U. S. ARMY

Technical Memorandum 8-72

TWO CONTEMPORARY PROBLEMS IN MULTIDIMENSIONAL SCALING

Richard M. Fenker, Jr.

March 1972

AMCMS Code 502F.11.81900

HUMAN ENGINEERING LABORATORY

ABERDEEN RESEARCH & DEVELOPMENT CENTER

ABERDEEN PROVING GROUND, MARYLAND

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Institute for the Study of Cognitive Systems Texas Christian University Fort Worth, Texas	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

TWO CONTEMPORARY PROBLEMS IN MULTIDIMENSIONAL SCALING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Richard M. Fenker, Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1972	23	12

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAAD05-68-C-0176 b. PROJECT NO. 1T061102B81A c. Program Element 61102A d. THEMIS Proposal No. 367	Technical Memorandum 8-72 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Human Engineering Laboratory U. S. Army Aberdeen Research & Development Ctr Aberdeen Proving Ground, Maryland 21005

13. ABSTRACT

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DD FORM 1 NOV 68 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Feature Space Judgment Models Multidimensional Scaling Multivariate Applications Matrix Sampling						

AMCMS Code 502F.11.81900

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ABSTRACT

Two contemporary problems related to the application of multidimensional scaling techniques are discussed and possible solutions presented. The first of these problems concerns the prohibitive number of judgments required in paired comparison tasks when the number of stimuli is large. Several procedures are proposed for reducing the necessary number of comparisons by using standard or reference stimuli. The second problem is that methods are needed for relating the psychological spaces obtained from scaling analyses to other behavioral data. One method, which involves embedding novel stimuli in previously defined psychological spaces, is described. Both problems are considered from a metric as well as nonmetric standpoint.

CONTENTS

TWO CONTEMPORARY PROBLEMS IN MULTIDIMENSIONAL SCALING

INTRODUCTION

REDUCING THE NUMBER OF JUDGMENTS REQUIRED IN PAIRED COMPARISON TASKS

A major difficulty associated with the application of multidimensional scaling (MDS) techniques is that current methods of collecting proximity data severely limit the feasible sample sizes. The number of comparisons required for judgments on all possible pairs or triplets of stimuli (or any large fraction of this number) varies as the square or cube of the sample size. A sampling constraint of this type is particularly deleterious in the case of MDS since the interpretability and utility of the scaling solutions depends to a large extent on the representativeness or exhaustiveness of the stimulus sample. Unlike most other multivariate models, which assume that the relevant psychophysics is under experimental control, in MDS analyses the orderings of the sample stimuli along the psychological dimensions are used to identify the corresponding physical properties. The smaller the sample size the more difficult it is to obtain an unambiguous interpretation ordering especially when the orientation of the axes can be arbitrary.

There have been several methods discussed in the literature for circumventing the restrictions on sample size. The most straightforward of these involves obtaining some fraction of all possible comparisons, then generating a solution based on the incomplete data matrix. A second alternative would require that each subject judge all possible pairs by extending the experimental period to the necessary number of sessions or; different subjects might judge portions of the total comparison matrix. It would be important to use marker stimuli in the latter two cases so that estimates of consistency or reliability could be obtained.

None of these alternatives has been systematically evaluated, and it is readily apparent that there are difficulties with each. In the first approach there is the problem of determining which pairs to sample. Also, many of the available MDS procedures cannot be applied to an incomplete comparisons matrix. The question of the number of pairs to be sampled is also important but as that is a problem common to all selection procedures it will be discussed later. In the repeated session approach, there are of course no problems with missing data; however, these difficulties may in many cases be eliminated at the expense of experimental practicality. Also, it may be unreasonable to expect subjects to be consistent in their use of a numerical scale over several experimental sessions. Although the use of marker pairs would at least indicate whether or not there was intergroup or intersession consistency, it is not at all clear what procedure should be followed when inconsistency is discovered.

One purpose of the present paper is to suggest a third procedure for effectively enlarging the sample size in paired comparison tasks, a procedure which does not involve multiple experimental sessions, and which attempts to circumvent some of the difficulties associated with other partial sampling approaches.

Briefly, the procedure requires that proximity estimates be obtained between each of N "experimental stimuli" and each of M "standards," as well as between all possible combinations of standards. While the M by M standard matrix can be scaled directly, in order to scale the N experimental stimuli it is necessary either to obtain estimates of the distances or scaler products for all pairs of the stimuli using only the given data. Several analytical procedures which can be used to generate these estimates are presented later.

ENHANCING THE UTILITY OF MDS ANALYSIS IN APPLIED RESEARCH SETTINGS

A second major purpose of the paper is to discuss procedures for embedding a novel stimulus in a previously defined multidimensional space. The author feels that the development of efficient techniques for determining the location of novel stimuli in known spaces is of vital importance to researchers who are interested in applying scaling techniques to practical problems. It has been demonstrated time and time again that regardless of the stimulus domain, MDS techniques yield interpretable, predictive psychological spaces. Questions concerning the utility of these spaces, however, have been virtually ignored with the exception of a few recent articles (1, 2, 3, 6, 9, 11).

One problem has concerned the fact that often the psychological dimensions have no clear cut physical interpretation and many researchers have balked at the suggestion of using psychopsychics (for example, using psychological variables to predict responses). It is ironic that the development of a predictive psychophysics should be of major concern in this particular area since one of the major contributions of the MDS techniques was to free the psychologist from the need for determining the relevant physics prior to collecting behavioral data. MDS procedures made it possible for researchers to ask "what dimensions do you normally use in perceiving this stimulus domain" rather than "how do you order this domain along dimension x." It is time that the concept of a psychological space be accepted as a useful behavioral construct independent of whether or not the dimensions have clear cut physical interpretations. In this case, the validity of the construct should be evaluated on the basis of whether the psychological spaces have predictive potential in the behavioral rather than physical domain. In line with this reasoning, the current paper, in proposing methods for embedding stimuli in predetermined psychological spaces, is suggesting that the location of a stimulus in such a space can potentially predict some forms of behavior related to the stimulus.

Several examples should clarify this point. Humans typically are more efficient feature extractors than machines when complex patterns are involved (8). In order to classify complex patterns it is therefore advantageous to make use of of "behavioral" feature spaces. Such spaces are analogous to the psychological spaces derived with MDS techniques. Classes of objects can be represented as regions in the feature space; hence, a decision as to the classification of a new stimulus can be based on its proximity to the various class regions. If the new stimulus could somehow be embedded in the feature space, then a decision concerning its class membership would be a straightforward statistical problem. This procedure bypasses the question of finding physical definitions for the relevant features and instead simply requires that subjects use the features in a ordinary, intuitive manner. For a more detailed explanation of this approach to pattern classification see Fenker & Evans (5).

In a more practical setting, there are many cases in which it would be useful to know where one type of item would be located in a space defined by another type of items. When the two sets of items are individuals and stimuli, this problem is essentially the individual differences problem discussed by Carroll & Chang (2) and others. In the individual differences case, the solution space contains not only the stimuli ordered along the underlying psychological dimensions, but the orientation of subject vectors as well. The subjects' locations in the space not only reflect their relative weighting of the dimensions, but in some cases the distance between a subject point and the stimulus items can be used to predict such things as preference for or concern about the stimuli.

In a final example, the space might contain relevant political issues. Here the psychological dimensions would represent dimensions of political concern. Various political or social groups would also have a location in the space based on their interest and concern for the different issues. Politicians could similarly be projected into the space. This three-way space would be useful for predicting which candidate a particular political group will support or what location (issue profile) a politician should represent in order to influence a particular group.

SELECTING A SET OF STANDARDS

It was mentioned above that in order to reduce the number of paired comparisons judgments required for MDS, a set of "standard" stimuli must be selected. Before outlining the procedures to be considered, it will be useful to define what we mean by "standards."

The standards represent a collection of M stimuli selected so as to comprehensively exhaust the dimensionality of the multidimensional psychological space. In other words, if the underlying psychological space (for the given sample of N stimuli) is R-dimensional, the space necessary to explain the distance relationships between the standards should also be R-dimensional. If the M standards can be embedded in an R-1 dimensional hyperplane of the psychological space, then a poor collection of standards was picked. Although the issue sounds circular at this point since to know the dimensions on which to select the standards is to have solved the scaling problem, in practice it should be possible to select "enough" widely varying types of stimuli from the population to designate as standards in order to sample all relevant types of variation. The minimum number of standards which can be selected is given by the number of degrees of freedom in psychological space. Thus, if the underlying space has R dimensions, then for N stimuli a solution would have N x R degrees of freedom. Hence, the number of standards M, should be somewhat larger than the expected number of dimensions, R, so that $N \times M \geq N \times R$.

A second important issue concerns the determination of the appropriate distance function, which is assumed to map coordinate differences into interstimulus distances. In the current paper we will not consider this problem in any depth. When metric scaling solutions are discussed, it will be assumed that the psychological space is Euclidean. When non-metric scaling procedures are being considered, then the distance function is arbitrary to the extent permitted by the procedures (this usually is the class of functions known as Minkowski R-metrics).

The remainder of the paper is organized as follows. Section II presents a brief description of the data which must be obtained in order to apply the distance estimation or scaling procedures. Sections III and IV describe, respectively, metric and non-metric versions of the procedures. Each of these sections is subdivided into two parts. Part A considers the case where a joint scaling solution for the M+N stimuli is required. Techniques developed in Part A are intended to be applied when a well defined standard space is not available and the N experimental stimuli are to be used in defining the underlying space. Part B describes techniques which are useful when the underlying psychological space is assumed to be known and adequately defined by the standards. The problem in this latter case consists of embedding the experimental stimuli in a well defined space rather than making use of them to determine the space.

THE EXPERIMENTAL DATA

The basic data required for the present approach are proximity estimates between the members of a selected sample of stimulus pairs. The specific sampling required is represented schematically in Figure 1. Out of a set of L = M+N stimuli, M stimuli are designated as standards while the remaining N stimuli comprise the experimental sample. Since we require that proximity estimates be obtained for all possible pairings of the standards as well as for all combinations of the standards and the experimental stimuli, the total number of required paired comparisons, T, will be

$$T = \frac{(M)\ (M-1)}{2} + M \cdot N . \qquad [1]$$

This can be compared to, G, the total number of comparisons required when all possible pairs are sampled, where

$$G = \frac{(M + N)\ (M + N - 1)}{2}. \qquad [2]$$

The difference, G–T, represents the savings in the number of comparisons required using the current procedure, where

$$G-T = \frac{N(N-1)}{2}. \qquad [3]$$

The reader will notice that the savings is equivalent to the number of unordered comparisons required for a set of N stimuli. Hence, the savings will increase as the square of N. A useful statistic for estimating the efficiency of this sampling is the ratio of the number of paired comparisons required using the procedure to the total number of possible comparisons or some fraction of the total number based on another sampling procedure. This ratio is given by

$$E = \frac{(M)\ (M-1) + 2M \cdot N}{(M+N)\ (M+N-1)\ P} \qquad [4]$$

where P is the fraction of the total number of possible comparisons actually obtained. Table 1 contains the values of E for the following sampling conditions.

a. M+N = 15,20,25,30,40,50,60,80,100

b. M = 6,12

c. P = 1, .8, .6, .4, .2

Table 1 permits a comparison of the savings using the current sampling procedure with the savings obtained by random or stratified sampling of the M+N by M+N proximity matrix. The values of E may be interpreted as the precentage of judgments required using the current procedure relative to other sampling procedure is more efficient.

In the next section it will be assumed that the proximity judgments either correspond directly to distances defined on an interval scale or can be transformed into distances. In Section IV this restriction will be relaxed and we will assume only that the proximity measures are monotonically related to the interstimulus distances.

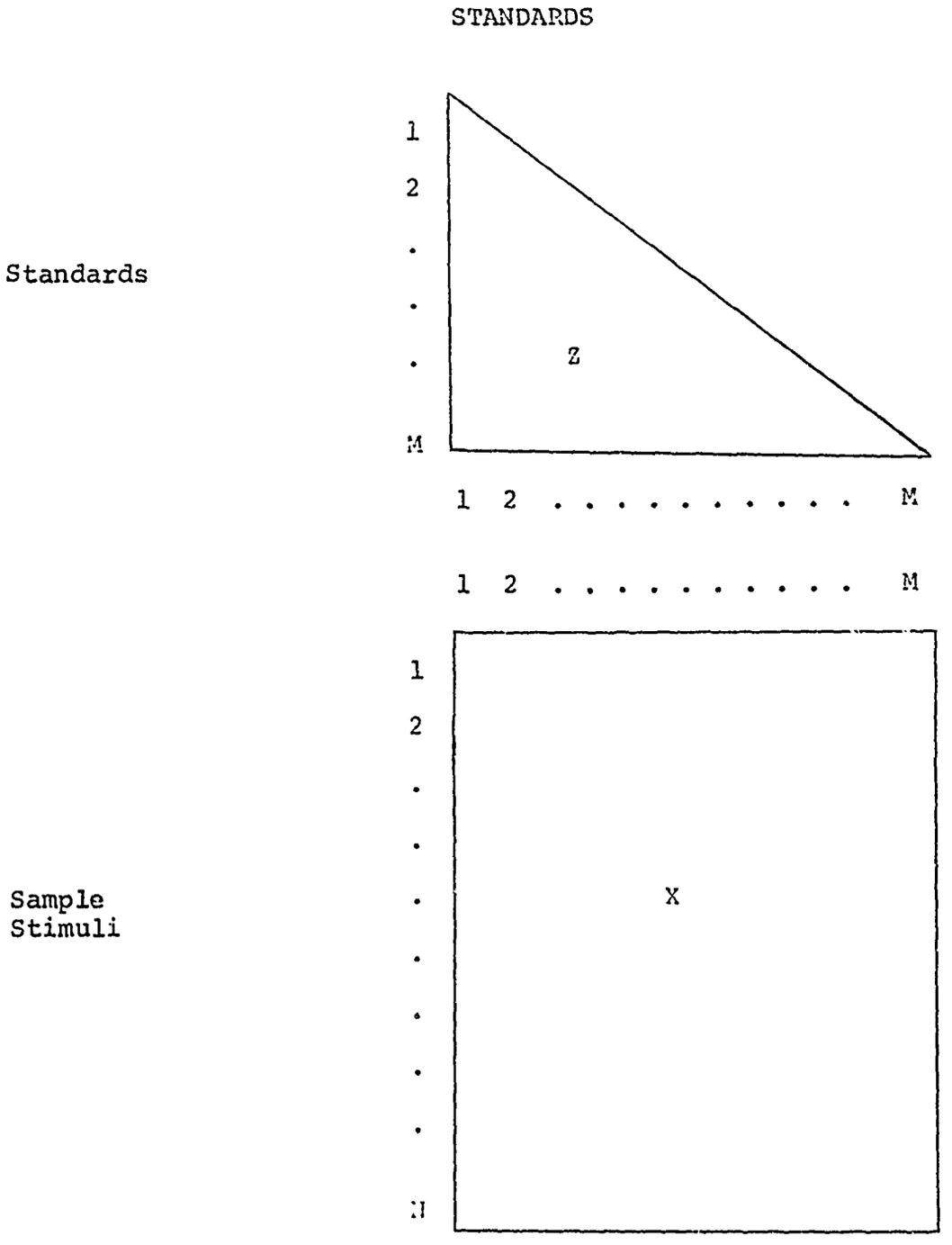

Fig. 1. SCHEMATIC REPRESENTATION OF THE BASIC DATA MATRICES REQUIRED USING THE SAMPLING PROCEDURE

TABLE 1

Values of the Efficiency Coefficient, E, for Various Sample Sizes and Sampling Conditions

M+N= (total sample size)	15		20		25		30		40		50		60		80		100	
M= (number of standards)	6	12	6	12	6	12	6	12	6	12	6	12	6	12	6	12	6	12
P (percentage of all possible pairs sampled)																		
100%	.657	.971	.521	.853	.430	.740	.366	.648	.281	.516	.228	.426	.192	.363	.145	.279	.117	.227
80%	.821	1.214	.651	1.066	.538	.925	.458	.810	.351	.645	.285	.533	.240	.454	.181	.349	.146	.284
60%	1.095	1.618	.868	1.422	.717	1.233	.610	1.080	.468	.860	.380	.710	.320	.605	.242	.465	.195	.378
40%	1.643	2.428	1.303	2.133	1.075	1.850	.915	1.620	.703	1.290	.570	1.065	.480	.908	.363	.698	.293	.568
20%	3.285	4.855	2.605	4.265	2.150	3.700	1.830	3.240	1.405	2.580	1.140	2.130	.960	1.815	.725	1.395	.585	1.135

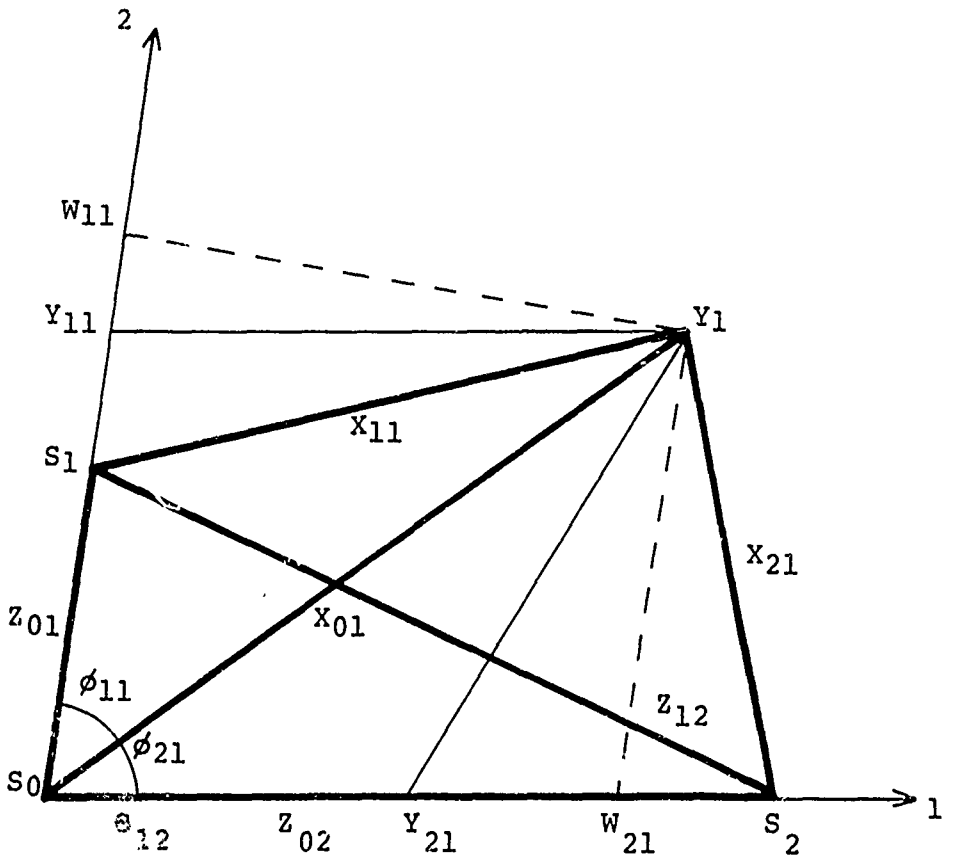

Fig. 2. DISTANCE RELATIONSHIPS BETWEEN STIMULUS Y_1 AND THE TWO STANDARD AXES DEFINED BY S_1, S_2 AND THE ARBITRARY ORIGIN S_0

THE METRIC CASE

The experimental data for the metric case are estimates of the absolute distances (or proximity data which can be transformed into distances with the appropriate scaling model) in psychological space between all possible pairs of standards or experimental stimuli and standards. It is assumed that both the M standards and N stimuli can be represented as points in a K-dimensional psychological space such that Euclidean distances between points in the space correspond to the perceived distances between stimuli. Since each experimental stimulus is paired only with the standards, no interpoint distance estimates are available for pairs of experimental stimuli. Our problem is to take the information which relates these stimuli to the standards and use it either to help define the underlying space containing the standards or to locate the stimuli in the space already well defined by the standards.

JOINT SOLUTION FOR M STANDARDS AND N STIMULI

In this case it is assumed that the number of standards is insufficient to permit an experimental determination of the underlying psychological space. This is not an unusual situation, for the robustness of solutions generated by current MDS scaling procedures depends largely on the estimation of the correct dimensionality and the number of stimuli in the sample. Also, with a small standard sample it may be difficult to identify the dimensions of a MDS solution. This issue becomes especially important when the Euclidean metric is used since the axis orientation is somewhat arbitrary.

For the first case we are interested in only the relative location of the stimuli and standards, and while both are assumed to be embedded in the underlying K-dimensional space, the space has not been determined at this point. Each stimulus has projections on the axes connecting an arbitrary origin with the standards. In the technique to be described, estimates of interstimulus distances can be obtained from these projections. When the projections are considered in conjunction with interstandard distances and standard-stimulus distances, one obtains an M+N by M+N distance matrix which can be scaled using current MDS techniques. The joint stimulus-standard space can then be used to define the underlying psychological dimensions.

Since all the techniques to be presented have several aspects in common, the notation conventions we adopt in this section will be used throughout the paper.

The M standards and N stimuli can be represented as points in the K-dimensional psychological space. The arbitrary origin of the space is represented either by the centroid of the M standards or a designated standard. Although it is perhaps better to use the centroid as the origin since it avoids any bias associated with a particular standard, it will simplify the discussion below if we assume that one of the M standards is selected as the origin. The vectors connecting each of the M-1 standards with the Mth standard (which corresponds to an arbitrary origin in the space) define an oblique set of reference axes which span the space. In order to make use of these reference axes, it is necessary to transform the distance judgments between the standards and stimuli into projections on the axes defined by the standards. Figure 2 illustrates the problem geometrically.

Figure 2 presents the distance relationships for one experimental stimulus and two arbitrary reference axes. Note that estimates of the distances outlined by the thic' lines will be available for each stimulus for each pair of reference axes. Also, notice that whil the standards S_1 and S_2 uniquely define the reference vectors I and II, respectively, an indefinite number of other pairs of standards could also define the same reference vectors.

The terms in Figure 2 can be defined as follows:

S_o : standard stimulus arbitrarily assumed to represent the origin of the space.

S_1, S_2: standards which when taken in conjunction with S_0 define reference axes I and II.

Y_1: experimental stimulus 1.

Z_{01}, Z_{02}, Z_{12}: distance estimates between standards.

X_{11}, X_{01}, X_{21}: distance estimates between S_0, S_1, S_2 and Y_1, respectively.

Y_{11}, Y_{21}: projections of Y_1 on the oblique reference axes.

W_{11}, W_{21}: coordinates of Y_1 on the oblique axes.

The projections of Y_1 on axes 1 and 2 are given by

$$W_{11} = X_{01} \cos \phi_{11}$$
$$W_{21} = X_{01} \cos \phi_{21} \qquad [5]$$

The values for $\cos \phi_{11}$ and $\cos \phi_{21}$ can be found using the cosine law for

$$\cos \phi_{11} = \frac{Z_{01}^2 + X_{01}^2 - X_{11}^2}{2 Z_{01} X_{01}}, \qquad [6]$$

and

$$\cos \phi_{21} = \frac{Z_{02}^2 + X_{01}^2 - X_{21}^2}{2 Z_{02} X_{01}}. \qquad [7]$$

Thus,

$$W_{i1} = \frac{Z_{01}^2 + X_{01}^2 - X_{11}^2}{2Z_{01}}, \quad [8]$$

and

$$W_{11} = \frac{Z_{02}^2 + X_{01}^2 - X^2_{21}}{2Z_{02}}, \quad [9]$$

or, in general, the projection of stimulus k on reference axis i is given by

$$W_{ik} = \frac{Z_{0i}^2 + X_{0k}^2 - X_{ik}^2}{2Z_{0i}} \quad [10]$$

The matrix containing the projections of the N stimuli on the M-1 reference axes can be determined using Equation 10.

Once the projections of the N experimental stimuli on the M-1 standard axes have been obtained, the next problem is to somehow use this information in order to obtain the distances or scalar products for pairs of experimental stimuli. There are several possible approaches to this problem, but for the sake of brevity I will consider only the two which appear most promising.

a. In this approach we assume that it is possible to convert stimulus projections on the reference axes into coordinates. In order to do this, it is necessary to find the transformation matrix V which contains the cosines of the angles between the reference axes. Once V is determined, the coordinate matrix Y can be computed for $Y = WV^{-1}$.

Estimates of V can be obtained directly from the interpoint distances between standards since

$$\cos \phi_{ij} = \frac{Z_{0i}^2 + X_{0j}^2 - Z_{ij}^2}{2Z_{0i}\ Z_{0j}} \quad [11]$$

and Z_{0i}, Z_{0j}, Z_{ij} are available for any pair of reference axes. Unfortunately, unless this matrix is of rank M-1, it will be singular and hence, V^{-1} is undefined. Thus this approach would be useful only when the number of standards is less than or equal to the number of dimensions in the underlying psychological space. It would be possible to circumvent this problem by choosing a sufficiently small number of standards, however, with so few standards defining the underlying space the positions of the new stimuli would be very sensitive to judgment errors or errors due to the selection of a nonrepresentative set of standards.

In the case where the rank of V is R and $R < M-1$, then a solution based on a principle components analysis of V is possible. This procedure could be used to define an R-dimensional orthogonal basis of V since a factor analysis of V would give us X where

$$\underset{M-1,M-1}{V} = \underset{M-1,R}{X}\ \underset{R,M-1}{X'} \quad [12]$$

and the elements of X represent cosines of the angles between the M standard and R orthogonal axes. Projections of an experimental stimulus on the M standard axes would in a sense (see below) define the location of the stimulus on the R orthogonal axes.

We see that the problem at this point has been reduced to one of embedding the stimuli in the R dimensional space defined by the standards. The analytical solution to this problem is presented in part B; however, adopting this approach eliminates the need for computing interpoint distances between experimental stimuli, since their location on the underlying axes completely determines their location in the R-dimensional space. In fact, the underlying space is at worst defined by a rigid rotation of the R axes to orientations which are more interpretable. Any redundancy in the estimates of distances between the standards and the experimental stimuli will be "eliminated" (for example, because the standard space is already defined, the positions of the standards in this space are in no way influenced by the experimental stimuli which are projected into the space on the basis of a least squares criterion) in the process of projecting the experimental stimuli on the orthogonal axes. While this redundancy may be of use in determining the "true" location of the stimuli on the underlying reference axes, it cannot help determine what these reference axes will be since they are already completely determined by the principle components analysis of V.

Since we would expect that, in general, $R \leq M-1$, this first approach does not seem particularly useful or practical except as a method for deriving a standard space into which novel stimuli can be projected.

b. The second approach to scaling the experimental stimuli should be familiar to factor analyst. If we assume that the M axes connecting the standards with an arbitrary origin represent orthogonal reference axes (just as each stimulus can be assumed to define a reference axis in N space for factor analysis), then the correlation between two stimuli computed across the loadings on the M reference axes is equivalent to the cosine of the angle between the stimuli in the underlying R-space. (I am assuming that all stimuli are completely defined, except for error, by the common factor space. No consideration of unique stimulus variance is given.)

Now, the scalar product of stimulus vector i and stimulus vector j is given by

$$b_{ij} = X_{0i} \cdot X_{0j} \cdot \alpha_{ij} \qquad [13]$$

where X_{0i} and X_{0j} represent the lengths of the vectors connecting the origin i and j, respectively, and α_{ij} represents the angle between the vectors. Since X_{0i} and X_{0j} are assumed known, and cos α_{ij} can be computed as the correlation discussed above, estimates of the scalar products between the vectors representing experimental stimuli can be obtained. Thus the N x N matrix of scalar products between stimuli can be estimated from the data. The M x M matrix of scalar products between standards, and the N x M matrix of scalar products between standards and experimental stimuli can both be computed applying the law of cosines to the distance estimates given by subjects. The scalar product between standards i and j is given by

$$b_{ij} = Z_{0i}Z_{0j}\cos \phi_{ij} \; ; \qquad [14]$$

and the scalar product between standard i and stimulus j is given by

$$b_{ij} = Z_{0i}X_{0j}\cos \phi_{ij} \qquad [15]$$

where the terms are defined as in Figure 2. Once the (N + M) x (N + M) matrix of scalar products has been obtained, standard factor analytic procedures can be used to find the underlying R-dimensional psychological space. This approach assumes that an absolute scale of distance can be estimated from the data. Also, as mentioned above, it would be convenient to select the origin as the centroid of the standards rather than a particular standard. Both of these latter points are discussed in detail by Torgerson (12).

SOLUTION FOR EMBEDDING THE EXPERIMENTAL STIMULI IN THE STANDARD SPACE (2)

In this case we assume that the M-1 x M-1 matrix of distances between standards has been reduced to an M-1 x R matrix, T, containing coordinates of the M-1 standards on the R-psychological dimensions. The coordinates of the jth standard are defined by $\underline{t}_j = \{t_{j1}, t_{j2}, \cdots t_{jR}\}$. We are also given estimates of the distances between N experimental stimuli and the M standards. For stimulus i, these distances are given as $(X_{i1}, X_{i2} \ldots X_{iM})$. The unknown coordinates of i on the psychological axes are given by $\underline{Y}_i = \{Y_{i1}, Y_{i2}, \ldots Y_{iR}\}$ where $R \leq$ M-1. The Euclidean distances between stimulus i and the jth standard can be written as

$$X_{ij}^2 = \sum_{K=1}^{R} (t_{jk} - Y_{ik})^2 \qquad [16]$$

If we introduce a vector of constants, $\underline{W}$, which weight the R axes according to their psychological utility[a] and rewrite Equation 16 in matrix notation, we have

$$X_{ij}^2 \equiv \underline{T}_j W \underline{T}_j' - 2\underline{Y}_i W \underline{T}_j' + \underline{Y}_i W \underline{Y}_i' , \qquad [17]$$

where $\underline{W}$ is a diagonal matrix containing the weights in $\underline{W}$.

Since for a given standard the term $\underline{Y}_i W \underline{Y}_i$ is constant, as is $\underline{Y}_i W$, we can write

$$x_{ij}^2 = \underline{T}_j W \underline{T}_j' - 2\underline{B}_i \underline{T}_j' + G_i \qquad [18]$$

Equation 18 is a quadratic function of T which can be solved for estimates of W and $\underline{B}_i$ using multiple regression analysis. To illustrate this, consider the case where R = 2. Expanding Equation 18 for each of the M standards gives the following linear system.

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[a] This permits us to deal with psychological spaces in which the dimentions have been scaled in an arbitrary manner. A still more general procedure which utilizes a different set of weights and a specific orthogonal transformation of the R axes for each $\underline{X}_j$ was developed by Carroll and Chang (2).

$$x_{i1} = t_{11}^2 W_{11} + t_{12}^2 W_{22} + b_{i1}t_{11} + b_{i2}t_{12}$$

$$x_{i2} = t_{21}^2 W_{11} + t_{22}^2 W_{22} + b_{i1}t_{21} + b_{i2}t_{22}$$

. .

. . [19]

. .

$$x_{iM} = t_{M1}^2 W_{11} + t_{M2}^2 W_{22} + b_{i1}t_{M1} + b_{i2}t_{M2}$$

As Carroll (2) mentions, the t_{ij}^2 represent "dummy" independent variables because of their obvious relation to the t_{ij}. Estimates of W and B_i can be obtained from the beta coefficients of the multiple regression analysis. Having estimates W and $\underline{B}_i$, we can solve for the coordinates of $\underline{Y}_i$ using the equation

$$\underline{Y}_i = -\frac{1}{2}\underline{B}_i W^{-1} \qquad [20]$$

The problem of embedding the N stimuli in the metric space defined by the standards can be solved by using this procedure. The coordinates of each stimulus can be defined by solving the system of equations given above with the appropriate value of i.

Although both of the above approaches utilize the same data and hence the same potential information, the former approach avoids a pitfall which is present in the latter but may be ignored either out of expediency or because of supporting research. The likelihood of obtaining a degenerate MDS solution in the latter case is high relative to the former approach because the sample being scaled is comparatively small. When only the standards are scaled it is possible that the "correct" two-dimensional space could become unidimensional in the scaling output (or in general that the solution might represent a hyperplane of the true space). This could occur if the stimuli were arranged in non-random manner (for example, if the standards fell approximately on a line or a semicircle in the space). Hopefully this type of problem would be overcome by careful sampling of standards, but often it is unreasonable to expect a priori knowledge of the location of the standards in the psychological space.

By placing additional stimuli into the space before scaling, bias is reduced since if there are M standards a new stimulus imposes M – 1 additional constraints on the location of the standards.

THE NONMETRIC CASE

The term nonmetric as used in this section refers to the properties assumed to hold for the experimental data, not the scaling technique employed in generating the final configuration. Thus, we do not include the case where distances, generated on the basis of the metric assumptions in the previous section, are analyzed with a nonmetric scaling procedure. This would represent a trivial extension of the metric case discussed above.

The data for this section are assumed to be the rank order of the distances between an experimental stimulus and each of the standards. Such data might be obtained by having subjects select the standard most similar to the stimulus, then the next most similar standard and so forth.

Wtihout the metric distance infromation available in the previous section, it is not possible to obtain estimates of the distances between experimental stimuli without first embedding these stimuli in a metric space defined by the standards and then using the axes of the standard space as reference axes.

It is, however, possible to use current nonmetric MDS procedures to jointly scale both stimuli and standards.

JOINT SOLUTION FOR M STANDARDS AND N STIMULI

The data required for this section are the rank order of the N x M psychological distances (usually obtained as similarity or dissimilarity judgments) between experimental stimuli and standards as well as the M x M (M−1)/2 distances between standards. All rankings must be on the same ordinal scale. This implies that the distance between a stimulus and a standard is not only compared to the distances between that stimulus and other standards, but also to all other distances between stimuli and standards or standards and standards as in typical MDS tasks. The ordinal scale of distances described above represents one possible sample of proximities which could be obtained from the (M + N) x (M + N) proximity matrix. This incomplete set of proximities can be analyzed with most of the available nonmetric MDS procedures, thus giving a joint solution for the M + N points.[a]

Although we would not expect these programs to have difficulty in scaling this incomplete set of distances, solutions may be unsatisfactory in certain important respects.

[a] It is interesting to note that the even more restrictive case (which is likely to occur in practice) where only the stimulus by standard proximity data is available can still be analyzed with a recent version of the Kruskal program (10). The analysis in this instance essentially represents a multidimensional unfolding analysis as described by Coombs (4).

Since there are not data directly relating pairs of experimental stimuli, the relationship between two experimental stimuli (which is unknown) can obviously have no influence on the final solution. Thus points will be "moved" in the space solely on the basis of constraints existing between two standards or a stimulus and a standard. In a sense then, we are embedding one stimulus at a time since the location of a second stimulus is largely independent of the location of the first. If this were literally the case there would be little reason not to compare each stimulus to the standards individually since the judgments required of subjects would be much less demanding.

There is, however, one potentially useful characteristic of joint solutions in which all experimental stimuli are considered simultaneously. If the sample of standards is small or the rank estimates likely to be in error, the distances between experimental stimuli and standards can be used to refine the relative positions of the standards. This is possible because these latter distance estimates represent redundant but useful constraints on the locations of the standards. For example, if i and j represent standards, and k is an experimental stimulus then according to the triangle inequality

$$d_{ij} \leq d_{ik} + d_{kj}$$

By similar logic we can establish a lower bound for d_{ij} since

$$d_{kj} \leq d_{ij} + d_{ik}$$

and,

$$d_{ij} \geq d_{kj} - d_{ik}$$

Hence, if our solution is based on a metric distance function we have d_{ij} bounded by the distances between stimulus k and standards i and j for

$$d_{kj} - d_{ik} < d_{ij} < d_{ik} + d_{kj} \qquad [21]$$

Similar constraints are imposed on every pair of standards by each experimental stimulus.

Since the nonmetric scaling algorithms "force" the stimuli and standards into the same metric space, minimizing the deviations between the distances computed from the space and the ranking of the distance estimates, the contraints on d_{ij} implied by Equation 21, will have considerable influence on the final solution.

Because this first nonmetric approach utilizes current MDS programs in a straightforward manner, we will not discuss it further. Questions concerning the efficiency and effectiveness of this approach will be discussed in a companion paper which deals in some detail with the computational aspects of the scaling procedures presented in the current work.

SOLUTION FOR EMBEDDING THE EXPERIMENTAL STIMULI IN THE STANDARD SPACE

In the former procedure, the distances between the experimental stimuli and the standards were used to help define the locations of the standards. In the current procedure it is assumed that the nature of the standard space is known and fixed and the problem is one of locating the experimental stimuli in this predetermined space. This means that in determining the location of a stimulus in the standard space, adjustments will be made only in the location of the stimulus: the relative positions of the standards will remain fixed. No constraints based on the inter-relationships between standards will be used. The criterion to be maximized will be the monotonicity between the estimated and computed (in the standard space) distance between the stimulus and standards.

The following experimental data is assumed to be available.

a. The M x R matrix containing coordinates of the M stimuli on the R psychological dimensions. This can be used to compute the distances between all possible pairs of standard stimuli. The distance between standards i and j in this space is denoted by Z_{ij}. Since the Z_{ij} are computed on the basis of the projections of the standards on the axes of the psychological space, they in essence define the psychological space. It does not matter what type of procedure (metric or nonmetric) or data were used to generate the psychological space since all current procedures produce metric representations which are based on a particular distance function.

b. The rank order of the distances between each of the experimental stimuli and the M standards. Since it is assumed that the rank order relationships would be reflected in any type of proximity judgments, the actual form of the data is not too important.

One solution to the problem of embedding an unknown stimulus i into the R-dimensional standard space would be to apply the nonmetric version of the regression procedure discussed for the metric case. Instead of the distances between stimulus i and the M standards, we are given the rank order of the distances. Let H represent the monotonic function relating the ordinal proximity data to distances in the underlying metric space; then, Equation 16 can be rewritten as

$$\tilde{x}_{ij} = H(x_{ij}) = \sum_{K=1}^{R} (t_{jk} - Y_{ik})^2 \qquad [22]$$

where $\tilde{x}_{ij}$ represents the proximity between stimulus i and standard j.

Carroll's (2) approach to this problem is based on Kruskal's (7) nonmetric scaling algorithm and involves the following steps.

a. The metric regression approach described above is used to obtain estimates of $x_{ij}^{(1)}$.

b. Kruskal's algorithm is used to estimate the monotonic function $H^{(1)}$ relating the $\tilde{x}_{ij}$ to the $x_{ij}^{(1)}$.

c. New estimates of $\tilde{x}_{ij}$ are computed according to the formula

$$\tilde{x}_{ij}^{(1)} = H^{(1)} (x_{ij}^{(1)})$$

d. The regression procedure is applied to the $\tilde{x}_{ij}^{(1)}$ to obtain estimates of $x_{ij}^{(2)}$ and these in turn are used to estimate $H^{(2)}$.

e. The iterative process is continued until no change occurs in the parameters of the regression equation.

The goal of this process is to obtain estimates of $\underline{y}_i$, the coordinates of stimulus i on the psychological axes. For more detail on the iteration procedure the reader is referred to the article by Carroll and Chang (2).

While in theory the above procedure may represent the optimal approach to the problem of nonmetric embedding, in practice it is possible to accomplish this goal using a modified version of Kruskal's (7) popular MDS program. The Kruskal program constructs an arbitrary metric representation of a given set of data points in R-dimensional space (where R is specified by the user) and then uses a nonmetric scaling algorithm to move these points in the space in such a manner as to maximize the monotonic correspondence between the interpoint distances and the ordinal proximity data. For the current problem, we have assumed that the locations of the standards in the space are fixed; hence, we would like to adjust the positions of the experimental stimuli (without affecting the positions of the standards) in such a way as to maximize the monotonic relationship between the given proximity data and the estimated distances. To accomplish this goal one would use the Kruskal procedure in the following manner.

a. The "configuration" start option would be utilized. The input configuration would contain coordinates of the M standards on the R psychological axes as well as estimates of the location of the N stimuli on these axes. The latter estimates might be obtained either by applying another scaling analysis or by simply using the coordinates of the standard to which each stimulus is closest. Additional input data would, of course, include the rank order of the proximities between each of the experimental stimuli and the M standards.

b. The monotonicity requirements in this case depend only on the proximity relationships between the N stimuli and the M standards. No interstimulus or interstandard constraints are present since these latter proximity data are not fed into the procedure. The stress criterion which is to be minimized is computed solely on the basis of the stimulus by standard proximity matrix.

c. Although the Kruskal procedure will compute increments in the position of the M standards on the R axes which are intended to lower stress, these increments will be ignored. This alteration, so that the program does not adjust the positions of stimuli designated as standards, is the only change required in the current versions of the Kruskal procedure.

d. The positions of the N stimuli will be incremented along the R axes in such a way as to lower stress (improve fit) at each step of the iteration. These increments represent useful adjustments in that the stimuli are moved closer to their "true" position in the space relative to the standards.

e. The output configuration will consist of an R-dimensional space with the standards represented in their original locations and the stimuli located in their "true" position relative to the standards.[a]

[a] In theory, the location of a given stimulus in the multidimensional space is not unique, but rather is defined by an isotonic region (4), where any point within the region satisfies the ordinal constraints imposed by the distance relations between the stimulus and the standards. If the location of stimulus is defined by a closed region (a region bounded on all sides by decision hyperplanes characterizing different proximity orderings), then the solution for the location of a stimulus within the region will be approximately unique (e.g., the region is small relative to the dimensions of the space). On the other hand, if the stimulus lies in an open region, then its position could be changed considerably without affecting the proximity ordering. If the number of standards is several times as large as R, and the range of variation in the standards spans the R-dimensional space, there should be no experimental stimuli located in open isotonic regions, however, when these conditions are not met some problems may arise.

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