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PROBLEMS OF THE SPECTRAL THEORY OF NON SELF ADJOINT OPERATORS

by

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13. ABSTRACT

The class of non self adjoint operators, for which the unconditionally converging expansion of the Eigenfunctions is correct, has not yet been fully defined (for example, it is not known whether ellipital differential operators with partial derivatives belong to this class). However, it is now clear that the spectral expansion converging on the norm is not a necessary characteristic of the general linear operator. Apparently, further development of the theory will be achieved by establishing the generalized spectral expansion. We note that considerable material has been accumulated in the theory of non self adjoint problems, and it is characteristic that in recent years the theory has been supplemented with a number of new and important studies. Successes have been particularly great in the area of operators with discrete spectrum. We will dedicate the first three sections in our review to this theme.

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PROBLEMS OF THE SPECTRAL THEORY OF NON SELF ADJOINT OPERATORS

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Introduction

One basic method of production of expansions for the Eigenfunctions of linear Operator A, Operating in Hilbert space $\mathfrak{H}$, is a method utilizing the representation of the operator by a contoured integral of its resolvent. This method was used by O. Cauchy, who applied it to the investigation of series of Eigenfunctions of ordinary differential equations.

This method is based on the following formula, correct for any limited operator A:

$$Ah = \frac{1}{2\pi i}\int_C \lambda (A-\lambda E)^{-1}h\,d\lambda; \tag{1}$$

The integral is taken with respect to the contour containing all specifics of the operator resolvent. Equation (1) is established by calculating the residue where $\lambda=\infty$.

Suppose, for example, A is a fully continuous operator. Then, as we know, its resolvent $R\lambda=(A-\lambda E)^{-1}$ is a meromorphic function with the point of concentration of poles at 0. Suppose there is a sequence of closed contours C_k approaching 0, such that

$$\lim_{k\to\infty} \| \int_{C_k} \lambda (A-\lambda E)^{-1}h\,d\lambda \| = 0. \tag{2}$$

Then, using the fact that the residue of the resolvents in each pole is equal to the projection operator P_k on the corresponding root space, we produce the converging expansion

$$Ah = \sum_{k=1}^{\infty} \lambda_k P_k h. \tag{3}$$

In particular, when all poles are simple, expansion (3) becomes

$$Ah = \sum_{k=1}^{\infty} \lambda_k (h, \psi_k) \varphi_k. \qquad (4)$$

where ϕ_k are the Eigenvectors of Operator A, while ψ_k is the adjoint operator A^1.

We note that when A is a self-adjoint (or in the more general case, normal), fully continuous operator, the secret of contours C_k always exists. This is explained by the specifics of the self-adjoint operator, stating in particular that the resolvent of this operator grows slowly as parameter λ approaches the spectrum. Equation (4) in the self-adjoint case is well-known as the "Hilbert-Schmidt theorem."

In the case of a general self-adjoint limited operator, when the resolvent is not meromorphic and the spectrum[2] may be continuous, integral (1), can still be represented as

$$Ah = \int_{\sigma} \lambda dP_{\lambda} h \qquad (5)$$

and we can produce a spectral expansion[3] generalizing formula (3).

Whereas in the theory of self-adjoint operators, general results of a final nature were produced relatively long ago, for linear non self adjoint operators, an expansion has only been produced in a few particular cases.

The superficial reason for this lies in the difficulties arising in estimation of the resolvent. The true reason, probably, is the complex spectral structure of the non self adjoint operator.

The class of non self adjoint operators, for which the unconditionally converging expansion of the Eigenfunctions is correct, has not yet been fully defined (for example, it is not known whether elliptical differential operators with partial derivatives belong to this class). However, it is now clear that the spectral expansion converging on the norm is not a necessary characteristic of the general linear operator.

Apparently, further development of the theory will be achieved by establishing the generalized spectral expansion.

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[1] In the case of multiple poles in formula (4) the attached vectors appear in addition to the Eigenvectors.

[2] The spectrum means the set of all irregular points of the resolvent.

[3] Expansion (5), first produced by Hilbert [1] (1904), has been produced from integral (1) by Hellinger [2] (1909).

We note that considerable material has been accumulated in the theory of non self adjoint problems, and it is characteristic that in recent years the theory has been supplemented with a number of new and important studies. Successes have been particularly great in the area of operators with discrete spectrum[1]. We will dedicate the first three sections in our review to this theme.

§. Completeness of System of Eigenvectors and Attached vectors

Since D. Birkhoff produced an expansion of the Eigenfunctions of the non self adjoint boundary problem for an ordinary, linear, nth order differential equation with regular homogeneous conditions at the ends of the finite interval [a, b], in 1908, a number of studies have appeared [4, 5, 6]. In these works, the results of Birkhoff have been extended to the case of other boundary problems for ordinary differential equations and systems studied over a finite interval. The expansions in all cases were produced by the Cauchy method described in the Introduction.

The problem is that in problems for ordinary differential equations, the asymptote of the solutions can be found if x changes over a finite interval, while λ is great. Using the asymptote of the solution, we can estimate the Green function of the corresponding problem and prove the existence of an appropriate sequence of contours.

The structure of the Green function is more complex in problems with partial derivatives, apparently the reason why works dedicated to the boundary non self adjoint problem for partial derivatives were extremely scarce for some time.

The significant contribution to this area was made by the well-known works of T. Carleman [7] (1936). In this work, in the case of a boundary problem for an elliptical type equation

$$L(u) = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{s=1}^{n} b_s(x)\frac{\partial u}{\partial x_s} + c(x)u = \mu u \tag{6}$$

$$u|_\Gamma = 0 \tag{7}$$

where n=3, the primary term of the asymptote of the Eigenvalues of μ_k was found as $k \to \infty$. In his proof, Carleman developed a new method for producing the asymptote of the Eigenvalues, based on estimation of the track of the iterated Green function with subsequent application of the theorem of Tauber.

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[1] As the operators are called, the spectrum of which consists of Eigenvalues of finite multiplicity having only one collection point.

In his proof, Carleman also utilized his own preceding results, produced in [8] (1921), in which he studied the resolvent of an integral equation with a kernel having an integrable square (Hilbert-Schmidt kernel)[1].

The works of Carleman played a leading role in the development of the theory of non self adjoint problems. His methods allowed the resolvent to be estimated as a function of the parameter in the case of problems with partial derivatives. It is also significant that they can be used in the investigation of operators acting in an abstract Hilbert space.

However, one of the main problems, namely the question of the completeness of the system of Eigenvectors and attached vectors remained open in the case of problems with partial derivatives for some time following the works of Carleman.

In 1951, M. V. Keldysh succeeded in finding broad conditions of completeness in [10].

In this same work, a theorem was proven concerning the asymptote of the Eigenvalues of operators acting in an abstract Hilbert space. It followed from this theorem, in particular that if L_1 is an elliptical, self-adjoint differential operator with discrete spectrum, then when it is perturbed by a differential operator of lower order, the main term of the asymptote of the Eigenvalues is retained. Although we cannot discuss this problem in detail, let us formulate the theorem of M.V. Keldysh of completeness. We will present it in the following abbreviated form:

Suppose fully continuous operator A, for which 0 is not an Eigenvalue, has the form

$$A = H(E + Q), \tag{8}$$

where Q is a fully continuous operator, while H is a self-adjoint, fully continuous operator, such that with a certain $\rho > 0$

$$\sum_{k=1}^{\infty} |\nu_k|^{\rho} < \infty \tag{9}$$

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[1] Estimates of the resolvents for Hilbert-Schmidt kernels were then produced by another method in an important work by Hille and Tamarkin [9]. It was first shown in this work that the Fredholm determinant of the convolution of two Hilbert-Schmidt kernels has 0 order, and a number of other results were produced.

(ν_k are the Eigenvalues of H).

Then the system of Eigenvectors and attached vectors of operator A

$$\varphi_1, \varphi_2, \ldots \varphi_n, \ldots \tag{10}$$

is complete in Hilbert space $\mathfrak{H}$.

In other words, no matter what might be element h and no matter what might be ε, a N and a number c_k are found, such that

$$\left\| h - \sum_{k=1}^{N} c_k \varphi_k \right\| < \varepsilon. \tag{11}$$

The system of Eigenvectors and attached vectors will be referred to as the system of main vectors in the rest of this paper.

The theorem of completeness of the system of main vectors was established not only for problem (6), but for the case of the boundary problem for elliptical equations of any order studied in a finite area of a space of n measurements. Actually, under these conditions the differential operator L is

$$L = L_1 + L_2, \tag{12}$$

where L_1 is the self-adjoint elliptical operator, while L_2 is the operator of lower order, so that operator $L_2 L_1^{-1}$ is fully continuous. Equation $(L_1+L_2)u=\mu u$ can be represented in the form $(E+L_2L_1^{-1})L_1u=\mu u$, from which

$$u = L_1^{-1}(E + L_2 L_1^{-1})^{-1}\mu u. \tag{13}$$

Assuming here $L_1^{-1}=H$, $(E+L_2L_1^{-1})^{-1}=E+Q$ and $\mu^{-1}=\lambda$, we arrive at the problem of the Eigenvalues for operator (8). Condition (9) in this case is fulfilled [1].

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[1] The completeness of the system of Eigenvectors and attached vectors of the elliptical differential operator was proven by F. Brauder in [45] without referring to the theorem of M. V. Keldysh.

Let us now present the proof of the theorem of M. V. Keldysh concerning completeness, retaining in essence the initial proof (cf.[45] and [48]).

a) For the case of fully continuous operators of the form

$$A = KH \tag{14}$$

where H is a self-adjoint limited operator, satisfying condition (9), while K is any limited operator, the theory of Fredholm determinants can be applied.

Suppose $\lambda_1, \lambda_2, \ldots$ are not equal to 0 and are numbered considering the multiplicity of the corresponding values of operator A; $\nu_1, \nu_2, \ldots$, as before, are the Eigenvalues of H.

We find that the following inequality is always correct:

$$\sum_{k=1}^{\infty} |\lambda_k|^2 \leqslant \|K\|^2 \sum_{k=1}^{\infty} |\nu_k|^2.$$

Considering this fact and assuming $\mu_k = \lambda_k^{-1}$, [1] let us study the following integral function (Fredholm determinant of operator A):

$$\Delta_A(\mu) = \prod_{k=1}^{\infty} \left(1 - \frac{\mu}{\mu_k}\right) \exp \sum_{k=1}^{n} \left(\frac{\mu}{\mu_k}\right)^k \cdot \frac{1}{k}.$$

Here n is the least integer satisfying inequality $n+1 \geq \rho$. Obviously, with this selection of $\Delta_A(\mu)$, the operator function

$$D_A(\mu) = \Delta_A(\mu)(E - \mu A)^{-1} \tag{15}$$

is also integral. On the strength of known theorems from the theory of functions, $\Delta_A(\mu)$ is an integral function of order not over ρ.

We find that when condition (9) is fulfilled, the order of the integral function $D_A(\mu)$ is also not over ρ.

Thus, the meromorphic operator function $(E-\mu A)^{-1}$, where A is an operator of the form of (14) and the Eigenvalues of H satisfy condition (9), can be represented as the ratio of integral functions, each of which is of order not over ρ.

$$|\Delta_A(\mu)| \leqslant \exp C_1 |\mu|^{\rho}, \quad |D_A(\mu)| \leqslant \exp C_2 |\mu|^{\rho} \tag{16}$$

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[1] The inverse values of Eigenvalues are generally called characteristic numbers of an operator.

b) Estimates (16), of course, do not indicate the existence of a sequence of contours over which condition (2) would be fulfilled. However, there is no need to prove the completeness of the system of main vectors in this sequence.

The problem is, and this is very significant for our further discussion, that investigation of the completeness of the system of main vectors of the fully continuous operator can be reduced to study of a certain fully continuous operator with the unique point of the spectrum as a 0. This allows us to avoid the difficulties which arise in the investigation of the meromorphic resolvent of an operator, and reduce the problem to the study of a certain integral function.

For greater generality, we can perform the corresponding discussion in the case of an arbitrary, fully continuous operator, although in the investigation of completeness of operators of a special form (8) it is not used in full volume.

Suppose A is an arbitrary, fully continuous operator. Let us represent by Ω_1 the closed linear envelope of the main vectors of this operator

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots \tag{17}$$

relating to the non 0 Eigenvalues. Suppose Ω_2 is the orthogonal complement of Ω_1. Since Ω_1 is the invariant space A, then Ω_2 is the invariant space of the adjoint operator A^*.

Let us represent by V the operator induced by A^* in Ω_2.

We can now show that for completeness of system (17) in the area of values of operator A, it is necessary and sufficient that

$$V = 0. \tag{18}$$

Actually, if system (17) is complete and, therefore, $Ah \in \Omega_1$ with any h, then for any $g \in \Omega_2$ we have $(Ah, g)=0$. Consequently,

$$0 = (Ah, g) = (h, A^*g) = (h, Vg) \tag{19}$$

with any h, and therefore V=0. Conversely, if condition (18) is fulfilled, then, tracking equation (19) from right to left, we conclude that $Ah \in \Omega_1$ with any h, and, consequently, system (17) is complete.

Let us now show that fully continuous operator V has a unique spectral point at 0 or, as it is sometimes stated, it is a Walter operator.

Let us assume the opposite; then $Vg-\lambda_0 g=0$, $\lambda_0 \neq 0$. Applying scalar multiplication by arbitrary vector h, we produce

$$(Vg-\lambda_0 g, h)=(g, (A-\lambda_0 E)h)=0. \tag{20}$$

It is known that the direct complement to the subspace of all vectors such as $(A-\lambda_0 E)h$ lies in $\mathfrak{N}_1$. Therefore, it follows from (20) that g=0, and we arrive at a contradiction.

Thus, proof of the completeness of the system of main vectors of a fully continuous operator can be reduced to proof of equality of a certain Walter operator to 0.

c) Under the conditions of the theorem in question, equation (18) is proven as follows.

Let us study the function

$$\omega(\mu)=((E-\mu V)^{-1}g, h), \tag{21}$$

where g and h $\in \Omega_1$. Since V is a Walter operator, then $\omega(\mu)$ is an integral function. Representing by P the projection operator in Ω_1, we have

$$(E-\mu V)^{-1}=P(E-\mu A^*)^{-1}P.$$

since operator A^* has the form of (14), then, according to (16), $\omega(\mu)$ is an integral function of order not over ρ.

We shall now show that where $\mu \to \infty$ along each ray differing from the real axis, function $\omega(\mu)$ remains limited. From this, on the strength of the Fragman-Lindelhoff theorem, it follows that $\omega(\mu)=\text{const}$. Since further the fraction $\left.\frac{d\omega}{d\mu}\right|_{\mu=0} = (Vg, h)$, consequently, $(Vg, h)=0$ with all g and h $\in \Omega_1$, and therefore equation (18) actually obtains.

Thus, it is sufficient to show that

$$|\omega(re^{i\varphi})|<C. \tag{22}$$

when $r \to \infty$. Let us prove this fact. We have

$$\begin{aligned}(E-\mu A^*)^{-1}&=(E-\mu(E+Q^*)H)^{-1}=(E+S-\mu H)^{-1}(E+Q^*)^{-1}=\\&=(E+(E-\mu H)^{-1}S)^{-1}(E-\mu H)^{-1}(E+Q^*)^{-1}.\end{aligned} \tag{23}$$

where S represents the fully continuous operator such that $E+S=(E+Q^*)^{-1}$. Assuming $\mu = \rho e^{i\alpha}$, let us estimate the right portion of (23). We note that the Eigenvalues of $(E-\mu H)^{-1}$ are $(1-re^{i\alpha}\nu_k)^{-1}$. Since

$$|(1-re^{i\alpha}\nu_k)^{-1}|^2 \leq \frac{1}{(\cos\alpha-r\nu_k)^2+\sin^2\alpha} \leq \frac{1}{\sin^2\alpha},$$

operator $(E-\mu H^{-1})$ is evenly limited. Furthermore, for fixed f we have

$$\|(E-\mu H)^{-1}f\|^2 = \sum_{k=1}^{N} \frac{|c_k|^2}{|1-re^{i\alpha}\nu_k|^2} + \sum_{k=N+1}^{\infty} \frac{|c_k|^2}{|1-re^{i\alpha}\nu_k|^2}. \tag{24}$$

Selecting N sufficiently high, we can first make the second sum $<\varepsilon/2$ after which, by selecting r, we can make the first sum $<\varepsilon/2$. Thus, as $r\to\infty$

$$\|(E-\mu H)^{-1}f\|\to 0. \tag{25}$$

Using this fact, we can show

$$\lim_{r\to\infty}\|(E-\mu H)^{-1}S\|=0. \tag{26}$$

Using the fixed ε, let us represent the fully continuous operator S as

$$S=S_1+S_2,$$

where $||S_1||<\varepsilon/2\sin\alpha$, while S_2 is a finite-dimensional operator. Suppose element h is such that $||h||<1$. We then have

$$\|(E-\mu H)^{-1}Sh\| \leq \|(E-\mu H)^{-1}S_1h\|+\|(E-\mu H)^{-1}S_2h\| \leq$$
$$\leq \frac{\varepsilon}{2}\|h\|+\|(E-\mu H)^{-1}S_2h\|. \tag{27}$$

And since set S_2h is a finite-dimensional and limited set, on the strength of (25) the second component in (17) with sufficiently large r is also $\leq\varepsilon/2\,||h||$. Thus, formula (26) actually obtains. Representation (23) now directly indicates limited norm $(E-\mu A^*)^{-1}$ as $r\to\infty$. Consequently, $|u(re^{i\alpha})|<C$, which was stated.

Let us separate the essential element contained in this proof: finite order of the resolvent of Walter operator V, resulting from inequality (9) allows us -- on the strength of the Fragmen-Lindelhoff theorem -- on the basis of the behavior of the resolvent in a spectrum not containing the spectrum of operator A^*, where it is comparatively easily estimated, to draw a conclusion concerning the resolvent of operator V as a whole.

This theorem on completeness was subsequently developed in a number of works, which we will discuss later.

The later works were also influenced by a work of M. S. Livshits [11], in which a triangular model was produced for a limited operator of the form

$$A = A_R + iA_I, \tag{28}$$

where the imaginary Hermith component A_I is fully continuous and has a track[1]. In particular, this model leads to an integral representation of the Walter operator. M. S. Livshits estabiished also the following fact.

If operator (20) is fully continuous and $A_I \geq 0$, then it is necessary and sufficient for completeness of the system of attached and Eigenvectors that

$$\sum_{k=1}^{\infty} Im\lambda_k = Sp A_I. \tag{29}$$

This theorem, produced by M. S. Livshits using a triangular model, was then proven significantly more simply by B. R. Mukminob [12].

It can be shown that formula (29) immediately indicates equality of Walter operator V, acting in $\mathfrak{S}_+$ to 0; the reverse is also true.

§2. Further Theorems on Completeness, Triangular Representation of Walter Operators

Let us now go over to later results. Suppose A is a fully continuous operator. Let us refer to the Eigenvalues s_n of operator $\sqrt{A^*A}$ as the singular values of operator A.

Obviously, always $s_n \to 0$. We will study only those operators A for which with a certain $\rho > 0$

$$\sum_{n=1}^{\infty} s_n^{\rho} < \infty. \tag{30}$$

Exponent ρ characterizes the degree of deviation of operator A from a finite-dimensional operator. The lower the value of ρ, the more rapidly number s_n approaches 0, and the better the operator is approximated by a finite-dimensional operator.

---

[1] It is stated that fully continuous operator A has a track if the series of Eigenvalues s_n of the non-negative operator $\sqrt{A^*A}$: $\sum s_n < \infty$ converges. Here, the track refers to $\sum_{k=1}^{\infty} (Ax_k, x_k)$, where x_k is a certain orthonormalized base in $\mathfrak{H}$.

If $\rho=2$, operator A is called a Hilbert-Schmidt operator. Integral operators of this type were studied by Carleman [8]. Where $\rho=1$, operator A is called a kernel operator (concerning kernel operators, see [46]). Let us introduce one more characteristic of operator A. It is known that the set of values of the quadratic form (Ah, h) in the complex plane fills either a certain angle $\varkappa$ with its tip at the origin of the coordinates, or the entire plane.

If operator A is self-adjoint and non-negative, the values of (Ah, A) fill the positive half axis. In the general case, multiplying the operator by an appropriate complex constant, it can be arranged that the bissectrix of the angle of values of form (Ah, h) is the positive half axis. The aperture of this angle can be used as a characteristic of the deviation of the operator from a non-negative self-adjoint operator. The following theorem is correct.

If operator A satisfies condition (30) where $\rho\geq 1$ and if

$$-\frac{\varkappa}{2\rho}\leqslant \operatorname{Arg}(Ah,\ h)\leqslant\frac{\varkappa}{2\rho}, \tag{31}$$

then the system of main vectors of operator A is complete in $\mathfrak{H}$.

This fact was initially established in a number of particular cases by various methods by V. B. Lidskiy. For the case $\rho=2$ in [13] using the results of T. Carleman [8]; for the case $\rho=1$ in [14] based on the formula of tracks

$$\sum_{k=1}^{\infty}(A\chi_k,\ \chi_k)=\sum_{k=1}^{\infty}\lambda_k. \tag{32}$$

which, as was proven in [14], is correct for any kernel operator [in formula (32), λ_k are the Eigenvalues of A, while χ_k is an arbitrary orthonormalized base].

However, after minimality of the first Fredholm minor $D_A(\lambda)$ was proven under condition (30)[1], the theorem formulated above was proven by a strong method, applying the Fragmen-Lindelhoff theorem to function (21).

As B. Ya. Levin and V. I. Matsayev proved, the conditions of completeness of (30) and (31) are precise: with the given convergence indicator of series (30) ρ and a broader range of values of the quadratic form then (31), we can indicate an operator with an incomplete system of main vectors.

Further progress in the investigation of completeness was achieved in the works of M. G. Kreyna, L. A. Sakhnovich and M. S. Brodskiy,

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[1] See [43]. As the authors have learned, V. I. Matsayev showed that if V is a Walter operator and $s_n=o(n^{-1/\rho})$, then

$$\ln\|(E-\mu V)^{-1}\|=o(|\mu|^{\rho}).$$

L. A. Sakhnovich and M. S. Brodskiy produced new triangular representations of Walter operators. Let us discuss these works briefly.

L. A. Sakhnovich [15], [16], generalizing the results of M. S. Livshits, constructed a triangular model of the limited operator

$$A = A_R + iA_I, \tag{33}$$

having the property that no matter what the two invarient subspaces H_1 and H_2 of operator A, $H_1 \in H_2$ and dim $H_2 \ominus H_1 > 1$ invarient subspace H_3 of operator A is found, such that $H_1 \in H_3 \in H_2$ and $H_1 \neq H_3 \neq H_2$.

In particular, as L. A. Sakhnovich demonstrated, this property is shown by any operator (33) if A_I is a Hilbert-Schmidt type. In this case, when spectrum A consists only of the 0. and A_I is a Hilbert-Schmidt type, operator (33) is uniquely equivalent to the operator

$$\vec{A} = \int_0^\infty f(t) N(x,t)\, dt, \tag{34}$$

where f(t) is a vector function, generally infinite-dimensioned, and N(x,t) is the matrix kernel satisfying the condition

$$\int_0^1 \int_0^1 \sum_{i,j=1}^{\infty} |n_{ij}(x,t)|^2 \, dx\, dt < \infty. \tag{35}$$

It immediately follows from representation (35) that if A is a Walter operator and A_I is a Hilbert-Schmidt operator, then A is also of Hilbert-Schmidt type.

This fact has significantly influenced a number of later works (see below). In particular, it allowed L. A. Sakhnovich to strengthen the theorem of V. B. Lidskiy concerning completeness in the case of Hilbert-Schmidt operators in the following form.

If A is a fully continuous operator, $A_R \geq 0$ and $A_I \geq 0$, and if A_I is a Hilbert-Schmidt operator, then completeness obtains.

Another triangular presentation for the Walter operator was produced by M. S. Brodskiy [17]. The triangular presentation of M. S. Brodskiy is effective in the same Hilbert space as operator A, and corresponds with operator A fully, not with an accuracy to a supplementary component, as occurs in the models of M. S. Livshits and L. A. Sakhnovich.

Going over to a presentation of this problem, let us assume initially that A is a linear transform in an n-dimensional space, all Eigenvalues of which are equal to 0.

Suppose

$$e_1, e_2, \ldots, e_n \tag{36}$$

is an orthonormalized base, in which the conversion matrix is a triangle[1]. Then

$$Ae_1=0;\quad Ae_2=e_{21}e_1,\ \ldots.\ Ae_n=e_{n1}e_1+e_{n2}e_2+\cdots+e_{n,n-1}e_{n-1}. \tag{37}$$

Let us represent by P_k the projection operator onto the space stretched onto the first k base vectors (36), and suppose $\Delta P_k=P_k-P_{k-1}$. It then immediately follows from formula (37) that with any h

$$Ah=\sum_{k=1}^{n}P_{k-1}AP_kh. \tag{38}$$

We note also that according to (37), $\sum_{k=1}^{n}\Delta P_kAP_{k-1}=0$. Going over in this equation to adjoint operators and assuming $A_I=1/2i(A-A^*)$, we can write (38) as

$$Ah=2i\sum_{k=1}^{n}P_{k-1}A_I\Delta P_kh. \tag{39}$$

This representation, as M. S. Brodskiy has shown, is generalized in the case of any Walter (fully continuous) operator A, acting in $\mathfrak{H}$. Namely, any Walter operator can be represented as

$$A=2i\int_{\mathfrak{M}}P(x)A_IdP(x). \tag{40}$$

Here A_I, as always, is the imaginary component of operator A, $\mathfrak{M}$ is a certain closed set of sector [0,1], $P(x)$, $x\in\mathfrak{M}$, is a chain of projection operators, continuous in $\mathfrak{M}$ and monotonically increasing, projecting on the invariant subspaces of operator A, where $P(0)=0$, $P(1)=E$, and if (α, β) is the complimentary integral to set $\mathfrak{M}$, operator $P(\beta)-P(\alpha)$ is unidimensional.

Integral (39) is understood as the limit of the sequence of partial sums in the ordinary operator norm.

We note that proof of the existence of the chain of projectors P(x) is based on the Neuman-Aronshein theorem [18] on the existence of a nontrivial invariant subspace with a fully continuous operator acting in $\mathfrak{H}$. A chain of this form was constructed independently by L. A. Sukhnovich in [15], [16], and is the basis of the results produced there.

Representation (39) has been found quite convenient in the study of Walter operators. New, important representations concerning the convergence

[1] Existence of this type of base is established by the well-known theorem of I. Shur.

of integrals such as (40) under conditions when P(x) is a monotonic chain of projectors, not necessarily generated by the fixed Walter operator, while A_I is a certain self-adjoint, fully continuous operator, were produced by I. Ts. Gokhoyerg, M. G. Kreyn and V. I. Matsayev [19, 20, 21, 22]. These authors, using triangular representations, established the following fact, generalizing the theorem of L. A. Sakhnovich presented above.

Suppose $V=V_R+iV_I$ is a Walter operator and suppose τ_k are the Eigenvalues of $_I$, while σ_k are the Eigenvalues of V_R. Then where $\rho>1$, the series

$$\sum|\tau_k|^\rho \tag{41}$$

$$\sum|\sigma_k|^\rho \tag{42}$$

converge and diverge simultaneously.

Let us emphasize that the statement formulated allows us to judge the growth of integral functions $D_A(\mu)$ and $\Delta_A(\mu)$ in the case of a Walter operator, with information only concerning the imaginary or real component of the operator. The order of these functions with $\rho>1$ is not over ρ. This allows us to strengthen the completeness theorem formulated above on page 10.

If operator A is such that its imaginary portion $A_I=1/2i(A-A^*)$ satisfies condition (30) where $\rho>1$ and if condition (31) is fulfilled, completeness occurs.

Where $\rho=1$, convergence of series (41) does not gene.ally produce convergence of series (42). One example is the operator $Af=\int_0^x f(t)dt$, for which the imaginary component is unidimensional, while the Eigenvalues $\sigma_n=n(n=\pm1,\pm2,\ldots)$.

Walter operators, the imaginary components of which have tracks, were subjected to detailed study in the works of M. G. Kreyn [23], [24]. M. G. Kreyn relates the Walter operator $V=V_R+iV_I$ to the analytic function

$$f(z)=\operatorname{Det}\{(E-zV_R)(E-zV)^{-1}\}. \tag{43}$$

Since $(E-zV_R)(E-zV)^{-1}=E+izV_I(E+zV)^{-1}$ and V_I has a track, the determinant in the right portion of (43) converges evenly and is an integral function (we recall that V is a Walter operator and, consequently, $(E-zV)^{-1}$ is an integral function); the nulls of f(z) are the numbers σ_k^{-1}. As M. G. Kreyn proved, function f(z) within the upper and lower half planes can be represented as the ratio of 2 limited holomorphic functions. From this, based on the theorem of M. G. Kreyn [25] and the theorem of Levenson [26], it follows that there is a general finite limit

$$\lim_{r\to\infty}\frac{n_+(r,V_R)}{r}=\lim_{r\to\infty}\frac{n_-(r,V_R)}{r}=\frac{h}{\pi}. \tag{44}$$

Here, $n_+(r, V_R)$ and $n_-(r, V_R)$ represent the number of characteristic numbers σ_k^{-1} of operator V_R in the intervals (0, r) and (-r, 0) respectively. Formulas (44) contain the asymptote of the Eigenvalues of the real component of the Walter operator, the imaginary component of which has a track, and supplement the preceding result of Kreyn, Gokhyerg, and Matsayev. It is remarkable that in the case when $V_I \geq 0$, in formula (44)

$$h = SpV_I. \tag{45}$$

If therefore $V_I \geq 0$ and the general limit in (44) is equal to 0, then $V_I = 0$, and all of operator V, being a self-adjoint Walter operator, is equal to 0.

This established fact relative to Walter operators leads to the following completeness theorem.

If fully continuous operator $A = A_R + iA_I$ is such that $A_I \geq 0$ and if one of the two conditions

$$\lim_{r \to \infty} \frac{n_+(r, A_R)}{r} = 0 \quad , \tag{46}$$

or

$$\lim_{r \to \infty} \frac{n_-(r, A_R)}{r} = 0, \tag{47}$$

is fulfilled, then the system of main vectors A is complete.

This theorem contains the results of V. B. Lidskiy concerning completeness of operators having a track ($\rho = 1$) as a particular case, since if operator A has a track, then both conditions (46) and (47) of the theorem of M. G. Kreyn are fulfilled.

Further, M. G. Kreyn finds a necessary and sufficient condition of completeness for fully continuous operators A, such that $A_I \geq 0$, $SpA_I < \infty$.

Completeness occurs when and only when

$$\int_0^{\infty} \frac{n(r, A_R)}{r} dr - \int_0^{\infty} \frac{n(r, A)}{r} dr = o(\ldots \tag{48}$$

under the condition that $\rho \to \infty$, bypassing a certain set of finite logarithmic length.

Here n(r, A) is the number of characteristic numbers in a circle of radius r. Simultaneously with the work of M. G. Kreyn, an important study appeared by B.Ya. Levin [27], in which the following estimate was produced under the same assumptions ($A_I \geq 0$ and $SpA_I < \infty$)

$$\int_0^\rho \frac{n(r, A_R)}{r}\,dr - \int_0^\rho \frac{n(r, A)}{r}\,dr \leqslant \rho S_\rho \Delta_I + 0(\rho) \tag{49}$$

as $\rho \to \infty$, bypassing a set of finite logarithmic length, as well as a number of other results.

In all of these works concerning completeness of the system of main vectors in a fully continuous operator, conditions were stated under which the resolvent of the operator is represented as a ratio of finite order integral functions. Incidently, an attempt to remove condition (9) from the theorem of M. V. Keldysh, as yet unsuccessful, produces infinite order integral functions. In connection with this, there is great interest in a recent result by V. I. Matsayev [21], according to which the system of main vectors of operator A=H(E+Q) [cf. (8)] is complete if only

$$\sum_{k=1}^{\infty} \frac{s_k}{2k+1} < \infty.$$

where s_k are the Eigenvalues of $\sqrt{Q^*Q}$. Condition (9) can be discarded. Under these assumptions, the resolvent is generally not represented by the ratio of finite order integral functions.

We have not touched upon an interesting study by D. E. Allakhverdiev [40] concerning the conditions of completeness in the case of weakly perturbed normal operators, in which the author succeeded in extending the theorem of M. V. Keldysh to this case; we have also not mentioned the new, deep theorems of V. I. Matsayev, based on precise estimates of the integral functions, or a number of other studies.

However, even our complete review shows that the problem of the conditions of completeness has been greatly advanced in recent years.

This progress has been achieved by a combination of geometric and analytic methods.

§ 3. Theorems on Integrability and Convergence of Series with Respect to Main Vectors

It must be emphasized that since the system of main vectors is not orthogonal, its completeness does not indicate convergence of the Fourier series of elements of this system. Furthermore, as examples have shown, under the conditions of completeness found, formally described series such as (3) and (4) generally diverge. It therefore becomes a pressing problem to define the coefficients of linear combination (11) of the attached and Eigenelements approximating a given element f with predetermined accuracy.

For one class of operators, this problem was solved in the work of V. V. Lidskiy [28], in which he set forth the idea of summation of series with respect to main vectors by the method of Abel. Let us briefly discuss this problem.

Suppose A is a fully continuous operator and suppose s_k are its singular values (natural values of operator $\sqrt{A^*A}$). Let us assume that operator A satisfies condition (30) with a certain $\rho \geq 1$.

$$\sum_{k=1}^{\infty} s_k^{\rho} < \infty \tag{50}$$

and with a certain $\rho' > \rho$, the condition

$$-\frac{\pi}{2\rho'} \leqslant \operatorname{Arg}(Ah, h) \leqslant \frac{\pi}{2\rho'}. \tag{51}$$

Assuming for simplicity that all characteristic numbers μ_k of operator A are simple, we represent by ϕ_k the Eigenvectors of A, by ψ_k the Eigenvectors of A*, normalized by the condition $(\phi_k, \psi_k)=1$.

Suppose f=Ah, where h is an arbitrary element in a Hilbert space. The formally written series (4) for vector f=Ah generally diverges. However, the following theorem is correct.

If the fully continuous operator A satisfies conditions (50) and (51), with any t>0, the series

$$u(t) = \sum_{s=1}^{\infty} \left(\sum_{n=N_{s-1}+1}^{N_s} e^{-\mu_n^{\alpha} t} (f, \psi_n) \varphi_n \right) \tag{52}$$

converges and

$$\lim_{t \to +0} u(t) = f. \tag{53}$$

In formula (52), a is any number satisfying the condition $\rho' > \alpha \geq \rho$; N_s is a certain subsequence of numbers in the natural series, independent of t.

Thus, by replacing condition (31) with the somewhat more rigid condition (51), we can guarantee not only completeness of the system of main vectors, but integrability of the corresponding expansions.

It can further be shown that under conditions (5) and (51), with any f=Ah, the following estimates are correct:

$$\left| u(t) - \sum_{k=1}^{N_s} e^{-\mu_k^{\alpha} t} (f, \psi_k) \varphi_k \right| \leqslant \exp a^{\alpha} \left(-t |\mu_{N_s}|^{\alpha} + |\mu_{N_s}|^{\rho} 0(1) \right) \cdot \|f\| \tag{54}$$

and

$$\|f - u(t)\| \leqslant e^{\alpha}\|A\| \quad (\alpha > 0). \tag{55}$$

These formulas allow us, using a fixed ε, to select first a sufficiently small $t>0$, then with the selected t, a sufficiently large N_s, so that using the coefficients contained in formula (54), we satisfy inequality (11).

Proof of the theorem is presented by converting the integrating factor to a Cauchy integral.

Suppose[1]

$$u(t) = \frac{1}{2\pi i}\int_{\gamma} e^{-\mu^a t}(E - \mu A)^{-1} h \frac{d\mu}{\mu^2}, \tag{56}$$

where γ is an infinite contour, encompassing all bands of the integrand, and containing the function $\exp(-\mu^a)t$, in the decreasing sector. Using estimates (16) and considering the minimal nature of type $D_A(\mu)$ and $\Delta_A(\mu)$ we can prove the existence of a sequence of contours γ_k, which diverges at infinity, in which the integrand approaches 0. This allows us to represent the integral by a series of the residues of (52).

In connection with formula (52) let us touch upon one problem which is of independent significance.

Where a=1, the expansion of (52) becomes

$$u(t) = \sum_{s=1}^{\infty}\left(\sum_{k=N_s+1}^{N_{s+1}} e^{-\gamma_k t}(f, \psi_k)\varphi_k\right) \tag{57}$$

and, as we can easily see, is a solution of the cauchy problem for the equation

$$\frac{\partial u}{\partial t} + Lu = 0 \quad (L^{-1} = A), \tag{58}$$

with the initial condition

$$u|_{t=0} = f. \tag{59}$$

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[1] Integral (56) is converted to integral (1) if we make the replacement $\lambda=\mu^{-1}$ and assume t=0.

Convergence of series (57) where t>0 means, therefore, that if operator L in equation (58) has a fully continuous inverse satisfying condition (50-51), the solution of the Cauchy problem (58-59) can be expanded into a Fourier series converging where t>0 with respect to the main vectors of operator L (cf. [49]).

These conditions are satisfied, for example, by the differential elliptic operators of order 2m, greater than the number n of independent variables. Consequently, in these cases the solution of the Cauchy problem can be found by the Fourier method. As concerns equation (58) with its elliptical operator, this result apparently can be strengthened, since it was produced using a very general estimate of resolvent (16), not considering the special form of the operator.

We note that for a resolvent of elliptical operator (6) with two independent variables, the following estimate is correct, produced by V.B. Lidskiy [44]:

$$\|(L-\mu E)^{-1}\| \leqslant \exp\left\{a^2|\mu|\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}\,|\mu-\mu_k|}\right\} \tag{60}$$

with all μ. In this formula, μ_k are the Eigenvalues of operator (6). Inequality (60) is more precise than the general estimate given by formula (16), and allows us to extend the result formulated above on convergence of the Fourier series to the case of elliptical operator (6) where n=2.

The problem of convergence of series (57) t=0 even in the case of differential operators with partial derivatives, remains open. Generally, convergence of expansions with respect to main vectors has been established with respect to a very narrow class of operators, as was noted in the introduction. In addition to the well-known old studies on convergence of series in the case of the problem for ordinary differential equations, we can note also the results of B. R. Mukminov [12], I. M. Glazman [29], A. S. Markus [30], in which operators were studied, acting in an abstract Hilbert space $\mathfrak{H}$. Let us discuss briefly the results of I. M. Glazman. The infinite system of elements ϕ_k(k=1, 2,...) is called the Riss base of its closed linear envelope, if with certain m and M and all N and c_k, the following inequality is correct:

$$m\sum_{k=1}^{N}|c_k|^2 \leqslant \sum_{i,j=1}^{N}(\varphi_i,\varphi_j)c_ic_j \leqslant M\sum_{k=1}^{N}|c_k|^2. \tag{61}$$

We will not discuss the fact that when condition (61) is fulfilled, system ϕ_k is linearly independent and actually forms a base[1]. We note only that condition (61) is obviously fulfilled, when the angles between the vectors of the system are near a right angle.

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[1] It can be proven that if system ϕ_k forms a Riss base, there is a limited, continuously inversable operator C which converts system ϕ_k to an orthonormalized base.

We find that if ϕ_k are the Eigenvectors of a certain dissipative A (i. e., $A_I \geq 0$), the angles between them can be estimated using the corresponding Eigenvalues. Namely, the following inequality is correct:

$$|(\varphi_i, \varphi_j)|^2 \leq \frac{4 l_m \lambda_i l_m \lambda_k}{|\lambda_i - \bar{\lambda}_k|^2} \tag{62}$$

(it is assumed that $||\phi_i|| = ||\phi_j|| = 1$.) Using this inequality, the following theorem is proven.

If ϕ_k is an infinite system of normalized Eigenvectors of a limited dissipative operator A and if

$$\sum_{i \neq k} \frac{l m \lambda_i l m \lambda_k}{|\lambda_i - \bar{\lambda}_k|^2} < \infty, \tag{63}$$

then system ϕ_k is a Riss base of its closed linear envelope.

This theorem was produced earlier under more limiting assumptions by B. R. Mukminov by another method.

A. S. Markus [30] generalized the theorem of Mukminov and Glazman, introducing the concept of the Riss base from subspaces. Estimating the angle between subspaces by an inequality similar to (62), A. S. Markus established that with certain limitations on the Eigenvalues of dissipative operator A, its root subspaces form a Riss base of their closed linear envelope.

In conclusion, we note that condition (63) and similar conditions place rigid limitations on the Eigenvalues, so that the class of operators for which they are fulfilled in quite narrow.

§ 4. Results of a General Nature and Singular Problems for Differential Operators

Of the general problems, let us first discuss works which develop the results of M. S. Livshits [11] and are dedicated to conversion of limited operator A to triangular form.

We have already indicated, in connection with the representation of Walter operators, that L. A. Sakhnovich [15] succeeded in constructing a triangular model of the limited linear operator, having a sufficient reserve of invarient subspaces. In this case, the model of L. A. Sakhnovich has the form

$$\hat{A}f = \frac{d}{dx} \int_0^x N(x, t) f(t)\, dt. \tag{64}$$

Operator (64) acts in Hilbert space $\mathfrak{H}$ of vector functions $f(t)=\{f_1(t), \ldots\}$, satisfying the condition

$$\sum_{n=1}^{\infty}\int_0^{\infty}|f_n(t)|^2\,dt<\infty.$$

In formula (64), N(x,t) is a certain matrix kernel. Operator $\tilde{A}$ is uniquely equivalent to the initial operator A with an accuracy to a certain invariant subspace relative to $\tilde{A}$ and $\tilde{A}^*$, in which the equality $\tilde{A}\tilde{A}^*=\tilde{A}^*\tilde{A}$.

Under certain additional conditions placed on operator A, differentiation can be performed following the integral sign in formula (64), thus simplifying the model. For example, if in formula (33), A_I is a Hilbert-Schmidt type operator and the spectrum of operator A is real, formula (64) becomes

$$\tilde{A}f=H(x)f(x)+\int_0^{\infty}N(x,t)f(t)\,dt, \tag{65}$$

where H(x) is the Hermith operator, while N(x,t) is a matrix kernel satisfying condition (35). We have already indicated the effectiveness of the triangular presentation in the case of Walter operators.

M. S. Brodskiy, in [31](1960), generalizing his earlier result [17], produced a triangular representation of limited operator A with real spectrum and imaginary, fully continuous component A_I, under an additional assumption concerning the structure of the invariant subspaces of operator A. The triangular representation is as follows:

$$A=\int_{\mathfrak{M}}a(x)\,dP(x)+\int_{\mathfrak{M}}P(x)A_I\,dP(x). \tag{66}$$

In this formula, P(x) is a monotonic chain of projection operators, projecting onto invariant subspaces of operator A [cf. (40)], while a(x) is a certain real function, the values of which correspond with the spectrum of A.

As Yu. I. Lybich and V. I. Matsaev showed [32], the conditions place on the invariant subspaces of operator A by M. S. Brodskiy are fulfilled if

$$\int_0^{\infty}\ln_+\ln_+M(t)\,dt<\infty,\quad M(t)=\sup_{|\lambda|=t}\|(A-\lambda E)^{-1}\|.$$

This condition is quite broad; as V. I. Matsayev proved, it is fulfilled if the series $\sum\frac{|\tau_n|}{2n+1}$ converges where τ_n are the Eigenvalues of A_I and, consequently, practically with any fully continuous[1] A_I.

[1] We note that on the assumption that $\Sigma|\tau_n|^p<\infty$, triangular representation (66) was produced earlier by I. Ts. Gokhbergom and M. G. Kreyn.

A further improvement of the triangular presentations can be produced, apparently, by simplifying the Walzer component in formula (65) and (66). Interesting results in this direction were produced by L. A. Sakhnovich [33]. It is also desirable to avoid the condition of reality of the spectrum of the operator. Although a non-real operator spectrum with a fully continuous imaginary portion is discrete, separation of the corresponding invariant subspaces is a far from trivial problem.

Let us now touch upon another trend in the theory of linear operators, the theory of spectral operators of N. Danford [35], (1954). It is assumed in this theory that limited operator T, acting in a Banach space has the set of projectors E(δ)(δ is any set in the complex plane measurable after Borel). Set E (δ) is assumed to be evenly limited with respect to δ:

$$\|E(\delta)\| < M \tag{67}$$

as well as denumerably additive: for each sequence of non-intersecting Borel sets δ_n,

$$E\left(\bigcup_1^\infty \delta_n\right) f = \sum_1^\infty E(\delta_n) f. \tag{68}$$

where the series on the right converges strongly.

Under certain natural additional assumptions concerning set E(δ), it has been established that the corresponding operator T called the spectral operator, can be represented as

$$T = S + N, \tag{69}$$

where

$$S = \int \lambda E(d\lambda), \tag{70}$$

while N is the generalized 0-power operator in the sense of I. M. Gel'fand

$$\lim_{n\to\infty} \sqrt[n]{\|T^n\|} = 0,$$

and the operators N and S are commutative. Representation (68) is a full analogue of the Jordan form. As N. Danford shows, for any single-valued function f(λ), analytic n spectrum T, the formula

$$f(T) = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int \lambda^n f(\lambda) E(d\lambda). \tag{71}$$

well-known in the finite-dimensional case, is correct.

Works on further development of the theory of spectral operators were included in the objective review of N. Danford [35], (1958). We can see from this review that the mathematicians working in this direction have directed their efforts toward the production of sufficient conditions imposed on operator T and its resolvent, under which the operator is spectral. The conditions produced to date contain a requirement of not over exponential growth of the resolvent as parameter λ approaches the point of the spectrum. Furthermore, it is required that for any two elements x and y such that the functions $R_\lambda x$ and $R_\lambda y$ have no common points of irregularity, the inequality $||x|| \leq c||x+y||$, be fulfilled with a certain constant c, independent of x and y.

As is stated in the review, all differential operators with ordinary derivatives and regular boundary conditions (operators studied by D. Birkgof) are spectral operators. The work of N. Danford also presents certain singular problems. For example, it is stated that the operator

$$l(y) = -\frac{d^2y}{dx^2} + q(x)y. \tag{72}$$

studied by M. A. Naymark in [36], defined in a variety of functions $y(x) \in L_2(0, +\infty)$, $y'(0) = hy(0)$, under the condition that

$$\int_0^\infty (1+x^2)|q(t)|\,dx < \infty \tag{73}$$

is spectral (q(x) and h are generally not real).

Among the differential operators for which no expansion into a Fourier integral was produced earlier, this review states, the following operator is spectral

$$l(y) = -\frac{d^2y}{dx^2} + a\frac{dy}{dx} + q(x)y. \tag{74}$$

where $\mathrm{Re}\, a \neq 0$, $q(x+\omega) \equiv q(x)$. With real q(x) and a=0, operator (74) is self-adjoint: its spectrum, as is well-known, is an infinite series of intervals moving off to $+\infty$. All points in the spectrum are double.

As M. I. Serov [41] has shown, in the case of the complex-valued function q(x) and a=0, the picture changes little: the intervals are deformed into curved sectors, asymptotically retaining their length and distance between neighboring n's. If, however, we assume in (74) that $\mathrm{Re}\, a \neq 0$, the spectrum changes significantly. Several of the first intervals are split into ovals; all remaining lacunas are extended and the twice-added ray is split into a curve asymptotically close to a parabola.

M. I. Serov, studying operator (74) on the suggestion of I. M. Gel'fand, estimated the resolvent of the operator with approximation of the parameter to the spectrum. However, he did not succeed in producing expansion into a Fourier integral. It is even more interesting that this problem is solved from general considerations.

It should be noted that proof of the results announced by N. Danford has unfortunately never been published. However, the incomplete formulation of the results and the absence of proof lead to disagreements. For example, in contrast to a statement contained in the review, B. S. Pavlov [42] has shown, but constructing a contradictory example, that operator (72) under condition (73) is not spectral. The corresponding statement is incorrect even if

$$\int_0^\infty (1+x^{2n})|q(x)|\,dx<\infty,\quad (n>1). \tag{75}$$

In connection with the theory of spectral operators, we note an interesting attempt undertaken by V. E. Lyantse [37] to construct a theory of spectral operators under conditions of completeness of the system of invariant subspaces, without assuming even limitation of the spectral set (67) or denumerable additiveness (68). It is to be hoped that this theory will be applied.

In conclusion, let us discuss the problem of expansion with respect to Eignefunctions of an ordinary differential operator in the case of an unlimited area of definition of the functions.

We have mentioned the well-known theorem of M. A. Naymark [36] of expansion with respect to Eigenfunctions of the Shturm-Liouville equation with unreal potential q(x), satisfying condition (73). This theorem of M. A. Naymark was extended by V. N. Funtakov [38] to the case of an even order differential operator

$$l(y)=y^{(2n)}+p_2(x)\,y^{(2n-2)}+\ldots+p_{2n}(x)\,y.$$

acting in $L_2(0,+\infty)$, on the assumption that the coefficients $p_s(x)$ decrease exponentially as $x\to\infty$.

A new approach to problems of expansion with respect to Eigenfunctions of a differential operator was suggested in a work by V. A. Marchenko [39].

Suppose

$$l(y)=\frac{d^2y}{dx^2}-q(x)\,y \tag{76}$$

is a differential operator, defined in $L_2(0,\infty)$ in a manifold of functions satisfying the boundary condition

$$y'(0)=hy(0). \tag{77}$$

q(x) is an arbitrary complex function, integrable in each finite interval, while h is a complex number.

Suppose $\omega(s, x)$ is the solution of equation $l(y)+s^2y=0$, satisfying the initial conditions

$$\omega(s, 0)=1, \quad \omega'_x(s, 0)=h. \tag{78}$$

Let us compare each finite function f(x) to a Fourier ω transform

$$E_f(s)=\int_0^\infty f(x)\omega(s, x)\,dx. \tag{79}$$

If $E_g(s)$ is the Fourier ω transform of function g(x), then in the case of real q(x) and h, as we know, the following equation of Parseval is correct:

$$\int_0^\infty f(x)g(x)\,dx=\int_{-\infty}^{+\infty} E_f(\sqrt{\lambda})E_g(\sqrt{\lambda})\,d\rho(\lambda). \tag{80}$$

where $\rho(\lambda)$ is a non-decreasing real function. The right portion of this formula can be interpreted to mean that the Parseval equation is retained with arbitrary q(x) and h, i. e.,in the non self adjoint case.

Going over to the presentation of this problem, we note that $\omega(s,x)$ and cos sx are related by the transforms

$$\omega(s, x)=\cos sx+\int_0^x K(x, t)\cos st\,dt \tag{81}$$

$$\cos st=\omega(s, t)+\int_0^t H(x, t)\omega(s, t)\,dt, \tag{82}$$

where K and H are smooth kernels. Substituting $\omega(s,x)$ from formula (81) into (79), it is easy -- on the basis of the Paley-Wiener theorem -- to see that $E_f(s)$ is as even, exponential-type function with integrable square on the real axis. Let us represent by Z the topological space of all integral even functions, integrable on the real axis with the following definition of convergence: $F_n(\lambda)\to F(\lambda)$, if

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}|F(\lambda)-F_n(\lambda)|\,d\lambda=0$$

and the power σ_n of functions $F_n(\lambda)$ are limited as a set. It is easy to see that the product $E_f(\sqrt{\lambda})E_g(\sqrt{\lambda})$ belongs to Z, and it can be shown that this set of such derivatives is compact in Z.

The right portion in formula (80) can therefore be looked upon as a linear functional in Z, fixed in a compact manifold. The latter can be extended to all of space Z.

Thus, in the self-adjoint case, operator (76) generates a certain continuous functional in Z for which formula (80) is correct. As V. A. Marchenko proves, this affirmation retains its force in the general case, that is, operator (76) can always be related to continuous functional $(R, F(\lambda))$, $F(\lambda) \in Z$, for which the following formula is correct:

$$\int_0^\infty f(x) g(x)\, dx = (R, E_f(\sqrt{\lambda})\, E_g(\sqrt{\lambda})). \tag{83}$$

It is remarkable that V. A. Marchenko succeeded in solving the reverse problem: restore function q(x) and h on the basis of fixed functional $(R, F(\lambda))$.

We note, however, that determination of the analytic expression for functional R can be fully performed only with certain additional limitations placed on function q(x). For example, under condition (73) it can be shown that

$$(R, F(s)) = \int_0^\infty \frac{F(\sqrt{\lambda})}{B^+(\lambda) B^-(\lambda)}\, d\left(\frac{2}{\pi}\sqrt{\lambda}\right) - \sum \operatorname{Res} m_1(s) F(\sqrt{s}),$$

where

$$B^\pm(\lambda) = g(\lambda \pm i0, 0) h - g'(\lambda \pm i0, 0),$$

$$m_1(s) = \frac{m(s)}{1 + hm(s)};$$

$m(s)$ is an analogue of the Weil function.

In the more general case, the functional can be represented by an integral with respect to the contour encompassing the spectrum of the operator. This contour has not yet been successfully extended to the spectrum.

It can be shown that the idea of comparison of a linear operator of a functional in a certain topological space of analytic functions with subsequent study of the carrier of this functional can be applied in the case of a general linear operator. However, up to now this has been realized only in the case of a problem for one ordinary differential second order equation and system (see [47]).

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