IDENTIFICATION OF PARAMETERS IN NONLINEAR BOUNDARY CONDITIONS OF DISTRIBUTED SYSTEMS WITH LINEAR FIELDS

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13. ABSTRACT

A method is presented for the formulation of unknown parameters in nonlinear boundary conditions in distributed parameter dynamic systems. In contrast to other available techniques, this method requires only as many measurement sensors within the field as there are unknown boundary conditions. Results are presented for simulated data from an example of heat conduction with radiation boundary and for experimental data from a cantilever beam with a nonlinear moment at the boundary. The method may be applied to partial differential equations which are linear, one-dimensional, and have known time invariant coefficients. The nonlinear boundary conditions are specified up to a set of unknown constant parameters which appear linearly in the boundary conditions.

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PREFACE

This technical report is a reprint of a paper submitted to ASME for publication and results from a Ph.D. thesis done at the Automatic Control Center by E. Dawson Ward under the direction of Professor R.E. Goodson. This paper has been accepted for presentation at the Joint Automatic Control Conference of the American Automatic Control Council. Research support for the work was provided in part by a National Science Foundation Science Faculty Fellowship and by the Office of Naval Research under contract N00014-67-A-0226-0012, NR 041-423.

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Abstract

A method is presented for the formulation of unknown parameters in nonlinear boundary conditions in distributed parameter dynamic systems. In contrast to other available techniques, this method requires only as many measurement sensors within the field as there are unknown boundary conditions. Results are presented for simulated data from an example of heat conduction with radiation boundary and for experimental data from a cantilever beam with a nonlinear moment at the boundary. The method may be applied to partial differential equations which are linear, one-dimensional, and have known time invariant coefficients. The nonlinear boundary conditions are specified up to a set of unknown constant parameters which appear linearly in the boundary conditions.

Introduction

Distributed system identification must address itself to four interrelated problems. They are:

I. Measurement Restrictions

Since the system is distributed in space, measurements can be taken anywhere within the field and at the boundaries. However, for a method to be practical, successful identification should be accomplished with a minimum number of sensors. Furthermore, measurement locations in the field must be selected such that nodes of the dominant modes of the system are avoided(1)[1].

II. Simulation Method

Closely tied to the measurement problem above, is the selection of a simulation method. For purposes of either digital or analog simulation, the infinite order partial differential equation must be reduced to a finite order, or equivalent lumped parameter, model. The simulation method selected should not require an excessive number of measurements and if implemented digitally should be fast enough to be practical.

[1]Numbers in parentheses designate References at end of paper.

III. Performance Criterion

In order to accomplish the identification, some criterion must be selected to measure the error in the method. For a practical method, selection of this error again must not require an excessive number of sensors. In particular, an error defined over the entire spatial domain is impractical.

IV. Optimization Technique

Finally an optimization technique must be selected to minimize a performance criterion in some stable fashion by identifying a "best" set of parameters.

Systems considered in this paper are those which can be adequately represented by known one-dimensional, linear, stationary field dynamics but have boundary conditions which are nonlinear algebraic equations in the field energy variables. Heat conduction in a continuum with a radiation boundary condition is an example of such a system. The method presented in this paper identifies the unknown parameters in these nonlinear algebraic relations between the energy variables at the boundaries.

Development of the Identification Procedure

The method presented in this paper, as shown in Figure 1, takes advantage of several properties of distributed parameter systems. In simulating partial differential equations one can separately simulate the field dynamics and the boundary conditions and then combine them to obtain the interactive response(2). Furthermore, since internal spatial data appropriately located are adequate along with the field model to describe response throughout the entire field, one can use information at a sensor located within the field to determine the energy variable response at the unknown boundary. Thus, the appropriate energy variable can be simulated at the unknown boundary using only as many appropriately located sensors as there are unknown boundary conditions. Having recovered the boundary energy variable by such simulation, the boundary forcing then can be expressed as a function of only the unknown boundary parameters. If the unknown boundary condition is appropriately modeled, this boundary forcing will be a linear function of the unknown parameters even though nonlinear in the field variables. For the full power of the proposed method to be utilized a linear function in the unknown parameters is necessary.

$$\frac{\partial u}{\partial t} = - \frac{\partial q}{\partial x}$$
$$q = - \frac{\partial u}{\partial x} \tag{1}$$

The symbols are defined in the nomenclature section of this paper. Since the heat conduction is assumed symmetrical only half of

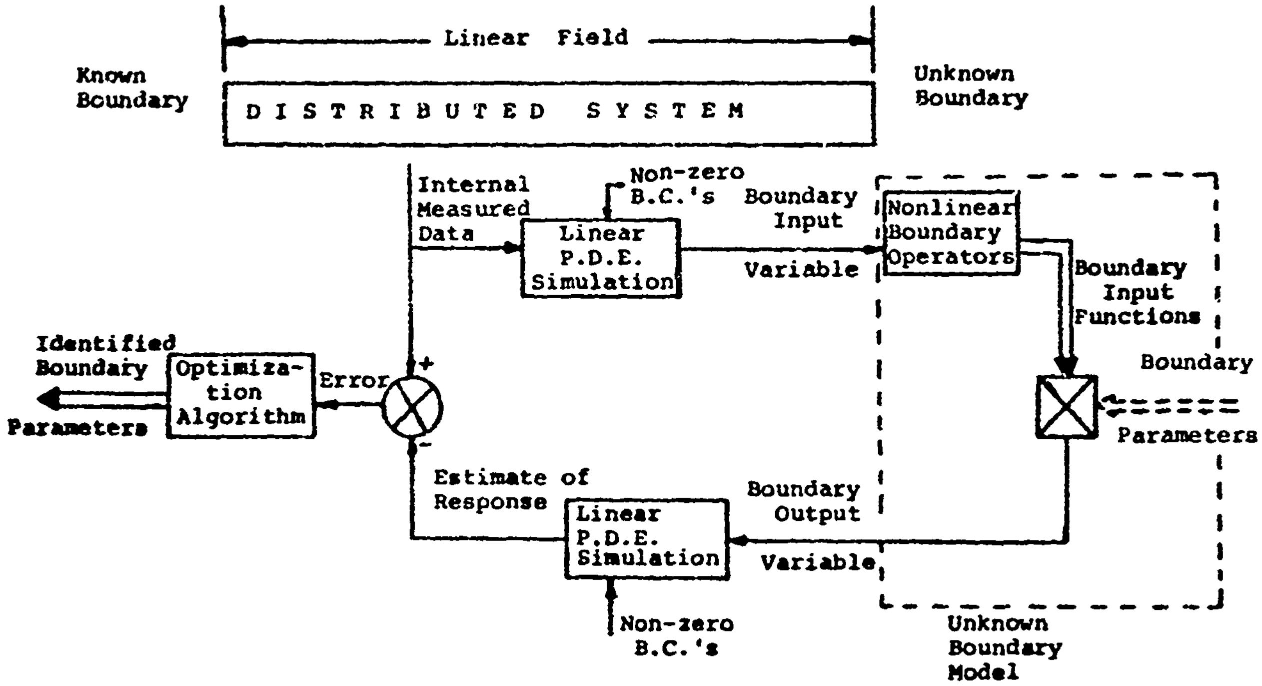

Figure 1 Block Diagram of the Basic Identification Method

After forming the boundary forcing at the unknown boundary as a linear function of the unknown parameters, an error is defined at the original measurement location between the actual measured response and the simulated response using the unknown boundary condition as forcing. An ISE performance index is then minimized yielding a set of linear algebraic equations which are solved digitally for the unknown parameters (3,4). Since a digital computer is used a steep descent technique is not necessary to solve the optimization problem.

The method is best explained by example.

Application of the Identification Method

Example 1. Heat conduction with a radiation and forced convection boundary condition.

The problem considered in this first example is one-dimensional heat conduction in a homogeneous slab with boundary radiation and forced convection. Constant diffusivity, environment temperature, and the initial uniform constant temperature are known. The field dynamics are the slab is simulated and the boundary at the center, $x = 0$, is equivalent to an insulated boundary. At the unknown boundary condition, $x = 1$, heat transfer occurs by combined radiation and forced convection. Thus, the boundary conditions are

$$q(0,t) = 0 \tag{2}$$

$$q(1,t) = \beta[(u(1,t) + 1)^4 - \theta_c^{\,4}] + \eta[(u(1,t) + 1) - \theta_c] \tag{3}$$

Data were available in the literature for this problem with $\eta = 0$, $\beta = 1.0$(3) and for $\eta = 1.0$, $\beta = 1.0$(4). These data in the literature were generated at $x = 1$ by an iterative numerical scheme given the η and β values. By further simulation, values of $u(.5,t)$ were generated as the measurement data for the identification method. Figure 2 shows the necessary steps in applying the identification method to this example.

By applying the Laplace transform to the partial differential equation, an ordinary differential equation is obtained in terms of $\hat{u}(x,s)$ and x where s is the Laplace variable. If the actual measured internal

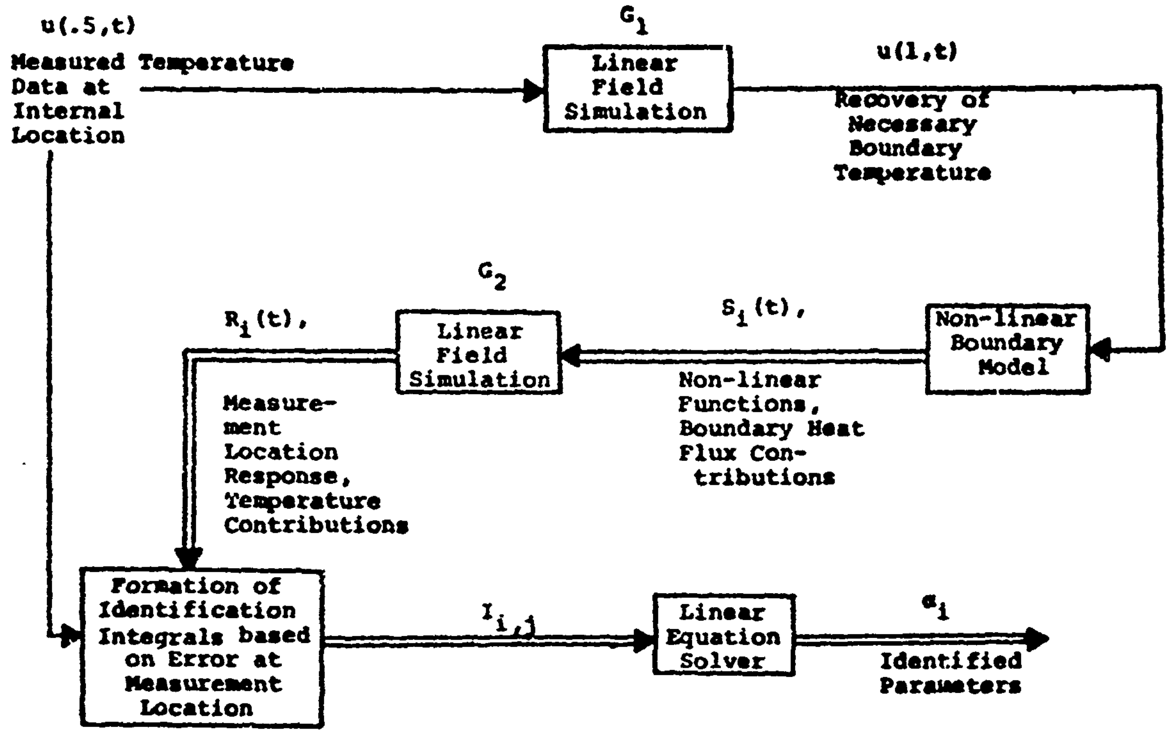

Figure 2 Boundary Condition Identification Procedure

data at $x = x_m$ is $u_m(t)$, then the spatial conditions used to recover $u(1,t)$ are Equation (2) and

$$u(x_m,t) = u_m(t) \tag{4}$$

Using these spatial conditions, a transcendental transfer function, G_1, is obtained relating $u(1,t)$ to $u_m(t)$

$$\hat{u}(1,s) = G_1[\hat{u}_m(s)] \tag{5}$$

To simulate this G_1 transfer function, defined in the Appendix, truncated infinite product expansions were used as proposed by Goodson[5]. To avoid noise problems due to lead network simulation involved in G_1, a tenth order least squares curve fit was used to smooth the input data without introducing phase shift. This transfer function was then digitally simulated using a matrix exponential routine with a tenth order hold on the input as developed by Krouse and Ward[6].

Having recovered boundary temperature, $u(1,t)$, the unknown boundary heat flux is formed as

$$q(1,t) = \beta S_1(t) + \eta S_2(t) \tag{6}$$

where $$S_1(t) = [(u(1,t)+1)^4 - \theta_c^4] \tag{7}$$

and $$S_2(t) = [(u(1,t)+1) - \theta_c] \tag{8}$$

Now the response at the measurement location can be estimated as a function of the unknowns β and η using $q(t)$ as a spatial condition. The spatial conditions for this simulation are Equation (2) and $q(1,t)$ in Equation (6). This yields

$$\hat{u}(x_m,s) = G_2[\hat{q}(1,s)] \tag{9}$$

Since $q(1,t)$ itself is not known, but $S_1(t)$ and $S_2(t)$ are known, the estimate of $u(.5,t)$ is expressed as a function of the unknown parameters β and η.

$$\hat{u}(.5,s) = \beta G_2[\hat{S}_1(s)] + \eta G_2[\hat{S}_2(s)] \tag{10}$$

To identify β and η, an error, $\epsilon(t)$, is defined as

$$\epsilon(t) = R_3(t) - [\beta R_1(t) + \eta R_2(t)] \tag{11}$$

where

$$\hat{R}_3(s) = \hat{u}_m(s), \text{ and } \hat{R}_i(s) = G_2[\hat{S}_i(s)] \quad i = 1,2 \tag{12}$$

In a manner similar to Rubin[7] and Kohr[8] an ISE performance index is minimized to find the "best" β and η.

$$IP = \frac{1}{2}\int_{t_a}^{t_b} \epsilon^2(t)\,dt \tag{13}$$

Minimization of the performance index results in a set of linear algebraic equations

$$\begin{aligned} \beta I_{11} + \eta I_{12} &= I_{13} \\ \beta I_{21} + \eta I_{22} &= I_{23} \end{aligned} \tag{14}$$

where

$$I_{i,j} = \int_{t_a}^{t_b} R_i(t)\,R_j(t)\,dt \tag{15}$$

Since a digital computer was used in this method, this set of linear algebraic equations was solved directly using a Gaussian elimination routine.

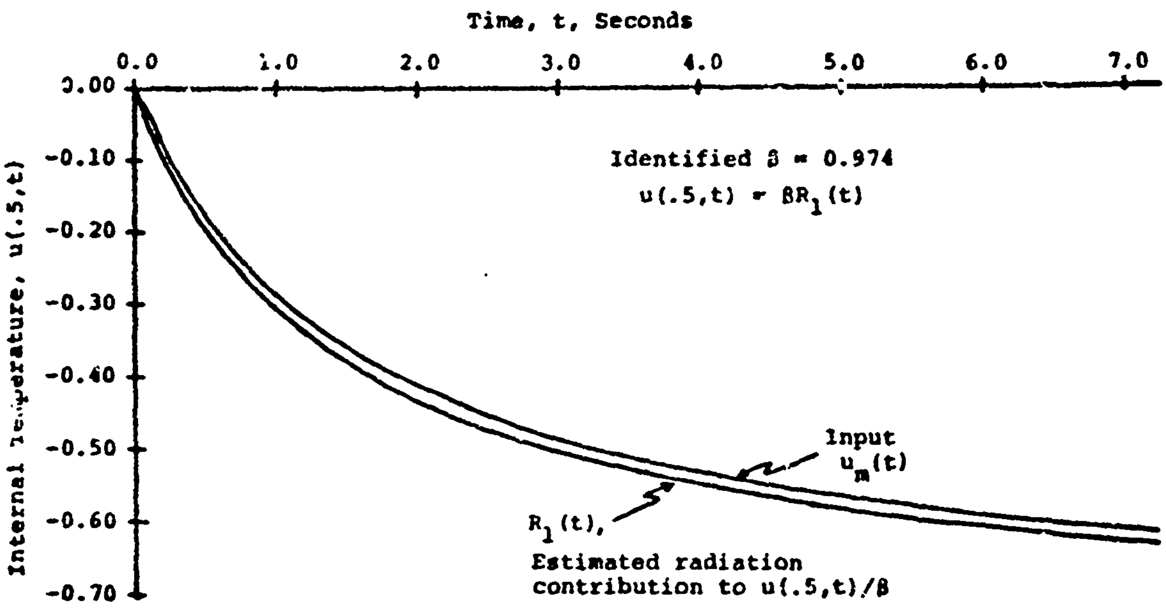

Figure 3 Simulated $R_1(t)$ Compared with Internal Input Temperature, $u_m(t)$
$x_m = 0.500$

Using data at $x_m = 0.500$ from Crosbie and Viskanta for radiation alone at the boundary ($\eta = 0, \beta = 1.0$) the simulated results in Figure 3 were obtained and $\beta = 0.974$ was identified. Thus the radiation coefficient was identified to within 3% of the correct value. Using this identified value of β the agreement with the actual measured data is quite good.

In Figure 4 the results are shown using data at $x_m = 0.500$ for combined radiation and forced convection at the boundary ($\beta = 1.0$, $\eta = 1.0$). In this case the sensitivity to the radiation contribution is much less than the convection contribution. Thus, the radiation coefficient, β, was identified only to within 20% of the correct value but the convection coefficient, η,

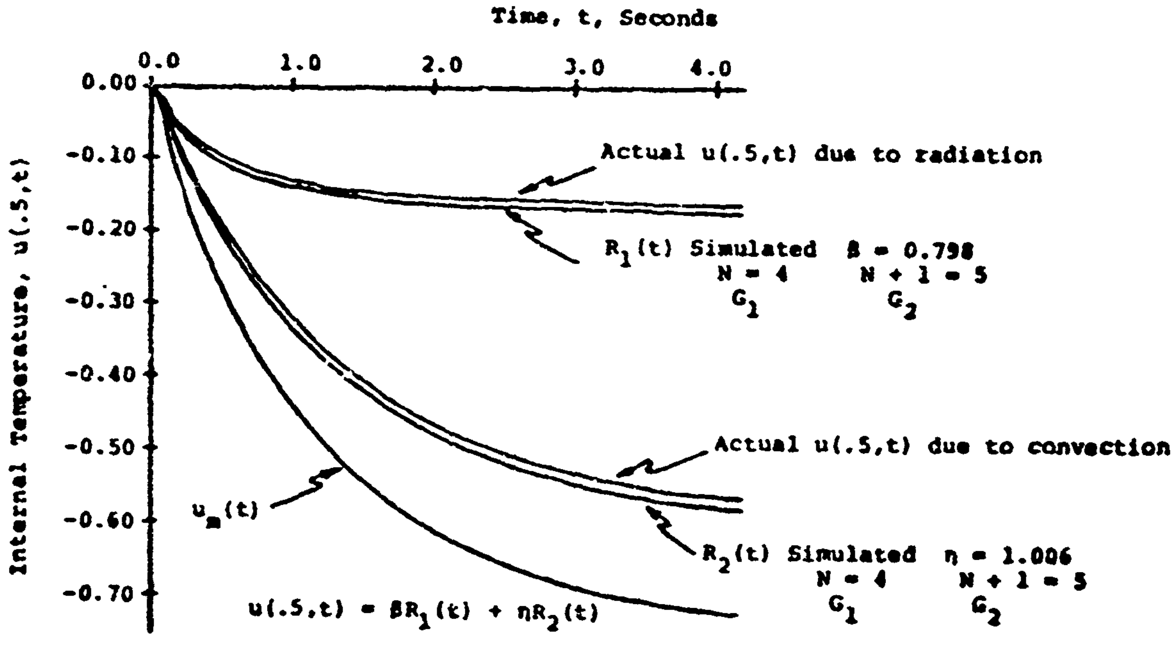

Figure 4 Simulated $R_1(t)$ and $R_2(t)$ Compared with Actual Values and Total Input Temperature, $u_m(t)$, at $x_m = 0.500$

was correct to within 1%. More importantly, the estimated u(.5,t) resulting from these identified parameters virtually coincides with the measured data. It was also observed that a higher order model for G_1 would allow better identification of β (within 5-10%) because of inclusion of less significant eigenvalue terms.

Engineering judgment must be used to select the order of the simulation model. A more complete and therefore more costly simulation model results in the more accurate quantitative parameter identification. However, a much simpler field model, while resulting in less accurate parameter identification, may still yield equally accurate state estimation results. In actual practice, the agreement between measured and estimated responses would generally be the final measure of a successful identification. Since the field dynamics are known, well known linear system techniques such as frequency response can be used to determine which eigenvalues are significant and therefore where the model should be truncated. A more complete consideration of field model selection is found in reference(2).

Example 2. Transverse beam vibration with a nonlinear support.

In the second example, a long slender beam, sketched in Figure 5, was excited in transverse motion by an electrodynamic shaker table clamped to the right end of the beam as shown in Figure 6 and Figure 7. The boundary condition at x = 0 was the unknown nonlinear one which was identified. Furthermore, the energy variables at this boundary could not be measured directly due to the interference of the knife edges. The input motion of the shaker table, z(t), was measured with a reluctance distance detector and the measurement internal to the field was obtained from a set of strain gages mounted to provide a temperature compensated measurement of bending moment.

A partial differential equation model for this system, neglecting shear and rotatory

Figure 6 Test Beam Mounted in Clamped-Unknown Boundary Configuration

inertia, is

$$\frac{\partial^4 y}{\partial x^4} + \frac{mL^4}{EI}\frac{\partial^2 y}{\partial t^2} + \frac{DL^4}{EI}\frac{\partial y}{\partial t} = 0 \qquad (16)$$

Equation (16) assumes, as a first approximation, that an equivalent viscous damping term is a valid approximation for internal structural damping. Since the system was started from rest both initial conditions

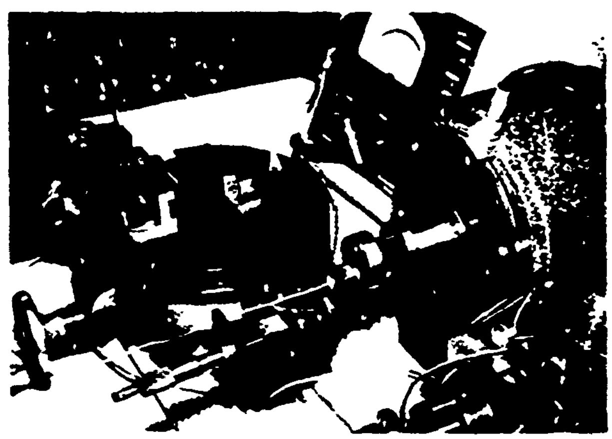

Figure 7 Beam Test Stand

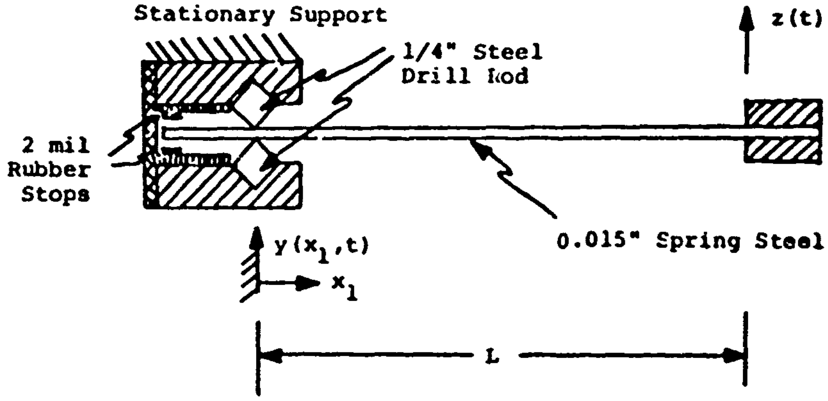

Figure 5 Test Beam Mounting Configuration

are zero. The known spatial conditions at $x = 1$ are

$$y(1,t) = z(t) \tag{17}$$

and

$$\frac{\partial y}{\partial t}(1,t) = 0 \tag{18}$$

At $x = 0$, the knife edges restricted vertical motion such that

$$y(0,t) = 0 \tag{19}$$

The problem is to formulate and identify the fourth remaining boundary condition.

Since the section of the beam inside of the boundary at $x_1 = 0$ was only 10 per cent of the total beam length, the displacement of the tip of the beam was assumed to be proportional to the slope at $x_1 = 0$, $\partial y/\partial x_1(0,t)$. Because of the eventual contact with the rubber stops, the force exerted on the tip of the beam would be similar to a hardening spring force. Furthermore, the moment at $x_1 = 0$, $EI\ \partial^2 y/\partial x_1^2(0,t)$, would be proportional to the force on the beam tip. Thus a cubic, hardening spring relationship was assumed for this unknown boundary condition model:

$$M(t) = \frac{\partial^2 y}{\partial x^2}(0,t) = \alpha_1 S_1(t) + \alpha_2 S_2(t) + \alpha_3 S_3(t) \tag{20}$$

where

$$S_1(t) = \frac{\partial y}{\partial x}(0,t) \tag{21}$$

$$S_2(t) = \left(\frac{\partial y}{\partial x}(0,t)\right)^2 \tag{22}$$

$$S_3(t) = \left(\frac{\partial y}{\partial x}(0,t)\right)^3 \tag{23}$$

The α_2 term allows for a nonsymmetrical boundary condition. This boundary model is able to fit a dead band reasonably well.

Having completed the important modeling step, the application of the identification

$$\alpha_1 = 2.513$$
$$\alpha_2 = -5.550$$
$$\alpha_3 = 218.50$$

$$P = \frac{1}{2}\int_{0.15}^{0.25} \varepsilon^2(t)\, dt = 1.839 \times 10^{-6}$$

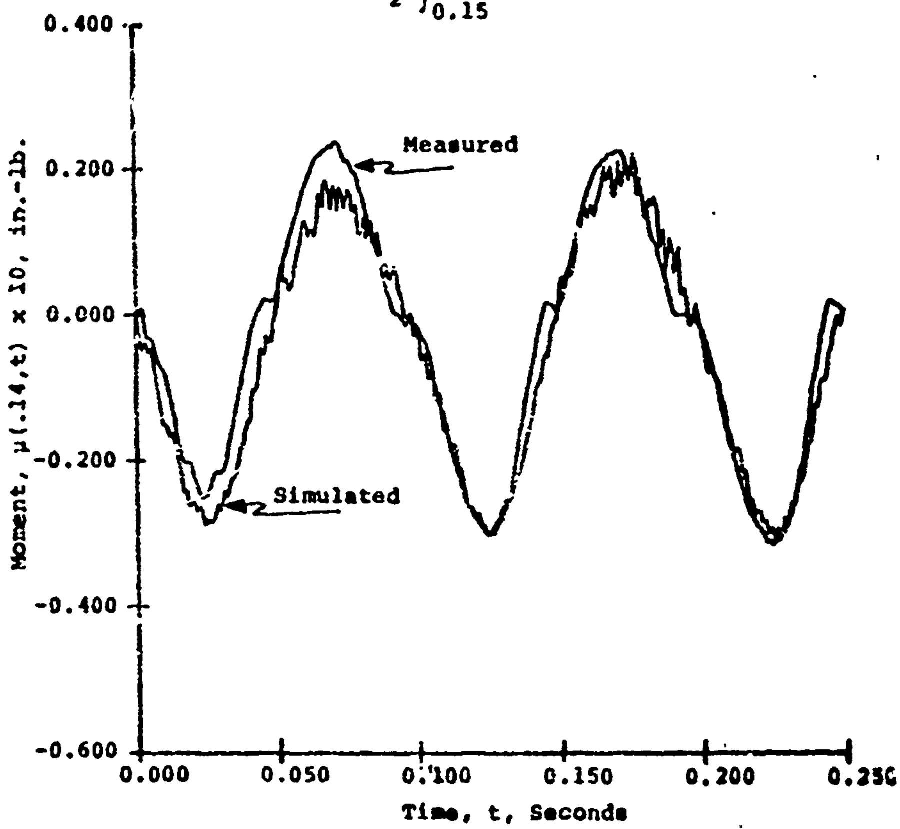

Figure 8 Simulated $\mu(.14,t)$ Given α's for $M = 4$, $\zeta_2 = 4\zeta_1$ Identification Model, $\zeta_2 = 4\zeta_1$ Damping Model

method proceeds in a fashion similar to Example 1. Laplace transforms are applied to the partial differential equation in (16) above. The three known spatial conditions and the internal measured bending moment data, $\mu_m(t)$, at $x = x_m$ were used to simulate the boundary slope at $x = 0$ yielding

$$\frac{\partial \hat{y}}{\partial x}(0,s) = G_1[\hat{z}(s)] + G_2[\hat{\mu}_m(s)] \qquad (24)$$

where (^) denotes Laplace transformed variables and again G_1 and G_2 are defined in the Appendix.

Having recovered the boundary slope using Equation (24), the unknown boundary condition, $M(t)$, is formed as a function of the unknown parameters using Equation (20). Using $M(t)$, and the three known boundary conditions, an estimate of the bending moment at $x = x_m$, $\mu(x_m,t)$, is simulated from:

$$\hat{\mu}(x_m,s) = G_3[\hat{z}(s)] + G_4[\hat{M}(s)] \qquad (25)$$

Now the error at $x = x_m$ is

$$\hat{\epsilon}(s) = \hat{\mu}_m(s) - (\alpha_1 G_4[\hat{S}_1(s)] + \alpha_2 G_4[\hat{S}_2(s)] + \alpha_3 G_4[\hat{S}_3(s)] + G_3[\hat{z}(s)]) \qquad (26)$$

Linear regression analysis using an ISE performance index again yields a set of linear algebraic equations which are solved for the desired α's as in Example 1.

Some difficulties were encountered during the simulation due to the equivalent viscous damping model selected in Equation (16). Thus, the damping ratio of the second mode was increased by a factor of four over the first mode damping ratio after comparison with experimental data. After the boundary parameters were identified a separate closed loop simulation was used to check the results against the actual measured data. These results are presented in Figure 8 for the identified boundary condition which is plotted in Figure 9. Although ringing still exists in the simulation model due to an imprecise damping model, it is significant to note that the identification method was nevertheless able to successfully match the peak amplitudes after the first two peaks. These peak amplitudes are important since they aid in predicting failure of a part. Distortion of the first two peaks was caused by FM tape recording of the measured data.

Summary and Conclusions

In summary, to implement the method presented in this paper, the user

1. Models the linear field with the known boundary conditions.
2. Formulates the unknown nonlinear boundary condition as an algebraic relationship linear in the unknown parameters.
3. Solves, by simulation, for the boundary variable necessary as input in (2) using internal measured data for the missing spatial condition in the model in (1).

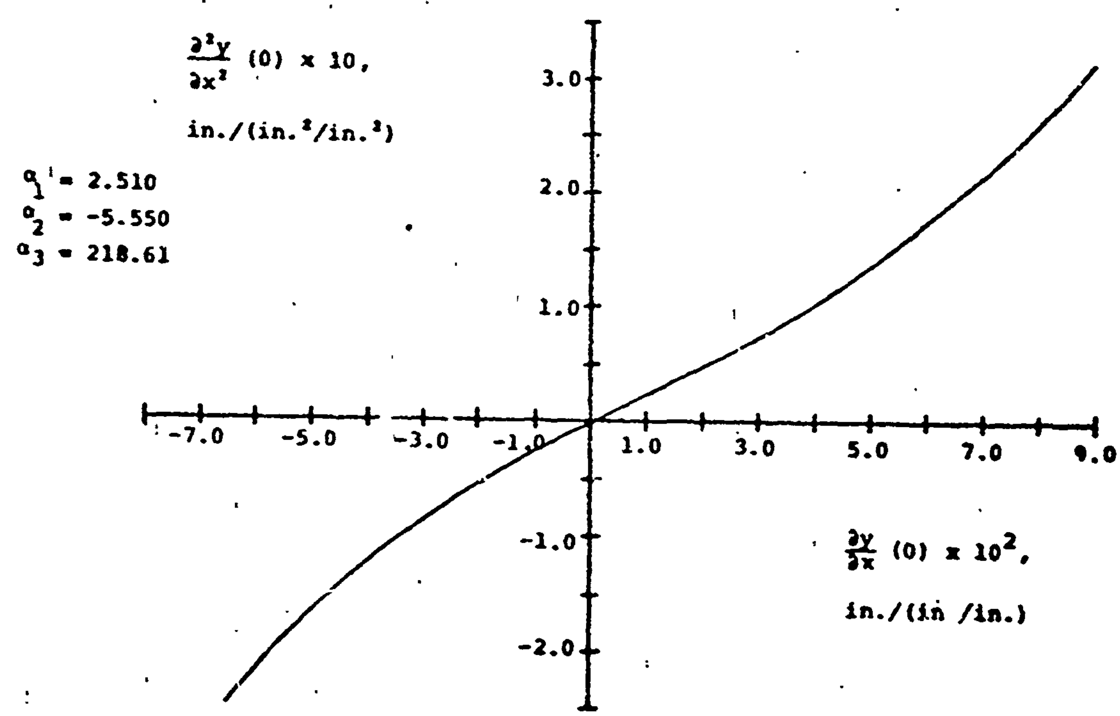

Figure 9 Plot of Identified Boundary, $\frac{\partial^2 y}{\partial x^2}(0)$ vs. $\frac{\partial y}{\partial x}(0)$, for Best Identified α's

4. Generates the unknown boundary output variable as a function of the unknown parameters using the boundary model in (2) and the results of (3).

5. Solves, by simulation, for the response at the original measurement location as a function of the unknown parameters using the results of (4) for the missing spatial condition in the model in (1).

6. Identifies the unknown parameters using regression analysis applied to an error between the actual measured data and the estimate in (5).

This parameter identification method was successfully applied to a heat conduction example using numerically generated data from the literature and to a beam equation example using experimentally measured data. In both cases the estimated response using the identified parameters was in good agreement with the actual measured data. In the heat conduction example, where a comparison with the correct parameter values was possible, the correct parameters were identified to within 3 per cent in one case and in the other case to within 1 per cent on one parameter and due to low sensitivity 10 to 20 per cent on the other parameter.

A particularly important result is the indication that a single measurement not at the boundary includes enough information for identification of parameters in the boundary given some knowledge of the field dynamics.

Acknowledgements

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References

1. Goodson, R.E. and Klein, R.E., "A Definition and Some Results for Distributed Systems," IEEE Transactions on Automatic Control, Vol. AC-15, No. 2, April 1970, pp. 165-174.

2. Ward, E.D., "Identification of Parameters in Nonlinear Boundary Conditions of Distributed Systems with Linear Fields," Ph.D. Thesis, Purdue University, August 1971.

3. Crosbie, A.L. and Viskanta, R., "Transient Heating or Cooling of One-Dimensional Solids by Thermal Radiation," 3rd International Heat Transfer Conference, Proceedings, Vol. V, August 7-12, 1966, p. 146.

4. Crosbie, A.L. and Viskanta, R., "Transient Heating or Cooling of a Plate by Combined Convection and Radiation," 3rd International Journal of Heat and Mass Transfer, Vol. 11, February 1968, pp. 305-317.

5. Goodson, R.E., "Distributed System Simulation Using Infinite Product Expansions," Simulation, Vol. 15, No. 6, December 1970, pp. 255-263.

6. Krouse, C.L. and Ward, E.D., "Improved Linear System Simulation by Matrix Exponentiation with Generalized Order Hold," Simulation, Vol. 17, No. 4, October 1971, pp. 141-146.

7. Rubin, A.I., "Continous Regression Techniques Using Analog Computers," IRETEC, Vol. EC-11, October 1962, pp. 691-699.

8. Kohr, R.H., "On the Identification of Linear and Nonlinear Systems," Simulation, March 1967, pp. 165-174.

APPENDIX

Transfer Functions and Infinite Product Approximations

Example 1 Heat Conduction

$$G_1 = \frac{\cosh\sqrt{s}}{\cosh\sqrt{s}\,x_m} \simeq \frac{\prod_{m=1}^{N-1}\left[1+\frac{s}{\left(\frac{2m-1}{2}\right)^2\pi^2}\right]}{\prod_{n=1}^{N}\left[1+\frac{s}{\left(\frac{2n-1}{2}\right)^2\frac{\pi^2}{x_m^2}}\right]} \tag{27}$$

For $x_m = 0.500$ and a fourth order model ($N = 4$),

$$G_1 \simeq \frac{\left(\frac{s}{2.47}+1\right)\left(\frac{s}{22.2}+1\right)\left(\frac{s}{61.7}+1\right)}{\left(\frac{s}{9.87}+1\right)\left(\frac{s}{88.8}+1\right)\left(\frac{s}{246.7}+1\right)\left(\frac{s}{483.6}+1\right)} \tag{28}$$

$$G_2 = \frac{\cosh\sqrt{s}\,x_m}{\sqrt{s}\,\sinh\sqrt{s}} \simeq \frac{\prod_{m=1}^{N-1}\left[1+\frac{s}{\left(\frac{2m-1}{2}\right)^2\frac{\pi^2}{x_m^2}}\right]}{s\prod_{n=1}^{N}\left[1+\frac{s}{n^2\pi^2}\right]} \tag{29}$$

For $x_m = 0.500$ and a fifth order model, ($N = 4$),

$$G_2 \simeq \frac{\left(\frac{s}{9.87}+1\right)\left(\frac{s}{88.8}+1\right)\left(\frac{s}{246.7}+1\right)}{s\left(\frac{s}{9.87}+1\right)\left(\frac{s}{39.4}+1\right)\left(\frac{s}{88.8}+1\right)\left(\frac{s}{157.8}+1\right)} \tag{30}$$

Example 2. Beam Equation

$$G_1 = \frac{\gamma[\sinh\gamma + \sin\gamma)(\sinh\gamma x_m + \sin\gamma x_m)]}{\text{Denom.}} - \frac{\gamma[(\cosh\gamma - \cos\gamma)(\cosh\gamma x_m + \cos\gamma x_m)]}{\text{Denom.}} \tag{31}$$

$$G_2 = \frac{\frac{2L^2}{EI\gamma}(1 - \cosh\gamma\cos\gamma)}{\text{Demon.}} \quad (32)$$

Denom. =

$$(\cosh\gamma(1-x_m) + \cos\gamma(1-x_m))(\sinh\gamma - \sin\gamma)$$

$$+(\sinh\gamma(1-x_m) + \sin\gamma(1-x_m))(\cos\gamma - \cosh\gamma) \quad (33)$$

and

$$\gamma^4 = -[\frac{mL^4}{EI}s^2 + \frac{DL^4}{EI}s] \quad (34)$$

For $x_m = 0.14$ and a fourth order model, infinite products yield[2]

$$G_1 = \frac{1.367\,(1 + \frac{\gamma^4}{1116.04})}{(1 - \frac{\gamma^4}{348.08})(1 - \frac{\gamma^4}{3947.53})} \quad (35)$$

$$G_2 = \frac{-0.316\,\frac{L^2}{EI}(1 - \frac{\gamma^4}{500.56})}{(1 - \frac{\gamma^4}{348.08})(1 - \frac{\gamma^4}{3947.53})} \quad (36)$$

$$G_3 = \frac{\gamma^2(\cos\gamma\sinh\gamma x + \cosh\gamma\sin\gamma x)}{(\sinh\gamma\cos\gamma - \cosh\gamma\sin\gamma)} \quad (37)$$

$$G_4 = \frac{(\cos\gamma - \cosh\gamma)(\sinh\gamma(1-x) + \sin\gamma(1-x))}{2(\sinh\gamma\cos\gamma - \cosh\gamma\sin\gamma)}$$

$$+ \frac{(\sinh\gamma - \sin\gamma)(\cosh\gamma(1-x) + \cos\gamma(1-x))}{2(\sinh\gamma\cos\gamma - \cosh\gamma\sin\gamma)} \quad (38)$$

For $x_m = 0.14$ and a fourth order model, infinite products yields

$$G_3 = \frac{-0.42\,EI/L^2\,(1 + \frac{\gamma^4}{25.36})}{(1 - \frac{\gamma^4}{237.72})(1 - \frac{\gamma^4}{2496.49})} \quad (39)$$

$$G_4 = \frac{0.790\,EI/L^2\,(1 - \frac{\gamma^4}{348.08})}{(1 - \frac{\gamma^4}{237.72})(1 - \frac{\gamma^4}{2496.49})} \quad (40)$$

<u>Nomenclature</u>

D	=	internal damping term $(lbf/(\frac{in}{sec}\,in)$
E	=	beam modulus of elasticty (psi)
G_1,G_2,G_3,G_4	=	transfer functions defined in Appendix
I	=	beam moment of inertia(in^4)
I_{ij}	=	identification integral
IP	=	ISE performance index
L	=	beam length (inches)
$M(t)$	=	signal proportional to boundary moment $(in/(\frac{in^2}{in^2}))$
m	=	beam mass/L $(\frac{lbf\text{-}sec^2/in}{in})$
$q(x,t)$	=	dimensionless heat flux
$R_i(t)$	=	measurement location response functions
$S_i(t)$	=	boundary input functions
s	=	Laplace variable
t	=	time (dimensionless in Example 1, seconds in Example 2)
$u(x,t)$	=	dimensionless temperature scaled to zero at $t = 0$
$u_m(t)$	=	measured dimensionless temperature data at $x=x_m$
x	=	dimensionless spatial distance
x_1	=	beam spatial distance (inch)
$y(x,t)$	=	beam deflection (inch)
$z(t)$	=	shaker table motion (inch)
α_i	=	boundary parameters to be identified in beam example
β	=	unknown radiation coefficient
γ	=	Laplace function defined by Equation (34)
ϵ	=	measurement location response error
η	=	unknown forced convection coefficient
θ_c	=	known constant dimensionless environment temperature
$\mu(x_m,t)$	=	$\frac{EI}{L^2}\frac{\partial^2 y}{\partial x^2}(x_m,t)$ = bending moment (in-lbf)
$\mu_m(t)$	=	measured bending moment at $x = x_m$