SEQUENTIAL DETERMINATION OF INSPECTION EPOCHS FOR RELIABILITY SYSTEMS WITH GENERAL LIFE TIME DISTRIBUTIONS.

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13. ABSTRACT

The problem of determining the optimal inspection epoch is studied for reliability systems in which N components operate in parallel. Life time distribution is arbitrary but known. The optimization is carried with respect to two cost factors: the cost of inspecting a component and the cost of failure. The inspection epochs are determined so that the expected cost of the whole system per time unit per cycle will be minimized. The optimization process depends in the general case on the whole failure history of the system. This dependence is characterized. The cases of Weibull life time distributions are elaborated and illustrated numerically. The characteristics of the optimal inspection intervals are studied theoretically.

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Increasing Failure Rate						
Decreasing Failure Rate						
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1. Introduction.

In the present study we investigate the problem of determining the optimal inspection epochs of a reliability system which is comprised of N components, operating independently (in parallel) and having the same life time distributions. The life time distribution is known. An inspector visits the system at a predetermined inspection epoch and finds a certain number of components which have failed. The exact times of failure are unknown. All the components which have failed during the interval between inspections are replaced by new components. Components which have not failed are left in the system. We consider two types of cost factors: (i) The cost of inspection, which depends on the number of components in the system; and (ii) The cost of failure per unit time. This cost component measures the loss due to a failure of components. The objective is to determine an inspection policy that would be optimal with respect to the criterion of minimizing the total expected (discounted) cost for the entire future. However, since we are dealing with cases of general life time distributions (not necessarily exponential) the dynamic programming solution is excessively complicated, even in the truncated case (when the number of inspections should not exceed a prescribed bound). Therefore, we are considering in the present paper a sequential myopic procedure. Accordingly, after each inspection the epoch of the next inspection is determined, as a function of the whole past failure history of the system. The aim is to minimize the conditional expected cost per time unit from the present time until the next inspection epoch. In the case of exponential life time distributions (constant failure rates) the optimal inspection interval (time interval between inspections) does not depend on the past history of the system. As shown in the present study if the life time distribution is not exponential this dependence might be very strong, especially if N is not large and the life time distribution is of a decreasing failure rate (DFR). The dependence of the optimal inspection intervals on the observed number of failures, and on the number of components that were replaced at previous inspections and are still operating, will be explicitly characterized. We start in Section 2 by formulating the model and the associated distributions. In Section 3 we develop a general formula for the sequential determination of the optimal length of the inspection intervals. In Section 4 we derive the corresponding formulas for life time distributions of the Weibull family; and illustrate the process with a numerical example. In Section 5 we try to explain the complex process illustrated in the example of Section 4 by further theoretical development.

There are numerous papers in the reliability literature on inspections epochs and optimal maintenance. For the general theory see Chapter 4 of Barlow and Proschan [1]. Articles which are close to the present study are those of Kamins [4], Kander [5], Kander and Naor [6] and Kander and Rabinovitch [7]. The present study provides further elaboration of a chapter in the thesis of Fenske [3]. The main difference between the present study and the articles mentioned above is in the basic model. The present study is concerned with multicomponent systems while the other studies treat the whole system as one component. The study of Ehrenfeld [2] was based on a model similar to ours, but Ehrenfeld considered the problem of determining the inspection interval for the estimation the mean time between failures in the exponential case.

2. The model and associated distributions.

Consider a reliability system which consists of N, $N \geq 1$, components. These components operate independently (in parallel). Let T designate the life time of a component. This is a random variable having a known distribution function (c.d.f) $F(t)$. We assume that $F(t)$ is absolutely continuous, with a positive density function $f(t)$, $0 < f(t) < 0$, and $F(0) = 0$. We further assume that the expected value of T, according to $F(t)$ is finite. Let $S_0 \equiv 0$ and let $S_0 < S_1 < S_2 < \ldots < S_m < \ldots$ designate a sequence of inspection epochs. Let J_m $(m = 1,2,\ldots)$ designate the number of components that failed during the time interval (S_{m-1}, S_m). All the J_m components are replaced at the inspection epoch S_m. The $N - J_m$ components which have not failed during $(S_{m-1} > S_m)$ are classified into m disjoint subsets $A_0^{(m)}, A_1^{(m)}, \ldots, A_{m-1}^{(m)}$. The subset A_j $(j = 0, \ldots, m-1)$ contains all the components that were replaced at epoch S_j and did not fail throughout the time interval $[S_j, S_m)$. Let $n_j^{(m)}$ designate the number of elements of $A_j^{(m)}$. Obviously, $A_j^{(m+1)} \subset A_j^{(m)}$ and $n_j^{(m+1)} \leq n_j^{(m)}$ for each $j = 0, 1, \ldots$ and $m = j, j+1, \ldots$ Let $n_m^{(m)} = J_m$, and $\underset{\sim}{n}^{(m)} = (n_0^{(m)}, n_1^{(m)}, \ldots, n_m^{(m)})$ for each $m = 0, 1, \ldots;$ $n_0^{(0)} \equiv N$.

If a component belongs to the subset $A_j^{(m)}$ then its conditional life time distribution at time t is:

$$(2.1) \qquad F_j^{(m)}(t) = P\{T \leq t-S_j \mid T \geq S_m-S_j\}$$

$$= \begin{cases} 0 & , \text{ if } t \leq S_m \\ \dfrac{F(t-S_j)-F(S_m-S_j)}{} & , \text{ if } t > S_m \end{cases}$$

In particular, $F_m^{(m)}(t) = F(t-S_m)$, if $t > S_m$ and zero otherwise. The conditional densities of $U = T - (S_m - S_j)$, corresponding to the life time T of a component which belongs to $A_j^{(m)}$ play an important role in our procedure. We can call U the remaining life time. If a component is chosen at random at time $t = S_m + 0$ its remaining life time U has a conditional density function

$$(2.2) \qquad h_m(u|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)}) = \frac{1}{N} \sum_{j=0}^{m} n_j^{(m)} \frac{f(u+S_m-S_j)}{1-F(S_m-S_j)}, \quad u \geq 0$$

where $\underset{\sim}{S}^{(m)} = (S_1, \ldots, S_m)$.

We notice that if T has a negative exponential distribution i.e., $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$, for any $0 < \lambda < \infty$, then $h_m(u|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)}) = f(u)$ for <u>all</u> $m = 1, 2, \ldots$ and <u>all</u> $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$. This is a well known property of the negative exponential distributions. Let $H_m(u|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$ designate the c.d.f. corresponding to (2.2).

3. Sequential determination of inspection epochs.

We will consider in the present section the problem of deriving an inspection policy which attains a certain economic objective. We assume therefore that the cost of inspecting the system is \$ C_o per inspection and on the other hand, if an element fails, then the cost associated with its failure is \$ C_f per time unit. The inspection policy adopted here is the following. Given the history connected with the past m inspection intervals, i.e., $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$ determine the $(m+1)$ st inspection epoch so that the average expected cost per time unit of inspection and of failure, over the $(m+1)$ st inspection interval will be minimized. We remark in this connection that this policy is in essence a myopic policy, which minimizes the expected time average costs for each inspection interval individually. A Dynamic Programming determination of the inspection epochs could attain a more global optimization. However, attempts at Dynamic Programming solutions lead to complicated sets of recursive functional equations. The solution of these equations is generally very tedious.

Let Δ designate the length of the $(m+1)$ st inspection interval. That is, $\Delta = S_{m+1} - S_m$. Given $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$ the conditional expected average cost per time unit, under Δ, is

$$(3.1) \qquad R_m(\Delta; \underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)}) = \frac{C_o}{\Delta} + \frac{C_f}{\Delta} \sum_{j=0}^{m} n_j^{(m)} \int_0^{\Delta} (\Delta-u) \frac{f(u+S_m-S_j)}{1-F(S_m-S_j)} du$$

Or in terms of the conditional distribution of the remaining life time U we can express (3.1) in the form

$$(3.2) \qquad R_m(\Delta;\underset{\sim}{S}^{(m)},\underset{\sim}{n}^{(m)}) = \frac{C_o}{\Delta} + NC_f H_m(\Delta|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)}) - \frac{C_f}{\Delta} N \int_0^{\Delta} uh_m(u|\underset{\sim}{S}^{(m)},\underset{\sim}{n}^{(m)})\, du.$$

The optimal $(m+1)$ st inspection epoch is defined as $S_{m+1} = S_m + \Delta^o$, where Δ^o is a positive real value, Δ, for which the infimum of (3.2) is attained.

Let

$$(3.3) \qquad \mu_m = \int_0^{\infty} uh_m(u|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})\, du,$$

be the expected remaining life, given $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$. According to the assumption of the previous section, $\mu_m < \infty$. Differentiating $R_m(\Delta;\underset{\sim}{S}^{(m)},\underset{\sim}{n}^{(m)})$ with respect to Δ we obtain that if $\mu_m \le C_o/NC_f$ then $\Delta^o = \infty$.

This is a case in which no more inspections are warranted. On the other hand, if $\mu_m > C_o/NC_f$, there exists a unique solution, Δ^o, to the equation:

$$(3.4) \qquad \int_0^{\Delta} u h_m(u \mid \underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})\, du = C_o/NC_f .$$

We realize from (3.4) that S_{m+1} is a function of the statistic $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$ of the system.

As we have already mentioned in cases of exponential life time distributions the optimal length of the inspection intervals is the same for <u>all</u> $m = 1, 2, \ldots$. If $\theta = \lambda^{-1}$ is the mean time between failures (MTBF) in the exponential case then $\mu_m = \theta$ for all m, and the condition for a finite Δ^o is that $C_o < NC_f\theta$; i.e., the cost of inspecting an element is smaller than the expected cost of failure of an element. If this condition is satisfied then, letting $\gamma = C_o/NC_f\theta$, it is easy to show that

$$(3.5) \qquad \Delta^o = \frac{\theta}{2} \chi^2_\gamma [4] ,$$

where $\chi^2_\gamma[4]$ designate the γ-fractile of a chi-square distribution with 4 degrees of freedom.

4. Optimal inspection epochs for Weibull distributions.

Suppose that the life time of an element, T, follows a Weibull distribution, with a density function

$$(4.1) \qquad f(t;\theta,\alpha) = \begin{cases} 0 & , \text{ if } t \le 0 \\ \frac{\alpha}{\theta} t^{\alpha-1} \exp\{-t^{\alpha}/\theta\} & , \text{ if } t > 0; \end{cases}$$

where α and θ are positive real parameters. We notice that if $0 < \alpha < 1$ then the distribution has a decreasing failure rate (DFR), and if $1 < \alpha < \infty$ its failure rate is increasing (IFR). When $\alpha = 1$ the distribution is exponential. Given $(\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)})$ the density function of the remaining life U assumes the special form

$$(4.2) \qquad h_m^{(\theta,\alpha)}(u|\underset{\sim}{S}^{(m)},\underset{\sim}{n}^{(m)}) = \frac{1}{N} \sum_{j=0}^{m} n_j^{(m)} \exp\left\{(S_m - S_j)^{\alpha}/\theta\right\} \cdot \frac{\alpha}{\theta} (u+S_m-S_j)^{\alpha-1} \cdot$$

$$\cdot \exp\left\{-(u+S_m-S_j)^{\alpha}/\theta\right\},$$

for $0 \le u \le \infty$. When $m = 0$ (4.2) reduces to (4.1). Following the procedure given in the previous section we realize that $S_1 < \infty$ if and only if,

$$(4.3) \qquad C_o/N < C_f\, \theta^{1/\alpha}\, \Gamma(\frac{1}{\alpha} + 1) \quad .$$

$\theta^{1/\alpha}\,\Gamma(\frac{1}{\alpha} + 1)$ is the expected life time. If (4.3) is satisfied then the optimal value of S_1 is

$$(4.4) \qquad S_1 = \left\{\frac{\theta}{2}\, G^{-1}(\gamma|\frac{1}{2}, \frac{1}{\alpha} + 1)\right\}^{1/\alpha} \quad ,$$

where $G^{-1}(\gamma|p,\nu)$, is the γ-fractile of the Gamma distribution $G(p,\nu)$, with scale parameter p, and where $\gamma = C_o/(NC_f\theta^{1/\alpha}\Gamma(\frac{1}{\alpha} + 1))$. We notice that if $2/\alpha$ is a positive integer j then

$$(4.5) \qquad S_1 = \left\{\frac{\theta}{2}\chi^2_{\gamma}\ [2+j]\right\}^{j/2} \quad .$$

We determine now a general expression for the left hand side of (3.4). According to (4.2),

$$(4.6) \qquad \int_0^{\Delta} u.h_m^{(\theta,\alpha)} \; (u|\underset{\sim}{S}^{(m)}, \underset{\sim}{n}^{(m)}) \; du =$$

$$\frac{1}{N} \sum_{j=0}^{m} n_j^{(m)} \exp\{S_m - S_j)^{\alpha}/\theta\}. \quad \frac{\alpha}{\theta} \int_0^{\Delta} u(u+S_m-S_j)^{\alpha-1} \exp\{-(U+S_m-S_j)^{\alpha}/\theta\} \; du.$$

By a proper change of variable we obtain

$$(4.7) \quad \frac{\alpha}{\theta} \int_0^{\Delta} u(u + S_m - S_j)^{\alpha-1} \exp\{-(u+S_m-S_j\}^{\alpha}/\theta\} \; du =$$

$$\int_{(S_m-S_j)^{\alpha}/\theta}^{(S_m-S_j+\Delta)^{\alpha}/\theta} \left[\theta^{1/\alpha} w^{1/\gamma} - (S_m-S_j)\right] \exp\{-w\} \; dw =$$

$$\theta^{1/\alpha}\Gamma(\frac{1}{\alpha}+1) \left[G\left(\frac{(S_m-S_j+\Delta)^{\alpha}}{\theta}; 1, \frac{1}{\alpha}+1\right) - G\left(\frac{(S_m-S_j)^{\alpha}}{\theta}; 1, \frac{1}{\alpha}+1\right)\right]$$

$$- (S_m-S_j) \left[\exp\{-\frac{(S_m-S_j)^{\alpha}}{\theta}\} - \exp\{-\frac{(S_m-S_j+\Delta)^{\alpha}}{\theta}\}\right]$$

Substituting (4.7) into (4.6) we obtain that $S_{m+1} = S_m + \Delta$, where Δ is the root of the equation:

$$(4.8) \quad \frac{1}{N} \sum_{j=0}^{m} n_j^{(m)} \exp\{(S_m-S_j)^{\alpha}/\theta\} \left[G\left(\frac{S_m-S_j+\Delta)^{\alpha}}{\theta}; 1, \frac{1}{\alpha}+1\right) - \right.$$

$$\left. G\left(\frac{(S_m-S_j)^{\alpha}}{\theta}; 1, \frac{1}{\alpha}+1\right)\right] = \gamma +$$

$$\frac{1}{\theta^{1/\alpha}\Gamma(\frac{1}{\alpha})} \frac{1}{N} \sum_{j=0}^{m} n_j^{(m)} (S_m-S_j) \left[1 - \exp\left\{-\frac{1}{\theta}\left[(S_m-S_j+\Delta)^{\alpha} - (S_m-S_j)^{\alpha}\right]\right\}\right]$$

γ is as before.

We notice that for $m = 0$ the solution of (4.8) is reduced to the one given by (4.4). In Figure 1 we illustrate the solution of (4.8) for three Weibull distributions, where the $n_j^{(m)}$ sequences were generated by Monte Carlo simulation. The cases under consideration have the following parameters: $C_f = \$10.$, $C_o = \$200 \cdot N$, $\theta =$ [hr] 100 and $\alpha = 3/4$, 1 and 5/4. The case of $\alpha = 1$ corresponds to the exponential distribution with mean $\theta = 100$, According to (3.5) the optimal inspection interval for $\alpha = 1$ is of length [hr.] 50 $\chi^2_\gamma[4]$ where $\gamma = C_o/N \cdot C_f\theta = 0.2$. One can find in any statistical tables that $\chi^2_{.2}[4] = 1.65$. Hence, the optimal interval between inspections is in the exponential case is length 82.5 hours. The case of $\alpha = 5/4$ represents an IFR distribution. We see in Figure 1 that the optimal inspection intervals are of length which vary very little around 59 hours. It is interesting to notice that in the present case of an IFR distribution the optimal inspection intervals do not depend strongly on the number of components, N, in the system. This is not the case when the Weibull distribution is a DFR (α=3/4). As illustrated in Figure 1 the optimal intervals for DFR distributions, as obtained from (4.8), are sensitive to N. When N = 10 there are considerable fluctuations of the solution of (4.8). When N = 100 these fluctuations diminish. The general trend of growth in the length of the inspection intervals is however the same. An explanation of this phenomenon will be provided in the next section. Finally we remark that the numerical solution of equation (4.8) in the case discussed here has been attained following the Newton-Raphson iterative corrections to an initial solution. For further details see Fenske [3].

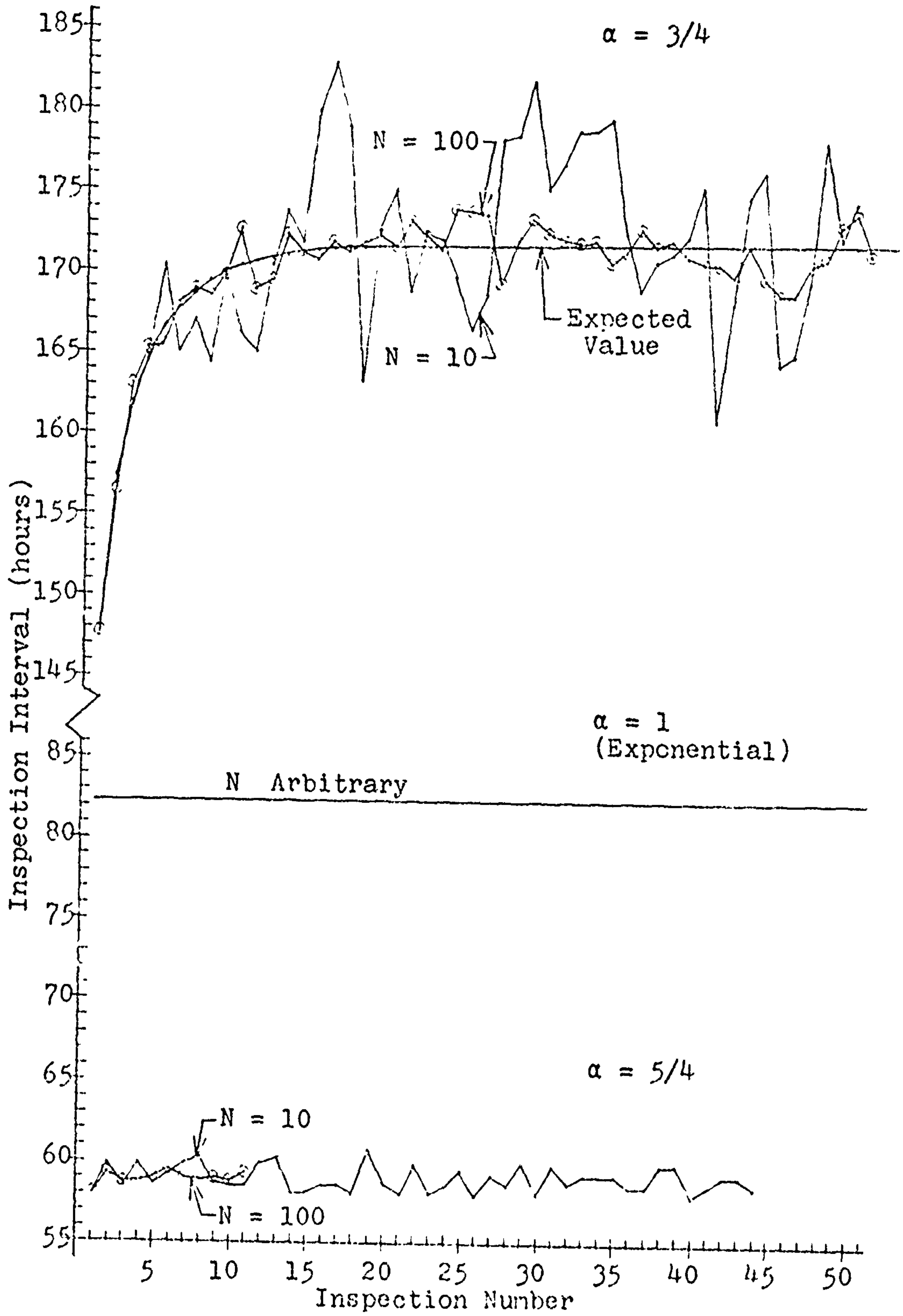

Figure 1. Optimum Inspection Intervals for Weibull Distribution With C_r = \$10, C_o = \$200N, and θ = 100(hrs)

The variance of W_m is

$$(5.7)\qquad \mathrm{Var}\{W_m\} = \frac{1}{N}\left\{\sum_{j=0}^{m} \frac{\theta_j^{(m)}(D_j^{(m)})^2}{[1-F(S_m-S_j)]^2} - \left(\sum_{j=0}^{m} \frac{\theta_j^{(m)}D_j^{(m)}}{1-F(S_m-S_j)}\right)^2\right\} .$$

We have shown that for any sequence of inspection epochs Var $\{W_m\} = O(N^{-1})$ as $N \to \infty$. This explains why the fluctuations of the roots of (4.8) are relatively larger when $N=10$ and small when $N=100$. We consider now a particular sequence of inspection epochs which consists of values of S_m obtained by the repeated solution (for each m) of the equation $\omega_m = \gamma$, i.e.,

$$(5.8)\qquad \int_0^{\Delta} uf(u)du + \sum_{j=1}^{\cdot} \left[1 - \sum_{i=0}^{j-1} \theta_i^{(j)}\right] \int_0^{\Delta} uf(u+S_m-S_j)du = \gamma$$

S_1 is the root Δ of $\int_0^{\Delta} uf(u)du = \gamma$, and for each $m = 1, 2, \ldots$, the $(m+1)$st inspection epoch is given by $S_{m+1} = S_m + \Delta$. The sequence of fixed inspection epochs determined by this procedure corresponds to the expected values of $n^{(m)}$ and we therefore label this procedure as the Procedure Of Averages. In Table 1 we provide the inspection intervals determined by the Procedure of Averages, and the corresponding multinomial probabilities $\theta_j^{(m)}$ (j=, ..., m,), for the two cases represented in Figure 1. The graph of the corresponding inspection intervals for the case of $\alpha = 3/4$ (DFR) is also plotted in Figure 1. As is demonstrated in Table 1, in the IFR case ($\alpha=5/4$) the significant contribution to the solution is expected to be that of $n_m^{(m)}$ and $n_{m-1}^{(m)}$, or of their corresponding expected values. Furthermore, the optimal length of the inspection intervals varies very little with the number of inspections, m, and

$$\theta_o^{(m)} = 1 - F(S_m)$$

(5.2) and

$$\theta_j^{(m)} = (1 - \sum_{i=0}^{j-1} \theta_i^{(j)}) [1 - F(S_m - S_j)], \quad j = 1, \ldots, m.$$

It follows that for any fixed sequence of inspection epochs and for each $m = 0, 1, \ldots$

$$(5.3) \qquad \begin{aligned} E\{n_j^{(m)}\} &= N\, \theta_j^{(m)} \\ Var\{n_j^{(m)}\} &= N\, \theta_j^{(m)}(1-\theta_j^{(m)}) \end{aligned} \qquad j = 0, 1, \ldots, m$$

and

$$(5.4) \qquad \mathrm{cov}(n_j^{(m)}, n_k^{(m)}) = - N\theta_j^{(m)}\, \theta_k^{(m)}, \quad \text{all} \quad 0 \le j < k \le m.$$

From (5.2) and (5.3) we conclude that if the length of each inspection interval is not smaller than Δ_o then for any distribution F, $\lim_{m \to \infty} n_j^{(m)} = 0$ for each j.

The variable W_m is a linear combination of multinomial random variables. Its expectation is

$$(5.5) \qquad \omega_m = E\{W_m\} = D_o^{(m)} + \sum_{j=1}^{m} \left[1 - \sum_{i=0}^{j-1} \theta_i^{(j)}\right] D_j^{(m)},$$

where

$$(5.6) \qquad D_j^{(m)} = \int_0^{\Delta} u\, f(u+S_m-S_j)\, du.$$

its expectation reaches in the present example a stable situation after two inspections. This is not the case, however, in the DFR distribution (α=3/4). The probabilities $\theta_j^{(m)}$ approach zero, as m grows, very slowly. This is reflected in a steady increase in the length of the inspection intervals as m grows, and a stable situation is reached in the present example only after 10 inspections.

To insure that the inspection intervals discussed in Sections 3 and 4 will have similar properties to those determined by procedures of fixed inspection epochs we could consider the following adjustment. First, determine for each $m = 1, 2, \ldots$ two fixed sequences of inspection epochs which will constitute upper and lower (confidence) limits for the solution of (3.4) (or (4.8)). This can be done by utilizing formulae (5.5) and (5.7). The lower confidence limits could be obtained by repeated solution (for the root Δ) of the equation

$$(5.9) \qquad \omega_m + 3.[\mathrm{Var}\{W_m\}]^{1/2} = \gamma \,; \qquad m = 1, 2, \ldots$$

The upper limit can be obtained by solving the equation

$$(5.10) \qquad \omega_m - 3.[\mathrm{Var}\{W_m\}]^{1/2} = \gamma \,, \qquad m = 1, 2, \ldots$$

In the second phase of computation solve equation (3.4). If the solution lies between the roots of (5.9) and (5.10) proceed; otherwise truncate the solution to either the lower limit or to the upper limit, whichever is closer to the actual solution. Such an adjustment will guarantee that every inspection interval will be bounded by lower and upper values which are determined by fixed sequences of inspection epochs, and will therefore have general characteristics as established here.

Table 1: Values of optimal inspection intervals Δ [hrs] and multimonial probabilities under the Procedure Of Averages for Weibull distributions with $\theta = 100$ [hrs] and cost components $C_o = \$200N$, $C_f = \$10$.

Case I: $\alpha = 5/4$ (IFR)

m	opt. Δ	j=0	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8	j=9	j=10
1	58.0	0.2017	0.7983									
2	59.3	0.0211	0.1543	0.8246								
3	59.1	0.0016	0.0161	0.1600	0.8223							
4	59.1	0.0010	0.0013	0.0167	0.1595	0.8225						
5	59.1	0	0.0001	0.0013	0.0166	0.1595	0.8225					
6	59.1	0	0	0.0001	0.0013	0.0166	0.1595	0.8225				

Case II: $\alpha = 3/4$ (DFR)

m	opt. Δ	j=0	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8	j=9	j=10
1	147.6	0.6548	0.3452									
2	156.8	0.4825	0.2216	0.2959								
3	161.8	0.3666	0.1624	0.1880	0.2830							
4	164.8	0.2839	0.1231	0.1372	0.1787	0.2771						
5	166.6	0.2229	0.0953	0.1039	0.1302	0.1742	0.2735					
6	167.9	0.1769	0.0748	0.0803	0.0984	0.1267	0.1716	0.2713				
7	168.8	0.1416	0.0594	0.0630	0.0761	0.0957	0.1246	0.1698	0.2698			
8	169.4	0.1142	0.0475	0.0500	0.0600	0.0739	0.0941	0.1233	0.1686	0.2685		
9	169.9	0.0927	0.0383	0.0401	0.0473	0.0580	0.0726	0.0930	0.1223	[illegible]	0.2678	
10	170.3	0.0756	0.0311	0.0323	0.0379	[illegible]	0.0569	0.0718	0.0924	[illegible]	[illegible]	[illegible]

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