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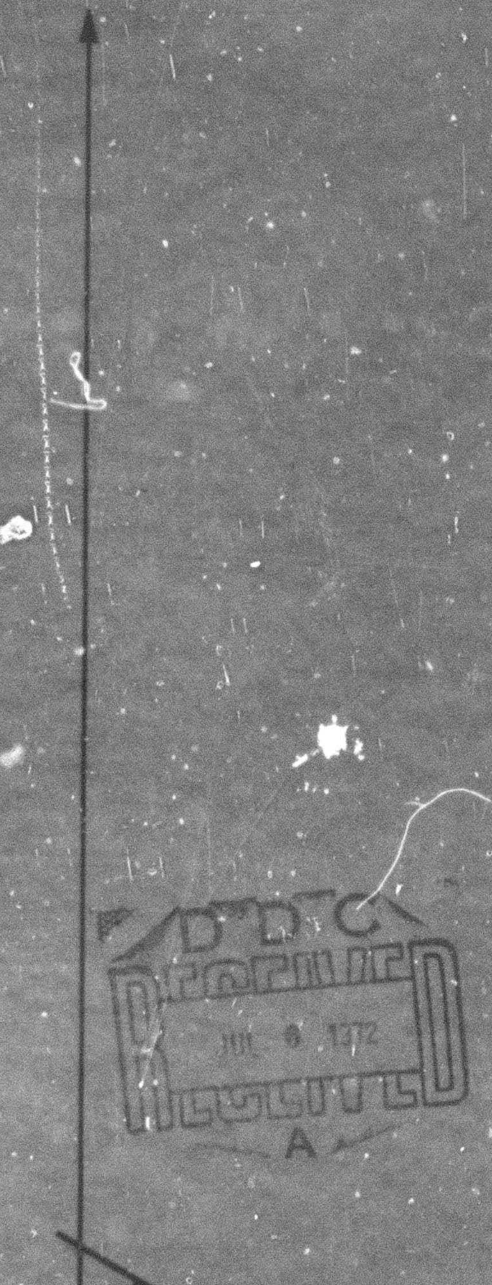
SOLVING THE FIXED CHARGE PROBLEM BY RANKING THE EXTREME POINTS

by

Katta G. Murty



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13. ABSTRACT

An algorithm for ranking the basic feasible solutions corresponding to a linear programming problem in increasing order of the linear objective function is described. An application of this algorithm for obtaining the minimal cost solution to a fixed charge problem is given. This algorithm can be applied in general to solve any fixed charge problem. However, the algorithm works efficiently when the problem is nondegenerate and the range in the values of the variable costs is large compared to the fixed charges. This algorithm can also be applied when the fixed charge part of the cost function is replaced by a concave function.

I

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II

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III

INTRODUCTION†

The fixed charge problem was formulated by G. B. Dantzig and W. Hirsch in 1954 [1]. It arises in situations which involve the planning of several interdependent activities, some or all of which have set-up charges (or fixed charges independent of the activity level as long as it is positive) associated with them. The problem may be formulated as follows:

$$\text{Min } \xi(x) = D(x) + Z(x)$$

$$\text{subject to } Ax = b$$

$$x \geq 0 \qquad (1)$$

where

$$Z(x) = \sum_{j=1}^{n} c_j x_j$$

$$D(x) = \sum_{j=1}^{n} d_j (1 - \delta_{0,x_j})$$

and

$$\delta_{0,x_j} = 1 \text{ if } x_j = 0$$

$$= 0 \text{ if } x_j > 0 \quad .$$

$A_{m \times n}$, $b_{m \times 1}$, $c_{1 \times n} = (c_1, \ldots, c_n)$, $d_{1 \times n} = (d_1, \ldots, d_n)$ are given real matrices and $x_{n \times 1} \in R^n$. Corresponding to any feasible solution x , $D(x)$ is known as the fixed charge component of the cost and $Z(x)$ the variable cost.

G. B. Dantzig and W. Hirsch have shown that min $\xi(x)$ is attained at an extreme point of the convex polyhedral set determined by (1) (See [1,2]). [3,4,5] discuss some approximative algorithms for solving the fixed charge problem, especially when the underlying structure of the restrictions (1) is of the transportation type.

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† The author is indebted to Professor R. Van Slyke, Mr. R. Chandrasekaran, and Professor Alan S. Manne for their suggestions and criticisms.

The algorithm described in this paper applies in general to any fixed charge problem. Since only the extreme points of (1), which are finite in number, have to be scanned, it leads to the optimal solution in a finite number of steps. However, the algorithm works efficiently when (1) is nondegenerate and the range in the value of $Z(x)$ for feasible x is large compared to the fixed charges.

Algorithm for the Fixed Charge Problem Assuming that the Vertices of (1) can be Ranked in Increasing Order of the Variable Costs Z(x)

An algorithm for ranking all the vertices of (1) in increasing order of the linear functional $Z(x)$ is given in Section 2. For an application of this algorithm we have to assume that $\min\limits_{[x|(1)]} Z(x)$ is finite. Here, $\min\limits_{[x|(1)]}$ indicates that the minimization is over all x satisfying (1).

Case 1: $\min\limits_{[x|(1)]} \xi(x) = -\infty$.

Assuming that d is finite, it is clear that $Z(x)$ is unbounded below. Hence by Problem 19, page 146 of [6], we know that $\min\limits_{[x|(1)]} \xi(x) = -\infty$ iff the system of equations

$$\begin{aligned} Ax &= 0 \\ x &\geq 0 \\ cx &< 0 \end{aligned} \qquad (2)$$

has a feasible solution.

Hence in all subsequent discussions we shall assume that $\min\limits_{[x|(1)]} Z(x) > -\infty$ and hence that $\min\limits_{[x|(1)]} \xi(x) > -\infty$.

Case 2: $\min\limits_{[x|(1)]} \xi(x) > -\infty \Rightarrow \min\limits_{[x|(1)]} Z(x) > -\infty$.

So in this case it is possible to rank all the extreme points of (1) in increasing order of $Z(x)$.

Let $S_1, S_2, \ldots, S_k, \ldots$ be such a ranking and let $Z(S_k) = Z_k$. Then we have $Z_1 \leqq Z_2$. Let $\Delta_k = Z_k - Z_1$ and let $D_k = D(S_k)$. Let D_0 be a lower bound on the fixed charge component of the total cost at any vertex of (1); i.e., $D_0 \leqq D_k \ \forall k$ method for obtaining D_0 is discussed at the end of this section. The efficienty of the algorithm improves with the nearness of D_0 to the greatest lower bound of $D_k \ \forall k$ Suppose we have determined some S_r. Then it is clear that the optimal solution to the fixed charge problem must be one of the vertices $S_1, \ldots, S_{k_r}$ where k_r is such that

$$Z_{k_r} - Z_r \leqq D_r - D_0$$

and

$$Z_{k_r+1} - Z_r > D_r - D_0 \quad . \tag{3}$$

$\because$ for any $k > k_r$ we have

$Z_k + D_k = Z_r + (Z_k - Z_r) + D_k > Z_r + D_r + (D_k - D_0)$ by (3). And since we know by the choice of D_0 that $D_k \geqq D_0 \ \forall k$

$$Z_k + D_k > Z_r + D_r \quad \text{for} \quad k > k_r \quad .$$

Hence it is not necessary to rank all the extreme points of (1) to solve the fixed charge problem. As soon as S_1 is found we get an upper bound on the extent of the values of $Z(x)$ to which we may have to carry on the ranking by using the above result.

In general, suppose we have determined S_r. Then it may be necessary to rank the extreme points of (1) to the extent that $Z(x) \leqq Z_r + D_r - D_0$. Now the stages in the algorithm can be described.

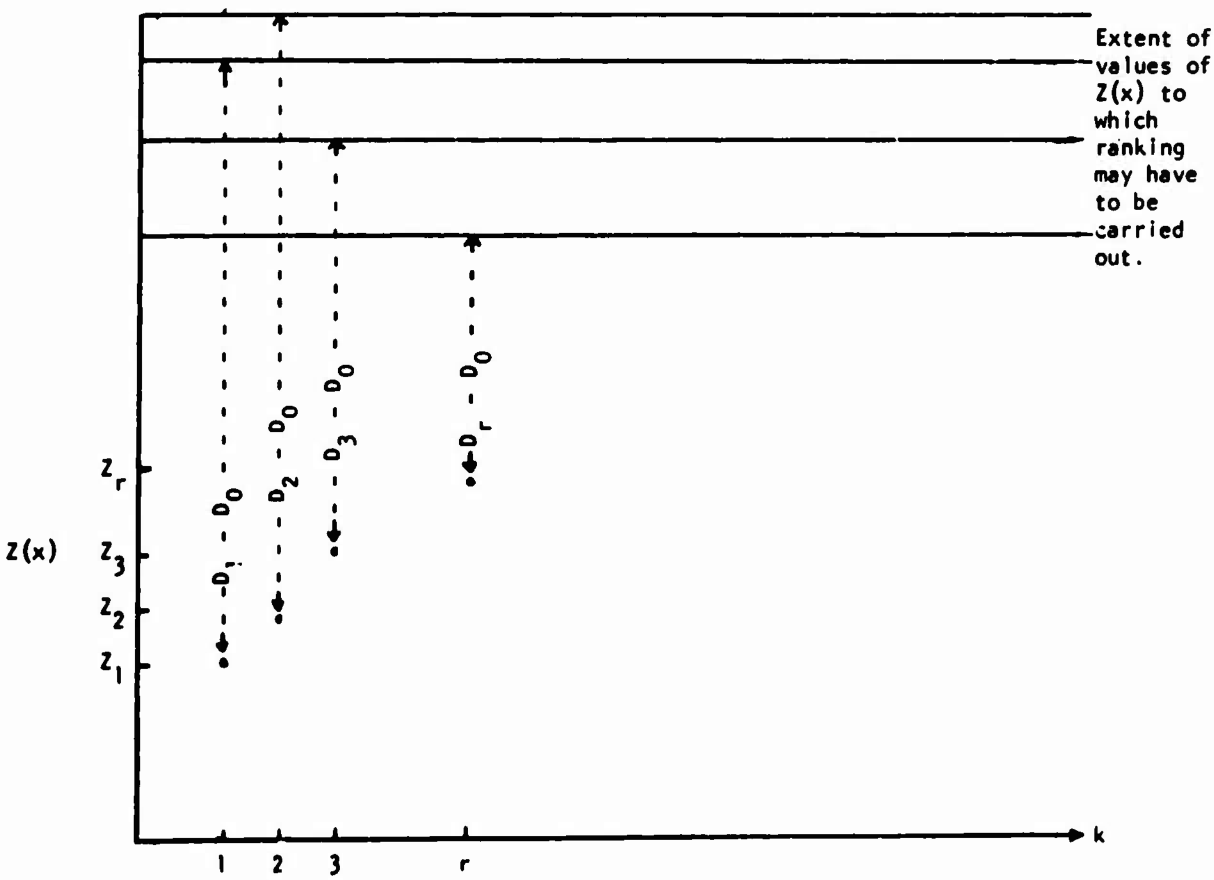
<u>Stage r</u>: In this stage $S_1, \ldots, S_r$ have been determined. Let

$$\delta_r = \min_{k=1,\ldots,r} [\Delta_k + D_k - D_0] \quad .$$

If $\Delta_r \geqq \delta_r$, the algorithm terminates and the optimal solution is given by the extreme point corresponding to

$$\min_{k=1,\ldots,r} [Z_k + D_k] \quad .$$

If $\Delta_r < \delta_r$, then it is necessary to determine Z_{r+1} and proceed to stage $r+1$. However, we know that the ranking algorithm has to be carried on to find out vertices of (1) only to the extent of $Z(x) \leqq Z_1 + \delta_r$. The various stages of the algorithm may be represented geometrically as follows:



The lines parallel to the k-axis on the diagram indicate the extent of the value of $Z(x)$ to which ranking may have to be carried out. At each stage, the horizontal line nearest the k-axis applies. This limit is improved at each stage.

<u>To Find</u> D_0 , <u>a Lower Bound on the Fixed Charge Component of Cost at any Vertex of (1)</u>

Suppose the variables $x_1,\ldots,x_n$ are arranged in increasing order of the value of d_j , i.e.,

$$d_1 \leqq d_2 \cdots \leqq d_n \quad .$$

Each extreme point of (1) consists of m basic variables, but some of them may be at zero values if the problem is degenerate. If we know that there is no degeneracy in the problem we can take

$$D_0 = d_1+\ldots+d_m \quad .$$

Even if (1) is degenerate, it may be possible to obtain a lower bound, m_1 , on the number of basic variables which are positive at any vertex of (1). If (1) is not totally degenerate (i.e., $b \neq 0$), then none of its canonical equivalents can be totally degenerate anyway. This may help us to get some lower bound for m_1 .

If the constraints of (1) are of the transportation type it is very easy to determine m_1 . Then if all $d_j \geq 0$, we can take

$$D_0 = d_1+\ldots+d_{m_1} \quad .$$

A crude value for D_0 is 0 when all $d_j \geq 0$.

<u>Corollary 1</u>: The algorithm works equally efficiently if we replace $D(x)$ be any concave function and D_0 by a lower bound to $\min_{\{x|(1)\}} D(x)$.

Then $\xi(x) = Z(x) + D(x)$ is also a concave function and it is a well-known result that the minimum of a concave objective function over a convex set occurs at an extreme point.

II. An Algorithm for Ranking the Vertices of (1) in Increasing Order of Z(x)

Consider the linear programming problem in its standard form

$$\begin{aligned} \min \quad & Z = cx \\ \text{subject to } & Ax = b \\ & x \geq 0 \quad . \end{aligned} \tag{1}$$

We shall assume that this problem has a finite optimum, i.e., that $\min_{[x|(1)]} Z(x) > -\infty$. Then it is well known that there exists a vertex of (1) which is optimal for the above problem.

The algorithm developed here is an extension of the simplex algorithm. It helps in ranking the basic feasible solutions of (1) in increasing order of Z, after the optimal is obtained by the simplex method. It uses only one step pivot operations.

Let the letters B and T , with any subscripts or superscripts if necessary, denote basic feasible solutions of (1). Suppose in any basic feasible solution B the variables $x_{r_1},\dots,x_{r_m}$ are basic. We shall indicate this by

$$x_{r_i} \in B \quad i = 1,\dots,m$$

and

$$B = \{x_{r_1},\dots,x_{r_m}\} \quad .$$

Here we are defining a basic feasible solution by the set of variables which are basic in it.

Let S_1, S_{max} denote the minimal cost (w.r.t. Z(x)) basic feasible solution and the maximal cost basic feasible solution respectively. We have assumed that S_1 exists.

Consider any basic feasible solution B . Corresponding to any nonbasic variable $x_j \notin B$, let

$\bar{c}_j^B$ = the relative cost coefficient of the nonbasic variable x_j corresponding to the basic feasible solution B .

θ_j^B = the value with which the nonbasic variable x_j enters the basis in the canonical form of the basic feasible solution B .

T_j^B = the new basic feasible solution obtained by pivoting on the column of x_j in the canonical form of B .

From the simplex algorithm

$$Z(T_j^B) = Z(B) + \theta_j^B \bar{c}_j^B \quad .$$

The basic solutions T_j^B for j such that $x_j \notin B$ are adjacent vertices of the vertex B .

However, when (1) is degenerate, several basic feasible solutions may represent the same vertex and all the adjacent vertices of this vertex are given by the basic feasible solutions, T_j^B , corresponding to the various canonical forms B which represent the same vertex.

Let $\emptyset(B)$ denote the set of all the adjacent vertices of B with cost value not less than that of B, i.e.,

$$\emptyset(B) = \{T_j^B \ \forall j \text{ such that } x_j \notin B \ , \ \bar{c}_j^B \geq 0\} \quad . \tag{4}$$

It can be seen that the canonical form corresponding to any of the adjacent vertices of B can be obtained by pivot operations on the canonical form of B . If (1) is nondegenerate, then corresponding to each vertex of the polyhedron there exists a unique basis B which represents it, and equation (4) holds for each individual basis.

However, if B is a degenerate basic feasible solution of (1), let V_B denote the vertex represented by it. Let $B_1,\ldots,B_r$ be all the basic feasible solutions of (1) that represent the same vertex V_B . Then we should replace equation (4) by

$$\emptyset(V_B) = \bigcup_{p=1}^{r} \{T_j^{B_p}\ \forall j \text{ such that } x_j \notin B_p \text{ and } \bar{C}_j^{B_p} \geq 0\} \quad . \tag{4a}$$

where $\emptyset(B)$ of (4) and $\emptyset(V_B)$ of (4a) represent the set of all adjacent vertices of the vertex represented by the basic feasible solution B whose cost value is not less than that of B.

To Get all the Basic Feasible Solutions Representing a Degenerate Vertex

When the vertex V_B is degenerate, the canonical forms of all the basic feasible solutions $B_1,\ldots,B_r$, which represent it, may be obtained by looking at the canonical form of any one of them and then pivoting among the non-zero input-output coefficients in the rows corresponding to the basic variables which are zero.

Ranking the Vertices of (1)

Let $S_1,S_2,\ldots$ be a ranking of the basic feasible solutions of (1) in increasing order of Z . Suppose we already know the basic feasible solutions $S_1,S_2,\ldots,S_{k-1}$ in the sequence. It is intuitively clear that the next element in the sequence, S_k , must be a cost nondecreasing adjacent vertex of one of the vertices represented by the known basic feasible solutions $S_1,\ldots,S_{k-1}$. We shall prove this.

Proposition 1: Every basic feasible solution can be reached by taking a cost nondecreasing path from S_1 through the vertices of (1) .

PROOF: Consider any basic feasible solution B . From the proof of the simplex algorithm we know that there exists a cost nonincreasing path moving along adjacent vertices from B to S_1 . By taking the same path in the reverse direction from S_1 , we reach B from S_1 by moving along adjacent vertices along a cost nondecreasing path.

Proposition 2: Suppose $S_1,\ldots,S_{k-1}$ are already known. Let us define

$$\emptyset_p = \bigcup_{i=1}^{p-1} \emptyset\{V_{S_i}\} - \{S_1,\ldots,S_{p-1}\} \quad p = 2,3,\ldots \quad . \tag{5}$$

Then S_k = minimal cost solution in $\emptyset_k$.

By Proposition 1, S_k must be a cost nondecreasing adjacent vertex of one of the vertices $S_1,\ldots,S_{k-1}$. But S_k is the minimal cost vertex after $S_1,\ldots,S_{k-1}$ are excluded. Hence S_k = minimal cost solution in $\emptyset_k$.

Now the algorithm can be given. The method starts with the finding of S_1 by the simplex method.

<u>General Step</u>: Suppose $S_1,\ldots,S_{k-1}$ have already been obtained and we are trying to find out S_k . Then S_k is the minimal cost basic feasible solution among

$$\bigcup_{i=1}^{k-1} \{T_j^{S_i}: \ j \text{ such that } x_j \notin S_i \text{ and } \bar{C}_j^{S_i} \geq 0\} - \{S_1,\ldots,S_{k-1}\}$$

$$= \bigcup_{i=1}^{k-1} \emptyset(S_i) - \{S_1,\ldots,S_{k-1}\} \qquad . \qquad (6)$$

Of course, if any of S_i are degenerate we should replace $\emptyset(S_i)$ by $\emptyset(V_{S_i})$ as in equation (4a).

Thus S_k can be easily located by examining the values $Z(T_j^{S_i})$ for $i = 1,\ldots,k-1$ and j such that $x_j \notin S_i$ and $\bar{C}_j^{S_i} \geq 0$. S_k is that new basic feasible solution in (6) which is distinct from $S_1,\ldots,S_{k-1}$ and which has least cost value $\geq Z_{k-1}$. The algl-rithm is stepwise and in each step we determine an additional element in the sequence of ranked vertices $S_1, S_2,\ldots$.

<u>To Organize Computations</u>: Computationally, this may be done by storing at:

<u>Array 1</u>: All the $Z(T_j^{S_i})$ values for all S_i determined so far, $\forall j$ such that $x_j \notin S_i$ and $\bar{C}_j^{S_i} \geq 0$ and $T_j^{S_i} \neq$ any of the known S_i so far Of course, when any of S_i are degenerate, we should scan all basic feasible solutions which represent that same vertex.

Array 2: All the basic feasible solutions that have already been found and ranked. Each of the S_i's may be stored in terms of the subscripts of the basic variables in it, arranged in increasing order.

Array 3: The basic feasible solutions $T_j^{S_i}$ corresponding to the Z-values stored in Array 1. Whenever a basic feasible solution is to be stroed, store the subscripts of the basic variables in it in increasing order.

It is convenient to locate Array 1 and Array 2 in core memory, and Array 3 on tape. The computations required to get the next element in the sequence, i.e., S_k , are

i) to scan Array 1 completely and then determine the least value there;

ii) to identify the corresponding basic solution from Array 3. This is S_k . The values of the basic variables in S_k may be obtained by referring to the restrictions (1). If it is required to find out some more elements in the sequence, then

iii) delete $Z(S_k)$ from Array 1, S_k from Array 3, and add S_k to Array 2.

iv) find out the canonical form of S_k and all its cost nondecreasing adjacent vertices, i.e., $\emptyset(S_k)$ (or $\emptyset(V_{S_k})$ if S_k is degenerate). Store these basic feasible solutions at Array 3 and their Z-values at Array 1.

If the problem is only to rank all basic feasible solution for which $Z \leq \alpha$, then we can save space by storing in Arrays 1 and 3 only those solutions for which $Z \leq \alpha$.

A Numerical Example: We apply the algorithm to the following fixed cost transportation problem.

							$\sum_j x_{ij}$
$d_{ij}=6$ / $c_{ij}=16$	8 / 13	0 / 12	3 / 6	7 / 24	4 / 19	19 / 20	23
35 / 17	4 / 40	5 / 15	1 / 8	26 / 13	10 / 11	2 / 5	26
9 / 19	11 / 109	24 / 8	16 / [illegible]	2 / 26	5 / 5	4 / 25	38
12 / 92	36 / 29	6 / 2	31 / 20	19 / 42	8 / 6	5 / 17	75
6 / 23	9 / 27	10 / 14	5 / 17	43 / 114	12 / 38	18 / 26	56
$\sum_i x_{ij}$: 22	9	35	54	8	55	35	

Table 1

to minimize $\xi(x) = \sum_i \sum_j d_{ij}(1 - \delta_{0,x_{ij}}) + \Sigma\Sigma C_{ij}x_{ij}$ subject to row and column sum constraints.

As before, let $Z = \sum_i \sum_j C_{ij}x_{ij}$. Let us rank the extreme points of the transportation problem with respect to Z . On solving the transportation problem we find that min $Z = Z_1 = 2214$ and the fixed charge corresponding to this is $D_1 = 83$.

We know that in any basic feasible solution, at least 7 of the x_{ij}'s must be positive. Hence we can take for D_0 the sum of the least seven of the fixed charge cost coefficients = 16.

$$\therefore \qquad D_1 - D_0 = 67$$

and hence it may be necessary to rank the extreme points of the transportation problem only to the extent of $Z \leq 2214 + 67 = 2281$. The tableau corresponding to Z_1 is given below.

$C_{ij} = 4$ $Z(T_j^B) = 2270$	$x_{ij} = 9$	18	14	7 2270	21	11
9 2268	31	25 2250	6	8	17	18
6	89	7	16	2 2230	32	9
72	8 2262	35	6 2250	17	23	17
16	3 2241	9	40	86	29	6

$Z_1 = 2214$, $D_1 = 83$ $\quad \epsilon = 2297$

Solution S_1

Only x_{ij}-values corresponding to the basic cells are recorded in the middle of the cell. The $Z(T_j^{S_1})$ values are recorded only in those nonbasic cells where it is ≤ 2281, since we are only interested in extreme points which have $Z \leq 2281$.

Using equation (6) we find that S_2 can be obtained by introducing x_{35} into the basis.

$\bar{c}_{ij} = 4$	9	18	14	5	21	11
9	31	25	6	-2	17	26
6	89	7	16	8	24	9
72	8	35	6	15	31	9
16	3 $Z(T_j^{8}) = 2257$	9	40	84	29	6

$Z_2 = 2230$ $D_2 = 46$ $\xi = 2276$

Solution S_2

Now $D_2 - D_0 = 30$ and hence it is necessary to rank the extreme points only to the extent that $Z \leq 2230 + 30 = 2260$. In the tableau corresponding to S_2, only those non-basic cells which lead to $Z(T_j^{S_2}) \leq 2260$ have been marked.

Using equation (6) again, we find that S_3 is obtained by introducing x_{52} into the basis in the tableau of S_1.

C_{1j} = 4	-3	6	23	7	21	11
9	28	13	6	8	17	18
6	86	-5	16	2 $Z(T_j^B)=2257$	32	9
72	5	35	6	17	23	17
16	9	-3	31	86	29	6

$Z_3 = 2241$ $\qquad$ $D_3 = 84$ $\qquad$ $\xi = 2325$

Solution S_3

Using equation (6) again, we find that there is a tie for the next position in ranking. S_4, S_5 are obtained by introducing x_{24} and x_{44} respectively into the basis of S_1 .

$C_{ij}=4$	9	24	14	13	27	17
3	25	25	6	8	17	12
-6	83	7	10	2	38	9
66	12	35	0 $Z(T_j^B)=2250$	17	17	23
22	3	15	34	92	35	12

$Z_4 = 2250$ $D_4 = 75$ $\xi = 2325$

Solution S_4

$C_{ij} = 4$	9	24	14	13	27	17
3	25	25	0	8	17	18
-6	83	7	10	2	38	9
66	2	35	6	17	17	17
22	3	15	34	92	35	12

$Z_5 = 2250$ $D_5 = 80$ $\xi = 2330$

Solution S_5

Using equation (6) again, we find that S_6 is obtained by introducing x_{52} into the basis of S_2 .

$c_{ij} = 4$	-3	6	23	3	21	11
9	28	13	6	-4	17	26
6	86	-5	16	8	24	9
72	5	35	6	13	31	9
16	9	-3	31	79	29	6

$Z_6 = 2257$ $D_6 = 60$ $\rho = 2317$

Solution S_6

With this, all the extreme points with $Z \leq 2260$ has been ranked and hence the algorithm terminates. By inspection we find that S_2 gives the optimal solution to the fixed problem.

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