section I.C.1.b). Both the regular distributions and the distribution plus a percentage of direct hits were used. The percent of direct hits was that observed in the data. The parameters necessary for the comparison distributions were derived from the observed data using the estimators shown in Section I.C.1.a and I.C.1.b.

A sample size of 100 randomly selected data points was chosen as being representative of a small number of observations for most purposes, yet, a readily achievable number in a test environment. Fifty runs of 100 impacts each were conducted sequentially so that a total of 5000 random data points were drawn from the 1244 impacts available. As a test of the randomness of the samples, the mean radial error for each of the 50 runs were compared and no two were found equal when rounded to the nearest integer value.

The comparisons were conducted for r_e values of 30 through 80 feet in 10 foot increments. The interval length (w) chosen for the multinomial sampling distribution was 10 feet. The α (probability of target destruction) was chosen to be 0.95. The CEP chosen for the Bayes prior was of a nominal value and was over 15 feet different from the value observed in the 1244 impacts.

Figure 5 presents a plot of Δ versus r_e for the different measurement methodologies. Qualitatively, the circular normal yields the worst Δ values and the Bayesian method the best. The Bayesian method completely dominates for r_e values of 50 feet and greater. All the estimates approach a common value at r_e = 70 feet. An interesting point is that the Cauchy distribution appears to be a better estimator than the Cauchy plus direct hits distribution.

The average Δ values for r_e values equal to and greater than 30 feet are:

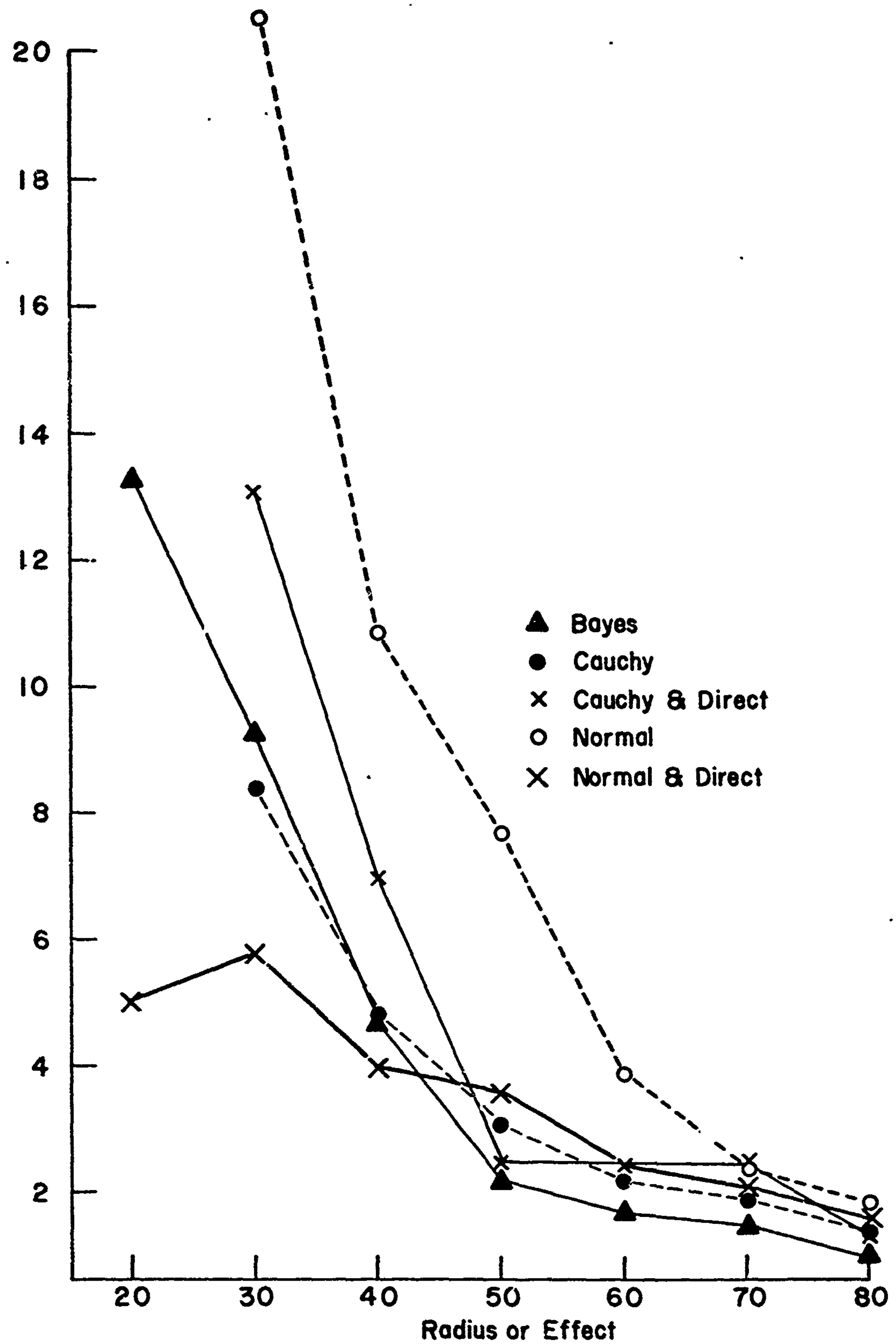


Figure 5.

Comparator Values Using A-7E Data.

1.	Normal plus direct	3.3
2.	Bayesian	3.4
3.	Cauchy	3.6
4.	Cauchy plus direct	4.8
5.	Normal	7.9

The slight advantage of the normal plus direct distribution results from the large difference at $r_e = 30$ feet. It can be concluded that the exact knowledge of the percent of direct hits was of considerable aid to the normal plus direct distribution. This is substantiated by the fact that the Δ value for the normal plus direct is smaller at $r_e = 20$ feet than at $r_e = 30$ feet. Removing the r_e value of 30 feet results in both the Bayesian and Cauchy moving ahead of the normal plus direct.

Though not presented in figure 5, several other techniques for calculating the multinomial parameters (p_i) were compared to the Bayesian. The maximum likelihood estimator of $p_i = x_i / N$ and several weighted averages of three and five adjacent intervals (i.e.,

$$p_i = \frac{w_{i-2}\, p_{i-2} + w_{i-1}\, p_{i-1} + w_i\, p_i + w_{i+1}\, p_{i+1} + w_{i+2}\, p_{i+2}}{w_{i-2} + w_{i-1} + w_i + w_{i+1} + w_{i+2}} \;)$$

yielded Δ values significantly greater than those resulting from the Bayesian technique.

B. SIMULATION

In order to test the Bayesian technique across a wider set of possible impact distributions, the desired distributions were simulated. The programs used to generate the distributions were those presented in Ref. 10 except for the Cauchy. The Cauchy was generated by solving equation (5) for r while using a random number generator to assign values between 0 and 1 to F(r). A CEP of 150 feet and three percent direct hits were used in the generation of data points. The distributions simulated were the regular and the regular plus direct hit forms of the exponential, circular normal, circular Cauchy and uniform and a mixture of two circular normal distributions. The mixture consisted of 30 percent with CEP = 60 feet and 70 percent with CEP = 200 feet. The simulation routines were verified by plotting and by comparing the generated statistics to the input values.

The same techniques were used for comparison as with the A-7E data (section IV.A). The percent of direct hits was assigned the known value of three percent. The r_e value chosen was 50 feet and the Bayes prior was CEP = 150 feet. One hundred runs of 100 samples each were conducted.

Figure 6 presents the Δ value for each of the techniques and each of the simulated distributions. The Bayesian technique provided the best estimator for the exponential, uniform and mixed normal distributions. Surprisingly, the Bayesian technique also provided the best estimator for the Cauchy plus direct hit distribution but by a very small margin.

Table I presents the same results in terms of the ranking of each technique for each distribution. The Bayesian was best overall.

To ensure that the test was not biased by the use of the exponential and uniform distributions, these were removed. Table I also presents these results and the Bayesian technique is still seen to be the best. Interestingly, both the Cauchy and normal plus direct surpass the Cauchy plus direct when the exponential and uniform distributions are removed.

It can be concluded that the Bayesian technique provides the most exact estimator of the techniques tested over a wide range of possible impact distributions.

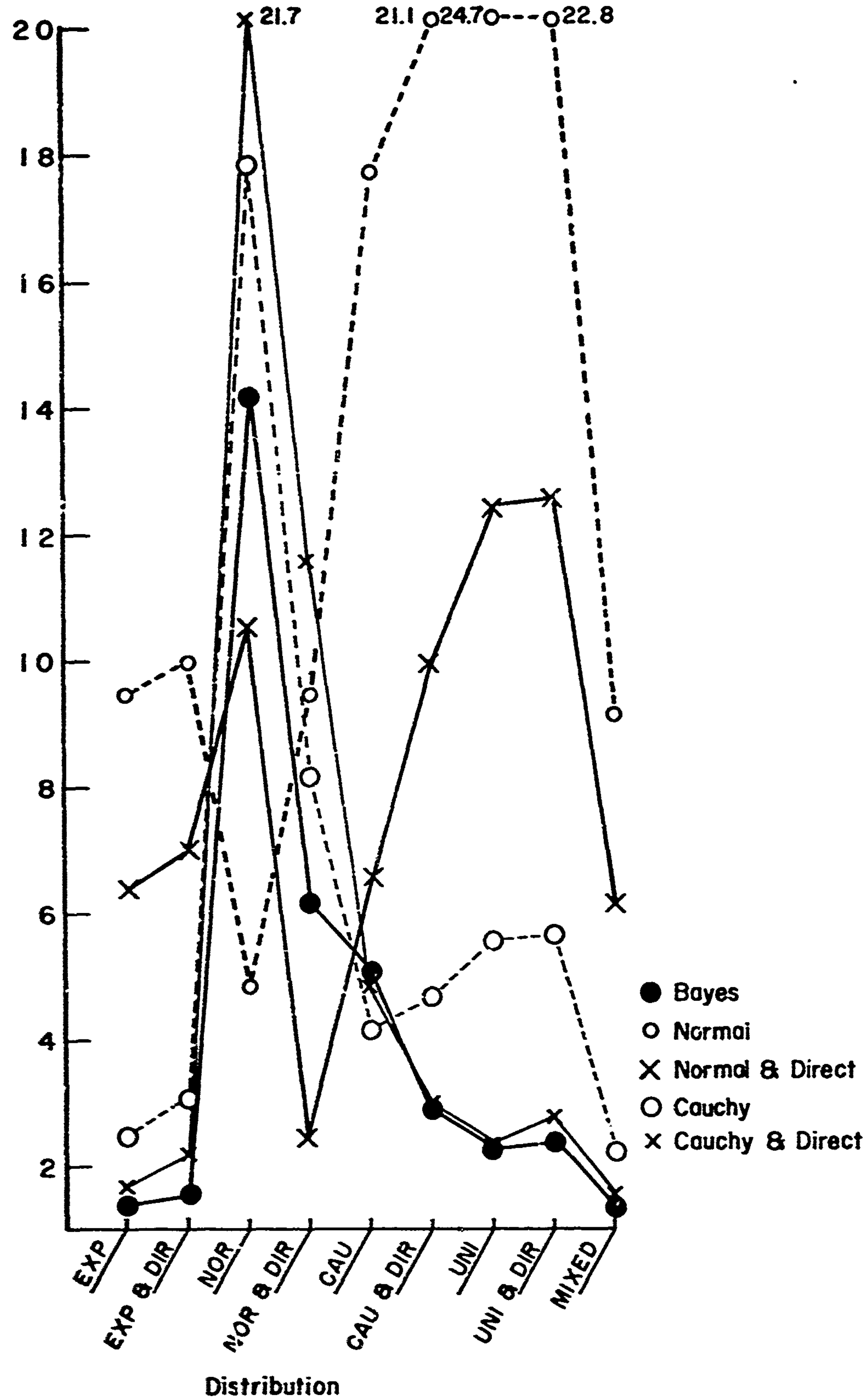


Figure 6.

Comparator Values for Simulation Data.

TABLE I

Ranking of Comparator Values for Simulation Data

	Exp	Exp+Dir	Nor	Nor+Dir	Cau	Cau+Dir	Unif	Unif+Dir	Mixed	Total	Average	Standing	Total	Average	Standing
													Without Exp and Unif Distributions		
BAYES	1	1	3	2	3	1	1	1	1	14	1.56	1	10	2.00	1
NORMAL	5	5	1	4	5	5	5	5	5	40	4.44	5	20	4.00	5
NOR+DIR	4	4	2	1	4	4	4	4	4	31	3.44	4	15	3.00	3
CAUCHY	3	3	4	3	1	3	3	3	3	26	2.89	3	14	2.80	2
CAU+DIR	2	2	5	5	2	2	2	2	2	24	2.67	2	16	3.20	4

C. SENSITIVITY

The A-7E data wa. used to test the sensitivity of the CEP value used for the Bayesian prior. As before, 50 runs of 100 samples each were conducted. The r_e value used was 50 feet. CEP values from 120 to 180 feet in increments of 10 feet were tested.

Figure 7 presents the Δ values versus the CEP values. The Δ values range from 2.1 to 3.1. The minimum Δ value does not occur at the true CEP. The Δ values at r_e = 50 feet, for three of the other techniques are presented for comparison.

It can be concluded that the Bayesian technique is relatively insensitive to the prior CEP value chosen when 100 data points are available, and the prior CEP value is reasonably close to the true CEP value.

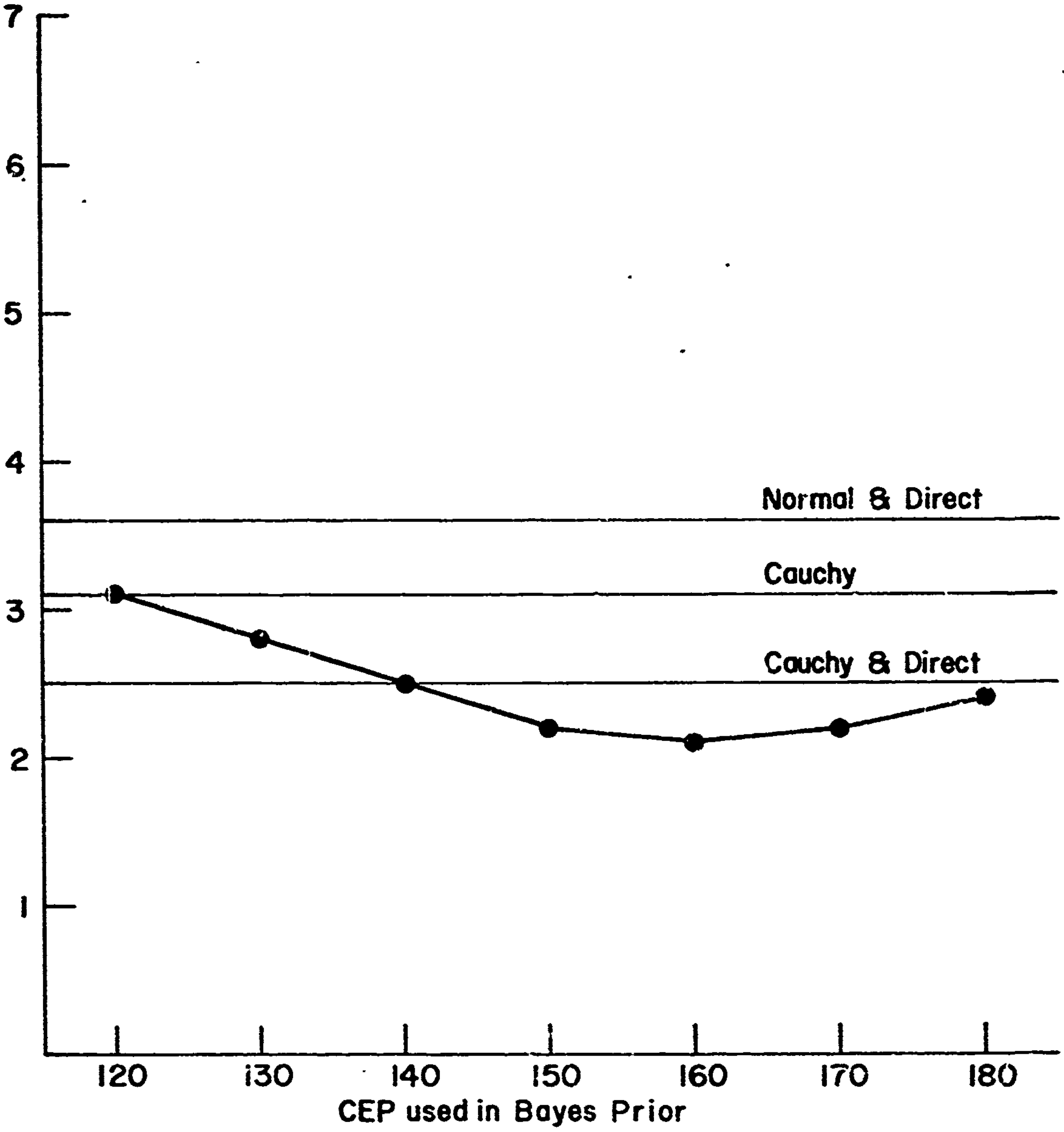


Figure 7.

Sensitivity of Bayes Prior A -- 7E Data.

V. COST-EFFECTIVENESS GUARANTEE

The purpose of this section is to attempt to employ the methodology of cost-effectiveness analysis in specifying and evaluating a contract guarantee for an A/G delivery system. Cost-effectiveness analysis has been defined as "a method for studying how to make the best of several choices. Cost-effectiveness is always used in relation to the effectiveness of alternative systems, organizations or activities." [11, p. 1] It is apparent that comparison of alternatives is the core of cost-effectiveness as applied to choice theory. If its techniques are applied to a single system, caution must be exercised.

It can be seen from section I.B.1 that present specifications sometimes guarantee accuracy in a restrictive sense which inhibits both the guarantee and the evaluation to determine specification compliance. It is also notable that most guarantees avoid the issue of costs except for penalty values.

The cost-effectiveness approach has proven valuable in the choice of a system from a set of alternatives and could prove valuable in guaranteeing a particular system. Additionally, the cost-effectiveness based guarantee provides a stepping-stone toward controlling cost overruns.

The MOE ($1/\bar{n}$) required in a cost-effectiveness analysis has been presented in section II.B.

A. COSTS

1. Types of Cost

The three types of total or life cycle costs are research and development (R&D), investment and operating costs. To fully account for the "cost" of a system, these must be expressed as "economic costs" or benefits lost due to the use of the resources required to develop, procure and operate a particular system. Thus, these costs include hardware, manpower, new facilities, supplies, dollar, etc.; everything directly related to the decision to achieve the system [12, p. 25,66-67].

Care must be taken when costing an A/G delivery system that is to be installed in an airframe so that the costs do not include those that are incurred by the airframe independent of the A/G system [13, p. 5]. Also the concept of sunk costs [12, p. 33] must be considered if some of the components of the system are currently developed or procured. These costs should not be included.

Another cost that is sometimes erroneously included in computing system costs is the attrition of systems due to combat or operational type losses. Once the system is developed and procured, the only pertinent cost is the operating cost.

The cost of the weapons expended to achieve the MOE are not included due to their negligible effect compared to the other costs. The MOE qualitatively reflects the increased weapons cost of an inefficient system.

The methods of measuring the costs are well detailed by Fisher [12] and many other authors and need not be repeated here. However, the total cost concept has serious connotations in regard to a guarantee that will be fair for many different systems and needs to be further discussed.

2. Total Cost

Total costs represent the total resource impact or full economic cost of the system. Necessarily, the magnitude of these costs is highly dependent on the number of systems purchased and the operating lifetime of the system. (The buy size and lifetime are normally estimated during the conceptual phase of a system and the determination of these values is not essential to this thesis.) If these total costs are used to determine a cost-effectiveness guarantee, systems with a large buy and/or a long lifetime will be unfairly penalized by the requirement for a higher effectiveness level than a similar (in performance) system of which only a few are purchased and have a shorter lifetime. The logical conclusion is that some form of normalization is in order.

Varying opinions exist as to whether or not costs should be normalized [6, p. 40 and 11, p. 29]. In general, normalization tends to hide what the total cost is and in the usual context of cost-effectiveness analysis it is important to be fully aware of the total cost. However, in the proposed context, the lack of normalization creates the inequity shown above. Thus, normalization is considered applicable in the guarantee context.

3. Cost Model

Close examination of the three incremental costs shows that:

a. The R&D cost is nearly independent of the buy size and system lifetime. Therefore, this cost can be used with no normalization.

b. The investment cost is most dependent on the buy size and it seems natural to normalize it over the number of systems resulting in units of dollars per system.

c. The operating cost is dependent on both lifetime and buy size. Normalizing over both yields units of dollars per unit of time per system. This will be expressed as an annual operating cost per system in this thesis.

Now, there are three different costs with three different dimensions. The cost-effectiveness model has been transformed from E^2 to E^4 (where E^n is Euclidean n-dimensional space).

The normalization of the costs in the cost model is unique to the guarantee context and these costs should be used with care in other contexts.

B. COST-EFFECTIVENESS RELATIONSHIP

The various techniques that could possibly be used to determine the functional relationship between effectiveness and cost are detailed in many sources such as Theil [14] and Raiffa and Schlaifer [17]. Prior to the use of any of these techniques, however, a data base is required.

1. Data Base

A proper, though not extensive, data base exists for determining the functional relationship. The cost data, reliability data, and weapon delivery accuracy data for present and past systems is historical and can be used [15, p. 11-13].

The data available needs to be transformed into a form compatible with the cost-effectiveness framework outlined above. The availability, dependability (based on a standard mission time) and capability are used to compute 1/n. The cost data needs to be partitioned in the appropriate accounts; i.e., R&D, investment per system, and annual operating per system. Other adjustments to the data may be necessary. Examples of these are well detailed in a RAND report [15, p. 17-32].

2. Technological Dynamics

With the data transformed into the proper framework, two other aspects requiring attention still exist. First, is the fact that the technology under which each of the systems was produced may not be the same. The natural assumption is that the technology is increasing chronologically and that later systems are more effective. Increased technology has historically carried an increased price tag. This is partially what the cost-effectiveness curve is representing. However, the possibility of cost decreasing, technological break-throughs is real. If during the determination of the functional relationship, one or more data points appear not to fit the others then these must be studied carefully to see if such a break-through did exist. When this is the case, a new cost-effectiveness frontier should be generated based on the new technology. Generally, omitting the data points representative of the old technology will be sufficient. It may be though, that there are too few data points from the new technology. In this case, the shape of the frontier could be determined using the old technology and the "height" (in an E^4 sense) from the new technology.

3. Inflation

The second aspect is that the dollar used to cost a particular system is not the same dollar used to cost chronologically future or prior systems. This is not serious though as well developed techniques exist [15, p. 23-32] to account for the time dependent value of the dollar. All that needs to be done is to select a date for the base value of the dollar and transform all dollar values to this base date. It may be desirable to change the base date periodically to keep it fairly close to the present.

4. Functional Relationship Model

The statistical technique used to determine the functional relationship between cost and effectiveness needs to include the ability to explicitly state the uncertainty incurred in the predicted form. A Bayesian regression technique might be reasonable, due to the ability to make probability statements concerning the estimated parameters but no extensive study of this has been conducted by the author.

The successful completion of the above steps will result in the depiction (mathematically) of a hypersurface in E^4 that represents a cost-effectiveness frontier.

5. Uncertainty Considerations

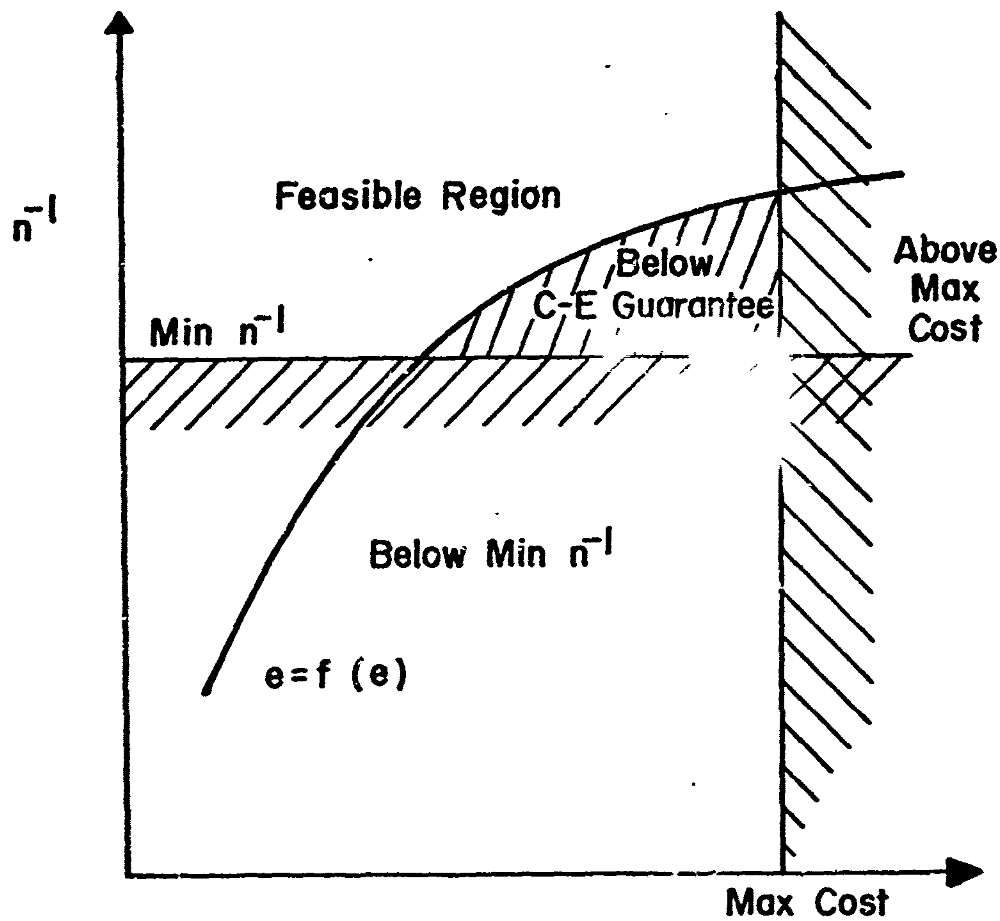
When estimating a hypersurface of unknown form from a few data points, there is a high degree of uncertainty. Though it would be appealing to use the generated hypersurface for the guarantee, it might be unfair. Thus, the prediction uncertainty is required. The direction of the prediction error is also uncertain and in fairness the applicable lower bound of the prediction error interval should be .

C. GUARANTEE

The cost-effectiveness guarantee is in E^4 and cannot be depicted graphically. However, an interpretation of the concept can be portrayed using a representative cost axis as shown in Figure 8.

Adding a minimum effectiveness and maximum cost to the guarantee defines the areas as shown in Figure 8 including the feasible region. The minimum effectiveness requirement is straight forward but the maximum cost has hidden implications. In order to guarantee a maximum total cost, that total cost must be apportioned among the modified incremental costs. Thus, though specifying a maximum total cost is preferable, it may be difficult. The alternative is to specify maximum incremental costs where the sum is not the desired maximum total cost.

The guarantee also must specify the mission time from takeoff to on target and the r_e value. If these values are altered, the entire cost-effectiveness frontier must be regenerated. Therefore, these values must be chosen wisely.



Cost
(R&D, Inv/Sys, Annual Op/Sys)

Figure 8.

Typical Cost-Effectiveness Guarantee.

1. Penalty Assessment

The historical method of assessing penalties is to assign a dollar penalty for each incremental unit outside the guaranteed value. In theory this penalty assessment seems acceptable but in practice it often happens that the penalty costs, when assessed, are charged back to the procuring agency disguised as increased overhead, ground support equipment or other type costs adding to the total system cost. The theory of a cost-effectiveness guarantee would eliminate this practice but the method of penalty assessment could also eliminate it. The manufacturer could be required to meet the guarantee by reducing cost or increasing effectiveness.

2. Guarantee Currency

A periodic review of the guarantee would be necessary to keep it current. A logical mechanism for ensuring that the review is accomplished would be to require a regeneration of the cost-effectiveness hypersurface each time new actual data becomes availabl[illegible] require a constructive review each time the guarantee is used. T[illegible]imum effectiveness and maximum cost portion of the guarantee could be unique to each system and these parameters could be defined during the latter type review.

3. Guarantee Parameters

The parameters for the cost-effectiveness guarantee are summarized as:

a. Minimum effectiveness

b. Maximum cost of

   1) R&D
   2) investment per system
   3) annual operating per system

c. Cost-effectiveness hypersurface

d. Penalty assessment

e. Mission time from launch to target

f. r_e value

Other parameters peculiar to some system or overlooked herein might also require specification.

APPENDIX A

DERIVATION OF THE ACCURACY ESTIMATORS

The detailed derivation of the accuracy estimators, based on Bayesian techniques, is presented below.

A. LOSS FUNCTION

The concept of a Bayesian estimator is minimization of the expected loss; thus, a loss function is needed.

A loss function is a function that assigns a number (called the loss) to each combination of decision and state of nature [8, p. 122]. In the problem at hand, the state of nature is the theoretic number of targets destroyed per sortie $(1/n)$. The decision is the estimate of $1/n$, symbolized as $1/\hat{n}$. Any nonnegative function of the error $(1/n - 1/\hat{n})$ can be used as a loss function for estimation [8, p. 225]. The most commonly used are absolute value of the difference and squared error (quadratic).

DeGroot shows [8, p. 227-228] that the quadratic loss is an acceptable approximation for a wide range of nonnegative loss functions. Principally because of this, the quadratic loss was chosen. An implicit assumption involved in this choice is that the disutility of an overestimate is identical to that of an underestimate.

The loss function is

(A.1) $L = a\,(1/n - 1/\hat{n})^2$ where a is some nonnegative constant.

Recalling equation (15) (from section II.C) with $z = 1$, the loss can be expressed as:

(A.2) $L = \dfrac{a}{\ln^2(1-p)}\,[\ln(1-p) - \ln(1-\hat{p})]^2$ where $\hat{p}$ is the estimator of p

Letting the decision be $D = \ln(1-\hat{p})$ and the state of nature be $\theta = \ln(1-p)$, $L = K(D-\theta)^2$ where $K = a/\ln^2(1-\alpha)$. DeGroot shows [3, p. 228] that, for this loss function, the Bayes decision against any given distribution of θ is:

(A.3) $$D^* = E[\theta]$$

B. SAMPLING DISTRIBUTION

One of the attributes listed in section I.D was that the methodology be distribution free. The use of a multinomial distribution to depict the impact pattern achieves this goal. The form of the multinomial probability mass function:

(A.4) $$f(x_1,\ldots,x_k|N,p_1,\ldots,p_k) = \frac{N!}{x_1!\cdots x_k!} p_1^{x_1}\cdots p_k^{x_k}$$

where $$\sum_{i=1}^{k} x_i = N$$

$$\sum_{i=1}^{k} p_i = 1$$

$$p_i \geq 0 \qquad i=1,\ldots,k$$

permits the cumulative distribution curve to take nearly any shape from convex to concave and many conbinations thereof.

Due to the desirable independence from run-in heading shown in section II.E, the radial miss distance was chosen as the variable for the density.

Figure A.1 relates the geometry of the impact pattern to the multinomial density. The p_i are the probability of an impact in the i^{th} interval and the x_i are the number of impacts in the i^{th} interval.

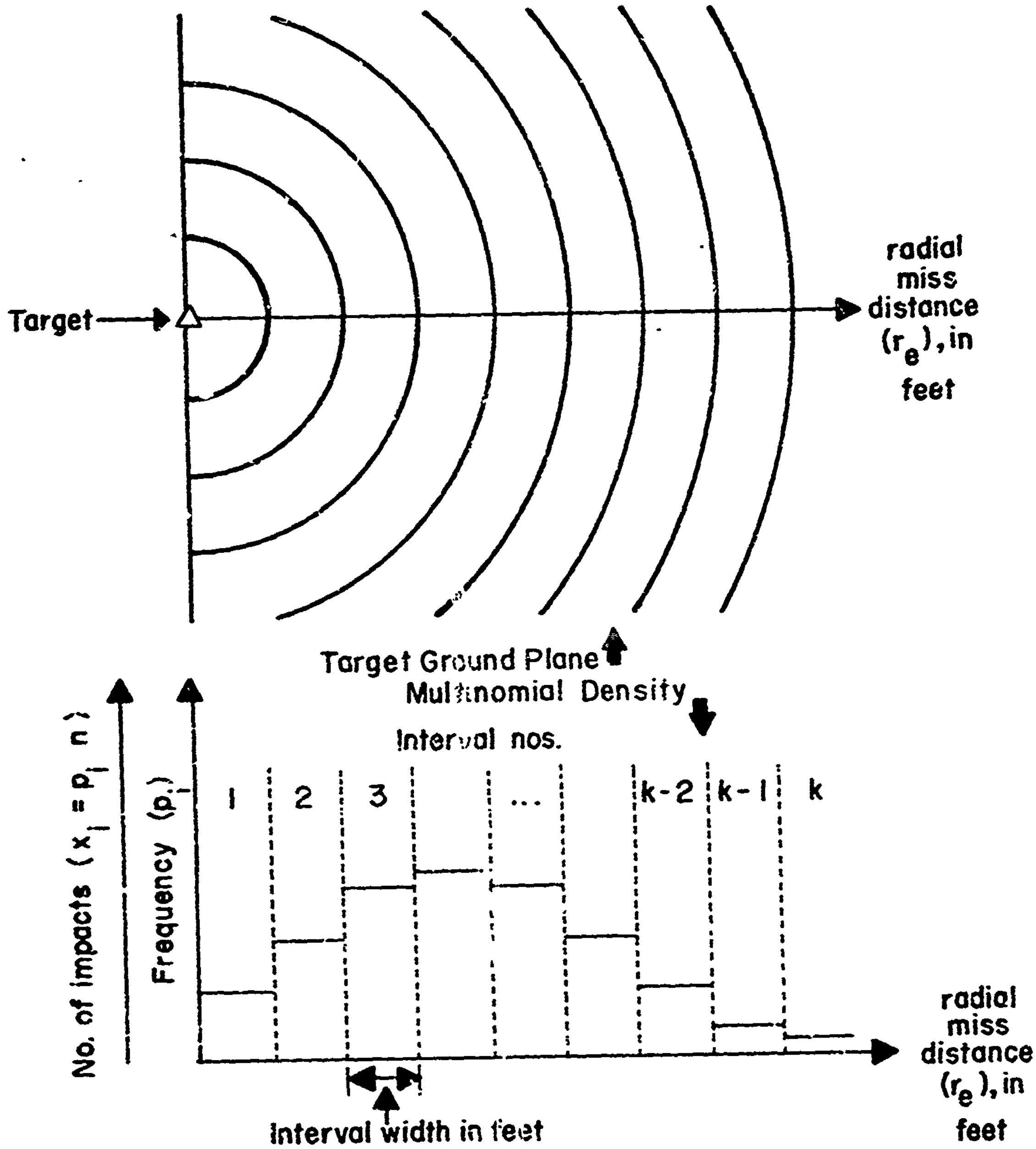


Figure A.1

Relationship of Multinomial Density to Target Geometry.

The length of the intervals (w) can be assigned as desirable. They need not be of uniform length, though it is assumed here that all but the k^{th} will be. The k^{th} interval will include the portion of the real line beyond the end of the $(k-1)^{th}$ interval. It would seem logical to choose an interval length that is some function of the measurement accuracy of the data collection method. A reasonable choice seems to be an interval length equal to twice the measurement error in the data collection. The data analyzed for this thesis was collected with a measurement error of ± 5 feet leading to w = 10 feet.

Another consideration is the number of intervals (k). The choice of k should be based on the number of data points available for analysis, the interval length and the maximal miss distance of interest. The observed distribution of impacts may also influence the choices of k and w. In general, a study of the system, observed data, and data collection method should enable one to assign reasonable values to k and w.

C. CONJUGATE FAMILY

The use of a conjugate family in Bayesian analysis simplifies the mathematical manipulation and ensures an estimate based on a sufficient statistic [8, p. 159].

The conjugate family for the multinomial is the multivariate Beta or Dirichlet distribution [8, p. 174]. The functional form of the Dirichlet density is:

$$\text{(A.5)} \quad f(p_1,\ldots,p_k|\alpha_1,\ldots,\alpha_k) = \frac{\Gamma(\alpha_1+\ldots+\alpha_k)}{\Gamma(\alpha_1)\cdots(\alpha_k)} p_1^{\alpha_1-1}\cdots p_k^{\alpha_k-1}$$

$$\text{where} \quad \sum_{i=1}^{k} p_i = 1$$

$$p_i > 0 \quad i=1,\ldots,k$$

$$\alpha_i > 0 \quad i=1,\ldots,k$$

The expected value and variance are:

$$\text{(A.6)} \quad E[p_i] = \frac{\alpha_i}{\alpha_0} \quad \text{where } \alpha_0 = \sum_{i=1}^{k} \alpha_i$$

$$\text{(A.7)} \quad V[p_i] = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$$

DeGroot shows [8, p. 174] that if the prior parameters are $\alpha_1,\ldots,\alpha_k$ and $x_1,\ldots,x_k$ are the number of observations in each interval, then the posterior parameters are $\alpha_1+x_1,\ldots,\alpha_k+x_k$.

D. ESTIMATORS

It was stated in paragraph A (Loss Function) that the Bayes estimator, D*, was equal to the expected value of $\theta = \ln(1-\rho)$. There is an obvious difference between the distribution of θ and the posterior distribution of the vector of p_i's. One assumption and one approximation are used to derive the Bayes estimators for the p_i's.

The assumption involves the relation $\rho = A^T DC$. C is the probability of an impact within r_e of the target given some state of the system. Thus, C is related to the posterior of the p_i's by:

$$\text{(A.8)} \quad C = \sum_{i=1}^{m} p_i \qquad \text{where } m \text{ is the smallest integer satisfying the inequality } m \geq r_e/w$$

Note that $E[C] = E[\sum_{i=1}^{m} p_i] = \sum_{i=1}^{m} E[p_i] = \sum_{i=1}^{m} \frac{\alpha_i + x_i}{\alpha_0}$.

Assuming the reliability of the system is unity ($A^T D = 1$) permits setting $E[\rho] = E[C]$.

The approximation involves the relation $\theta = \ln(1-\rho)$. For reasonable values of r_e, the values of ρ are relatively small (less than .2). For such values of ρ, $\ln(1-\rho)$ can be approximated by ρ with a maximum error of approximately 15 percent.

The approximation $\rho = \ln(1-\rho)$ permits setting $D^* = E[\theta] = E[\rho] = E[C]$. Thus, the Bayes estimators for the ρ_i are:

$$\text{(A.9)} \quad \hat{p}_i^* = \frac{\alpha_i + x_i}{\alpha_o} \qquad \text{and}$$

$$\text{(A.10)} \quad \hat{\rho}^* = \sum_{i=1}^{m} \frac{\alpha_i + x_i}{\alpha_o}$$

E. PRIOR PARAMETERS

For the subject problem several schemes for assigning the prior parameters were possible. The prime factor dictating the choice was to require a minimal number of parameter values to be assigned.

One of the most widely used and well known accuracy parameters is the CEP. The CEP is also one of the easiest parameters to assign a priori. The assignment can be based on system simulation, system design or some other method tempered by experience. The procedure for using CEP as a single assignment parameter, as discussed below, is Ad Hoc. Several assumptions are made, the justification of which is made in section III.B.

In terms of the p_i's, the CEP is defined as:

$$.5 = \sum_{i=1}^{j} p_i \quad \text{where } j = \text{CEP}/w.$$

Assuming $E[p_i]$ are equal for $i = 1,\ldots,j$ and a uniform length (w) for intervals $1,\ldots,j$:

$$E[p_i] = .5/j \; ; \quad i = 1,\ldots,j$$

and from equation (A.6), the α_i are equal for $i = 1,\ldots,j$. Allowing the k^{th} interval to be the remainder of the distribution, $E[p_k] = .5$. From equation (A.6) and the discussion above,

$$\alpha_o = \sum_{i=1}^{j} \alpha_i + \alpha_k = j\alpha_i + \alpha_k$$

and $$E[p_i] = \frac{\alpha_i}{j\alpha_i + \alpha_k}$$

thus,

$$\alpha_i = \frac{.5}{j}(j\alpha_i + \alpha_k) \qquad i = 1,\ldots,j$$

and

$$\alpha_k = .5\,(j\alpha_i + \alpha_k)$$

These equations are solved to find

$\alpha_k = j\alpha_i \quad i=1,\ldots,j$ as might have been expected.

Attempting to use the derived values and assigning values to the variances (equation (A.7)) leads to an inconsistent system of equations. This can be resolved by assuming that each p_i is distributed symmetrically about its expected value. Then the variance of the p_i for $i=1,\ldots,j$ can be set such that 95 percent of the probable p_i will be nonnegative. It follows that:

$$V[p_i] = (E[p_i]/1.64)^2$$

For $i=1,\ldots,j$;

$$V[p_i] = (1/3.28j)^2$$

Solving for the α_i yields;

$$\text{(A.11)} \qquad \alpha_i = \frac{2.69j - .845}{j} \qquad i=1,\ldots,j$$

and

$$\text{(A.12)} \qquad \alpha_k = j\alpha_i = 2.69j - .845$$

It has been shown, with several assumptions, that the prior parameters can be derived given the prior CEP.

APPENDIX B

SAMPLE DERIVATION OF $\hat{p}_i$

The procedures for using the Bayesian technique suggested in this thesis are illustrated by a simple, fictitious example involving a sample of 10 impacts. Suppose the observed radial miss distances (in feet) for dive maneuvers in the prime mode are:

7, 92, 56, 37, 23, 6, 88, 75, 29, 41.

An interval length of 10 feet and prior CEP of 50 feet will be used.

$$j = CEP/w = 50/10 = 5 \qquad \text{thus } k = 6$$

$$\alpha_i = \frac{2.69(5) - .845}{5} = 2.52 \qquad i=1,\ldots,5$$

$$\alpha_6 = 12.61$$

The values of interest can be tabulated as follows:

i	interval endpoints	α_i	x_i*	$\alpha_i + x_i$	$\hat{p}_i$**
1	0-10	2.52	2	4.52	.13
2	10-20	2.52	0	2.52	.07
3	20-30	2.52	2	4.52	.13
4	30-40	2.52	1	3.52	.10
5	40-50	2.52	1	3.52	.10
6	50-∞	12.61	4	16.61	.47

* x_i is the number of impacts observed in the i^{th} interval.

** $\hat{p}_i = \dfrac{\alpha_i + x_i}{\alpha_0}$; $\alpha_0 = \sum_{i=1}^{6} (\alpha_i + x_i)$.

APPENDIX C

EXAMPLE DERIVATION OF $\hat{\rho}$

The equation $\rho = A^T DC$ given in the WSEIAC report [7] is explained by the following fictitious example using the C vector component as derived in Appendix B.

Consider a system which has a prime mode and one degraded mode. The possible system states are defined as:

A - prime mode

B - degraded mode

C - inoperative

Suppose the availabilities (probability that the system is in some state at the beginning of the mission) are:

$$P\{A\} = 1/2$$

$$P\{B\} = 1/4$$

$$P\{C\} = 1/4$$

Thus, $A^T = (1/2, 1/4, 1/4)$.

Assume the system cannot be repaired in flight. The dependabilities (probability that the system is in some state at a specified time after takeoff given the system state at the beginning of the mission) are assumed to be:

Given state A at the beginning of the mission;

$$P\{A\} = 1/2$$

$$P\{B\} = 1/4$$

$$P\{C\} = 1/4$$

Given state B at the beginning of the mission;

$$P\{A\} = 0$$

$$P\{B\} = 3/4$$

$$P\{C\} = 1/4$$

Given state C at the beginning of the mission;

$$P\{A\} = P\{B\} = 0$$

$$P\{C\} = 1$$

Thus, $D = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 3/4 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^T D = (1/4, 5/16, 7/16)$

The $A^T D$ is constant for a constant mission time and in most systems will probably remain fairly constant over various mission times. Therefore, in the weaponeering usage this could be accepted as a set of values unique to each A/G delivery system. Then, only the final calculation (shown next) need be performed for various r_e values.

The capabilities (probability that weapon impacts within r_e of the target given the system state at the time of release) are as derived in Appendix B using r_e = 30 feet[1]:

$$P\{hit|A\} = \sum_{i=1}^{3} \hat{p}_i = .33 = 1/3$$

$$P\{hit|B\} = 1/6$$

$$P\{hit|C\} = 0$$

Thus, $\hat{C}^T = (1/3, 1/6, 0)$

---

[1]The P{hit|B} is assumed to be 1/6. The value for the degraded mode was not computed in Appendix B.

Solving for $\hat{\rho} = A^T D \ddot{C}$

$$\hat{\rho} = (1/4, 5/16, 7/16) \begin{bmatrix} 1/3 \\ 1/6 \\ 0 \end{bmatrix} = 13/96$$

Therefore, the estimated probability of placing a weapon within 30 feet of the target with this system is 13/96.

APPENDIX D

WEAPONEERING EXAMPLE

The weaponeering usage of the proposed measurement and methodology will be illustrated using the fictitious $\hat{\rho}$ calculated in Appendices B and C.

For the computed $\hat{\rho} = 13/96$ and assuming $\alpha = .95$, $\hat{n}$ can be calculated as:

$$\hat{n} = \frac{\ln(1-\alpha)}{\ln(1-\hat{\rho})} = \frac{\ln(.05)}{\ln(83/96)} = 21$$

A graph similar to that shown in Figure D.1 could be used, vice solving the above equation, to find $\hat{n}$. Entering the graph with $\hat{\rho}$ (called the cumulative percentage) and α, the $\hat{n}$ could be found.

In general usage, the weaponeering requirements would be:

1) Capability tables for the applicable system and degradation modes.
2) $A^T D$ vector for the applicable system.
3) Graph similar to Figure D.1.

If the $A^T D$ were negligibly dependent on mission time, requirements 1 and 2 could be combined into a single table of $\hat{\rho}$ values for the various values of r_e.

At the ship or airwing level, the weaponeer could use the individual squadron capability tables or an airwing table aggregated over particular systems in the airwing. (i.e., A-7 capability, F-4 capability, A-6 capability). In mixed strikes, the ship level weaponeer could use a ratio weighting factor for the $\hat{\rho}$ based on the scheduled mix. As an example consider a strike consisting of 8 A-7, 4 A-6 and 8 F-4. The $\hat{\rho}$ value for the strike could be $\hat{\rho}^* = .4\hat{\rho}_{A-7} + .2\hat{\rho}_{A-6} + .4\hat{\rho}_{F-4}$.

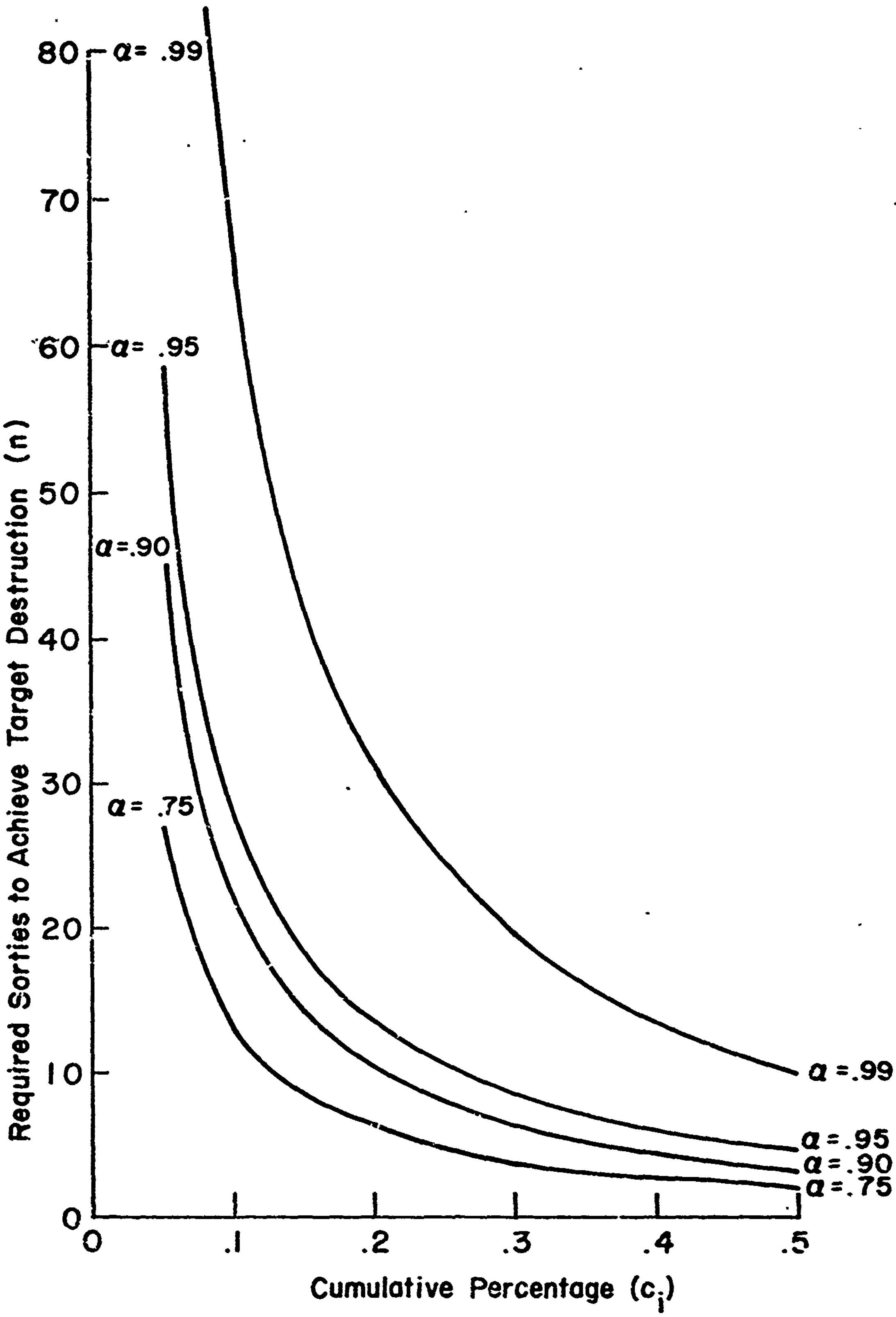


Figure D.1

Weaponeering Graph

BIBLIOGRAPHY

1. Headquarters United States Air Force, AF/OA Paper 70-4, by Capt. David Mastran, August 1970.

2. Operations Evaluation Group Study No. 625, Probability-of-Damage Problems of Frequent Occurrence, by J. M. Dobbie, 11 December [illegible].

3. Crow, E. L., Davis, F. A. and Maxfield, M. W., Statistics Manual, NAVORD Report 3369, 1960.

4. Naval Air Systems Command NAVAIR 00-130-ASR-3, Joint Munitions Effectiveness Manual (Air to Surface) Delivery Accuracy (U), 24 April 1970. CONFIDENTIAL document.

5. Shelby, S. M., Standard Mathematical Tables, 17 ed., The Chemical Rubber Co., 1969.

6. Headquarters, Air Force Systems Command, Final Report of Task Group IV, Weapon System Effectiveness Advisory Committee, January 1965.

7. Headquarters, Air Force Systems Command, Final Report of Task Group II, Weapon System Effectiveness Advisory Committee, January 1965.

8. DeGroot, M. H., Optimal Statistical Decisions, McGraw Hill, 1970.

9. Savage, L. J., "Bayesian Statistics," Recent Developments in Information and Decision Processes, Macmillan, 1962.

10. Naylor, T. H., Balintfy, J. L., Burdick, D. S. and Chu, K., Computer Simulation Techniques, Wiley, 1966.

11. Research Analysis Corporation Study [illegible], Guide for Reviewers of Studies Containing Cost-Effectiveness Analysis, by [illegible], July 1965.

12. Fisher, G. H., Cost Considerations in Systems Analysis, American Elsevier, 1971.

13. The RAND Corporation, Weapon System Cost Sensitivity Analysis as an Aid in Determining Economic Resource Impact, by [illegible], June 1960.

14. Theil, H., Principles of Econometrics, Wiley, [illegible].

15. The RAND Corporation, An Introduction to Equipment Cost Estimating, by C. A. Batchelder et al., December [illegible].

16. Lindgren, B. W., Statistical Theory, Macmillan, 1965.

17. Raiffa, H. and Schlaifer, R., Applied Statistical Decision Theory, Harvard University, 1961.

18.

19. Buchanan, J. M., Cost and Choice, Markham, 1969.

20. Corbisiero, B. A., Evaluating Weapon System Accuracy from a Classical-Bayesian Approach, Master of Science Thesis, Air Force Institute of Technology, June 1970.

21. McCall, J. J., "The Simple Economics of Incentive Contracting," The American Economic Review, v. 60 no. 5, p. 837-846, December 1970.

22. The RAND Corporation Memorandum RM-4567-PR, Fast-VAL: Target Coverage Model, by K. Harris, R. H. Snow and J. R. Lind, March 1966.

23. Naval Air Test Center Technical Report WST-0125R-67, Avionics and Armament Service Acceptance Trials of the A-7A Airplane (U), by R. C. Macke and J. L. Madel, 26 July 1967. CONFIDENTIAL document.

24. Naval Air Test Center Technical Report WST-03R-69, Evaluation of the Automatic Weapons Delivery System in the A-7A Airplane (U), by J. L. Madel and R. C. Macke, 3 February 1969. CONFIDENTIAL document.