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THE GEOMETRY OF THE RICCATI EQUATION

by

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13. ABSTRACT

A study of the matrix Riccati equation is presented without assuming that Q is positive semi-definite.

For the autonomous case, a general existence theorem for the equilibrium points of the Riccati equation is proven and their structure is given with respect to the partial ordering induced by the positive semi-definite matrices. Further, a method for finding all equilibria is demonstrated.

Necessary and sufficient conditions are given for a global existence of the solution of the Riccati equation, as a function of the initial condition.

In the time dependent case, the domain of global existence is shown to contain a certain cone.

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JOSE MARIA RODRIGUEZ-CANABAL

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CONTENTS

NOTATION CONVENTION

$\dot{A}$	$\frac{dA}{dt}$
$\|A\|$	absolute value of A
$\|\|A\|\|$	appropriate norm of A
I	identity matrix
Re A	real part of A (p. 15)
$\{a\|\ldots\}$	set of a's
$a \rightarrow b$	a tends to b, or a changes into b
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	(2 x 2) matrix
$\cap$	set intersection
ϕ	empty set
ϵ	belongs to
$\notin$	does not belong to
$\subseteq$	set inclusion

CHAPTER I

INTRODUCTION

The importance of the Riccati equation (RE) in itself and in connection with the optimal linear-quadratic control and Kalman-Bucy filtering theory, has motivated several works examining the behavior of its solution. In particular, results pertaining to sufficient conditions for global existence, monotoneity properties, asymptotic character, invariant sets and equilibrium points have been reported in the literature. A discussion of the interesting problem of a priori bounds for the solution is presented in [4].

As Willems [19] has pointed out, various aspects of the RE need further investigation. Specifically, one is interested in i) performance criterion which are not positive, ii) the cost linear functional in the control, iii) control with conflicting aims, and iv) conjugate point structure in variational problems.

This latter case is of particular importance in the numerical solution of the RE: although the solution of the difference matrix equation may be well behaved, in the sense of being finite for all time and convergent to an equilibrium solution, it does not pertain to the problem under consideration since the solution lies outside the interval of disconjugacy. The only indication that one gets from the numerical computation is a change of the definiteness of the solution between two consecutives iterations.

There exists a duality principle between the optimal linear-quadratic control problem and the optimal linear-gaussian filtering problem. The duality is achieved by a simple change of the dynamics, sensor, and actuator along with a reversal of time. See [10, Section 5]. This correspondance allows one to formulate results in either control or filtering form and obtain results for the dual problem by a reinterpretation within the appropriate context.

Henceforth we will be concerned with solutions of the RE with time running forward and the initial condition imposed at the starting time. We will also concentrate on the autonomous case, i.e., when the coefficient matrices are time independent. For this case, under certain regularity conditions, it has been shown [6] that there exists a fixed point of the RE, called P_-, which generates a convex cone in the partial ordering induced by the positive semi-definite matrices, such that if the initial condition belongs to this cone then the solution exists globally. The result was proved by the application of a theorem of Reid [16].

The importance of P_- in connection with the structure of the solution of the RE was also recognized in [19] whereas its physical interpretation as the steady state error variance of the present state given future observations was made clear in [4]. Also a solution [4, Corollary 3.2] of the RE is given when the initial condition tends to infinity; this is a particular case of the general form presented in Chapter III.

The main results determined in this analysis are:

i) Necessary and sufficient conditions for global existence of the RE as a function of the initial condition for the autonomous case.

ii) Sufficient conditions for the solution to be in the domain of global existence for the time dependent case.

iii) A general existence theorem for the equilibrium points P_+ and P_-. (The reader is referred to Chapter II for the definition of P_+, P_- and P_θ).

iv) A method for finding the equilibria.

There are several techniques available for the computation of the points P_+, P_- and P_θ. One vay to find P_+ is by the method of quasi-linearization or Newton's method in function space. What results is a numerical computation [5, Chapter 8], [12] which can efficiently compute P_+ ; P_- can also be determined by iterating to a related equation [6]. This method is not applicable for the determination of P_θ.

A second method makes use of information contained in the Hamiltonian matrix, $\mathcal{H}$, associated with the RE. One form of this technique, due to Potter [14], is valid when $\mathcal{H}$ is diagonalizable; Martensson [13] has extended the method to include a general Hamiltonian matrix. In the present work we extend another form of the Hamiltonian method, due to Bass and Roth, [1], [5], and [18], which is based on a partition of the characteristic polynomial of $\mathcal{H}$

The use of the Hamiltonian provides not only P_+ and P_-, but also any P_θ, if it exists, as well as results concerning the number of equilibria. In [19] a geometric method is presented which, however, is not amenable to computation.

The work on determination and structure of equilibria presented here was done independently of that by Willems [19] (see comment in [6]). We note that although he has presented some results concerning the equilibria of the RE, many of these assertions are without proof. Furthermore, we feel our proofs are simple and direct. In addition our method is numerically tractable because it is eigenvector free, whereas those of [13], [14]*, and [19] are numerically impractical. In fact we have a digital program which effectively computes all real symmetric equilibria and is quite accurate.

* Note that [14] concerns the determination of complex equilibria.

CHAPTER II

FIXED POINTS

For real symmetric matrices we will use positive definite, $A > 0$, positive semi-definite, $A \geq 0$, and strictly positive semi-definite, $A \geqslant 0$, to denote that the matrix A has all its eigenvalues greater than zero, greater or equal to zero, and greater or equal to zero but at least one equal to zero, respectively. The obvious dual definitions and notations are used for negative definite, semi-definite, and strictly semi-definite. Nonnegative definite will mean a real symmetric matrix which is either positive semi-definite or indefinite, rather than the normal use with the meaning of positive semi-definite in the present form.

The set of positive semi-definite matrices is a convex cone, $\mathcal{C}$ which induces the following partial ordering over the space of real symmetric matrices:

$$A \geq B \text{ iff } A-B \geq 0, \text{ i.e. } A - B \in \mathcal{C}$$

A maximal (minimal) element of the set $\mathcal{B}$ is an element $P \in \mathcal{B}$, such that there is no element in $\mathcal{B} - \{P\}$ which is greater (smaller) than P. Finally, the supremum (infimum) element of the set $\mathcal{B}$ is a $P \in \mathcal{B}$ such that P is greater (smaller) than any other element of $\mathcal{B}$.

Notice that in the space of symmetric matrices it is not possible to define the supremum as the least upper bound, since the common upper bounds of two given matrices may be not comparable among themselves, for instance, given

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}$$

then $C \geqq A$, $C \geqq B$ and $D \geqq A$, $D \geqq B$ but C and D are not comparable. See [2].

If A is a finite real symmetric matrix we define the pseudo-supremum of A and 0, the zero matrix, as:

$$\text{ps sup}(A,0) = T'[\lambda_i \vee 0]\,T$$

where $T'AT = \text{diag}[\lambda_i]$, $[\lambda_i \vee 0] = \text{diag}[\max(\lambda_i, 0)]$, and then

$$\text{ps sup}(A,B) = \text{ps sup}(B,A) = \text{ps sup}(A-B,0) + B$$

As is normal $\partial \mathcal{B}$ and $\mathring{\mathcal{B}}$ will denote the boundary and the interior of the set $\mathcal{B}$, respectively.

We consider now the autonomous RE:

$$\frac{dP}{dt} = PF' + FP - PH'HP + Q \qquad\qquad (2.1)$$
$$P(t_0) = \Gamma$$

where prime denotes transpose and with $M_{1,n}(R)$ the space of $(1 \times n)$ real matrices, $F \in M_{n,n}(R)$, $H \in M_{s,n}(R)$, $Q = Q' \in M_{n,n}(R)$, $\Gamma = \Gamma' \in M_{n,n}(R)$, $P = P' \in M_{n,n}(R)$. The right hand side of (2.1) will be denoted by $S(P)$, and let $\Pi(t,\Gamma)$ be the locally unique solution of (2.1), see [5, Chapter 5].

The Hamiltonian matrix associated with (2.1) is

$$\mathcal{H} = \begin{bmatrix} -F' & H'H \\ Q & F \end{bmatrix} \qquad\qquad (2.2)$$

For the definition of uniform asymptotic stability (u.a.s.)

see [5, Chapter 1].

A simple substitution gives the following:

IDENTITY 2.1. For $\lambda+\bar{\lambda}=1$, λ a real number and P_1, P_2 any two real symmetric matrices

$$S(\lambda P_1 + \bar{\lambda} P_2) = \lambda S(P_1) + \bar{\lambda} S(P_2) + \lambda\bar{\lambda}R(P_1, P_2)$$

where $R(P_1, P_2) \geq 0$ for every P_1, P_2.

DEFINITION 2.1.

$$S_+ = \{\Gamma = \Gamma' \mid S(\Gamma) \geq 0\}$$

$$S_- = \{\Gamma = \Gamma' \mid S(\Gamma) \geq 0\}$$

Notice that they are extensions of S_+ and S_- defined in [5, Chapter 5]. Here we do not make the restriction that $\Gamma \in \mathcal{C}$.

REMARK 2.1. S_+ is convex. It follows from Identity 2.1. In [19] the convexity is proved by the use of the linear matrix inequality (LMI) and a complicated string of equivalences. A direct proof from the LMI can be constructed by using results in [9].

REMARK 2.2. If $P_1 \in S_+$, $P_2 \in S_-$ ($P_2 \in \mathring{S}_-$) and $\lambda < 0$ then $P = \lambda P_1 + \bar{\lambda} P_2 \in S_-$, ($P \in \mathring{S}_-$) where $\lambda + \bar{\lambda} = 1$.

THEOREM 2.1. If $\Gamma \in S_+$, ($\Gamma \in S_-$) then $\Pi(t, \Gamma, t_o)$ a solution of (2.1) belongs to S_+, (S_-) for all time $t \in \mathcal{J}$ (interval of existence) and is monotone increasing (decreasing).

PROOF: See [5, Chapter 5].

THEOREM 2.2. If $\Pi(t, \Gamma_1, t_o)$ a solution of (2.1) exists in the interval $\mathcal{J}$ then for $\Gamma_2 \geq \Gamma_1$, $\Pi(t, \Gamma_2, t_o)$ exists in the interval $\mathcal{J}$ and

$\Pi(t,\Gamma_2,t_0) \geq \Pi(t,\Gamma_1,t_0)$ for $t\in \mathcal{J}$

PROOF: See [16, Lemma 2.3].

DEFINITION 2.2. $\bar{P} = \bar{P}' \in M_{n,n}(R)$ is an equilibrium point of (2.1) iff $S(\bar{P}) = 0$.

REMARK 2.3. If $\{\bar{P}\}$ is the set of equilibrium points of (2.1) then $\{\bar{P}\} \subseteq S_+ \cap S_-$.

THEOREM 2.3. If P_1, (P_2) is a maximal minimal element of S_+ then P_1, (P_2) is an equilibrium point of (2.1).

PROOF: Let P_1 be a maximal element of S_+, then by Theorem 2.1., valid with $Q = Q'$, $\Pi(t,P_1) \geq P_1$, hence $\Pi(t,P_1) = P_1$ for $\forall t \geq 0$ by maximality and therefore $P_1 \in \{\bar{P}\}$. For P_2 the proof follows by reversing the time i.e. $t \to -t$.

The following three lemmas are basic for the proof of the main theorem concerning the existence of maximal and minimal elements of S_+.

LEMMA 2.1. Let $\mathcal{C}$ be a closed convex cone with vertex at zero, and S_+ be a closed convex set such that $0 \in \mathring{S}_+$. If $S_+ \cap \mathcal{C}$ does not have a maximal element in the partial ordering induced by $\mathcal{C}$ then there exits an $A \in \mathcal{C}$ such that $\lambda A \in S_+ \cap \mathcal{C}$ for every real number $\lambda \geq 0$.

PROOF: Since $S_+ \cap \mathcal{C}$ does not have a maximal element it is possible to find a totally ordered sequence A_n with no finite upper bound, such that $A_n \in S_+ \cap \mathcal{C}$, for otherwise by Zorn's Lemma $S_+ \cap \mathcal{C}$ has a maximal element. Without loss of generality A_n may be taken increasing.

Since $0 \in \overset{\circ}{S}_+$, there exists a sphere of radius k about 0, $\mathcal{B}_k = \{A \mid \|A\| \leq k\} \subseteq S_+$ and $\partial\mathcal{B}_{k/2} \cap \mathcal{E}$ is compact. Define λ_n so that $C_n = 1/\lambda_n \; A_n$ and $\| C_n \| = k/2$.

By compactness of $\partial\mathcal{B}_{k/2} \cap \mathcal{E}$ there exists a subsequence C_m which converges to C i.e. $C_m \to C \in \partial\mathcal{B}_{k/2} \cap \mathcal{E}$;

Let $D_m = C_m + \mu_m (C - C_m)$ with $\mu_n \geq 1$ and $\| D_m - C \| = \epsilon \leq k/2$ with ϵ fixed. By construction $D_m \in \overset{\circ}{S}_+$ as D_m is in the ϵ-sphere about C which lies in the k-sphere about 0.

Consider the two one-parameter families

$$L(\beta_m) = D_m + \beta_m(A_m - D_m) \qquad \text{for } 0 \leq \beta_m \leq 1 \;.$$

$$M(\lambda) = \lambda C \qquad \text{for } \lambda \geq 0 \;.$$

$L(\beta_m) \in S_+$ for $\forall m$, since $\overset{\circ}{S}_+$ is convex and D_m , $A_m \in S_+$.

Suppose there exists a λ_1 , such that $M(\lambda_1) \notin S_+ \cap \mathcal{E}$, and let choose β_m such that:

$$\frac{1 - \beta_m}{\beta_m} = \frac{\lambda_m}{\mu_m - 1}$$

then $L(\beta_m) = \mu_m(1 - \beta_m) \; C \in S_+ \cap \mathcal{E}$, and since $\epsilon = (\mu_m - 1) \; \| C - C_m \|$ then $\mu_m \to \infty$.

Suppose there exists a subsequence $\mu_{m'}$ such that $\mu_{m'}(1 - \beta_{m'}) \to k_1 < \infty$, this implies that $\beta_{m'} \to 1$ and $\beta_{m'} \lambda_{m'} \to \infty$ contradicting the choice of $\beta_{m'}$.

Hence there exists m_o large enough, for which $\mu_{m_o}(1 - \beta_{m_o}) > \lambda_1$ and $\lambda_1 C \in S_+ \cap \mathcal{E}$ by convexity of S_+ , contradicting the supposition that $M(\lambda_1) \notin S_+ \cap \mathcal{E}$, therefore $\lambda C \in S_+ \cap \mathcal{E}$ for $\forall \lambda \geq 0$.

<u>LEMMA 2.2.</u> <u>A bounded monotone sequence converges.</u>

<u>PROOF:</u> See [17].

<u>LEMMA 2.3.</u> <u>If</u> S_+ <u>associated with</u> (2.1) <u>has a maximal element</u> P_1, <u>then there exists a matrix</u> $P_2 > P_1$ <u>such that</u> $P_2 \in \mathring{S}_-$.

<u>PROOF:</u> Since P_1 is a maximal element of S_+ then $S(\Gamma) \notin \mathcal{C}$ for $\forall \Gamma \geq P_1$.
Let $\Gamma_A = P_1 + A$ with $A \geq 0$ then:
$S(\Gamma_A) = S^*(A) = A\bar{F}_1' + \bar{F}_1 A - AH'HA \notin \mathcal{C}$ for $\forall A \geq 0$, where
$\bar{F}_1 = F - P_1 H'H$.

Consider the perturbed operator

$$S_1^*(A) = \frac{dA}{dt} = A\bar{F}_1' + \bar{F}_1 A - AH'HA + kI \qquad (2.3)$$

with k a positive real number and I the $(n \times n)$ identity matrix.

$\mathring{S}_+$ corresponding to (2.3) is not empty as $A = 0 \in \mathring{S}_+$.

Suppose that $S_+ \cap \mathcal{C}$ associated with (2.3) does not have a maximal element, then by Lemma 2.1. there exists an $A_1 \geq 0$ such that $S_1^*(\lambda A_1) \geq 0$ for $\forall \lambda \geq 0$, but

$$S_1^*(\lambda A) = \lambda S^*(A) + \lambda(1-\lambda)AH'HA + kI$$

with $S^*(A) \notin \mathcal{C}$. Let $\lambda_{min} = \min\{\lambda \mid \lambda \text{ eigenvalue of } S^*(A)\} < 0$
hence for $\lambda_1 > k/|\lambda_{min}|$ we have that $S_1^*(\lambda_1 A) \notin S_+$, contradiction, therefore the set $S_+ \cap \mathcal{C}$ associated with (2.3) has a maximal element $A_1 \geq 0$ and since (2.3) is completely controllable we can apply [5, Theorem 5.1] so that if $\Pi^*(t,0,t_o)$ is the solution of (2.3) with initial condition 0, then $\Pi^*(t,0,t_o) > 0$ for $t > t_o$ and by Theorem 2.2. $A_1 \geq \Pi^*(t,0,t_o)$ therefore $A_1 > 0$, and since $\Pi^*(t,0,t_o)$ is a bounded monotone sequence it converges to A^* such that $A_1 \geq A^* > 0$ and $S_1^*(A^*) = 0$, hence $P_2 = P_1 + A^*$

The next theorem shows that there is at most one maximal (minimal) element P_1 , (P_2) of S_+ and that

$S_+ \subseteq \{\Gamma = \Gamma' | \Gamma \leq P_1\} \cap \{\Gamma = \Gamma' | \Gamma \geq P_2\}$

THEOREM 2.4. If P_1 , (P_2) is a maximal (minimal) element of S_+ then it is the supremum (infimum) element of S_+ .

PROOF: Suppose $B \in S_+$ and B is not comparable with P_1 . By Lemma 2.3. there exists $A_1 > P_1$ and $A_1 \in \mathring{S}_-$. Then with $\lambda + \bar{\lambda} = 1, \lambda < 0$, we have $\lambda P_1 + \bar{\lambda} A_1 > P_1$ and for $|\lambda_1| > |\lambda_{max}(B - A_1) / \lambda_{min}(A_1 - P_1)|$, $\lambda_1 < 0$, then $\lambda_1 P_1 + \bar{\lambda}_1 A_1 > B$ and by Remark 2.2., $P_3 = \lambda_1 P_1 + \bar{\lambda}_1 A_1 \in \mathring{S}_-$.

$\Pi(t, P_3, t_o)$ is monotone decreasing as t increases, Theorem 2.1, and bounded below by P_1 , Theorem 2.2., hence converges to an equilibrium point which has to be P_1 , since otherwise P_1 will not be a maximal element of S_+ .

$\Pi(t, B, t_o)$ is monotone increasing and bounded above by $\Pi(t, P3, t\ )$ for all time, hence $P_1 \geq P_2$, contradicting the assumption that they were incomparable. The proof for the minimal element goes simmilarly by reversing the time.

Once the supremum and infimum of S_+ are characterized as equilibrium points, when they exist, will be denoted by P_+ , P_- , respectively. The use of this clasical notation will be justified latter.

We now give necessary and sufficient conditions for the existence of P_+ and of the P_- .

THEOREM 2.5. P_+ , (P_-) exists iff S_+ is non-empty and there exists a matrix $L_1 \in S_-$ $(L_2 \in S_-)$ such that $F - L_1 H'H$, $(-F + L_2 H'H)$ is u.a.s.

PROOF: Suppose $L \in S$. and $\bar{F}_L = F - LH'H$ is u.a.s. then for $\forall K \in S_+$

$$\bar{F}_L\Delta + \Delta\bar{F}_L' = \Delta H'H\Delta + A \qquad (2.4)$$

with $\Delta = K - L$ and $A \geq 0$.

Suppose Δ_0 is a solution of (2.4) and Δ_0 is not negative semi-definite, then

$$\bar{F}_L\Delta + \Delta\bar{F}_L' = \Delta H'H\Delta + A \qquad (2.5)$$

has by assumption at least one solution Δ_0, but the solution to (2.5) is unique as $\bar{F}_L$ is u.a.s. and negative semi-definite since

$$\Delta = -\int_0^\infty e^{\bar{F}_L t}\,[\Delta_0 H'H\Delta_0 + A]\; e^{\bar{F}_L' t}\,dt$$

Contradiction. Therefore $\Delta \leq 0$ and $L \geq K$ for $\forall K \in S_+$ i.e. L is an upper bound for S_+ which implies the existence of a maximal element of S_+ by Zorn's Lemma and by Theorem 2.4. of P_+ .

If there exists P_+ , then by Lemma 2.3 there exists and $L > P_+$ and such that $L \in \mathring{S}_-$, and it is easy to see that

$$\bar{F}_L\,(L - P_+) + (L - P_+)\,\bar{F}_L' = S(L)_. - (P_+ - L)\,H'H(P_+ - L) < 0$$

and $\bar{F}_L$ is u.a.s. by Liapounov's Theorem.

For P_- the proof goes simmilarly and is omitted.

The following two lemmas show why the notation P_+ and P_- has been used although no definiteness for them has been assumed.

<u>LEMMA 2.4.</u> $F - P_+H'H$, $-F + P_-H'H$ <u>are u.a.s. iff</u> $\mathring{S}_+ \neq \phi$

<u>PROOF:</u> It follows from Liapounov's Theorem. See [19, Theorem 5].

<u>LEMMA 2.5. With</u> $\mathring{S}_+ \neq \phi$, P_+ , <u>and</u> P_- <u>are the only equilibrium</u> solution of (2.1) that make $\bar{F}_+ = F - P_+H'H$, $-\bar{F}_- = -F + P_-H'H$ u.a.s.

<u>PROOF</u>: Suppose $K \neq P_+$ makes $\bar{F}_K$ u.a.s., then

$$\bar{F}_+(K-P_+) + (K-P_+)\bar{F}_+' - (K-P_+)H'H(K-P_+) = 0$$

implies $K - P_+ \leq 0$. (See proof of Theorem 2.5.), and

$$\bar{F}_K(P_+-K) + (P_+-K)\bar{F}_K' - (P_+-K)H'H(P_+-K) = 0$$

implies $P_+ - K \leq 0$, therefore $K = P_+$. See also [19, Lemma 2] for another proof.

REMARK 2.4. Theorem 2.4 or the proof of Lemma 2.5 shows that if P_θ is any other equilibrium point different from P_+ nd P_-, then $P_- \leq P_\theta \leq P_+$; in Chapter III we will show that $P_- \lneqq P_\theta \lneqq P_+$.

REMARK 2.5. In the case of $Q = GG'$, i.e., $Q \geq 0$, then $P = 0 \in S_+$ and hence if there exists P_+ it has to be positive semi-definite. With that constrain the following results are known.

i) If (F,H) is completely observable and (F,G) completely controllable, then there exists an unique positive semi-definite equilibrium point; it is positive definite and is P_+, and the matrix $F - P_+H'H$ is u.a.s. See [5], [10].

ii) If (F,H) is completely observable, then $\lim \Pi(t,0,t_0)$ exists as $t \to \infty$ and equal to $\bar{P}$. Further $S(\bar{P}) = 0$ and $\bar{P}$ is positive semi-definite. See [5], [10]. For $Q = Q'$, [19].

iii) If (F', H') is stabilizable and (G', F') is observable, then $\lim \Pi(t,0,t_0) = P_+$ as $t \to \infty$ and P_+ is the unique positive semi-definite solution of $S(P) = 0$. Further P_+ is positive definite and the matrix $F - P_+H'H$ is u.a.s. See [20].

iv) If (F', H') is stabilizable and (G', F') is detectable, then $S(P) = 0$ has at least one solution $\bar{P}$ positive semi-definite and the matrix $F - PH'H$ is u.a.s. See [20].

v) If (F', H') is stabilizable, then $\Pi(t,0,t_o)$ is bounded on $[t_o, \infty)$.

Notice that Theorem 2.5 includes the above results as particular cases but is more general since no assumption on the definiteness of Q has been made, as in the following example.

EXAMPLE 2.1. Let

$$F = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \qquad H = (1\ ,\ 0) \qquad Q = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}$$

S_+ is non-empty as $P = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \in S_+$, $L = 0 \in S_-$ and $\bar{F}_L$ is u.a.s., therefore there exists P_+, in fact $P_+ = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix}$. However S_+ is unbounded below since there is no $L \in S_+$ such that $-\bar{F}_L$ is u.a.s. Indeed $S_+ = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & c \end{bmatrix} \middle|\ c \leq -1.5 \right\}$

The next problem to be considered is how to effectively compute the equilibrium points of (2.1). Notice that any method depending on eigenvectors is numerically troublesome as is well known from perturbation theory, see [11, Chapter 1].

We will use following [1], [5] the Hamiltonian matrix $\mathcal{H}$.

LEMMA 2.6. $\det(\mathcal{H} - \lambda I) = R(\lambda) = (-1)^n \Delta(\lambda)\, \Delta(-\lambda)$ *where* $\Delta(\lambda)$ *is a polynomial of degree* n.

PROOF: See [5, Chapter 8].

DEFINITION 2.3. A factorization of $R(\lambda)$ is any set $(\lambda_1, \lambda_2, .. \lambda_n)$

of eigenvalues of $\mathcal{H}$, possibly repeated, such that $R(\lambda) = (-1)^n \Delta(\lambda)\,\Delta(-\lambda)$ where $\Delta(\lambda)$ is a polynomial of degree n with roots $(\lambda_1, \lambda_2, \ldots \lambda_n)$.

THEOREM 2.6. Every equilibrium point $\bar{P}$ of (2.1) satisfies:

$$(-\bar{P}\ ,\ I)\,\Delta(\mathcal{H}) = 0 \tag{2.6}$$

where $\Delta(\lambda)$ corresponds to some factorization and $\Delta(\lambda) = \det(F - \bar{P}H'H - \lambda I)$.

PROOF: If $\bar{P}$ is an equilibrium point of (2.1), let $T = \begin{bmatrix} 0 & I \\ I & P \end{bmatrix}$ then

$$T^{-1}\mathcal{H}\,T = \begin{bmatrix} F - \bar{P}H'H & 0 \\ H'H & -F' + H'H\bar{P} \end{bmatrix}$$

Let $\Delta(\lambda)$ be the characteristic polynomial of $F - \bar{P}H'H$ which corresponds to some factorization of $R(\lambda)$, then $\Delta(\mathcal{H}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and

$$T^{-1}\begin{bmatrix} A & B \\ C & D \end{bmatrix} T = \begin{bmatrix} -\bar{P}B + D & C - \bar{P}A + (D - \bar{P}B)\bar{P} \\ B & A + B\bar{P} \end{bmatrix} =$$

$$= \begin{bmatrix} \Delta(F - \bar{P}H'H) & 0 \\ * & \Delta(-F + H'H\bar{P}) \end{bmatrix}$$

therefore $D = PB$ and $C = PA$ and the theorem follows. See [5, Theorem 8.5].

COROLLARY 2.1. P_+, (P_-) is given by (2.6) when the factorization is such that $\Delta(\lambda)$ is a Hurwitz, (totally non-Hurwitz) polynomial, i.e., $\mathrm{Re}\,\lambda_i < 0$, ($\mathrm{Re}\,\lambda_i > 0$) for $\forall i$, where $\Delta(\lambda_i) = 0$

It is now possible to know how many equilibria may exist.

If all eigenvalues are distinct and real, from each it is possible to form two different factorizations, namely one with λ

the other with $-\lambda$, then there are 2^n different factorizations and therefore there are 2^n equilibrium points, since the half lower part of the eigenvectors of $\mathcal{H}$ expand R^n.

EXAMPLE 2.2.

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad H'H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

then $R(\lambda) = \lambda^4 - 5\lambda^2 + 4$, and $(-P, I)\,\Delta_i(\mathcal{H}) = 0$, $i = 1,2,3,4$ gives:

$$P_+ = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad P_- = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad P_{\theta_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad P_{\theta_2} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

If all eigenvalues are real and one of them is repeated, say of order s, then the (n - s) different ones can be combined in 2^{n-s} ways and the s repeated gives (s + 1) possible combinations, hence either there are $(s+1)2^{n-s}$ equilibria as in example 2.3. If however the minimal polynomial of $\mathcal{H}$ coincides with one $\Delta_i(\ )$ corresponding to some factorization, then (2.6) is satisfied for every $\bar{P}$, and yet $\bar{P}$ has to satisfy $\Delta(F - \bar{P}H'H) = 0$ and there is the posibility of parametric families of solutions as in Example 2.4.

EXAMPLE 2.3.

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad H'H = Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

From the different factorizations we get:

$$P_+ = I \qquad P_- = -I \qquad P_\theta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

However $R(\lambda) = (\lambda^2 - 1)^2$, since $\lambda^2 - 1$ is not a minimal polynomial of $\mathcal{H}$.

EXAMPLE 2.4.

$$F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad H'H = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case $P_+ = I$ and $P_- = -I$. For P_θ , $\Delta_\theta(\lambda)$ is the minimal polynomial of $\mathcal{H}$, although the only $\bar{P}$ that satisfy $\Delta_\theta(F - \bar{P}H'H) = 0$ are $P_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, a family of equilibria.

The following example shows that P_+, P_- may be singular and yet $P_+ - P_-$ be positive definite, i.e., $\bar{F}_+$ is u.a.s.

EXAMPLE 2.5.

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad H'H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then $P_+ = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $P_- = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ and $P_+ - P_- > 0$

If there is a complex eigenvalue and the others real and distinct, then each factorization has to include that eigenvalue and its complex conjugate as $\Delta(\lambda)$ is the characteristic polynomial, of a real matrix. Then there are 2^{n-1} different factorizations.

EXAMPLE 2.6.

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad H H = Q = I$$

Inthis case there are only $P_+ = I$ and $P_- = -I$

If finally, there is a complex repeated eigenvalue of order s and the others real and distinct, it is possible to have $(s + 1)2^{n-2s}$ different factorization and posibilities isolated euilibrium points or families of solutions.

The above consideration are simpl ookeeping of the combinatorial ways to factor the ... igenvalues.

In the general case of real ... ict, real and repeated, complex distinct and complex repeated, we can count the possible number of factorizations by considering the different classes.

REMARK 2.6. If $[-\bar{P}, I]\Delta(\mathcal{H}) = 0$ has no solution for a certain factorization, then there is no equilibrium point $\bar{P}$ of $S(\bar{P}) = 0$ such that $\Delta(F - \bar{P}H'H) = 0$, since (2.6) is in general a necessary condition.

The following are examples of those cases.

EXAMPLE 2.7.

$$F = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad H'H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

the eigenvalues are +1 , -1, repeated and for

$\Delta(\lambda) = \lambda^2 + 2\lambda + 1$ one gets $P_+ = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$\Delta(\lambda) = \lambda^2 - 1$ one gets, through the use of the characteristic polynomial of $\bar{F}_\theta$:

$$P_\theta = \begin{bmatrix} 0 & \theta \\ \theta & 1 - \dfrac{(1+\theta)^2}{2} \end{bmatrix}$$

For $\Delta(\lambda) = \lambda^2 - 2\lambda + 1$, (2.6) is a set of incompatible equations and there is no P_-. Indeed Theorem 2.5 asserted it and S_+ is unbounded below.

EXAMPLE 2.8.

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad H'H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The set of equations (2.6) is incompatible for the different factorizations and therefore there are no equilibrium points. It is interesting to note that S_+ is non-empty but $S_- = \phi$.

CHAPTER III

GEOMETRY OF THE RICCATI EQUATION

In the following we will consider mainly the solution of the RE, in the form (2.2) of the previous chapter, i.e.,

$$\frac{dP}{dt} = PF' + FP - PH'HP + Q$$
$$P(t_0) = \Gamma = \Gamma' \in M_{n,n}(R) \tag{3.1}$$

In this case it is possible to find necessary and sufficient conditions on Γ for the global existence of $\Pi(t,\Gamma)$. The importance of P_- in connection with this problem, it has gone unnoticed until very recently [6], [19], maybe due to the fact that the principal tool used was a solution connected with the Hamiltonian matrix associated with (3.1) and due to Radon [15].

ASSUMPTION A1. S_+ is bounded above and below

DEFINITION 3.1.

$$\mathcal{E}_{P_-} = \{\Gamma = \Gamma' \mid \Gamma - P_- \geq 0\}$$
$$\mathring{\mathcal{E}}_{P_-} = \{\Gamma = \Gamma' \mid \Gamma - P_- > 0\}$$
$$\partial\mathcal{E}_{P_-} = \{\Gamma = \Gamma' \mid \Gamma - P_- \geqq 0\}$$

similarly,

$$\mathcal{E}_{P_+}, (\mathring{\mathcal{E}}_{P_+}) = \{\Gamma = \Gamma' \mid \Gamma - P_+ \leq (<)\ 0\}$$

Finally let $(F_\theta)_{\theta \in K}$ be the set of all other equilibrium points of (3.1) different from P_+ and P_- and this set can be empty, see example 2.6.

DEFINITION 3.2.

$$\mathcal{B} \subseteq \{\Gamma = \Gamma' \mid \Gamma \in M_{n,n}(R)\}$$

is an invarian set under the transformation defined by the RE if for $\Gamma \in \mathcal{B}$ then $P(t,\Gamma) \in \mathcal{B}$ for $\forall t \geqslant 0$.

ASSUMPTION A2. $P_+ - P_-$ is non-singular.

LEMMA 3.1. Suppose A1 and A2 hold and $\Gamma - P_-$ is non-singular then the solution of (3.1) is given for $t \in \mathcal{J}$ (the interval of existence of $\Pi(t,\Gamma)$) by

$$P(t)=P_- + \{(P_+-P_-)^{-1} + \psi'^{-1}(t,t_o)\left[(\Gamma-P_-)^{-1}-(P_+-P_-)^{-1}\right]\psi^{-1}(t,t\ )^{-1}\}^{-1} \tag{3.2}$$

where $\psi(t,t_o)$ is the fundamental matrix of $\bar{F}_-=F-P-H'H$

PROOF: The proposed solution satisfies (3.1) since defferentation gives

$$\dot{P}=(P-P_-)\{\bar{F}_-'\ \psi'^{-1}[(\Gamma-P_-)^{-1}-(P_+-P_-)^{-1}]\psi^{-1}+\psi'^{-1}[(\Gamma-P_-)^{-1}-(P_+-P_-)^{-1}]\psi^{-1}\ \bar{F}_-\}(P-P_-)$$

or

$$\dot{P}=(P-P_-)\{\bar{F}_-'[(P-P_-)^{-1}-(P_+-P_-)^{-1}]+[(P-P_-)^{-1}-(P_+-P_-)^{-1}]\bar{F}_-\}(P-P_-)$$

where a rearranged form of (3.1) has been used, namely

$$\dot{P}=(P-P_-)\ \bar{F}_-' + \bar{F}_-\ (P-P_-) - (P-P_-)H'H(P-P_-)$$

and since $\dot{P}\ (t,P_+) = 0$ for $\forall t$ then (3.2) satisfies (3.1). The initial condition $P\ (t_o) = \Gamma$ is satisfied as $\psi(t_o,t_o) = I$

THEOREM 3.1. Under the conditions of Lemma 3.1

i) <u>if</u> $\Gamma \in \overset{\circ}{\mathcal{E}}_P$ <u>the solution</u> $\Pi(t,\Gamma)$ <u>exists globally and for</u> $t \to \infty$, $\Pi(t,\Gamma) \to P_+$

ii) <u>if</u> $\Gamma \notin \mathcal{E}_P$ <u>and</u> $\Gamma - P_-$ <u>is non-singular, then the solution</u> $\Pi(t,\Gamma)$ has a finite escape time.

<u>PROOF:</u> To prove i) we have to show that $P(t,\Gamma)$ is finite for all time t or equivalently, that

$$B(t) = (P_+-P_-)^{-1} + \Psi^{-1'}\left[(\Gamma - P_-)^{-1} - (P_+-P_-)^{-1}\right]\Psi^{-1}$$

is non-singular for $t \in [t_o,\infty)$.

Lemma 2.4 gives the u.a.s. condition of $-F_-$ hence $\Psi^{-1}(t,t_o) \to 0$ as $t \to \infty$

Let

$$C(t) = (P_+-P_-)^{-1} - \Psi^{-1'}(P_+-P_-)^{-1}\Psi^{-1}$$

then $C(t)$ is the solution of

$$\dot{C} = -\bar{F}_-' C - C\bar{F}_- + H'H$$

$$C(t_o) = 0$$

Therefore

$$C(t) = \int_{t_o}^{t} \Psi^{-1'}(t,s)\, H'H\, \Psi^{-1}(t,s)\, ds \geq 0$$

and

$$B(t) = \Psi^{-1'}(\Gamma - P_-)^{-1}\Psi^{-1} + C(t) > 0 \quad \text{as } \Gamma - P_- > 0$$

as $t \to \infty$ $\Psi^{-1} \to 0$, and $P(t,\Gamma) \to P_+$

To prove part ii) we note that $B(t_o)$ is not positive definite or semi-definite and for $t \to \infty$, $B(t)$ limits to a positive definite matrix, namely $(P_+-P_-)^{-1}$, hence as the eigenvalues are continuous functions of t, for some finite value $t = T$, $B(T)$ will be singular and the solution will escape to infinity.

We have investigated the behavior of the solution for $\Gamma = \Gamma' \in M_{n,n}(R)$ and $\Gamma - P_-$ non-singular. To study the case of singular $\Gamma - P_-$ we need some more results.

LEMMA 3.2. (Dual of Lemma 3.1.). Suppose A1 and A2 hold and $\Gamma - P_+$ is non-singular then the solution of (3.1) is given for $t \in \mathcal{J}$, by

$$P(t)=P_+ + \left\{ \Phi^{-1'}(t,t_o)\left[(\Gamma-P_+)^{-1}+(P_+-P_-)^{-1}\right]\Phi^{-1}(t,t_o)-(P_+-P_-)^{-1}\right\}^{-1} \tag{3.3}$$

where $\Phi(t,t_o)$ is the fundamental matrix of $F-P_+H'H$

PROOF: Same as Lemma 3.1.

This Lemma generalizes Corollary 3.2 of [4].

LEMMA 3.3. (Complementary) Let $[\![A]\!] = A-P_-$ suppose A1, A2 hold and $\Gamma \in \partial \mathcal{E}_{P_-}$ then the solution of (3.1) is given for $t \in \mathcal{J}$ by

$$P(t)=P_- + \Psi [\![\Gamma]\!]^{\frac{1}{2}}\left\{I + [\![\Gamma]\!]^{\frac{1}{2}}(\Psi'[\![P_+]\!]^{-1}\Psi - [\![P_+]\!]^{-1})[\![\Gamma]\!]^{\frac{1}{2}}\right\}^{-1}[\![\Gamma]\!]^{\frac{1}{2}}\Psi' \tag{3.4}$$

with Ψ and $\mathcal{J}$ as before.

For $\Gamma \in \{\Gamma = \Gamma' \mid \Gamma - P_- \leq 0\}$ the solution is given by (3.4) replacing $[\![\Gamma]\!]^{\frac{1}{2}}$ by $(-[\![\Gamma]\!])^{\frac{1}{2}}$.

PROOF: Essentially same method as in Lemma 3.1 with the use of the positive square root of positive semi-definite matrices.

REMARK 3.1. Similarly there is a dual complementary lemma for $\Gamma - P_+$ being singular.

LEMMA 3.4. Suppose A1, A2 hold then $(P_\theta) \subseteq \partial \mathcal{C}_{P_-}$

PROOF: For $\Gamma \in \mathring{\mathcal{C}}_{P_-}$, $P(t,\Gamma) \to P_+$ as $t \to \infty$ hence there cannot be equilibria other than P_+ in $\mathring{\mathcal{C}}_{P_-}$

For $\Gamma \notin \mathcal{C}_{P_-}$ and $\Gamma - P_-$ non-singular, $P(t,\Gamma)$ escapes to infinity in finite time, therefore there cannot be equilibria in this region. We are left then with the set of Γ's such that $\Gamma - P_-$ is singular. Consider the subset $\Gamma - P_- \leqq 0$ of that set, call it $\partial \mathcal{R} \mathcal{C}_{P_-}$ (boundary of the rear cone with vextex at P_-). Since by the change $t \to -t$ the structure of the equilibrium points does not change [8], however, the u.a.s. ones change to totally non-u.a.s. and vice-versa, then by considering the solution of this new equation, $\partial \mathcal{R} \mathcal{C}_{P_-}$ is inside of the region of convergence of the solution to the point P_-. Therefore there is not equilibria on $\partial \mathcal{R} \mathcal{C}_{P_-}$, hence $(P_\theta) \subseteq \partial \mathcal{C}_{P_-}$

LEMMA 3.5. Suppose A1, A2 hold, then $(P_\theta) \subseteq \partial \mathcal{C}_{P_+}$

PROOF: Note that the transformation $t \to -t$ makes this lemma the dual of the previous one.

THEOREM 3.2. Suppose A1, A2 hold, then $(P_\theta) \subseteq \partial \mathcal{C}_{P_-} \cap \partial \mathcal{C}_{P_+}$

PROOF: Follows from Lemma 3.4 and 3.5.

REMARK 3.2. If $\bar{P}_\theta \in (P_\theta)$ then $P_- \lneqq P_\theta \lneqq P_+$

There are several invariant sets, e.g S_+ and S_-. Also since A being a singular positive semi-definite matrix implies $A^{\frac{1}{2}}$ is singular, then from Lemma 3.3 and its dual, it follows that

$$\{\Gamma = \Gamma' \mid (\Gamma - P_-) \text{ singular}\}$$

and

$$\{\Gamma = \Gamma' \mid (\Gamma - P_+) \text{ singular}\}$$

are invariant sets.

<u>THEOREM 3.3.</u> <u>The set</u> $\partial \mathcal{C}_{P_-} \cap \partial \mathcal{C}_{P_+}$ <u>is invariant under the transformation defined by the</u> RE.

<u>PROOF:</u> Follows from the invariance of the above sets.

We can sumarize the previous result in a graphic way. Refer to Fig. 3.1, where the structure of a 2 x 2 RE is represented via the space R^3 with axis P_{11}, P_{12}, P_{22}. Fig. 3.1 is just a sugestion, since actually the cones are cones with a maximum angle of $\pi/2$ at the vertex and elliptical cross section with excentricity $e=1/\sqrt{2}$.

We can define the domain of attraction of a set $\mathcal{B}$, as the set of Γ's such that $P(t,\Gamma) \in \mathcal{B}$ as $t \to \infty$. Then the domain of attraction of P_+ is $\mathring{\mathcal{C}}_{P_-}$ and the one of $\partial \mathcal{C}_{P_-} \cap \partial \mathcal{C}_{P_+}$ is $\partial \mathcal{C}_{P_-}$

We see that if P_+, P_-, $(P_+ - P_-)^{-1}$ exist, the signature of Q and Γ are immaterial for the behavior of the solution.

<u>REMARK 3.3.</u> The previous results are a remarkable generalization of the one-dimensional case.

$$p = 2\, f\, p - h^2\, p^2 + q$$

where the domain of attraction of p_+ is (p_-, ∞) as depicted in Fig. 3.2.

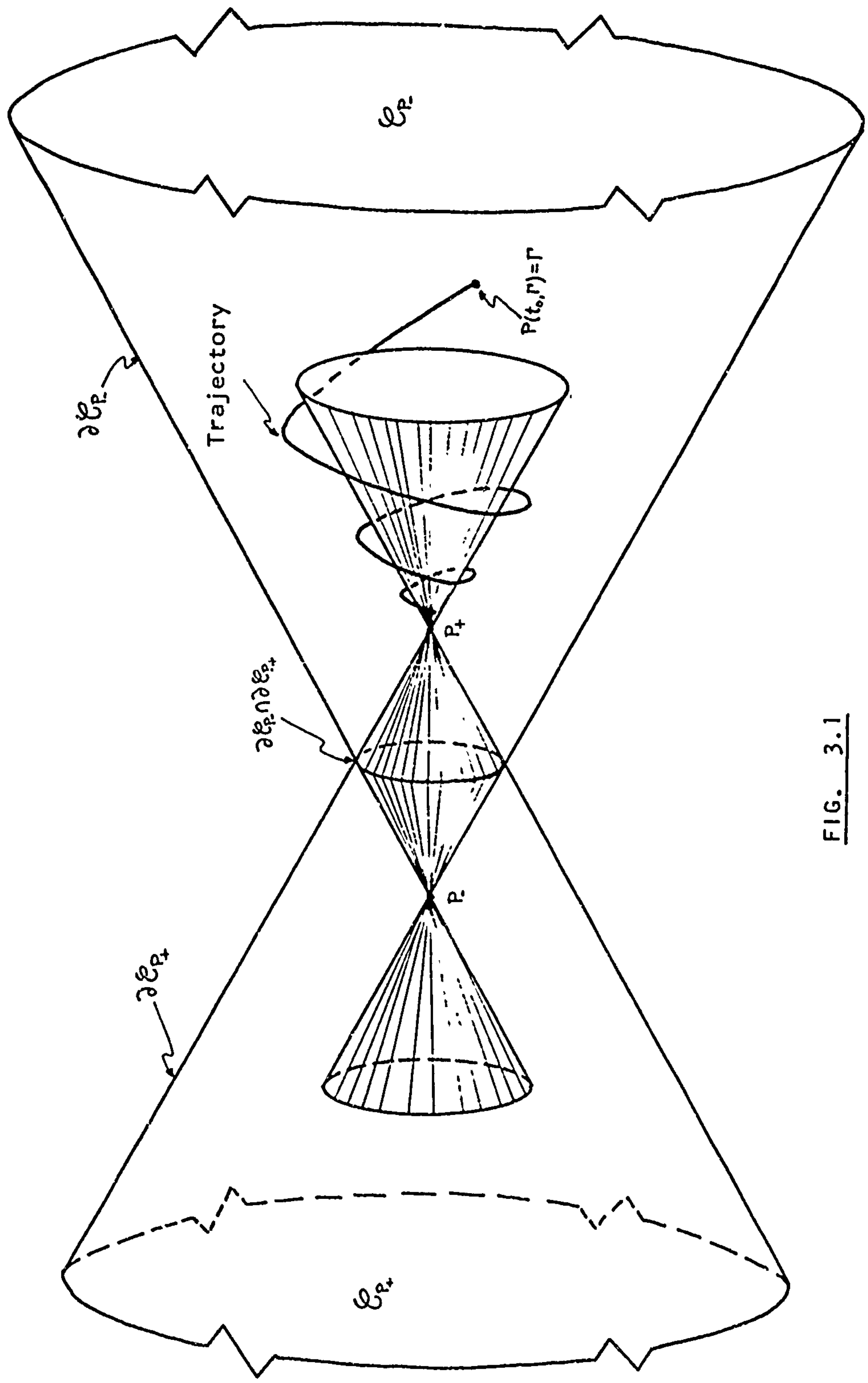

FIG. 3.1

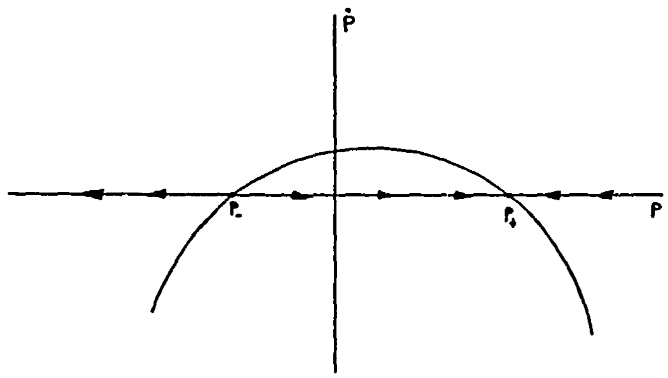

FIG. 3.2

We can now better understand some of the unusual features which appear in the numerical solution of the RE., see [13].

If $\Gamma \in \mathring{\mathcal{E}}_{P_-}$, $\Pi(t,\Gamma)$ will converge to P_+ as t gets large, in spite of round-off errors.

If $\Gamma \notin \mathcal{E}_{P_-}$ the solution will escape to infinity in a finite time.

If $\Gamma \in \partial \mathcal{E}_{P_-}$, theoretically $\Pi(t,\Gamma) \in \partial \mathcal{E}_{P_-}$ for $\forall t$ and tends to $\partial \mathcal{E}_{P_-} \cap \partial \mathcal{E}_{P_+}$ as $t \to \infty$; this is the only way of reaching a P_θ and thus provides a cure or in some cases even an analytic solution. See chapter IV. Excep for very special cases, the round-off error will throw the solution outside $\partial \mathcal{E}_{P_-}$, and will therefore escape in finite time or converge to P_+.

This explains why in [13] the numerical solution of (2.1) is unable to reach P_θ with initial values in the domain of attraction of P_θ.

We now consider a time dependent RE.

$$\dot{P}(t) = P(t)F'(t) + F(t)P(t) - P(t)H'(t)H(t)P(t) + Q(t)$$
$$P(t_o) = \Gamma = \Gamma' \epsilon M_{n,n}(R) \qquad (3.5)$$

with $Q(t) = Q'(t)$.

By using the fundamental matrix of F(t) it is possible to change (3.5) into:

$$\dot{P}_1(t) = -P_1(t)\ H_1'(t)\ H_1(t)\ P_1(t) + Q_1(t)$$
$$P_1(t_o) = \Gamma$$

where

$$W(t,t_o) = \int_{t_o}^{t} \Phi(s,t)H'(s)H(s)\Phi'(s,t)ds = \int_{t_o}^{t} H_1'(s)H_1(s)ds$$
$$C(t,t_o) = \int_{t_o}^{t} \Phi(t,s)\ Q(s)\ \Phi'(t,s)ds = \int_{t_o}^{t} Q\ (s)ds$$

with $W(t,t_o)$, $C(t,t_o)$ the observability and controllability matrix, respectively. See [3] .

Hence without loss of generality we will consider a RE with $F(t) = 0$

$$\dot{P}(t) = -P(t)\ M(t)\ P(t) + Q(t)$$
$$P(t_o) = \Gamma = \Gamma' \epsilon M_{n,n}(R) \qquad (3.6)$$

and $M(t) \geq 0$, $Q = Q'$ continuous functions of time, and let:

$$M^* = \underset{t\epsilon[t_o,\infty)}{\text{ps sup}}\ M(t)$$
$$Q^* = \underset{t\epsilon[t_o,\infty)}{\text{ps inf}}\ Q(t)$$

Notice that ps sup and ps inf can be replaced by any other convinient upper and lower bounds, respectively.

Consider the following constant RE:

$$\dot{P}^* = -P^*\ M^*\ P^* + Q^*$$
$$P^*(t_o) = \Gamma \qquad (3.7)$$

and assume there exists the minimum element of S_+ associated with it, $\mathcal{P}_{-}$.

THEOREM 3.4. If $M(t) \geq 0$, $Q(t) = Q'(t)$ are continuous functions of t and there exists $\mathcal{P}_{-}$, then $\Pi(t,\Gamma,t_0)$, solution of (3.6), exists in $t \in [t_0, \infty)$ for $\Gamma \geq \mathcal{P}_{-}$

PROOF: $\Pi^*(t,\Gamma,t_0)$ solution of (3.7) exists in $t \in [t_0,\infty)$, from Theorem 3.1 and with $\eta = \Pi(t,\Gamma,t_0) - \Pi^*(t,\Gamma,t_0)$, then η satisfies

$$\dot{\eta} = -\Pi^* M\eta - \eta M\Pi^* - \eta M\eta + (Q - Q^*) - \Pi^*(M - M^*)\Pi^*$$

$$\eta(t_0) = 0$$

therefore $\eta(t) \geq 0$ for all t, i.e., $\Pi(t,\Gamma,t_0) \geq \Pi^*(t,\Gamma,t_0)$.

CHAPTER IV

2-D RICCATI EQUATION

Through this research numerical computations and theoretical results for the 2-dimensional RE were useful since the space $M_{2,2}(R)$ is isomorphic with R^3 over the field of real numbers and addition of matrices and vectors, for symmetric matrices.

The equation of the motion of the solution of the RE with $\Gamma \in \partial\mathcal{P}_{P_-} \cap \partial\mathcal{P}_{P_+}$ will be presented here. It is remarkable the natural way in which improper rotations, [7], appear in the two-dimensional case. It should be expected that something similar will happen in a higner dimension, although we did not investigate this case any further.

IDENTITY 4.1. For (2 x 2) *real matrices the following identities are valid:*

i) $\det(A \pm B) = \det(A) + \det(B) \pm (\text{trace } A)(\text{trace } B) \mp \text{trace}(AB)$

or, if A *is non-singular:*

ii) $\det(A \pm B) = \det(A) + \det(B) \pm \det(A)\ \text{trace}(A^{-1}\ B)$

where det(C) = *determinant of* C.

LEMMA 4.1. If $P_+ - P_-$ *is non-singular then for* (2 x 2) *real symmetric matrices the solution of:*

$$\det(P - P_+) = 0$$
$$\det(P - P_+) = 0 \qquad (4.1)$$

is given by:

$$P_\theta = P_- + \frac{(I + T_\theta)}{\text{trace}\{(P_+ - P_-)^{-1} (I + T_\theta)\}} \qquad (4.2)$$

where:

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

PROOF: Let $P_\theta - P_- = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. To satisfy the second equation of (4.1) and $P_\theta - P_- \geq 0$ the following relations have to be met:

$$ac - b^2 = 0, \quad a, c \geq 0 \qquad (4.3)$$

This family of solutions can be represented by the two-parameter function:

$$P_\theta - P_- = \lambda(I + T_\theta)$$

where λ is a positive real number and $\theta \in [0, 2\pi]$.

Indeed the transformation:

$$a = \lambda(1 + \cos\theta)$$
$$b = \lambda \sin\theta$$
$$c = \lambda(1 - \cos\theta)$$

satisfies relations (4.3) and $(a, b, c) \rightarrow (\lambda, \theta)$ is one-to-one

$$\lambda = \frac{a + c}{2}$$
$$\theta = \sin^{-1}\left(\frac{2b}{a + c}\right)$$

where θ is fully determined from the other relations, and is well defined since $\left|\frac{2b}{a + c}\right| \leq 1$.

Now by the use of Identity 4.1. ii), it is possible to determine the value of λ so that the first equation of (4.1) is also satisfied and the proof is complete.

MOTION ON $\partial\mathcal{E}_{P_-} \cap \partial\mathcal{E}_{P_+}$:

Sustitution of (4.2) into the RE gives:

$$\frac{\dot{\theta}\, J\, T_\theta}{\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}} - \frac{\dot{\theta}(I + T_\theta)\ \text{trace}\{(P_+ - P_-)^{-1}J' T_\theta\}}{[\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}]^2} = \tag{4.4}$$

$$\frac{\bar{F}_-(I + T_\theta) + (I + T_\theta)\bar{F}_-'}{\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}} - \frac{(I + T_\theta)H'H(I + T_\theta)}{[\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}]^2}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since $\partial\mathcal{E}_{P_-} \cap \partial\mathcal{E}_{P_+}$ is an invariant set under the transformation defined by the RE, the three component equations derived from (4.4) should be non-independent; therefore we are permitted to choose the most convinient linear combination for the solution.

CASE I: $\text{trace}\{(P_+ - P_-)^{-1}J' T_\theta\} \neq 0$

By taking the trace of (4.4) the solution of the RE with initial condition $\Gamma \in \partial\mathcal{E}_{P_-} \cap \partial\mathcal{E}_{P_+}$ is given by:

$$P(t) = P_- + \frac{(I + T_\theta)}{\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}}$$

where θ satisfies:

$$\dot{\theta} = \frac{-\text{trace}\{\bar{F}_-(I + T_\theta)\}\ \text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\} + \text{trace}\{(I + T_\theta)H'H\}}{\text{trace}\{(P_+ - P_-)^{-1}J' T_\theta\}}$$

$\theta(t_0) = \theta_0$

where trace(AB) = trace(BA) and $\text{trace}(J' T_\theta) = 0$, have been used.

<u>CASE II:</u> $\text{trace}\{(P_+ - P_-)^{-1} J' T_\theta\} = 0$

In this case (4.4) reduces to

$$\dot{\theta} I = \bar{F}_-(I + T_\theta')T_\theta J' + (I + T_\theta)\bar{F}_-' T_\theta J' - \frac{(I + T_\theta) H' H (I + T_\theta) T_\theta J'}{\text{trace}\{(P_+ - P_-)^{-1}(I + T_\theta)\}}$$

or taking the trace and since $T_\theta J' T_\theta = J$, then the solution is given as in case I with θ satisfying:

$$\dot{\theta} = \text{trace}\{\bar{F}_-(I + T_\theta) J\}$$

$$\theta(t_\theta) = \theta_o$$

REFERENCES

1. R. W. BASS, "Machine Solution of High Order Matrix Riccati Equations", Douglas Aircraft, Missiles and Space Systems Division, Santa Monica (1967).

2. G. BIRKHOFF, Lattice Theory, American Mathematical Society, New York, 1948.

3. R. S. BUCY, "Global Theory of the Riccati Equation", Jour. of Computer and Systems Sciences, 1 (1967), 349-361

4. R. S. BUCY, "Bounds for the Riccati Equation", Jour. of Computer and Systems Sciences, 6 (1972)

5. R. S. BUCY and P. D. JOSEPH, Filtering for Stochastic Processes with Applications to Guidance, Interscience, New York, 1968.

6. R. S. BUCY and J. RODRIGUEZ-CANABAL, "A Negative Definite Equilibrium and Its Induced Cone of Global Existence for the Riccati Equation", Jour. S.I.A.M. Math. Anal., 3 (1972)

7. E. CARTAN, The Theory of Spinors, The M.I.T. Press, Cambridge, 1967

8. E. CODDINGTON and N. LEVINSON, The Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

9. E. V. HAYNSWORTH, "Determination of the Inertia of a Partitioned Hermitian Matrix", Linear Algebra and Its Applications, 1, (1968), 73-81.

10. R. E. KALMAN and R. S. BUCY, "New Results in Linear Filtering and Prediction Theory", Jour. Basic Eng. A.S.M.E.,

Ser. D, 83, (1961), 95-108.

11. T. KATO, Perturbation Theory For Linear Operators, Springer-Verlag, New York, 1966.

12. D. L. KLEINMAN, "On an Iterative Technique for Riccati Equation Computation", Trans. I.E.E.E. Automatic Control, 13,

13. K. MARTENSSON, "On the Matrix Riccati Equation", Information Sciences, 3, (1971), 17-49.

14. J. E. POTTER, "Matrix Quadratic Solution", Jour. S.I.A.M. Appl. Math., 14, (1966), 496-501.

15. J. RADON, Zum Problem von Lagrange, Hamburg. Math. Einzelschriten, 6. Heft, Leipzig, 1928.

16. W. T. REID, "Monotoneity Properties of Solutions of Hermitian Riccati Matrix Differential Equations", Jour. S.I.A.M. Math. Anal., 1, (1970), 195-213.

17. F. RIESZ and B. SZ.-NAGY, Lecons D'Analyse Fonctionnelle, Gauthier-Villars, Paris, 1955.

18. W. E. ROTH, "On the Matric Equation $X^2 + AX + BX + C = 0$", Proc. A.M.S., 1, (1950), 586-589.

19. J. C. WILLEMS, "Least Squares Stationary Optimal Control and the Algebraic Riccati Equation", Trans. I.E.E.E. Automatic Control, AC-16, (1971), 621-634.

20. W. M. WONHAM, "On a Matrix Riccati Equation of Stochastic Control", Jour. S.I.A.M. Control, 6, (1968), 681-697.