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DEVELOPMENT OF A FINITE ELEMENT MODEL FOR THE CLASS IV FLEXTENSIONAL UNDERWATER TRANSDUCER SHELL

BY

GEORGE GEORGOPOULOS
DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING
CENTER FOR ACOUSTICAL STUDIES
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA

LARRY H. ROYSTER
DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING
CENTER FOR ACOUSTICAL STUDIES
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA

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ABSTRACT

A finite element model is developed for the Class IV flextensional underwater transducer shell. The stiffness and mass matrices for a finite cylindrical type shell element of linearly varying radius of curvature are presented. The element has 36 degrees of freedom corresponding to the nine generalized coordinates $u, v, w, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}$ at each node. All six rigid body modes for the element are adequately represented by this model. The element is used to predict the free vibrations for the Class IV flextensional underwater transducer shell, and the results are verified experimentally.

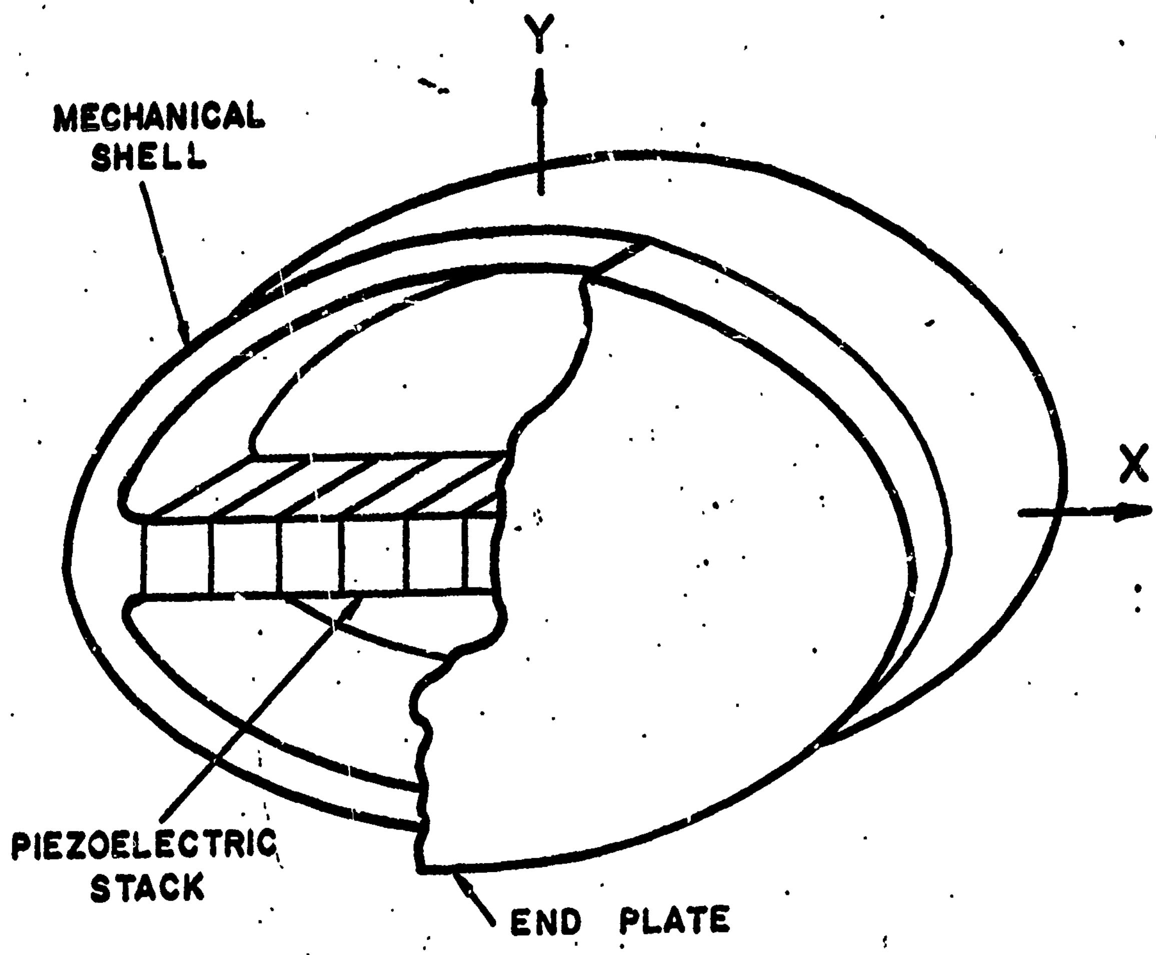


Figure 1. Components of Class IV

INTRODUCTION

The purpose of this investigation is to develop a finite element model for the Class IV flextensional underwater transducer shell. A picture of a Class IV transducer is shown in Figure 1. The Class IV flextensional transducer design was originally proposed as having possible applications as a sonobuoy transducer. A finite difference model of the Class IV transducer has been derived by Rutledge and Royster (1), (2), and (3). Due to the fact that plate effects were neglected by Rutledge and Royster, an attempt is made to develop a more complete and accurate analytical model. This paper describes such an improved linear analytical model developed for the calculation of axisymmetric modes of vibration and natural frequencies of a cylindrical type (oval) shell.

A finite element technique is utilized to construct the total shell stiffness and mass matrix by subdividing the shell into a set of axisymmetric shell components. Novozhilov's "Thin Shell Theory" (4) is utilized for describing the state of strains and stresses. The stiffness and mass matrices for the shell are obtained by superposition of the stiffness and mass matrices of the individual shell element (5) which are computed by using a Ritz approach. The superposition technique assures displacement compatibility and force equilibrium at the interface between components. After the systems for stiffness and mass matrices have been formulated, displacement boundary conditions are introduced by removing appropriate rows and columns corresponding to points on the shell which are rigidly restrained from motion. The shell's natural frequencies and node shapes are obtained from the eigenvalue equation constructed with the total stiffness and mass matrices.

$$\left[[K] - \omega^2 [M]\right]\{\delta\} = \{0\}$$

in which ω is the circular frequency of the system and $\{\delta\}$ is the modal vector where the nine generalized coordinate component of each node for all nodal points are represented.

ANALYTICAL MODEL

Displacement function

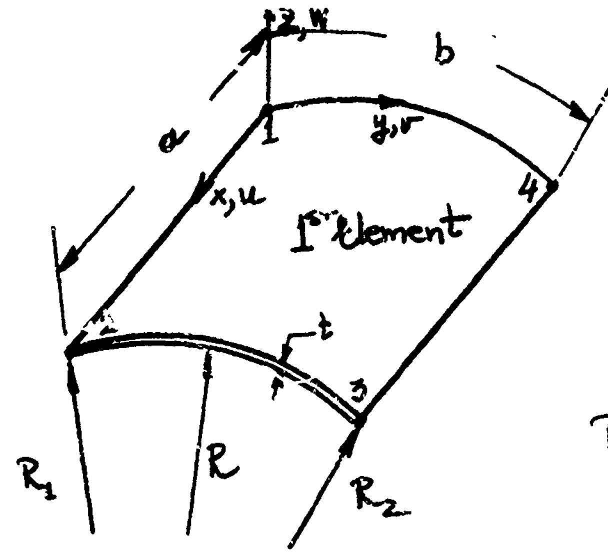
As pointed out on page 22 of reference (5), the first requirement of a finite element is that it must be capable of undergoing rigid body motions with little or no strain. Hence, the cylindrical type shell element will require special displacement functions in order to ensure that all six rigid body modes are included. First, it is clear that the twelve term polynomial in x and y used for the plate bending element may be used here for the radial, circumferential, and meridional displacements w,v.u (figure 2).

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_7 x^2 y + a_8 x y^2 + a_9 x^3 + a_{10} y^3 + a_{11} x^3 y + a_{12} x y^3 \tag{1}$$

$$v(x,y) = a_{13} + a_{14} x + a_{15} y + a_{16} xy + a_{17} x^2 + a_{18} y^2 + a_{19} x^2 y + a_{20} x y^2 + a_{21} x^3 + a_{22} y^3 + a_{23} x^3 y + a_{24} x y^3 \tag{2}$$

$$W(x,y) = a_{25} + a_{26} x + a_{27} y + a_{28} xy + a_{29} x^2 + a_{30} y^2 + a_{31} x^2 y + a_{32} x y^2 + a_{33} x^3 + a_{34} y^3 + a_{35} x^3 y + a_{36} x y^3 \tag{3}$$

The 36 constants in equation (1) to (3) are determined as functions of the 36 modal displacements $u_i, v_i, w_i, u_{xi}, v_{xi}, w_{xi}, u_{yi}, v_{yi}, w_{yi}, (i=1,4)$ for the element $\left(w_{x_i} = \frac{\partial w_i}{\partial x}, etc \right)$. These are then substituted back in equations (1) to (3) yielding expressions which completely difine the displacement anywhere on the element in terms of the modal displacements.



$$R = R_1 + (R_2 - R_1)\cdot y/b$$

Figure 2
Fini[illegible] cylindric[illegible]ne shell element of linearly varying [illegible] of curvature

Point 1 $\quad x=0,\ y=0 \quad u_1=u(0,0),\ v_1=v(0,0),\ w_1=w(0,0),\ u_{x_1}=u_x(0,0),\ v_{x_1}=v_x(0,0),$
$w_{x_1}=w_x(0,0),\ u_{y_1}=u_y(0,0),\ v_{y_1}=v_y(0,0),\ w_{y_1}=w_y(0,0).$

Point 2 $\quad x=a,\ y=0 \quad u_2=u(a,0),\ v_2=v(a,0),\ w_2=w(a,0),\ u_{x_2}=u_x(a,0),\ v_{x_2}=v_x(a,0),$
$w_{x_2}=w_x(a,0),\ u_{y_2}=u_y(a,0),\ v_{y_2}=v_y(a,0),\ w_{y_2}=w_y(a,0).$

Point 3 $\quad x=a,\ y=b \quad u_3=u(a,b),\ v_3=v(a,b),\ w_3=w(a,b),\ u_{x_3}=u_x(a,b),\ v_{x_3}=v_x(a,b),$
$w_{x_3}=w_x(a,b),\ u_{y_3}=u_y(a,b),\ v_{y_3}=v_y(a,b),\ w_{y_3}=w_y(a,b).$

Point 4 $\quad x=0,\ y=b \quad u_4=u(0,b),\ v_4=v(0,b),\ w_4=w(0,b),\ u_{x_4}=u_x(0,b),\ v_{x_4}=v_x(0,b),$
$w_{x_4}=w_x(0,b),\ u_{y_4}=u_y(0,b),\ v_{y_4}=v_y(0,b),\ w_{y_4}=w_y(0,b).$

Therefore, the 36 constants are determined easily from the above stated 36 equations.

$$a_1 = 1\cdot u_1,\quad a_2 = 1\cdot u_{x_1},\quad a_3 = 1\cdot u_{y_1},$$

$$a_4 = -\frac{1}{ab}\cdot u_1 - \frac{1}{b}u_{x_1} - \frac{1}{a}\cdot u_{y_1} + \frac{1}{ab}\cdot u_2 + \frac{1}{a}u_{y_2} - \frac{1}{ab}u_3 + \frac{1}{ab}u_4 + \frac{1}{b}u_{x_4}$$

$$a_5 = -\frac{3}{a^2}u_1 - \frac{2}{a}u_{x_1} + \frac{3}{a^2}u_2 - \frac{1}{a}u_{x_2}$$

$$a_6 = -\frac{3}{b^2}u_1 - \frac{2}{b}u_{y1} + \frac{3}{b^2}u_4 - \frac{1}{b}u_{y4}$$

$$a_7 = \frac{3}{a^2b}u_1 + \frac{2}{ab}u_{x1} - \frac{3}{a^2b}u_2 + \frac{1}{ab}u_{x2} + \frac{3}{a^2b}u_3 - \frac{1}{ab}u_{x3} - \frac{3}{a^2b}u_4 - \frac{2}{ab}u_{x4}$$

$$a_8 = \frac{3}{ab^2}u_1 + \frac{2}{ab}u_{y1} - \frac{3}{ab^2}u_2 + \frac{2}{ab}u_{y2} + \frac{3}{ab^2}u_3 - \frac{1}{ab}u_{y3} - \frac{3}{ab^2}u_4 + \frac{2}{ab}u_{y4}$$

$$a_9 = \frac{2}{a^3}u_1 + \frac{1}{a^2}u_{x1} - \frac{2}{a^3}u_2 + \frac{1}{a^2}u_{x2}$$

$$a_{10} = \frac{2}{b^3}u_1 + \frac{1}{b^2}u_{y1} - \frac{2}{b^3}u_4 + \frac{1}{b^2}u_{y4}$$

$$a_{11} = -\frac{2}{a^3b}u_1 - \frac{1}{a^2b}u_{x1} + \frac{2}{a^3b}u_2 - \frac{1}{a^2b}u_{x2} - \frac{2}{a^3b}u_3 + \frac{1}{a^2b}u_{x3} + \frac{2}{a^3b}u_4 + \frac{1}{a^2b}u_{x4}$$

$$a_{12} = -\frac{2}{ab^3}u_1 - \frac{1}{ab^2}u_{y1} + \frac{2}{ab^3}u_2 - \frac{1}{ab^2}u_{y2} - \frac{2}{ab^3}u_3 + \frac{1}{ab^2}u_{y3} + \frac{2}{ab^3}u_4 - \frac{1}{ab^2}u_{y4}$$

Due to symmetry if a_i where $1 \le i \le 12$ is given by a function f_i

$$a_i = f_i(u_1, u_{x1}, u_{y1}, u_2, u_{x2}, u_{y2}, u_3, u_{x3}, u_{y3}, u_4, u_{x4}, u_{y4})$$

then

$$a_{i+12} = f_i(v_1, v_{x1}, v_{y1}, v_2, v_{x2}, v_{y2}, v_3, v_{x3}, v_{y3}, v_4, v_{x4}, v_{y4})$$

and

$$a_{i+24} = f_i(w_1, w_{x1}, w_{y1}, w_2, w_{x2}, w_{y2}, w_3, w_{x3}, w_{y3}, w_4, w_{x4}, w_{y4})$$

The element with 4 nodes 1,2,3, and 4 thus possesses 36 degrees of freedom, determined by the element displacements.

$$\{\delta\}^e = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \quad \text{where} \quad \{\delta_1\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_{x1} \\ v_{x1} \\ w_{x1} \\ u_{y1} \\ v_{y1} \\ w_{y1} \end{Bmatrix} \quad \text{etc}$$

The displacement within the element is uniquely determined by the nodal displacements $\{\delta\}^e$ and the position x and y.

$$\{f\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N] \cdot \{\delta\}^e$$

where matrix $[N]$ is given in Appendix A.

Strains and Stresses

From Novoshilov's "Thin Shell Theory" (4), the strains and stresses for the thin orthotropic shell and for the case of an arbitrary radius of curvature shell is as follows

Strains

$$\epsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} - z\left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\cdot\frac{\partial v}{\partial y} + \frac{v}{R^2}\cdot\frac{\partial R}{\partial y}\right)$$

$$\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - 2z\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{1}{R}\cdot\frac{\partial v}{\partial x}\right)$$

Stresses

$$\sigma_x = \frac{E}{1-\nu^2}\cdot(\epsilon_x + \nu\epsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}\cdot(\epsilon_y + \nu\epsilon_x)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\cdot\epsilon_{xy}$$

A general elastic behavior is assumed, and the relationship between stresses $\{\sigma\}$ and strains $\{\epsilon\}$ is linear and of the form

$$\{\sigma\} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{Bmatrix} = [D]\cdot\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_{xy}\end{Bmatrix}$$

where $[D]$ is the elastic matrix containing the appropriate material properties and is given in Appendix A.

Since matrix $[N]$ is already calculated,

then

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} = [B]\{\delta\}^e$$

Matrix $[B]$ is easily obtained and is given in Appendix A.

The stiffness matrix becomes

$$[k]^e = \int_{-t/2}^{t/2}\int_0^b\int_0^a [B]^T \cdot [D] \cdot [B]\, dx\, dy \cdot dz$$

and the mass matrix becomes

$$[m]^e = \rho \int_{-t/2}^{t/2}\int_0^b\int_0^a [N]^T [N] \cdot dx\, dy\, dz$$

where ρ is the density of the shell.

A 5 point Causs-Legendre numerical integration is carried out through the use of an IBM 370/165 computer (6). The stiffness and mass matrices of the element will clearly always be square and of the form

$$[k]^e = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}, \quad [m]^e = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

in which $K_{11}, K_{12}, \ldots$ etc. and $m_{11}, m_{12}, \ldots$ etc. are submatrices which are again square and of the size 9×9. Then the total stiffness $[K]$ and total mass $[M]$ of the shell are obtained by proper superpoistion of the above mentioned submatrices. That is

$$[K]_{n \times n} = \sum_{m=1}^{m=n} \left(\sum [k_{im}] \right) \qquad [M]_{n \times n} = \sum_{m=1}^{m=n} \left(\sum [m_{im}] \right)$$

The "inside" summation is now taken over all the elements of the shell. If a particular element does not in fact include the node in question, it will contain no submatrices with an i suffix, and therefore, its contribution will simply be zero.

After introducing the boundary conditions to both matrices $[K]$ and $[M]$ by removing the appropriate columns and rows from them, then the problem is reduced to the simple form

$$\left[[K] - \omega^2 [M] \right] \{\delta\} = \{0\}$$

If the $[K]$ and $[M]$ matrices are of order $n \times n$, then different values of the angular frequency ω can be computed, and each of these frequencies will be associated with a particular node vector $\{\delta\}$ which also can be computed by utilizing a digital computer (6).

DISCUSSION

The approach adopted here is known as the displacement formulation which is equivalent to the minimization of the total potential energy of the system in terms of a prescribed displacement field. If this displacement field is defined in a suitable way, then convergence to the correct result must occur. The process is the equivalent to the Ritz procedure, and as a consequence, the so obtained approximate solution is always an upper bound to the exact solution which is very much desired. It should be emphasized that this bound is not available for a solutions in which discontinuity at interfaces arises (5).

The major characteristics of the presented analytical model formulation are: 1) an accurate representation of the shell geometry, 2) the dynamic mass matrix representation, and 3) the requirement of rigid body displacements.

The model described herein has been utilized to calculate eigenvalues and eigenvectors of the Class IV flextensional underwater transducer shell (6). Using a mess of 6 elements as described in reference (6), the first natural frequency (1295.25 cycles/sec.) is calculated within 3.62% as compared to the experimental value (1250 cycles/sec). Using a mess of 8 elements (4 x 2) the first natural frequency (1289.73 cycles/sec.) is calculated within 3.17% from the experimental value. This clearly indicates the accuracy and rapid convergence of the herein developed finite element model.

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APPENDIX A

$$D = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Also $\frac{\partial u}{\partial x} = \frac{1}{a} \cdot \frac{\partial u}{\partial (x/a)} = \frac{1}{a} \cdot \frac{\partial u}{\partial X}$, $\frac{\partial u}{\partial y} = \frac{1}{b} \cdot \frac{\partial u}{\partial Y}$, $X = \frac{x}{a}$, $Y = \frac{y}{b}$

$$[N] = \begin{bmatrix} N(1,1), & N(1,2), & \dots & N(1,35) & N(1,36) \\ N(2,1), & N(2,2), & \dots & N(2,35) & N(2,36) \\ N(3,1), & N(3,2), & \dots & N(3,35) & N(3,36) \end{bmatrix}$$

$$[B] = \begin{bmatrix} B(1,1), & B(1,2), & \dots, & B(1,35), & B(1,36) \\ B(2,1), & B(2,2), & \dots, & B(2,35), & B(2,36) \\ B(3,1), & B(3,2), & \dots, & B(3,35), & B(3,36) \end{bmatrix}$$

where

$$[M] = \rho \cdot t \cdot a \cdot b \cdot \int_0^1 \int_0^1 [N]^T \cdot [N] \cdot dX \cdot dY$$

$$[K] = \frac{E}{1-\nu^2} \cdot a \cdot b \int_{-t/2}^{t/2} \int_0^1 \int_0^1 [B]^T \cdot [D] \cdot [B] \, dX \, dY \, dz$$

$R = R_1 + (R_2 - R_1) y/b$, $\frac{\partial R}{\partial y} = (R_2 - R_1)/b$

$$N(1,1) = N(2,2) = N(3,3) = 1 - XY - 3X^2 - 3Y^2 + 3X^2Y + 3XY^2 + 2X^3 + 2Y^3 - 2X^3Y - 2XY^3$$

$$N(1,4) = N(2,5) = N(3,6) = a\cdot(X - XY - 2X^2 + 2X^2Y + X^3 - X^3Y)$$

$$N(1,7) = N(2,8) = N(3,9) = b(Y - XY - 2Y^2 + 2XY^2 + Y^3 - XY^3)$$

$$N(1,10) = N(2,11) = N(3,12) = XY + 3X^2 - 3X^2Y - 3XY^2 - 2X^3 + 2X^3Y + 2XY^3$$

$$N(1,13) = N(2,14) = N(3,15) = a(-X^2 + X^2Y + X^3 - X^3Y)$$

$$N(1,16) = N(2,17) = N(3,18) = b(XY - 2XY^2 + XY^3)$$

$$N(1,19) = N(2,20) = N(3,21) = -XY + 3X^2Y + 3XY^2 - 2X^3Y - 2XY^3$$

$$N(1,22) = N(2,23) = N(3,24) = a(-X^2Y + X^3Y)$$

$$N(1,25) = N(2,26) = N(3,27) = b(-XY^2 + XY^3)$$

$$N(1,28) = N(2,29) = N(3,30) = XY + 3Y^2 - 3X^2Y - 3XY^2 - 2Y^3 + 2X^3Y + 2XY^3$$

$$N(1,31) = N(2,32) = N(3,33) = a(XY - 2X^2Y + X^3Y)$$

$$N(1,34) = N(2,35) = N(3,36) = b(-Y^2 + XY^2 + Y^3 - XY^3)$$

Where the rest of the $N(I,J)$ not given above are all equal to zero.

$B(1,1)=\frac{1}{a}\cdot(-Y-6X+6XY+3Y^2+6X^2-6X^2Y-2Y^3)$,

$B(2,1)=0$, $B(3,1)=\frac{1}{b}(-X-6Y+3X^2+6XY+6Y^2-2X^3-6XY^2)$,

$B(1,2)=0$, $B(2,2)=\frac{1}{b}(1+\frac{z}{R})(-X-6Y+3X^2+6XY+6Y^2-2X^3-6XY^2)-z\frac{\partial R}{\partial y}\cdot\frac{1}{R^2}\cdot(1-XY-3X^2-3Y^2+3X^2Y+3XY^2+2X^3+2Y^3-2X^3Y-2XY^3)$, $B(3,2)=\frac{1}{a}\cdot(1+\frac{2z}{R})(-Y-6X+6XY+3Y^2+6X^2-6X^2Y-2Y^3)$,

$B(1,3)=-\frac{6z}{a^2}\cdot(-1+Y+2X-2XY)$, $B(2,3)=\frac{1}{R}\cdot(1-XY-3X^2-3Y^2+3X^2Y+3XY^2+2X^3+2Y^3-2X^3Y-2XY^3)-\frac{6z}{b^2}\cdot(-1+X+2Y-2XY)$, $B(3,3)=-\frac{2z}{ab}\cdot(-1+6X+6Y-6X^2-6Y^2)$

$B(1,4)=1-Y-4X+4XY+3X^2-3X^2Y$, $B(2,4)=0$,

$B(3,4)=\frac{a}{b}\cdot(-X+2X^2-X^3)$,

$B(1,5)=0$, $B(2,5)=\frac{a}{b}(1+\frac{z}{R})(-X+2X^2-X^3)-\frac{z\cdot a}{R^2}\frac{\partial R}{\partial y}(X-XY-2X^2+2X^2Y+X^3-X^3Y)$, $B(3,5)=(1+\frac{2z}{R})(1-Y-4X+4XY+3X^2-3X^2Y)$,

$B(1,6)=-\frac{z}{a}(-4+4Y+6X-6XY)$, $B(2,6)=\frac{a}{R}\cdot(X-XY-2X^2-2X^2Y+X^3-X^3Y)$, $B(3,6)=-\frac{2z}{b}\cdot(-1+4X-3X^2)$,

$B(1,7)=\frac{b}{a}\cdot(-Y+2Y^2-Y^3)$, $B(2,7)=0$, $B(3,7)=1-X-4Y+4XY+3Y^2-3XY^2$

$B(1,8)=0$, $B(2,8)=\left(1+\frac{z}{R}\right)\left(1-X-4Y+4XY+3Y^2-3XY^2\right)$,

$B(3,8)=\frac{b}{a}\cdot\left(1+\frac{2z}{R}\right)\left(-Y+2Y^2-Y^3\right)$,

$B(1,9)=0$, $B(2,9)=\frac{b}{R}\cdot\left(Y-XY-2Y^2+2XY^2+Y^3-XY^3\right)-$
$\frac{z}{b}\cdot\left(-4+4X+6Y-6XY\right)$, $B(3,9)=-\frac{2z}{a}\left(-1+4Y-3Y^2\right)$,

$B(1,10)=\frac{1}{a}\left(Y+6X-6XY-3Y^2-6X^2+6X^2Y+2Y^3\right)$, $B(2,10)=0$,

$B(3,10)=\frac{1}{b}\cdot\left(X-3X^2-6XY+2X^3+6XY^2\right)$,

$B(1,11)=0$, $B(2,12)=\frac{1}{b}\cdot\left(1+\frac{z}{R}\right)\left(X-3X^2-6XY+2X^3+\right.$
$\left.6XY^2\right)-z\frac{\partial R}{\partial y}\frac{1}{R^2}\cdot\left(XY+3X^2-3X^2Y-3XY^2-2X^3+2X^3Y+2XY^3\right)$,

$B(3,11)=\frac{1}{a}\cdot\left(1+\frac{2z}{R}\right)\left(Y+6X-6XY-3Y^2-6X^2+6X^2Y+2Y^3\right)$,

$B(1,12)=-6\frac{z}{a^2}\left(1-Y-2X+2XY\right)$, $B(2,12)=\frac{1}{R}\left(XY+3X^2-\right.$
$\left.3X^2Y-3XY^2-2X^3+2X^3Y+2XY^3\right)-\frac{6z}{b^2}\left(-X+2XY\right)$,

$B(3,12)=-\frac{2z}{ab}\cdot\left(1-6X-6Y+6X^2+6Y^2\right)$,

$B(1,13)=-2X+2XY+3X^2-3X^2Y$, $B(2,13)=0$,

$B(3,13)=\frac{a}{b}\cdot\left(X^2-X^3\right)$,

$B(1,14)=0$, $B(2,14)=\frac{a}{b}\left(1+\frac{z}{R}\right)\left(X^2-X^3\right)-\frac{za}{R^2}\frac{\partial R}{\partial y}\cdot\left(\right.$
$\left.-X^2+X^2Y+X^3-X^3Y\right)$, $B(3,14)=\left(1+\frac{2z}{R}\right)\left(-2X+2XY+3X^2-\right.$
$\left.3X^2Y\right)$,

$B(1,15)=-\frac{2z}{a}\cdot\left(-1+Y+3X-3XY\right)$, $B(2,15)=\frac{a}{R}\left(-X^2+X^2Y+X^3-X^3Y\right)$

$B(3,15) = -\frac{2z}{b}(2X - 3X^2)$,

$B(1,16) = \frac{b}{a}(Y - 2Y^2 + Y^3)$, $B(2,16) = 0$, $B(3,16) = X - 4XY + 3XY^2$,

$B(1,17) = 0$, $B(2,17) = (1+\frac{z}{R})(X - 4XY + 3XY^2) - \frac{zb}{R^2}\frac{\partial R}{\partial y}(XY - 2XY^2 + XY^3)$, $B(3,17) = \frac{b}{a}(1+\frac{2z}{R})(Y - 2Y^2 + Y^3)$,

$B(1,18) = 0$, $B(2,18) = \frac{b}{R}(XY - 2XY^2 + XY^3) - \frac{z}{b}(-4X + 6XY)$,

$B(3,18) = -\frac{2z}{a}(1 - 4Y + 3Y^2)$,

$B(1,19) = \frac{1}{a}(-Y + 6XY + 3Y^2 - 6X^2Y - 2Y^3)$, $B(2,19) = 0$,

$B(3,19) = \frac{1}{b}(-X + 3X^2 + 6XY - 2X^3 - 6XY^2)$,

$B(1,20) = 0$, $B(2,20) = \frac{1}{b}(1+\frac{z}{R})(-X + 3X^2 + 6XY - 2X^3 - 6XY^2) - \frac{z}{R}\frac{\partial R}{\partial y}(-XY + 3X^2Y + 3XY^2 - 2X^3Y - 2XY^3)$, $B(3,20) = \frac{1}{a}(1+\frac{2z}{R})(-Y + 6XY + 3Y^2 - 6X^2Y - 2Y^3)$,

$B(1,21) = -\frac{6z}{a^2}(Y - 2XY)$, $B(2,21) = \frac{1}{R}(-XY + 3X^2Y + 3XY^2 - 2X^3Y - 2XY^3) - \frac{6z}{b^2}(X - 2XY)$, $B(3,21) = -\frac{2z}{ab}(-1 + 6X + 6Y - 6X^2 - 6Y^2)$,

$B(1,22) = -2XY + 3X^2Y$, $B(2,22) = 0$, $B(3,22) = \frac{a}{b}(-X^2 + X^3)$,

$B(1,23) = 0$, $B(2,23) = \frac{a}{b}(1+\frac{z}{R})(-X^2 + X^3) - \frac{za}{R^2}\frac{\partial R}{\partial y}(-X^2Y + X^3Y)$,

$B(3,23) = (1+\frac{2z}{R})(-2XY + 3X^2Y)$,

$B(1,24) = -\frac{z}{a}(-2Y + 6XY)$, $B(2,24) = \frac{a}{R}(-X^2Y + X^3Y)$,

$B(3,24) = -\frac{2z}{b}(-2X + 3X^2)$,

$B(1,25) = \frac{b}{a}(-Y^2 + Y^3)$, $B(2,25) = 0$, $B(3,25) = -2XY + 3XY^2$,

$B(1,26) = 0$, $B(2,26) = (1 + \frac{z}{R})(-2XY + 3XY^2) - \frac{zb}{R^2}\frac{\partial R}{\partial y}(-XY^2 + XY^3)$,

$B(1,27) = 0$, $B(2,27) = \frac{b}{R}(-XY^2 + XY^3) - \frac{z}{b}(-2X + 6XY)$, $B(3,27) = -\frac{2z}{a}(-2Y + 3Y^2)$,

$B(1,28) = \frac{1}{a}(Y - 6XY - 3Y^2 + 6X^2Y + 2Y^3)$, $B(2,28) = 0$, $B(3,28) = \frac{1}{b}(X + 6Y - 3X^2 - 6XY - 6Y^2 + 2X^3 + 6XY^2)$,

$B(1,29) = 0$, $B(2,29) = \frac{1}{b}(1 + \frac{z}{R})(X + 6Y - 3X^2 - 6XY - 6Y^2 + 2X^3 + 6XY^2) - \frac{z}{R^2}\frac{\partial R}{\partial y}(XY + 3Y^2 - 3X^2Y - 3XY^2 - 2Y^3 + 2X^3Y + 2XY^3)$,

$B(3,29) = \frac{1}{a}(1 + 2\frac{z}{R})(Y - 6XY - 3Y^2 + 6X^2Y + 2Y^3)$,

$B(1,30) = -\frac{6z}{a^2}(-Y + 2XY)$, $B(2,30) = \frac{1}{R}(XY + 3Y^2 - 3X^2Y - 3XY^2 - 2Y^3 + 2X^3Y + 2XY^3)$, $B(3,30) = -\frac{2z}{ab}(1 - 6X - 6Y + 6X^2 + 6Y^2)$,

$B(1,31) = Y - 4XY + 3X^2Y$, $B(2,32) = 0$, $B(3,32) = \frac{a}{b}(X - 2X^2 + X^3)$

$B(1,32) = 0$, $B(2,32) = \frac{a}{b}(1 + \frac{z}{R})(X - 2X^2 + X^3) - \frac{za}{R^2}\frac{\partial R}{\partial y}(XY - 2X^2Y + X^3Y)$, $B(3,32) = (1 + \frac{2z}{R})(Y - 4XY + 3X^2Y)$,

$B(1,33) = -\frac{z}{a}(-4Y + 6XY)$, $B(2,33) = \frac{a}{R}(XY - 2X^2Y + X^3Y)$,

$B(3,33) = -\frac{2z}{b}(1 - 4X + 3X^2)$,

$B(1,34) = \frac{b}{a}(Y^2 - Y^3)$, $B(2,34) = 0$, $B(3,34) = -2Y + 2XY + 3Y^2 - 3XY^2$,

$B(1,35)=0$, $B(2,35)=\left(1+\frac{z}{R}\right)\left(-2Y+2XY+3Y^2-3XY^2\right)-\frac{zb}{R^2}\frac{\partial R}{\partial y}\left(-Y^2+XY^2+Y^3-XY^3\right)$, $B(3,35)=\frac{b}{a}\left(1+\frac{2z}{R}\right)\left(Y^2-Y^3\right)$,

$B(1,36)=0$, $B(2,36)=\frac{b}{R}\left(-Y^2+XY^2+Y^3-XY^3\right)-\frac{z}{b}\left(-2+2X+6Y-6XY\right)$, $B(3,36)=-\frac{2z}{a}\left(2Y-3Y^2\right)$.