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A DETERMINISTIC MULTIECHELON INVENTORY MODEL

by

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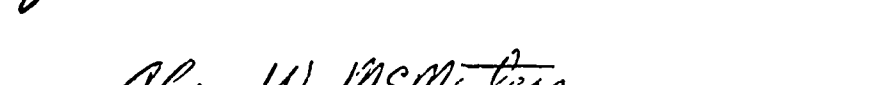
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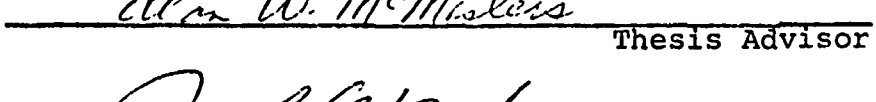
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I. INTRODUCTION

Traditional inventory analysis addresses the single echelon inventory problem. In this problem, customer demands are received and issues are made from a single outlet. The outlet in turn is replenished from a single source.

Unfortunately, most real world inventory systems are not this simple. A major manufacturer, for example, usually has a production and distribution system that includes (1) factories, (2) factory warehouses, (3) regional warehouses, (4) local warehouses, and (5) retail outlets [Hadley and Whitin]. Such a system is an example of a multiechelon inventory system. Each level of the system is a separate, distinct echelon. Figure 1 depicts graphically this organization. The United States Navy with its organization of inventory control points, supply centers, supply depots, shop stores, tenders and underway replenishment ships provides an example of a large multiechelon inventory system.

In defending their single echelon models Hadley and Whitin argue that, even though most real world systems are multiechelon, it is often true that the system need not be treated as multiechelon. They contend that a different organization frequently operates each level (echelon) of the system. In the example of Figure 1, a factory and its associated warehouses might be operated by an equipment manufacturer whereas the regional warehouses might be separate privately owned

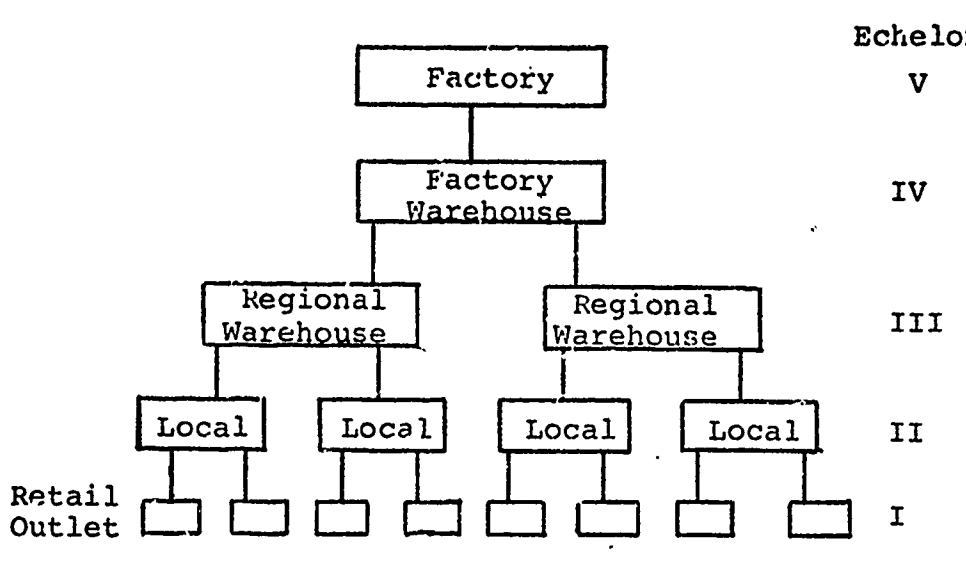

Figure 1. Typical Multiechelon Inventory System

distributorships not under the administrative control of the manufacturer. In like manner, the local outlets might be individually owned retail stores. Thus we would have a system comprised of several single echelons which are linked by a physical dependence but separated by administrative controls.

While it is clear that there are many examples of multiechelon systems that can be treated as a series of independent single echelon problems, it should be equally clear that there is a large class of systems which must be treated as strongly dependent multiechelon.

It is the objective of this thesis to develop and investigate the behavior of multiechelon models constructed from the classic single echelon inventory models. Specifically, the following deterministic single-echelon models of Hadley and Whitin will be restructured as two-echelon models:

a. No stockouts allowed (Economic Order Quantity),

b. Backorders allowed,

c. Finite production with no stockouts allowed.

Like their single echelon counterparts, these two-echelon models will be single item, single source models.

II. THE TWO ECHELON MODEL WITH NO STOCKOUTS

In any study of inventory theory, one of the first models investigated is the deterministic demand model with no stockouts allowed; often referred to as the Economic Order Quantity (EOQ) model. While it is true that complete deterministic demand is almost never known, it is felt that the mathematics of this model provide a good starting point for any inventory analysis. Further, it is felt that the deterministic model will provide an insight into the operation of a stochastic demand model. It is considered appropriate, therefore, to begin this multiechelon analysis by considering a two echelon extension of the classic EOQ model.

A. MODEL FORMULATION

The two echelon model is based on the following assumptions:

1. The upper echelon always replenishes its supply from the same outside source. The lower echelon always replenishes its supply from the upper echelon.

2. The upper echelon is always able to meet the demand of the lower echelon.

3. External customer demand always occurs at the lower echelon.

4. The external demand rate is deterministic, continuous, and constant with a value of λ units per year.

5. Neither backorders nor lost sales are allowed.

6. Procurement lead time (PLT) is negligible.

7. The order quantity at the upper echelon, Q2, is an integer multiple of the lower echelon order quantity, Q1. Let $n = Q_2/Q_1$; then this assumption requires $n \geq 1$ and integer.

The optimal policy throughout this thesis will be that which minimizes average annual variable system cost, subject to the constraint(s) of the model. The form of the average annual variable system cost will be developed by first determining the system cost per cycle, and then dividing this cost by the cycle length. The total system variable cost per cycle is the sum of the individual echelon variable cycle costs, $K_1(c)$ and $K_2(c)$. The cycle length is defined as the time between receipt of two successive orders at the upper echelon (echelon two).

The cost per cycle at echelon two is comprised of an ordering cost, an inventory holding cost (IHC) and a purchase cost. The ordering cost is assumed to be a constant cost per order which includes the administrative costs associated with inventory review and order (contract) preparation. This cost is independent of the quantity on hand or on order. The IHC includes a warehousing cost, an obsolescence cost and a foregone opportunity cost. It is assumed to be a function of the inventory on hand and to be expressable by

$$IHC = I_2 C \int_0^T \mathcal{I}_2(t)\,dt,$$

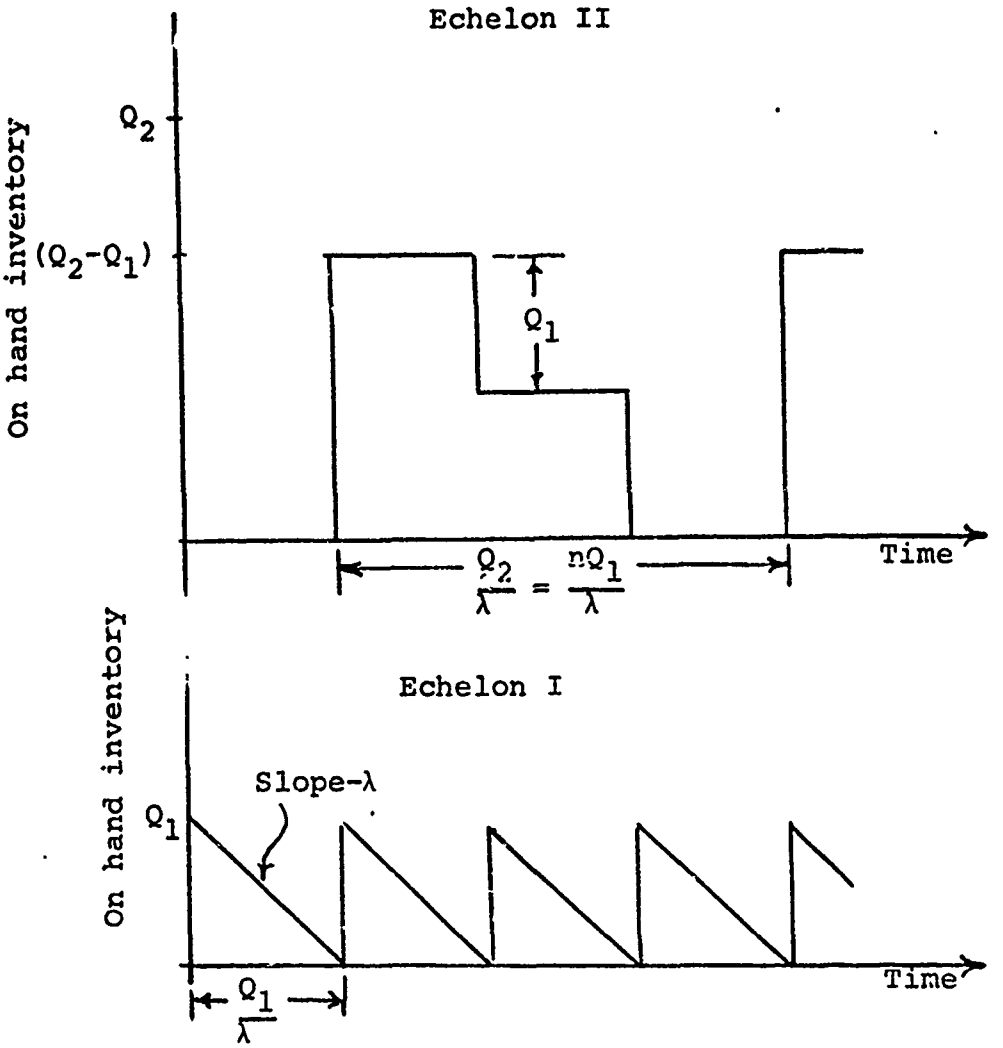

Figure 2. Two-Echelon EOQ Model.

where I_2 is the average cost per dollar invested in inventory per unit time, C is the item unit cost, T is the cycle length, and $\ell_2(t)$ is the on hand inventory at echelon two at time t. The purchase cost is assumed to be independent of the quantity produced, and can be expressed as Q_2C where Q_2 is the quantity procured.

The variable cost per cycle at echelon two can now be expressed as

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$$K_2(c) = A_2 + CQ_2 + I_2C \int_0^T \ell_2(t)dt \qquad (2\text{-}1)$$

The height of the on-hand inventory at echelon two is

$$Q_2 - Q_1 = [n-1]\ Q_1$$

at the beginning of the cycle since Q_2 is ordered, arrives immediately, and is partially used to meet an immediate demand of Q_1 units. The next demand at echelon two is for an amount Q_1 which occurs Q_1/λ time units later. Similarly, another demand of Q_1 occurs after Q_1/λ additional time units have passed. The on-hand level at echelon two continues to decrease in steps of Q_1 until it reaches zero. It remains at zero for the last Q_1/λ time units of T. The area under the on-hand inventory curve for one cycle of echelon two is therefore

$$\int_0^t \ell_2(t)dt = \frac{Q_1}{\lambda} \sum_{m=1}^{n-1} mQ_1 = \frac{n[n-1]}{2\lambda} Q_1^2 .$$

From (2-1), the total variable cost per cycle at echelon two is

$$K_2(c) = A_2 + CQ_2 + \frac{I_2C\ n[n-1]\ Q_1^2}{2\lambda} . \qquad (2\text{-}2)$$

From the assumption that $Q_2 = nQ_1$, where n is integer, it follows that the cycle length at echelon two is equal to n reorders at echelon one. Therefore, from Figure 2,

$$T = \frac{Q_2}{\lambda} = \frac{nQ_1}{\lambda} . \qquad (2\text{-}3)$$

The average annual cost at echelon two is obtained by dividing (2-2) by (2-3) which yields

$$K_2 = \frac{A_2\lambda}{Q_2} + C\lambda + \frac{I_2C[n-1]Q_1^2}{2Q_2} ,$$

or, as a function of Q_1 and n,

$$K_2 = \frac{A_2\lambda}{nQ_1} + C\lambda + \frac{I_2CQ_1[n-1]}{2} . \qquad (2\text{-}4)$$

Development of the cost per cycle at echelon one follows an argument that is analogous to the echelon two development, except that there is no purchase cost incurred at echelon one. Also, from the assumption that Q_2 is an integer multiple of Q_1 , it follows that in each cycle there are n reorders at echelon one, i.e., $K_1(c)$ is linear in n. Thus the cost per cycle can be written as

$$K_1(c) = nA_1 + \frac{nI_1CQ_1^2}{2} . \qquad (2\text{-}5)$$

When (2-5) is divided by (2-3) the average annual variable cost is

$$K_1 = \frac{A_1\lambda}{Q_1} + \frac{I_1CQ_1}{2} . \qquad (2\text{-}6)$$

The total average annual cost of the system, K, is the sum of (2-4) and (2-6).

$$K = \frac{A_1\lambda}{Q_1} + \frac{I_1CQ_1}{2} + \frac{A_2\lambda}{nQ_1} + C\lambda + \frac{I_2C[n-1]Q_1}{2} . \qquad (2\text{-}7)$$

The model has assumed that the unit price is independent of the quantity ordered. Therefore the C term in (2-7) is a constant, and can be dropped from (2-7) resulting in (2-8) being the average annual variable cost.

$$K = \frac{A_1\lambda}{Q_1} + \frac{I_1CQ_1}{2} + \frac{A_2\lambda}{nQ_1} + \frac{I_2C[n-1]Q_1}{2} . \qquad (2\text{-}8)$$

B. OPTIMAL POLICY

Determining the optimal inventory policy involves finding the values of Q_1 and n (call them Q_1^* and n*) which minimize (2-8) for a given set of model parameters. To find this policy, rewrite (2-8) as

$$K(Q_1,n) = [A_1 + \frac{A_2}{n}] \frac{\lambda}{Q_1} + \frac{[I_1 + I_2[n-1]]CQ_1}{2} . \qquad (2\text{-}9)$$

Then since $K(Q_1,n)$ is convex in Q_1 for fixed n, $Q_1^*(n)$ must satisfy the equation

$$\frac{\partial K}{\partial Q_1} = - [A_1 + \frac{A_2}{n}]\frac{\lambda}{Q_1^2} + \frac{[I_1 + I_2[n-1]]C}{2} = 0 ,$$

or

$$Q_1^*(n) = \left[\frac{2\lambda [A_1 + \frac{A_2}{n}]}{[I_1 + I_2[n-1]]C}\right]^{1/2} . \qquad (2\text{-}10)$$

After substituting (2-10) for Q_1 in (2-9) and collecting terms, (2-9) takes the following form:

$$K(Q_1^*(n),n) = \{2\lambda [A_1 + \frac{A_2}{n}] [I_1 + I_2[n-1]]C\}^{1/2} ,$$

or

$$K(Q_1^*(n),n) = \left[2\lambda A_1 I_1 C + \frac{2\lambda A_2 I_1 C}{n} + 2\lambda A_1 I_2 C[n-1] + 2\lambda A_2 I_2 C[\frac{n-1}{n}]\right]^{1/2} . \qquad (2\text{-}11)$$

Note that the first term in (2-11) is equivalent to K_w^2 (Eq. (2-11) [Hadley and Whitin]). In fact, it is readily seen that each term in (2-11) contains a form of this classic formula. Let

$$K_{ij}^2 = 2\lambda A_i I_j C .$$

Then (2-11) can be rewritten

$$K(n) = K(Q_1^*, n) = \{K_w^2 + \frac{K_{21}^2}{n} + K_{12}^2[n-1] + K_{22}^2 [\frac{n-1}{n}]\}^{1/2} . \tag{2-12}$$

It is immediately seen that (2-12) is positive for all feasible values of n ($n \geq 1$). Further,

$$\lim_{n \to 1} K(n) = [K_w^2 + K_{21}^2]^{1/2} ,$$

and for very large values of n,

1. $K(n) \rightarrow K_{12} n^{1/2} > 0$,

2. $\frac{\partial K(n)}{\partial n} \rightarrow \frac{K_{12}}{2n^{1/2}} > 0$,

3. $\frac{\partial^2 K(n)}{\partial n^2} \rightarrow \frac{-K_{12}}{4n^{3/2}} < 0$.

If the assumption that n is an integer is temporarily relaxed the slope of K(n) can be readily investigated by considering

$$\frac{\partial K}{\partial n} = \frac{\frac{-K_{21}^2}{n^2} + \frac{K_{22}^2}{n^2} + K_{12}^2}{2K(n)^{1/2}} . \tag{2-13}$$

If (2-13) is positive when n = 1, then it follows that it is positive over all n, and the optimal value of n is $n^* = 1$. If (2-13) is negative when n = 1, then there exist values of n such that K(n) is less than K(1). However, since it has been shown that (2-13) is positive for very large n, it follows that n^* is finite, and satisfies

$$\frac{\partial K}{\partial n} = 0 ,$$

or

$$n^* = \left[\frac{K_{21}^2 - K_{22}^2}{K_{12}^2}\right]^{1/2} . \qquad (2\text{-}14)$$

A further consideration of the case where (2-13) is negative at n = 1 reveals that

$$\frac{\partial^2 K}{\partial n^2} > 0$$

over the range of n values from 1 to beyond n*. Let the value of n where

$$\frac{\partial^2 K}{\partial n^2} = 0$$

be denoted by $\hat{n}$. Then in the range $1 \leq n \leq \hat{n}$ the function K(n) is convex. For values of $n > \hat{n}$, (2-12) is not convex. However, (2-13) will always be positive. Therefore n*, given by (2-14), is the optimal non-integer value of n. The reader will note that the above arguments indicate that (2-12) is pseudo-convex.

Since K(n) is pseudo-convex over all n, it follows that K(n) is also pseudo-convex on the integer values of n. If n* is not an integer, then the optimal integer value of n will be either $\overline{n}$ (the smallest value greater than n*) or $\underline{n}$ (the largest value less than n*). The integer optimal will be the value of n corresponding to $K^* = \min [K(\overline{n}) , K(\underline{n})]$.

In summary, the optimal inventory policy is found as follows:

1. Compute K_{21}^2, K_{22}^2, K_{12}^2 .

2. Test $K_{21}^2 - K_{22}^2 \geq K_{12}^2$. If this inequality holds then n optimal = 1. Go to step 8.

3. If $K_{21}^2 - K_{22}^2 < K_{12}^2$ then compute n* using (2-14).

4. If n* is an integer then n optimal = n*. Go to step 8.

5. If n* is not an integer, compute $\bar{n}$ and $\underline{n}$.

6. Compute $K(\bar{n})$ and $K(\underline{n})$ from (2-12).

7. $K^* = \min \{K(\bar{n}, K(\underline{n})\}$ and n optimal is the value of n corresponding to K* .

8. Compute Q_1^* from (2-10).

C. NUMERICAL EXAMPLE

As an illustration of the algorithm, consider the following problem where

$I_1 = 0.75$ $\quad$ $A_2 = \$200.00$

$I_2 = 0.50$ $\quad$ $\lambda = 100$ units/year

$A_1 = \$25.00$ $\quad$ $C = \$100$/unit

Begin by computing $K_{ij}^2 = 2\lambda A_i I_j C$.

$K_{21}^2 = 3.0 \times 10^6$

$K_{22}^2 = 2.0 \times 10^6$

$K_{12}^2 = 2.5 \times 10^5$

Next compare $K_{21}^2 - K_{22}^2 = 1.0 \times 10^6 > K_{12}^2 = 2.5 \times 10^5$,

so compute n* using (2-14). The value is 2.0 which is integer. Therefore n optimal = 2. Using (2-10) the optimal order quantities are

$$Q_1^* = 166.67 \text{ units}$$

and

$$Q_2^* = 166.67 \times 2 = 333.34 \text{ units.}$$

D. PARAMETER ANALYSIS

The primary objective of this analysis was to observe the effect of parameter variations on optimal n. The values of four of the five parameters were fixed and the value of the remaining parameter was allowed to vary. The process was then repeated for each of the other parameters so that the effect of each could be observed.

Representative graphs of optimal n versus a given parameter are illustrated in Figures 7 through 10. In Figure 7 it is seen that n optimal is inversely related to A_1. However, as noted in Figure 8, there is a direct relationship between n optimal and A_2. Further, a comparison of Figures 7 and 8 reveals that, for the parameters selected, n optimal is much more sensitive to A_2 than to A_1.

Figures 9 and 10 indicate the relationship of n optimal to I_1 and I_2, respectively. The results indicate that n optimal is directly related to I_1, inversely related to I_2, and that n optimal is more sensitive to a change in holding cost at echelon two than at echelon one.

An analysis of the effect of demand rate on n optimal revealed that n optimal was completely insensitive to λ over the range $5 \leq \lambda \leq 10{,}000$.

III. FINITE PRODUCTION AT THE SECOND ECHELON

An additional two echelon variation to the deterministic models treated [Hadley and Whitin] addresses the problem of finite production. This model would have application in any situation where the upper echelon manufactures as well as warehouses a given product.

A. MODEL FORMULATION

The model, shown graphically in Figure 3, assumes that echelon two produces the needed material at a constant continuous rate of ψ units per unit time ($\psi \geq \lambda$). The cost A_2 is now considered to be a set up cost which is incurred each time a new production run is initiated. This set up cost is independent of the quantity produced. The rest of the assumptions are identical to those of the EOQ model.

The model will, as in Section II, seek optimal values of Q_1 and Q_2 which minimize the average annual system cost. This cost is the sum

$$K = K_1 + K_2 ,$$

where K_i is the average annual cost at the ith echelon. In Section II it was shown that

$$K_1 = \frac{A_1\lambda}{Q_1} + \frac{1}{2}I_1CQ_1 .$$

To develop K_2 it is necessary to define the cycle length as

$$T = t_1 + T_p + T_d , \qquad (3\text{-}1)$$

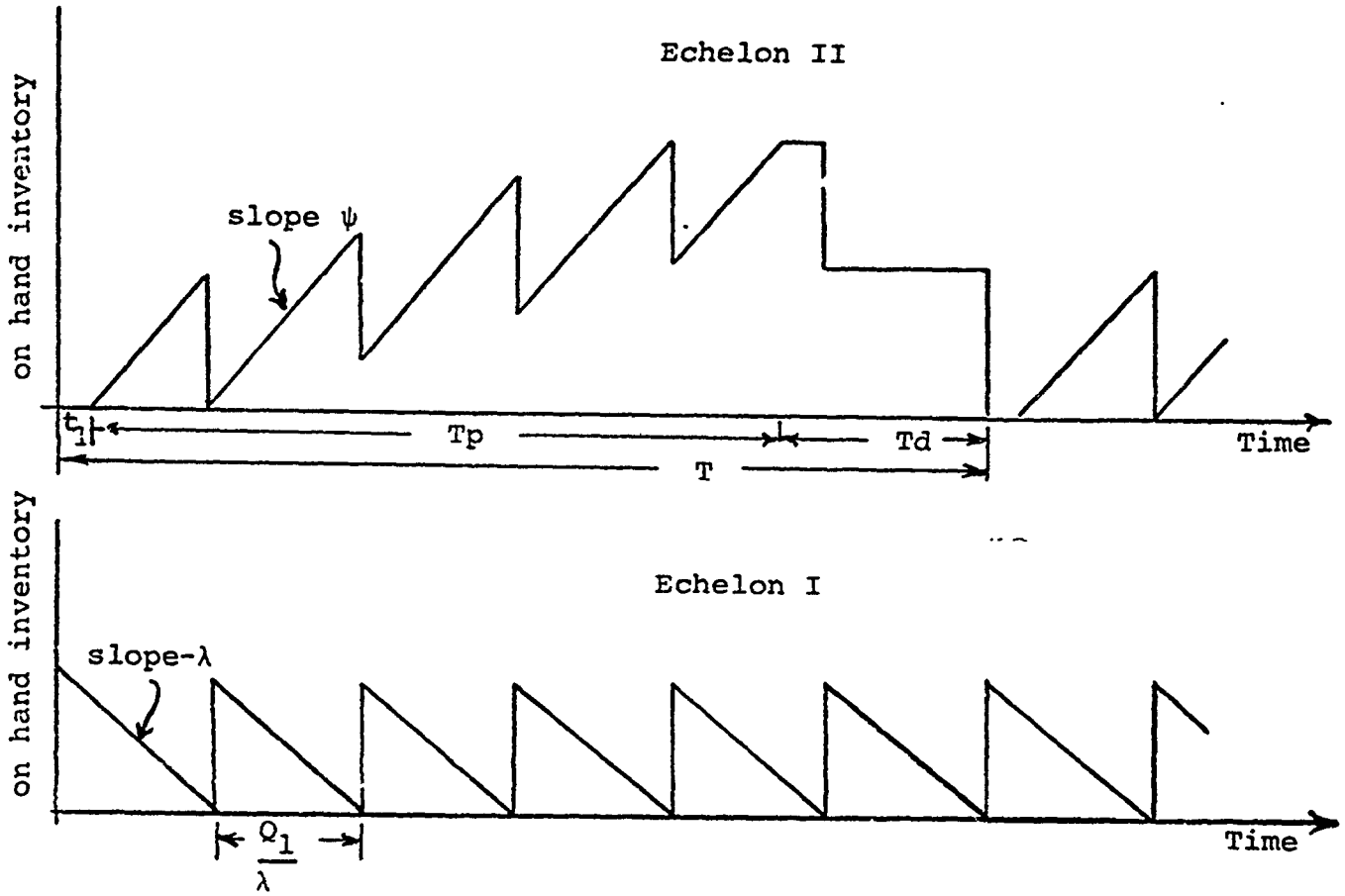


Figure 3. Two-Echelon Finite Production Model.

where t_1 is the time from the beginning of the cycle until the start of production, T_p is the production length, and T_d is the time required to deplete the inventory at echelon two after production stops (see Figure 3). Note that

$$t_1 = \frac{Q_1}{\lambda} - \frac{Q_1}{\psi},$$

$$T_p = \frac{Q_2}{\lambda},$$

and

$$T_d = \frac{(Q_2 - Q_1)}{\lambda} - \frac{(Q_2 - Q_1)}{\psi}.$$

Define i as the number of times that the quantity Q_1 is demanded at echelon two during the production period, T_p. It can be seen that

$$i = \left[\frac{t_1 + T_p}{t}\right]^{(-)} \tag{3-2}$$

where $t = \frac{Q_1}{\lambda}$.

As in Section II, the variable cost per cycle at echelon two, $K_2(C)$, will have the form

$$K_2(C) = A_2 + I_2 C \int_{t_1}^{t_1 + T_p + T_d} \mathcal{L}_2(t)\, dt \quad . \tag{3-3}$$

From Figure 4 it follows that

$$\int_{t_1}^{t_1 + T_p + T_d} \mathcal{L}_2(t)\, dt = f(i) + g(i + 1) + h(i + 2),$$

where f(i) is the area under the first i saw teeth, g(i + 1) is the shaded area shown in Figure 4 and h(i + 2) is the area remaining under the curve.

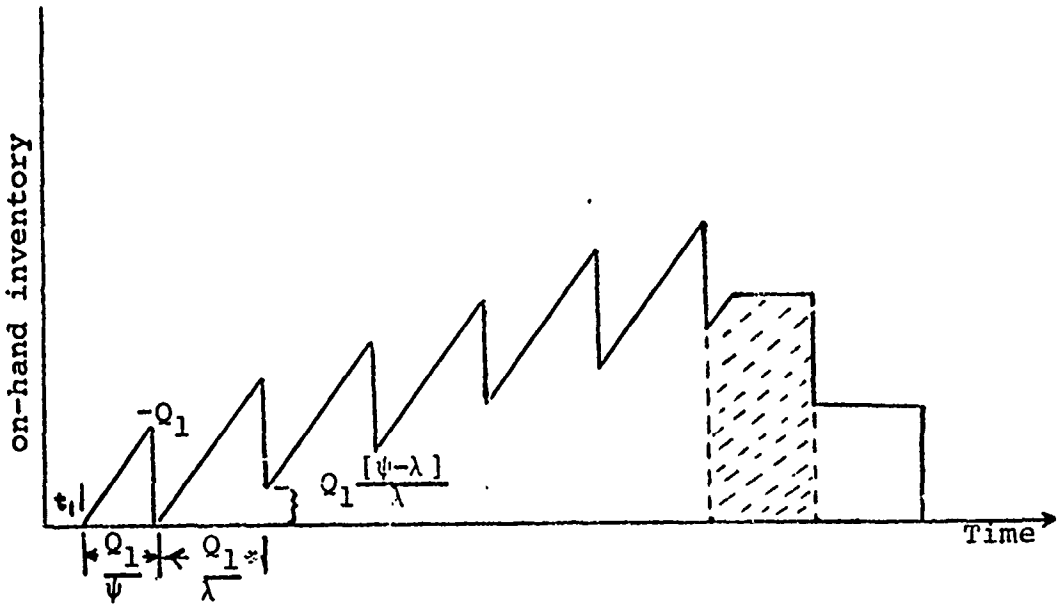


Figure 4. Finite Production Model, Area Under the Inventory Curve at Echelon Two.

From Figure 4 it can be shown that

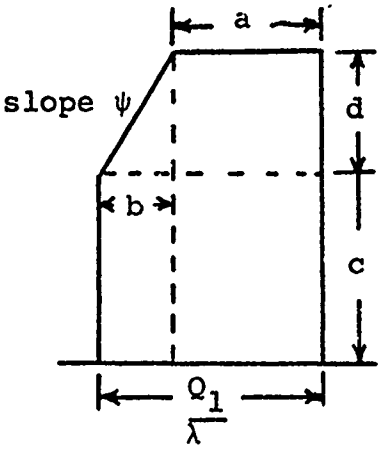
$$f(i) = \frac{Q_1^2}{2\psi} + \frac{[i-1]\ Q_1^2\psi}{2\lambda^2} + \frac{Q_1^2[\psi-\lambda][i-1][i-2]}{2\lambda^2} \qquad (3\text{-}4)$$

and

$$h(i) = \frac{Q_1^2\ [n-i][n-i-1]}{2\lambda} \qquad (3\text{-}5)$$

From Figure 5 the area g(i + 1) is found to be

$$g(i+1) = \frac{-Q_1^2\,[n-1]^2}{2\psi} - \frac{Q_1^2\psi[i^2-1]}{2\lambda^2} + \frac{Q_1^2\,[i[n-i]]}{\lambda} + \frac{Q_1^2\{[i-1][\psi-\lambda]\}}{\lambda 2}. \qquad (3\text{-}6)$$



$$a = [i\,\frac{Q_1}{\lambda} - \frac{Q_1}{\psi}\,[n-1]]$$

$$b = \frac{Q_1}{\psi}\,[n-1] - \frac{Q_1}{\lambda}\,[i-1]$$

$$c = Q_1\,[i-1]\,[\frac{\psi-\lambda}{\psi}]$$

$$d = \psi b$$

$$= Q_1[n-1] - Q_1\,\frac{\psi}{\lambda}\,[i-1]$$

Figure 5. Finite Production Model. Area Under $i + 1^{st}$ Saw Tooth.

Substituting (3-4), (3-5) and (3-6) into (3-3) yields

$$K_2(C) = A_2 + \frac{I_2 C Q_1^2}{2\psi} + \frac{I_2 C Q_1^2 \psi [i-1]}{2\lambda^2}$$

$$+ \frac{I_2 C Q_1^2 [n-i][n-i-1]}{2\lambda} - \frac{I_2 C Q_1^2 [n-1]^2}{2\lambda}$$

$$- \frac{I_2 C Q_1^2 \psi [i-1]}{2\lambda^2} + \frac{I_2 C Q_1^2 [i[n-i]]}{\lambda}$$

$$+ \frac{I_2 C [i-1][\psi-\lambda]}{\lambda^2} . \qquad (3\text{-}7)$$

After dividing (3-7) by the total cycle length and collecting terms K_2 can be written

$$K_2 = \frac{A_2 \lambda}{n Q_1} + \frac{I_2 C Q_1 [n-1]}{2} - \frac{I_2 C Q_1 \lambda [n-2]}{2\psi} . \qquad (3\text{-}8)$$

The system average annual variable cost is found by summing K_1 and K_2 .

$$K(Q_1, n) = [A_1 + \frac{A_2}{n}] + \frac{[I_1 + I_2[n-1] - I_2 \frac{\lambda}{\psi}[n-2]] C Q_1}{2} \qquad (3\text{-}9)$$

B. OPTIMAL POLICY

The development of an algorithm for determining the optimal inventory policy in the finite production model follows an argument that is identical to the one discussed in Section II. Since (3-9) is convex in Q_1 for fixed n, $Q_1^*(n)$ must satisfy

$$\frac{\partial K(Q_1, n)}{\partial Q_1} = 0 ,$$

2.1-

or

$$Q_1^*(n) = \left[\frac{2[A_1+\frac{A_2}{n}]\lambda}{C[I_1+I_2[n-1] - I_2\frac{\lambda}{\psi}[n-2]}\right]^{1/2} . \qquad (3\text{-}10)$$

When (3-10) is substituted for Q_1 in (3-9) then

$$K(n) = \left[K_{11}^2 + \frac{K_{21}^2}{n} + K_{12}^2\,[[n-1] - \frac{\lambda}{\psi}[n-2]] + \frac{K_{22}^2[[n-1] - \frac{\lambda}{\psi}[n-2]]}{n}\right]^{1/2} \qquad (3\text{-}11)$$

where $K_{ij}^2 = 2\lambda A_i I_j C$.

It is apparent that (3-11) is positive for all values of n. Further,

$$\lim_{n\to 1} K(n) = K_{11}^2 + K_{21}^2 + K_{12}^2\frac{\lambda}{\psi} + K_{22}^2\frac{\lambda}{\psi} .$$

and for very large values of n,

1. $K(n) \to K_{12} n^{1/2} > 0$,

2. $\frac{\partial K}{\partial n} \to \frac{K_{12}}{2n^{1/2}} > 0$,

3. $\frac{\partial^2 K}{\partial n^2} \to \frac{-K_{12}}{4n^{3/2}} < 0$.

If the assumption that n is an integer is temporarily relaxed the slope of K(n) can be investigated using

$$\frac{\partial K}{\partial n} = \frac{\frac{-K_{21}^2}{n^2} + K_{12}^2[1-\frac{\lambda}{\psi}] + \frac{K_{22}^2}{n^2} - \frac{2\lambda K_{22}^2}{\psi n^2}}{2\,K(n)^{1/2}} . \qquad (3\text{-}12)$$

If (3-12) is positive when n = 1, it is also positive over all n and it follows that the optimal value of n is $n^* = 1$. If (3-12) is negative when n = 1, then there exist values of n such that K(n) is less than K(1). However, since it has been shown that (3-12) is positive for very large n, n^* must be finite and satisfy

$$\frac{\partial K}{\partial n} = 0 ,$$

or

$$n^* = \left[\frac{K_{21}^2 - \hat{K}_{22}^2}{\hat{K}_{12}^2}\right]^{1/2} , \qquad (3\text{-}13)$$

where

$$\hat{K}_{22}^2 = K_{22}^2 [\tfrac{\psi - 2\lambda}{\psi}] ,$$

and

$$\hat{K}_{12}^2 = K_{12}^2 [\tfrac{\psi - \lambda}{\psi}] .$$

A further consideration of the case where (3-12) is negative for n = 1 will show that

$$\frac{\partial^2 K}{\partial n^2} > 0$$

over the range of n values from 1 to beyond n^*. Let the value of n where

$$\frac{\partial^2 K}{\partial n^2} = 0$$

be denoted by $\hat{n}$, and it follows that in the range $1 \leq n \leq \hat{n}$ K(n) is convex. For values of $n > \hat{n}$ K(n) is not convex, but

(3-12) is positive. Therefore n* is the optimal non integer value of n and (3-11) is pseudo-convex.

Since (3-11) is pseudo-convex over all n, it follows that it is also pseudo-convex on the integer values of n. So if n* is not an integer, then the optimal value of n will correspond to

$$K^* = \min[K(\bar{n}), K(\underline{n})] .$$

The optimal policy algorithm can now be written.

1. Compute K_{21}^2, $\hat{K}_{22}^2$, and $\hat{K}_{12}^2$.

2. If $K_{21}^2 \leq \hat{K}_{22}^2 + \hat{K}_{12}^2$ then n* = 1. Go to step 8.

3. If $K_{21}^2 > \hat{K}_{22}^2 + \hat{K}_{12}^2$ then compute n* using (3-13).

4. If n* is an integer then n optimal = n*. Go to step 8.

5. If n* is not an integer compute $\bar{n}$ and $\underline{n}$.

6. Compute $K(\bar{n})$ and $K(\underline{n})$ from (3-11).

7. $K^* = \min[K(\bar{n}), K(\underline{n})]$ and n optimal is the value of n corresponding to K*.

8. Compute Q_1^* from (3-10).

C. SPECIAL CASES OF ψ

When $\psi = \lambda$

$$K(n) = \frac{A_1\lambda}{Q_1} + \frac{I_2CQ_1}{2} + \frac{A_2\lambda}{nQ_1} + \frac{I_2CQ_1}{2},$$

and

$$\lim_{n\to\infty} K(n) = \frac{A_1\lambda}{Q_1} + \frac{[I_1+I_2]CQ_1}{2}. \qquad (3\text{-}14)$$

Thus it is seen that if the production rate is equal to the demand rate, the optimal policy is to start the production line and never let it stop.

As one would expect, when $\psi \to \infty$

$$\lim_{\psi\to\infty} K(n) = \frac{A_1\lambda}{Q_1} + \frac{I_1CQ_1}{2} + \frac{A_2\lambda}{nQ_1} + \frac{I_2CQ_1[n-1]}{2},$$

and it is immediately seen that this model degenerates to the EOQ model of Section II.

D. NUMERICAL EXAMPLE

Consider the problem where

$I_1 = 0.50$ $\qquad$ $C = \$100/\text{unit}$

$I_2 = 0.20$ $\qquad$ $A_1 = \$50/\text{unit}$

$\lambda = 25$ units/year $\qquad$ $A_2 = \$200/\text{unit}$

$\psi = 75$ units/year

First compute K^2_{ij}, $\hat{K}^2_{22}$ and $\hat{K}^2_{12}$.

$K^2_{21} = 5.0 \times 10^5$ $\qquad$ $K^2_{11} = 1.25 \times 10^5$

$K^2_{22} = 2.0 \times 10^5$ $\qquad$ $\hat{K}^2_{22} = 6.67 \times 10^4$

$K^2_{12} = 5.0 \times 10^4$ $\qquad$ $\hat{K}^2_{12} = 3.33 \times 10^4$

Next compare K_{21}^2 with $\hat{K}_{22}^2 + \hat{K}_{12}^2 - K_{21}^2 = 5.0 \times 10^5 > \hat{K}_{22}^2 + \hat{K}_{12}^2 = 1.0 \times 10^5$, so compute n* using (3-13). The value is 3.64 which is not an integer. Therefore $\bar{n} = 4$ and $\underline{n} = 3$.

From (3-11)

$K(\bar{n}) = \$685.00,$

$K(\underline{n}) = \$694.00,$

and n optimal is 4. Using (3-10)

$Q_1^* = 7.2$ units

and

$Q_2^* = 28.8$ units

E. PARAMETER ANALYSIS

Figures 11 and 12 illustrate the effect of production rate on optimal n. If the holding cost at echelon two is small then optimal n was relatively insensitive to a change in production rate (Figure 11). However, the results of Figure 12 show that this sensitivity increases as I_2 increases.

The responses of this model to parameter variations were identical to the responses observed in Section II.

IV. THE TWO ECHELON MODEL WITH BACKORDERS PERMITTED

A. MODEL FORMULATION

The EOQ and finite production models were based on the assumption that all demands would be immediately satisfied. This assumption will now be relaxed in order to investigate a system in which all demands are ultimately satisfied, but where it is permissible to accumulate backorders at the lower echelon. No backorders are allowed in echelon two.

In this model demands which occur when the lower echelon is out of stock are backordered against future procurement. When the procurement arrives these backordered demands are met, and any excess quantity is placed in stock. Each backorder at echelon one results in a cost of the form $\pi + \hat{\pi}$, where π is a fixed charge and $\hat{\pi}$ is a time dependent cost.

As noted, if there were no costs associated with incurring backorders [Hadley and Whitin], then it would be optimal to never have any inventory on hand. Conversely, if the backorders are sufficiently expensive, then the optimal policy would be to never incur any. However, for an intermediate range of backorder costs, it will usually be optimal to incur some backorders toward the end of the cycle.

With the exceptions noted in the preceding paragraphs, the model is predicated on the same assumptions used in Section II.

Define s as the number of backorders at echelon one when an order is received. Then from Figure 6 it follows that the

on hand inventory at echelon one immediately after receipt of an order is $(Q_1 - s)$ units. It is assumed that $Q_1 \geq s$. Then the echelon one on hand inventory varies from $(Q_1 - s)$ to zero.

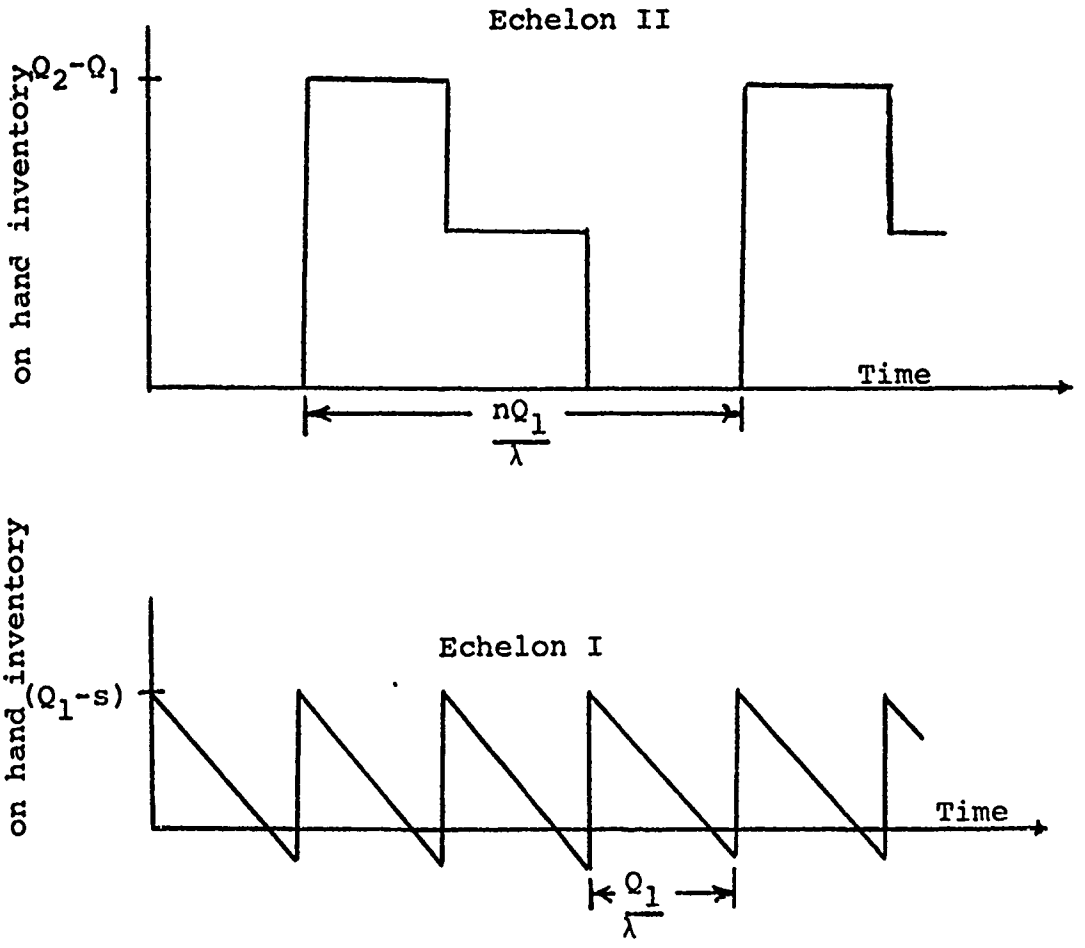


Figure 6. Two-Echelon Backorders Allowed Model.

The optimal policy continues to be that which minimizes average annual variable system cost subject to the model constraints. Since the model does not permit backorders at the second echelon, the average annual variable cost at echelon two will be identical to equation (2-4), which for convenience is restated here.

$$K_2 = \frac{A_2\lambda}{nQ_1} + \frac{I_2CQ_1[n-1]}{2} \qquad (4\text{-}1)$$

Development of the echelon one variable costs follow identically the single echelon backorders permitted model of Hadley and Whitin. In the interest of brevity, this development will be omitted and the results (equation 2-17) [Hadley and Whitin] merely stated.

$$K_1 = \frac{A_1\lambda}{Q_1} + \frac{I_1C[Q_1-s]^2}{2Q_1} + \frac{1}{Q_1}\left[\pi\lambda s + \frac{\hat{\pi}s^2}{2}\right]. \qquad (4\text{-}2)$$

Summing (4-1) and (4-2), the average annual variable cost is

$$K = \frac{A_1\lambda}{Q_1} + \frac{A_2\lambda}{nQ_1} + \frac{I_1C[Q_1-s]^2}{2Q_1} + \frac{1}{Q_1}\left[\pi\lambda s + \frac{\hat{\pi}s^2}{2}\right] + \frac{I_2CQ_1[n-1]}{2}. \qquad (4\text{-}3)$$

B. OPTIMAL POLICY

The optimal solution seeks the values of the decision variables Q_1^*, s^*, and n^* which minimize (4-3) subject to the constraint that all demands are ultimately satisfied. To do this, note that (4-3) can be rewritten as

$$K = \frac{A\lambda}{Q_1} + \frac{ICQ_1}{2} - I_1Cs + \frac{\pi\lambda s}{Q_1} + \frac{X\,s^2}{2Q_1}, \qquad (4\text{-}4)$$

where

$$X = I_1C + \hat{\pi},$$

$$A = A_1 + \frac{A_2}{n},$$

and

$$I = I_1 + I_2 [n-1] .$$

Then from

$$\frac{\partial K}{\partial s} = 0 ,$$

it follows that

$$s = \frac{I_1 C Q_1 - \pi\lambda}{X} , \tag{4-5}$$

or

$$Q_1 = \frac{\pi\lambda + Xs}{I_1 C} . \tag{4-6}$$

Next,

$$\frac{\partial K}{\partial Q_1} = \frac{-2A\lambda}{2Q_1^2} + \frac{IC}{2} - \frac{2\pi\lambda s}{2Q_1^2} - \frac{Xs^2}{2Q_1^2} = 0$$

results in

$$Q_1 = \left[\frac{2A\lambda + 2\pi\lambda s + Xs^2}{IC}\right]^{1/2} \tag{4-7}$$

To solve for Q_1^* and s^* in terms of n and the system parameters, substitute (4-7) into (4-5) and (4-6). After collecting terms, the resulting equations are

$$Q_1^*(n) = \left[\frac{2A\ X - \pi^2\lambda^2}{ICX - I_1^2\ C^2}\right]^{1/2} , \tag{4-8}$$

and

$$s^*(n) = \frac{I_1 C}{X}\left[\frac{2A\lambda X - \pi^2\lambda^2}{ICX - I_1^2\ C^2}\right]^{1/2} - \frac{\pi\lambda}{X} . \tag{4-9}$$

The expressions (4-8) and (4-9) can now be back substituted for Q_1 and s in (4-4) to yield an expression for the average

annual variable cost, K, as a function of n and the system parameters. After regrouping the terms this expression can be written

$$K(n) = \frac{[IC[I_1C+\hat{\pi}]-I_1^2C^2]^{1/2}\,[2A\lambda[I_1C+\hat{\pi}]-\pi^2\lambda^2]^{1/2}}{[I_1C+\hat{\pi}]} + \frac{I_1C\pi\lambda}{I_1C+\hat{\pi}} . \tag{4-10}$$

Equations (4-8), (4-9), and (4-10) are valid only if $2A\lambda IC - \pi^2\lambda^2 \geq 0$. Otherwise an analysis similar to that of Hadley and Whitin is required for the case when π and $\hat{\pi}$ are positive. Such an analysis is beyond the scope of this thesis.

Because of the complex form of K(n) given by (4-10) as a function of n no attempt was made to evaluate whether or not K(n) was convex or pseudo-convex in n when π and $\hat{\pi}$ are both positive. However, an example is presented later in which an optimal value of $n > 1$ is obtained for π and $\hat{\pi}$ positive.

If $\pi = 0$ (4-10) reduces to

$$K(n) = [2A\lambda IC - 2A\lambda I_1C\frac{I_1C}{X}]^{1/2} , \tag{4-11}$$

To expand (4-11), let

$$K_{ij}^2 = 2\lambda A_i I_j C \tag{4-12}$$

Then,

$$K(n) = \frac{1}{X^{1/2}} K_{11}^2X + K_{12}^2X[n-1] + \frac{K_{21}^2X}{n} + \frac{K_{22}^2X[n-1]}{n} - K_{11}^2 I_1 C - \frac{K_{21}^2 I_1 C}{n}^{1/2} ,$$

and since $x = I_1C + \hat{\pi}$, $K(n)$ reduces to

$$K(n) = \left[\frac{K_{11}^2\,\hat{\pi}}{I_1C+\hat{\pi}} + \frac{K_{21}^2\,\hat{\pi}}{n[I_1C+\hat{\pi}]} + K_{12}^2[n-1] + \frac{K_{22}^2[n-1]}{n}\right]^{1/2} . \tag{4-13}$$

Following the analysis of Section II, it is immediately seen that (4-13) is positive for all feasible values of n, and that

$$\lim_{n\to 1} K(n) = \frac{K_{11}^2\,\hat{\pi} + K_{21}^2\,\hat{\pi}}{I_1C + \hat{\pi}}^{1/2} .$$

Also, as n gets very large

1. $K(n)\to K_{12}\, n^{1/2} > 0$,

2. $\frac{\partial K}{\partial n}\to \frac{K_{12}}{2n^{1/2}} > 0$,

3. $\frac{\partial^2 K(n)}{\partial n^2}\to \frac{-K_{12}}{2n^{3/2}} < 0$.

Assume, for the sake of argument, that n is continuous for all $n \geq 1$. Then

$$\frac{\partial^2 K}{\partial n^2} = \frac{\frac{-K^2_{21}}{[I_1C+\hat{\pi}]n^2} + K_{12}^2 + \frac{K_{22}^2}{n^2}}{K(n)^{1/2}} \tag{4-14}$$

and it is obvious that for n = 1 (4-14) is non negative if and only if

$$K_{21}^2 \leq K_{12}^2 + K_{22}^2[1 + \frac{I_1 C}{\hat{\pi}}] \quad . \tag{4-15}$$

If (4-15) holds, then (4-14) is positive for all n. This implies that over all n, the slope of K(n) is positive, and $n^* = 1$.

IF (4-14) is negative at n = 1, then there exist values of n such that K(n) < K(1). However, it has been shown that for very large n, (4-14) is positive, and it follows that n^* is finite and satisfies

$$\frac{\partial K}{\partial n} = 0 \ ,$$

or

$$n^* = \left[\frac{K_{21}^2 \hat{\pi} - K_{22}^2 [I_1 C + \hat{\pi}]}{K_{12}^2 [I_1 C + \hat{\pi}]}\right]^{1/2} \tag{4-16}$$

To complete the argument, note that when (4-14) is negative at n = 1,

$$\frac{\partial^2 K}{\partial n^2} > 0$$

over the range of n values from 1 to beyond n^*. Let the value of n where

$$\frac{\partial^2 K}{\partial n^2} = 0$$

be noted by $\hat{n}$, and it follows that K(n) is convex over the range $1 \leq n \leq \hat{n}$. For values of $n > \hat{n}$, (4-13) is not convex, but in this range (4-14) is always positive. Therefore n^* given by (4-16) is the optimal non integer value of n. The

reader should again note that the above arguments indicate that (4-13) is pseudo-convex.

Since K(n) is pseudo-convex over all n, it is also pseudo-convex on the integer values of n. Therefore, if (4-16) does not yield an integer value then the optimal value of n can be found by evaluating $K(\bar{n})$ and $K(\underline{n})$ as in Section II.

An algorithm for finding the optimal policy (in the case where $\pi = 0$) can now be stated.

1. Compute K_{21}^2, K_{22}^2, and K_{12}^2 .

2. Test $K_{21}^2 \leq [K_{12}^2 + K_{22}^2][1 + \frac{I_1 C}{\hat{\pi}}]$. If the inequality holds then n optimal is 1. Go to step 8.

3. If $K_{21}^2 > [K_{12}^2 + K_{22}^2][1 + \frac{I_1 C}{\hat{\pi}}]$ then compute n* using (4-16).

4. If n* is an integer then n optimal = n*. Go to step 8.

5. If n* is not an integer compute $\bar{n}$ and $\underline{n}$.

6. Compute $K(\bar{n})$ and $K(\underline{n})$ from (4-13).

7. K* = min[$K(\bar{n})$ and $K(\underline{n})$] and n optimal is the value of n corresponding to K*.

8. Compute Q_1^* from

$$Q_1^*(n) = \left[\frac{2\lambda [A_1 + \frac{A_2}{n}]}{C[I_1 I_2 C[n-1] + I_1 \hat{\pi} + I_2 \hat{\pi}[n-1]]}\right]^{1/2},$$

and s* from

$$s^*(n) = \left[\frac{I_1 C\, Q_1^*}{I_1 C + \hat{\pi}}\right]^{1/2}$$

C. NUMERICAL EXAMPLE

Consider the problem where

$I_1 = 0.75$ $\quad\quad$ $\pi = \$5.00$

$I_2 = 0.25$ $\quad\quad$ $\hat{\pi} = \$100.00/year$

$A_1 = \$25.00$ $\quad\quad$ $\lambda = 100$ units/year

$A_2 = \$100.00$ $\quad\quad$ $C = \$100/unit$

Begin by computing $K_{ij}^2 = 2\lambda A_i I_j C$.

$$K_{21}^2 = 1.5 \times 10^6$$

$$K_{22}^2 = 5.0 \times 10^5$$

$$K_{12}^2 = 1.25 \times 10^5$$

Next compare $K_{21}^2 < [K_{12}^2 + K_{22}^2][1 + \frac{I_1 C}{\hat{\pi}}]$. The inequality does not hold so compute n* using (4-17). The value is 1.69 which is not an integer. From (4-13)

$$K(\bar{n}) = \$735.00$$

and

$$K(\underline{n}) = \$620.00$$

Therefore optimal n = 1. The values of Q_1^* , Q_2^* , and s* are now found to be

$$Q_1^* = 12.9 \text{ units}$$

$$Q_2^* = 12.9 \text{ units}$$

$$s^* = 8.16 \text{ units.}$$

D. PARAMETER ANALYSIS

As noted previously, computational difficulties preclude an analytical treatment of the general form of the backorder model. However, the response of the model to parameter variations can be approximated. This can be accomplished by considering separately the two functions (3-1) and (3-2) and ignoring, for now, the integer restriction on n.

Equation (4-1) can be rewritten

$$K_2 = \frac{A_2\lambda}{Q_2} + \frac{I_2C[Q_2 - Q_1]}{2} ,$$

and since the second term is never negative, it is obvious that K_2 is convex in Q_2 . Using the Hessian it can be shown that (4-2) is convex in Q_1 and s if and only if

$$2A_1\lambda [I_1C + \hat{\pi}] \geq [\pi\lambda]^2 . \qquad (4\text{-}18)$$

Then

$$K = \frac{A_1\lambda}{Q_1} + \frac{I_1C[Q_1 - s]^2}{2Q_1} + \frac{1}{Q_1}[\pi\lambda s + \frac{\hat{\pi}s^2}{2}]$$

$$+ \frac{A_2\lambda}{Q_2} \quad \frac{I_2C[Q_2 - Q_1]}{2} \qquad (4\text{-}19)$$

is convex if and only if (4-18) holds.

It can be shown that when (4-19) is not convex the optimal inventory policy will be realized when backorders are not allowed ($s^* = 0$). Thus if (4-18) does not hold, the problem can be solved by setting $s^* = 0$ and using the algorithm described in Section II to get Q_1^* and n^*.

For the case where (4-19) is convex, optimal Q_1, Q_2 and s can be found from

$$\frac{\partial K}{\partial Q_1} = \frac{\partial K}{\partial Q_2} = \frac{\partial K}{\partial s} = 0 . \qquad (4\text{-}20)$$

Equation (4-20) will be satisfied when

$$Q_1^* = \left[\frac{2\lambda [nA_1 + A_2][I_1C + \hat{\pi}] - n\pi^2\lambda^2}{n[I_1C + \hat{\pi} + I_1I_2C^2[n-1] + I_2C\hat{\pi}[n-1]}\right]^{1/2}, \qquad (4\text{-}21)$$

$$s^*(n) = \frac{I_1CQ_1^*(n) - \pi\lambda}{I_1C + \hat{\pi}}, \qquad (4\text{-}22)$$

and

$$Q_2^*(n) = nQ_1^*(n) , \qquad (4\text{-}23)$$

and the optimal policy can be found from the following algorithm.

1. Set n = 1 .

2. Compute $Q_1^*(n)$, $s^*(n)$, $Q_2^*(n)$ and K(n) using equations (4-21) through (4-23) and (4-19).

3. Set n = 2 and repeat step 2.

4. If K(1) < K(2), stop; n* = 1 and the optimal policy is known.

5. If K(2) < K(1), set n = 3 and repeat step 2.

6. Continue solving K(n) by increasing n in steps of 1 until K(n+1) $\geq$ K(n). Stop as soon as K(n) < K(n+1); n* = current value of n and the optimal solution has been found.

The algorithm described in the preceding paragraph was used to observe the effect of parameter variations on optimal n. In general, the results were similar to those observed in the EOQ model. For example, n optimal varied inversely with A_1 and I_2, but directly with A_2 and I_1.

The sensitivity of n to the various parameters was highly dependent on the selected values of the backorder costs. For example, Figure 13 illustrates the relationship between optimal n and I_2 in a situation where the backorder cost is small. Figure 14 illustrates this same example with a high time dependent backorder cost. The difference in the sensitivity is obvious.

Figures 15 and 16 illustrate the relationship of optimal n to a change in A_2. In Figure 15 the backorder costs were low, and optimal n was insensitive to a change in A_2. In this example optimal n remained at two, and s* increased as A_2 increased. When the time dependent backorder cost was high optimal n varied directly with A_2 (Figure 16). However, the change in optimal n was not nearly as large as in the EOQ model.

Figures 17 through 20 illustrate the relative magnitude of Q_1^* , Q_2^* , and s* as A_2 and I_2 are allowed to vary. The breaks in these curves occur at points where optimal n changes.

V. RECOMMENDATIONS FOR FURTHER STUDY

The analysis of the backorders permitted model should be completed for all values of the backorder costs π and $\hat{\pi}$ including the case where $\pi > 0$ and $\hat{\pi} = 0$. Hadley and Whitin discuss this case in their single echelon development and conclude that if $\hat{\pi} = 0$ then s^* is either 0 or infinite. Initial investigations indicated that in the two echelon model there could exist a finite value of $s^* > 0$ when $\hat{\pi} = 0$. However, this investigation was not completed and no conclusions were reached.

It is felt that the analytical argument used throughout the paper could be applied to the general backorder case where π and $\hat{\pi}$ are both greater than zero. However, the complexity of the equations would greatly complicate this development.

The models discussed in this thesis should be extended to more than one activity at each echelon. If it can be assumed that all activities within a given echelon order at the same time, then it is particularly easy to include multiple activities within an echelon. For example, assume that there are K activities in echelon one. Then in the EOQ model K_1 would take the form

$$K_1 = \sum_{i=1}^{k} \left[\frac{A_{1i} \lambda_i}{Q_{1i}} + \frac{I_{1i} C Q_{1i}}{2} \right].$$

If the values of the parameters A_i, λ_i, I_{1i}, and C are constant for all i, i = 1,...,k, then the expression for K_1 reduces to

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the trivial case of

$$K' = k K_i ,$$

where K_i is found in Section II. The expression for K_2 remains unchanged.

The problem of multiple activities at the top echelon is usually not a very interesting one, since each top echelon activity is usually responsible for supplying specified activities at the next lower echelon. Therefore, unless the model provides for lateral resupply actions, the problem of multiple activities at the top echelon can be reduced to the sum of a series of independent problems, each containing one top echelon activity.

A related problem which should be investigated is the extension of the models to more than 2 echelons. In the case where each echelon is limited to one activity, this extension would not be difficult. Indeed, given the assumption that there is an integer ratio between the quantity ordered by successive echelons, the model would probably take the form

$$K = \sum_{j=1}^{m} K_j ,$$

where the relationship between the Q_j's is

$$Q_2 = n_1 \cdot Q_1$$

$$Q_3 = n_2 \cdot Q_2 = n_2 \cdot n_1 \cdot Q_1$$

$$\vdots$$

$$Q_j = n_{j-1} \cdot Q_{j-1} = \prod_{i=1}^{j-1} n_i Q_1 ,$$

and nK is an integer for all k, k = 1,...,j-1. Because of the

integer property of the n_k's, the solution technique outlined in Chapter II could be used to solve recursively for K.

If the m-echelon problem allows k_i activities at the i^{th} echelon, i = 1,...,m, then for large values of k_i and m, the sheer magnitude of the problem would make its solution extremely difficult. In fact, it is doubtful that the procedures suggested in this paper could be utilized for a problem of this type. However, if the model is small enough, one could at least get a "feel" of the model's behavior.

VI. SUMMARY

Three of the deterministic models of Hadley and Whitin have been restructured as two echelon models. Equations for the average annual variable cost are derived, and standard mathematical programming techniques are utilized to find the optimal inventory policy. The optimal policy is defined as that which minimizes average annual variable cost subject to the constraints of the model.

General solutions are obtained for the EOQ and finite production models. Because of the complexity of the cost equation the analytical solution of the backorders permitted model is limited to the case where $\pi = 0$. However, a technique is developed which can be used to find the optimal policy in the general case.

A parametric analysis is conducted for each model in order to study the behavior of the models under parameter variations. The behavior of the curves of n optimal for the EOQ and finite production models were identical except that the finite production model enjoyed a significant cost advantage. The behavior of the backorder model was very dependent on the magnitude of the backorder cost. For example, if the model had a low time dependent backorder cost then parameter variations had little effect on the optimal behavior of the model. However, if the backorder cost was set sufficiently high then the model's behavior was similar to that of the EOQ model. Figures 7

through 20 illustrate the behavior of the three models under varying parameter values.

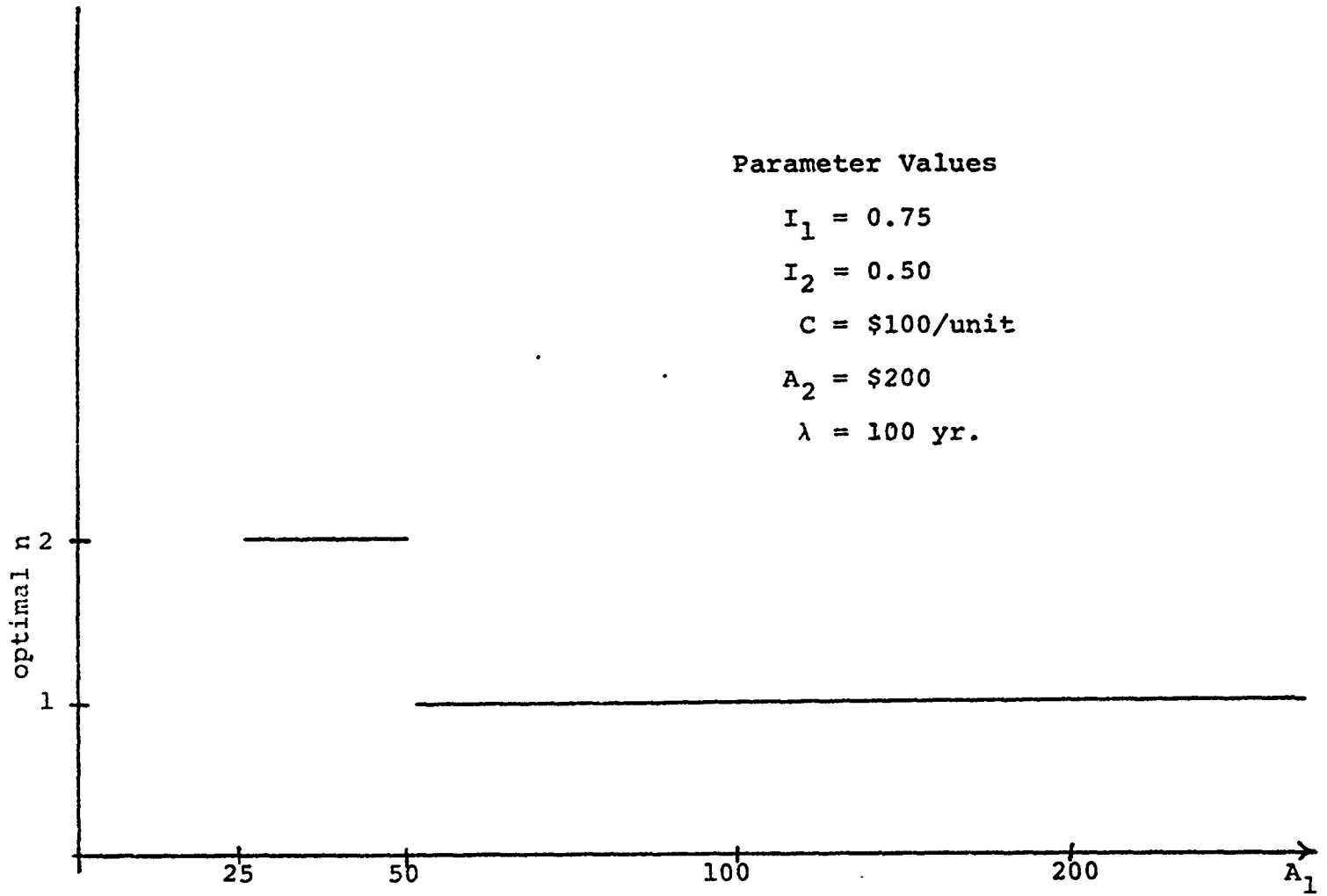

Figure 7. Optimal n : A_1 · EOQ Model.

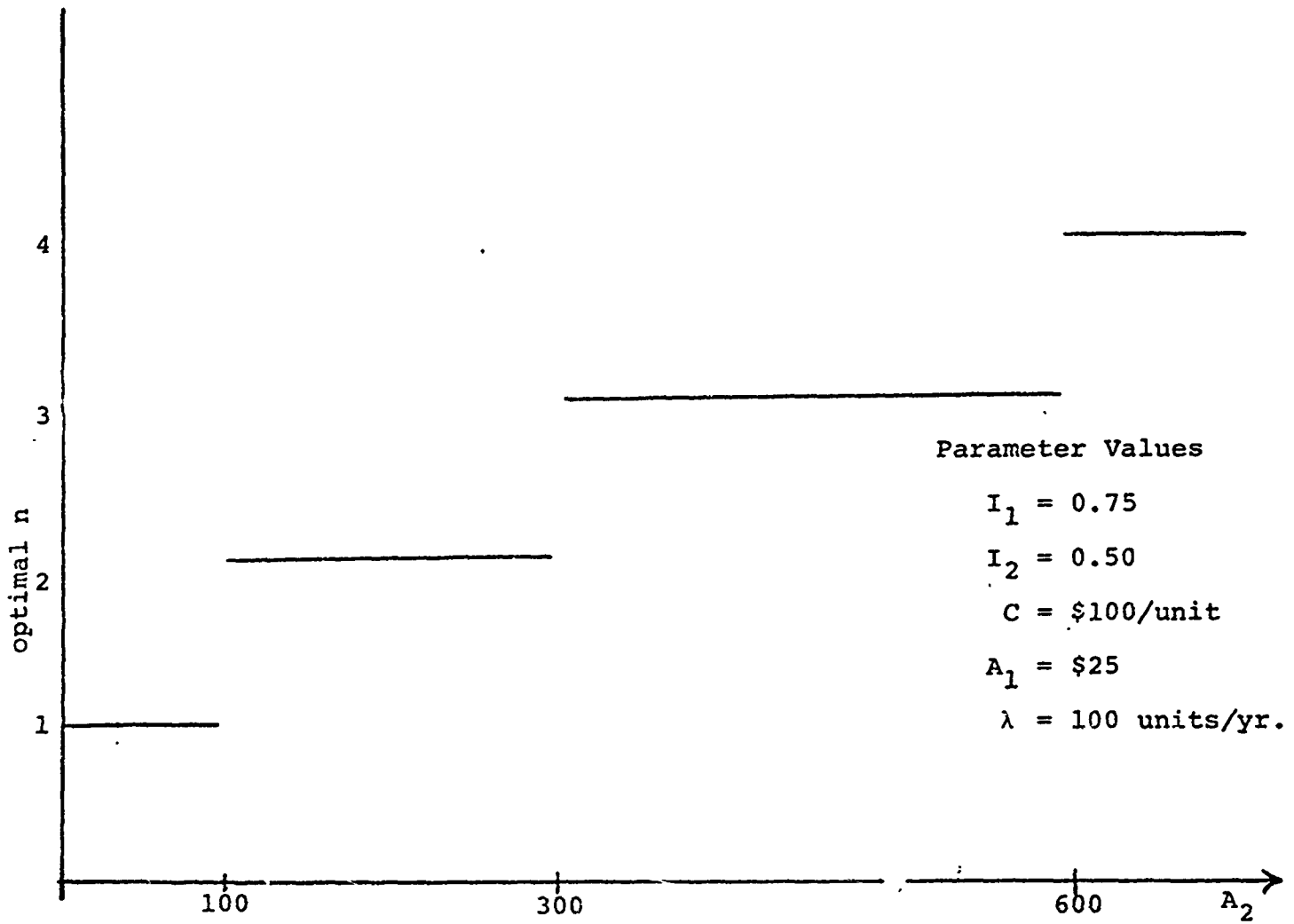

Figure 8. Optimal n : A_2 [illegible] el.

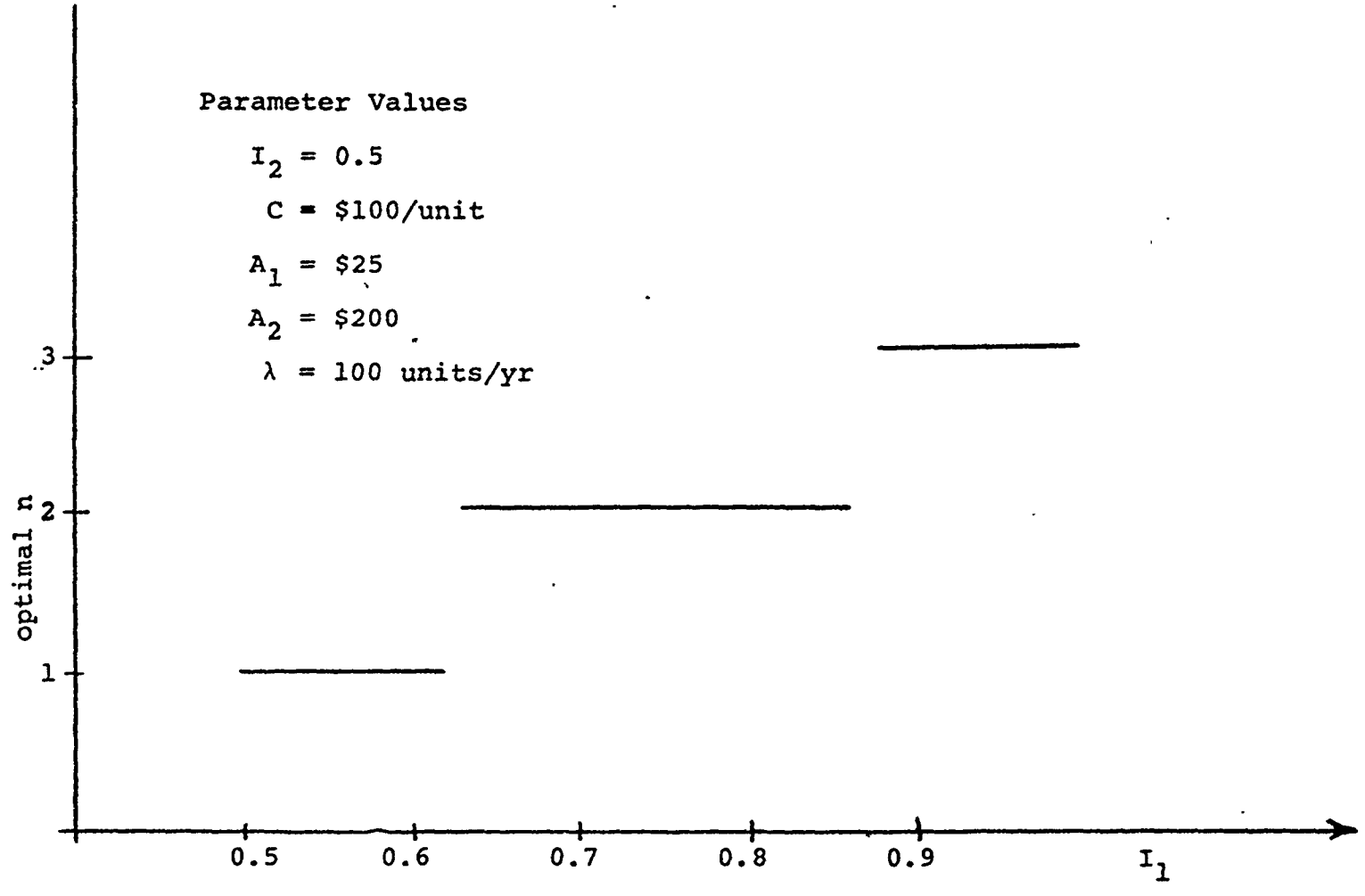

Figure 9. Optimal n : I_1 · EOQ Model.

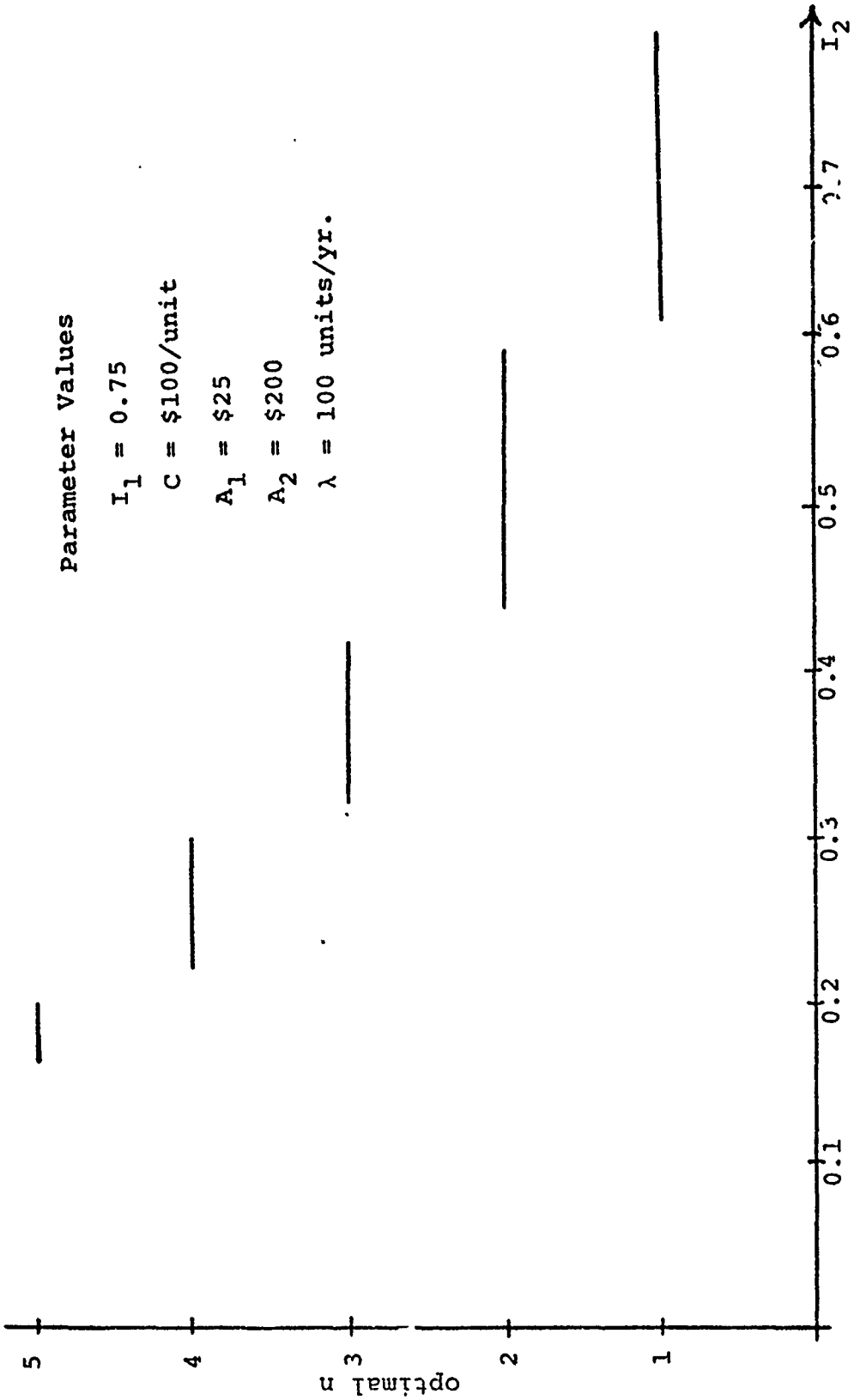

Figure 10. Optimal n : I_2 - EOQ Model.

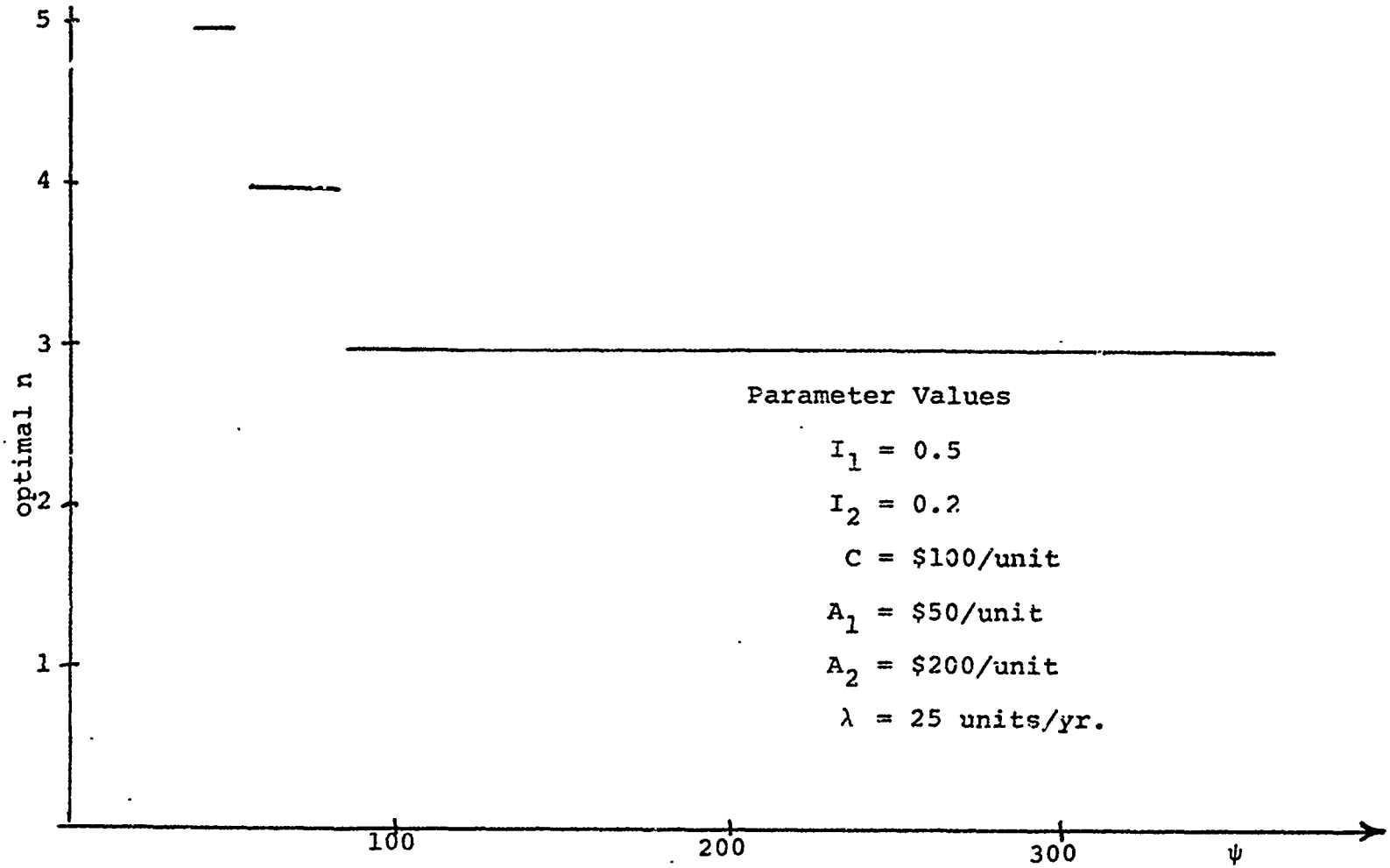

Figure 11. Optimal n : Production Rate. Finite Production Model.

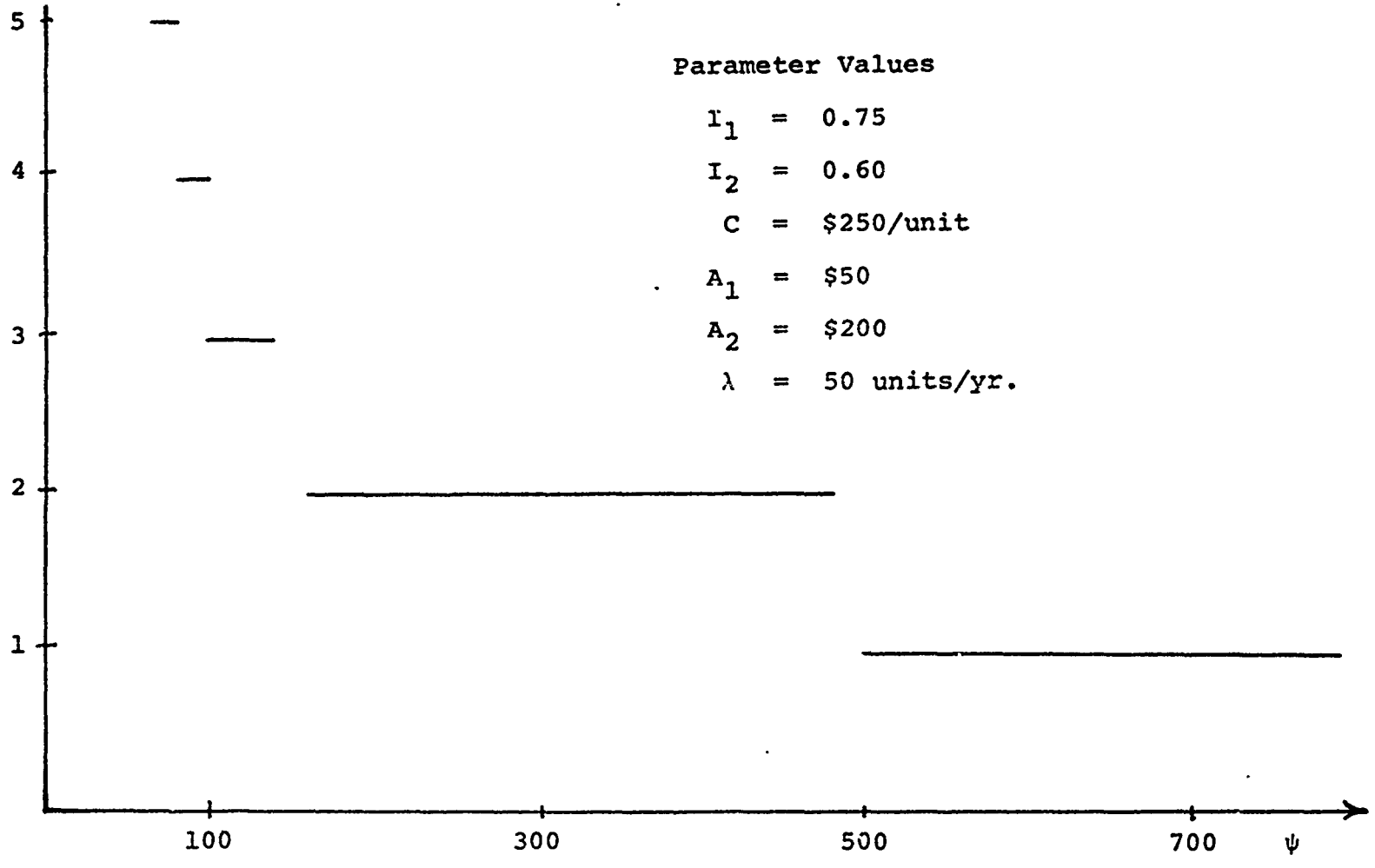

Figure 12. Optimal n : Production Rate. Finite Production Model.

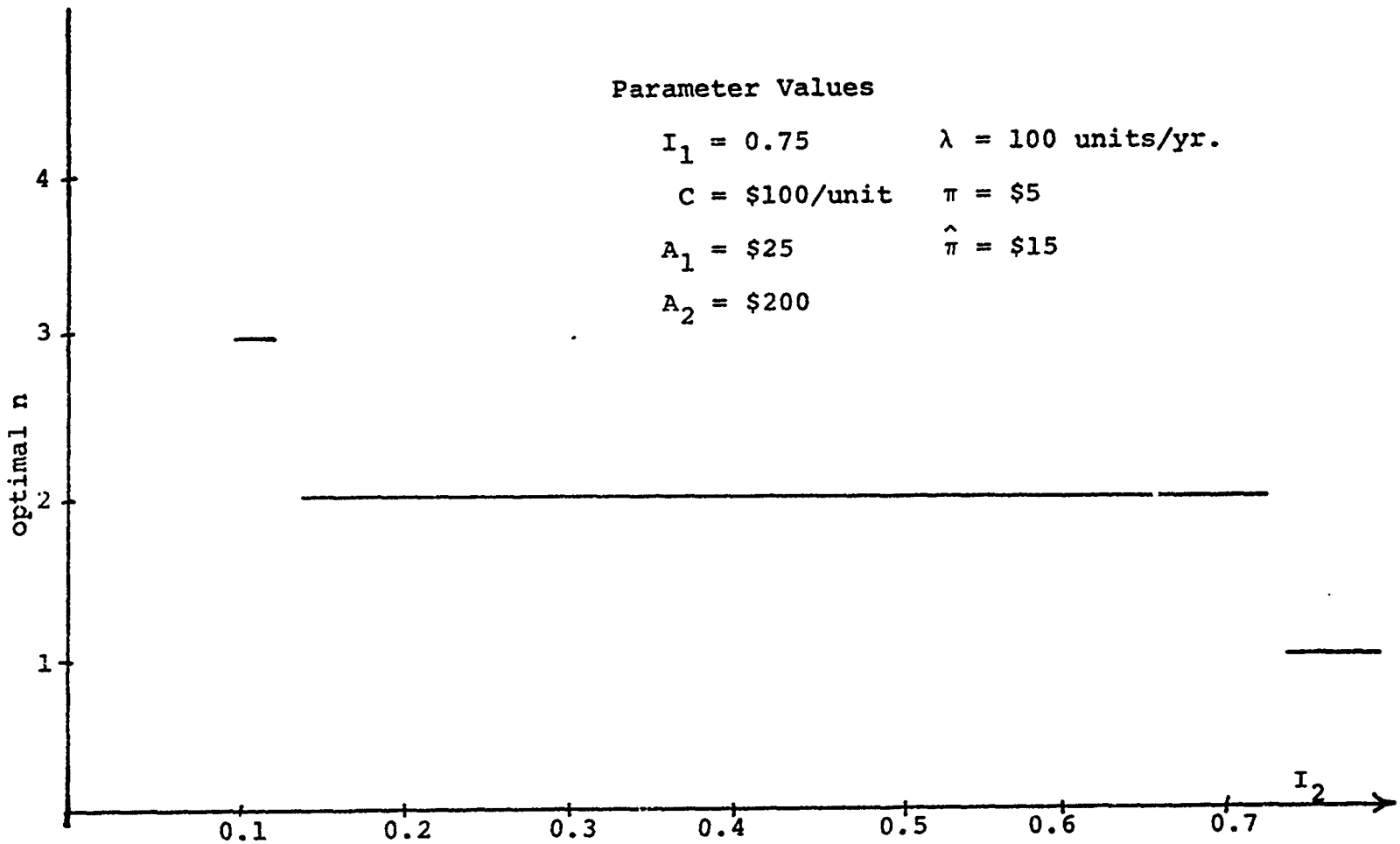

Figure 13. Optimal n : I_2 with Low Backorder Cost. Backorders Allowed Model.

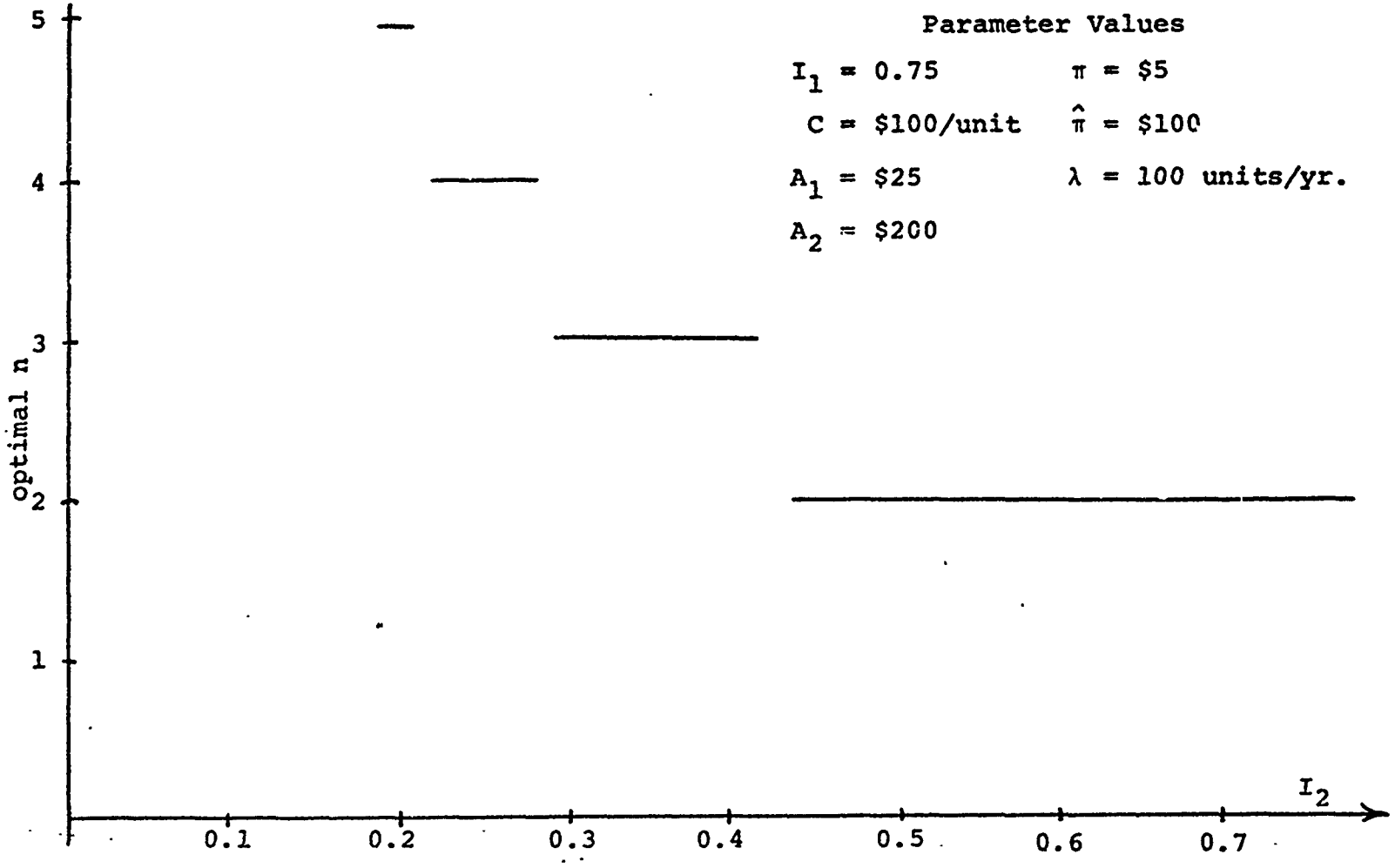

Figure 14. Optimal n : I_2 with High Backorder Cost. Backorders Allowed Model.

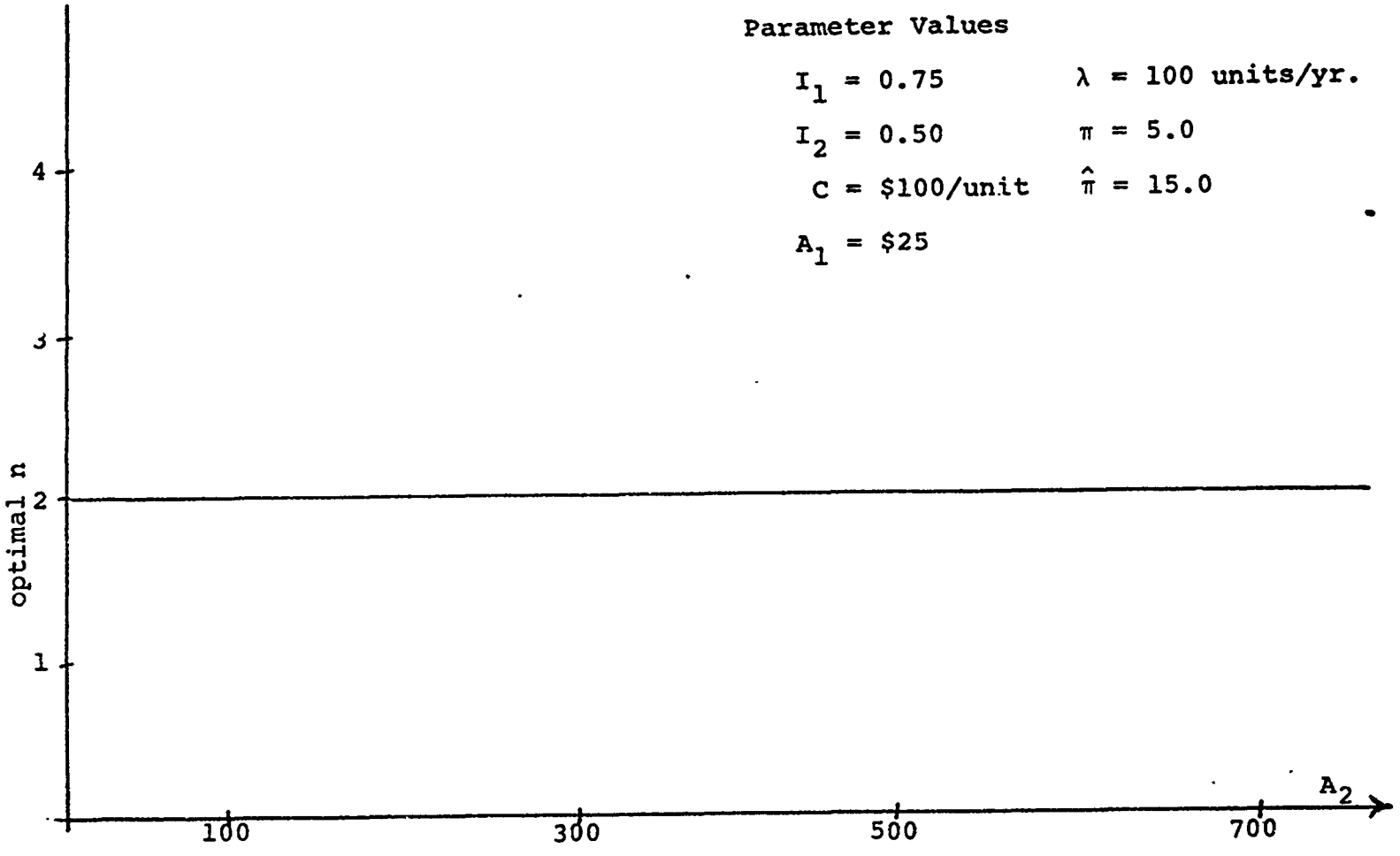

Figure 15. Optimal n : A_2 with Low Backorder Cost. Backorder Allowed Model.

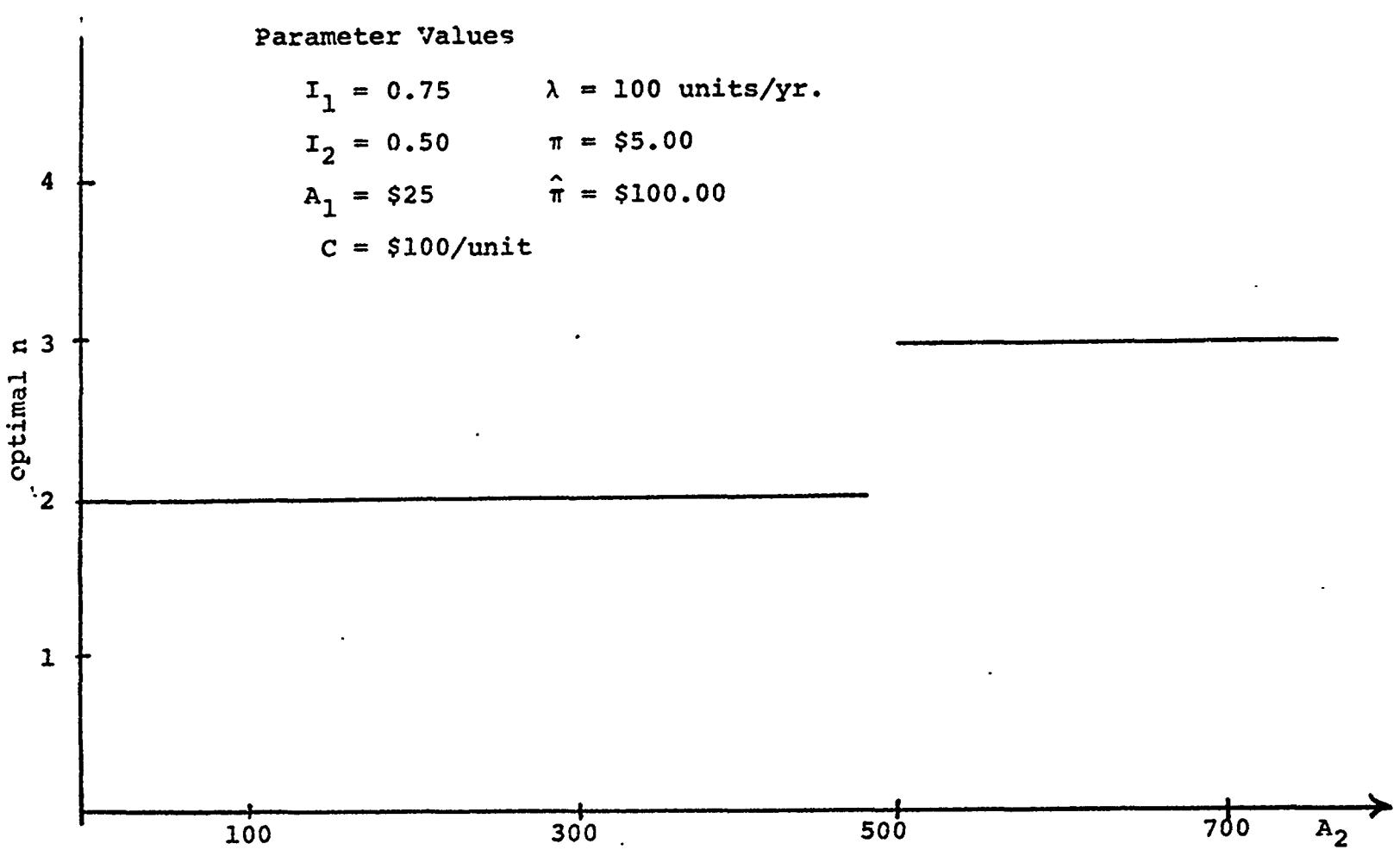

Figure 16. Optimal n : A_2 with High Backorder Cost. Backorder Allowed Model.

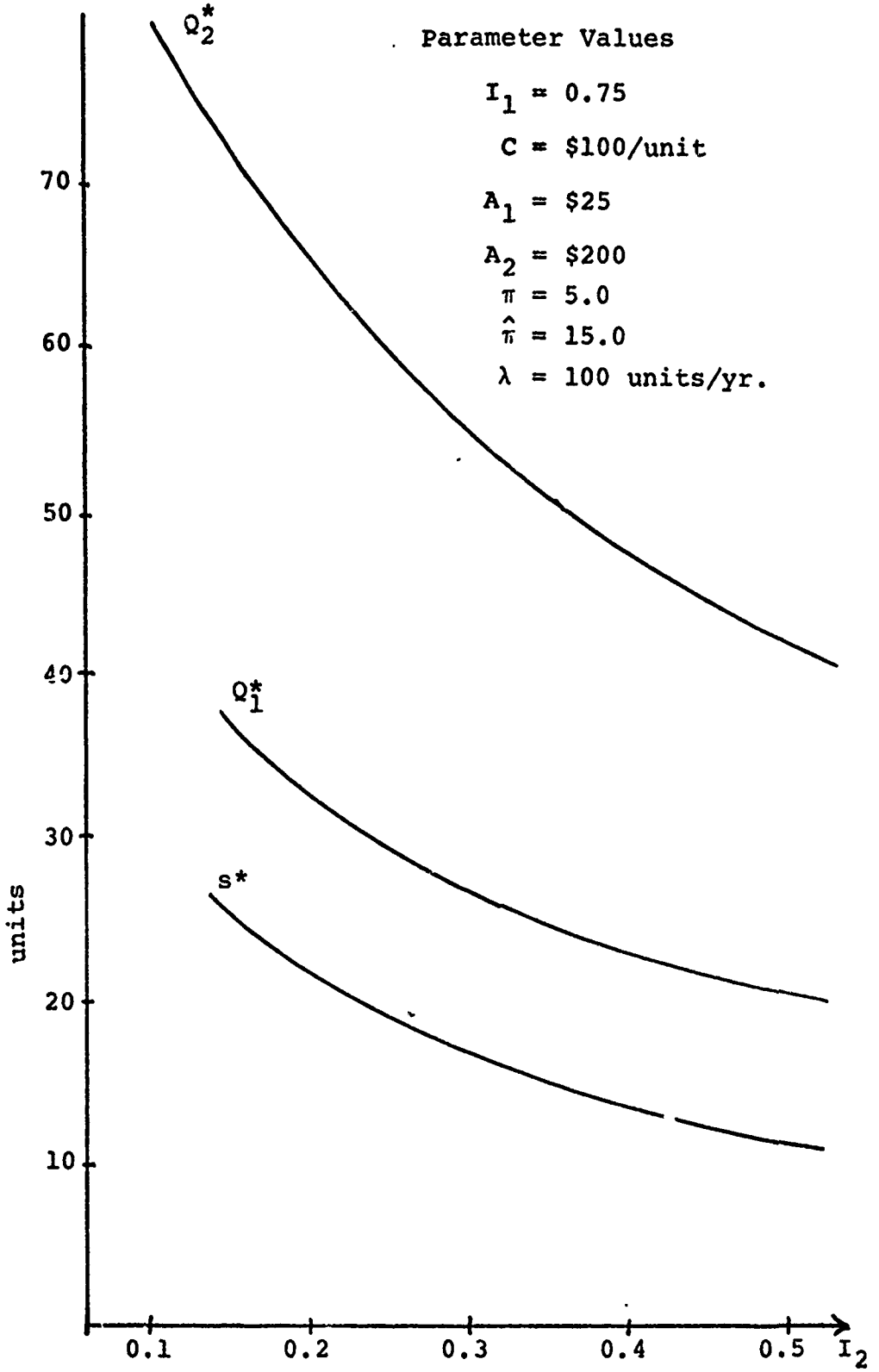

Figure 17. Q_1^*, Q_2^* and s* : A_2 Backorder Model with Low Backorder Costs.

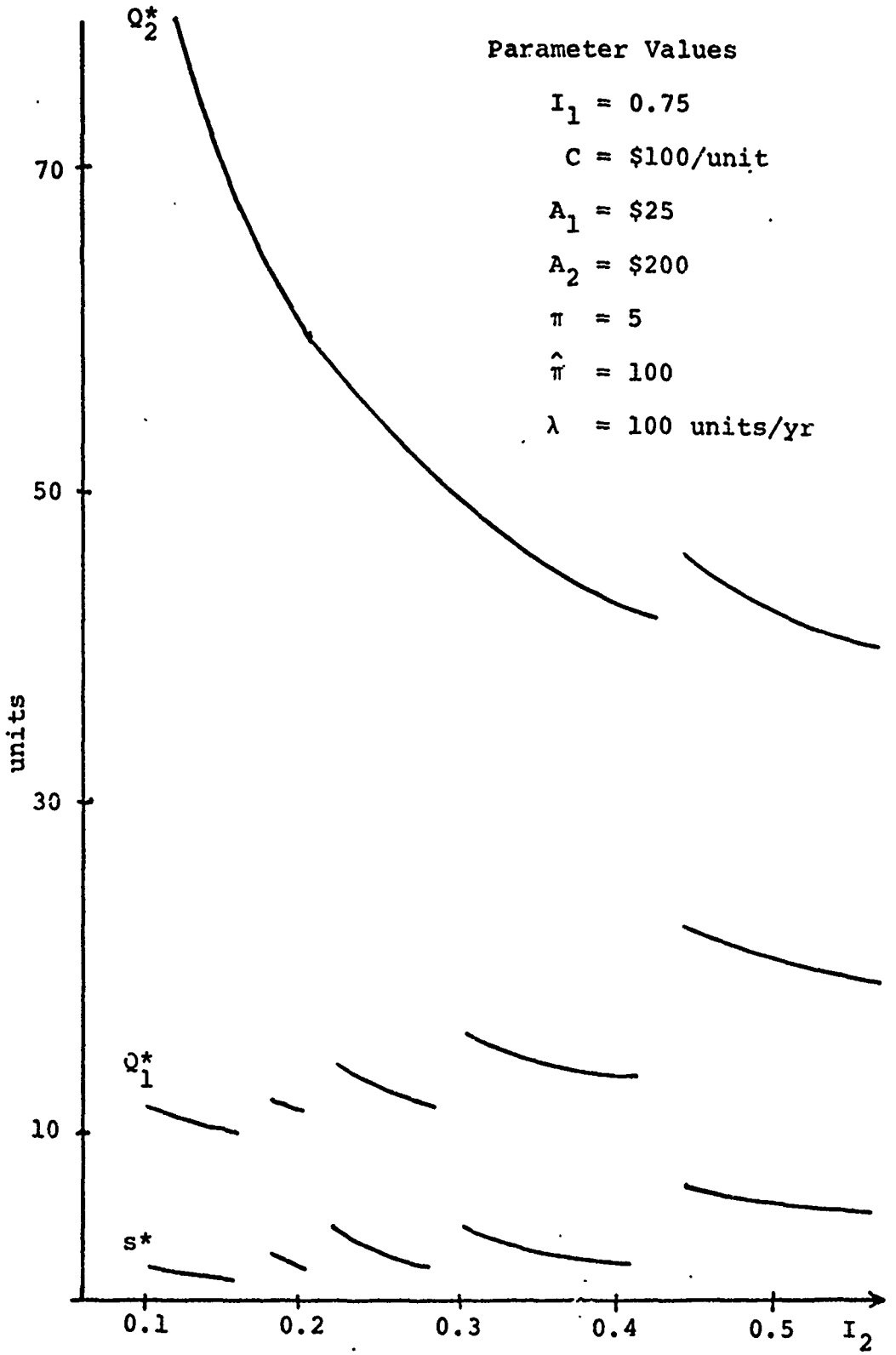

Figure 18. Q_1^*, Q_2^* and s^* : A_2 Backorder Model with High Backorder Costs.

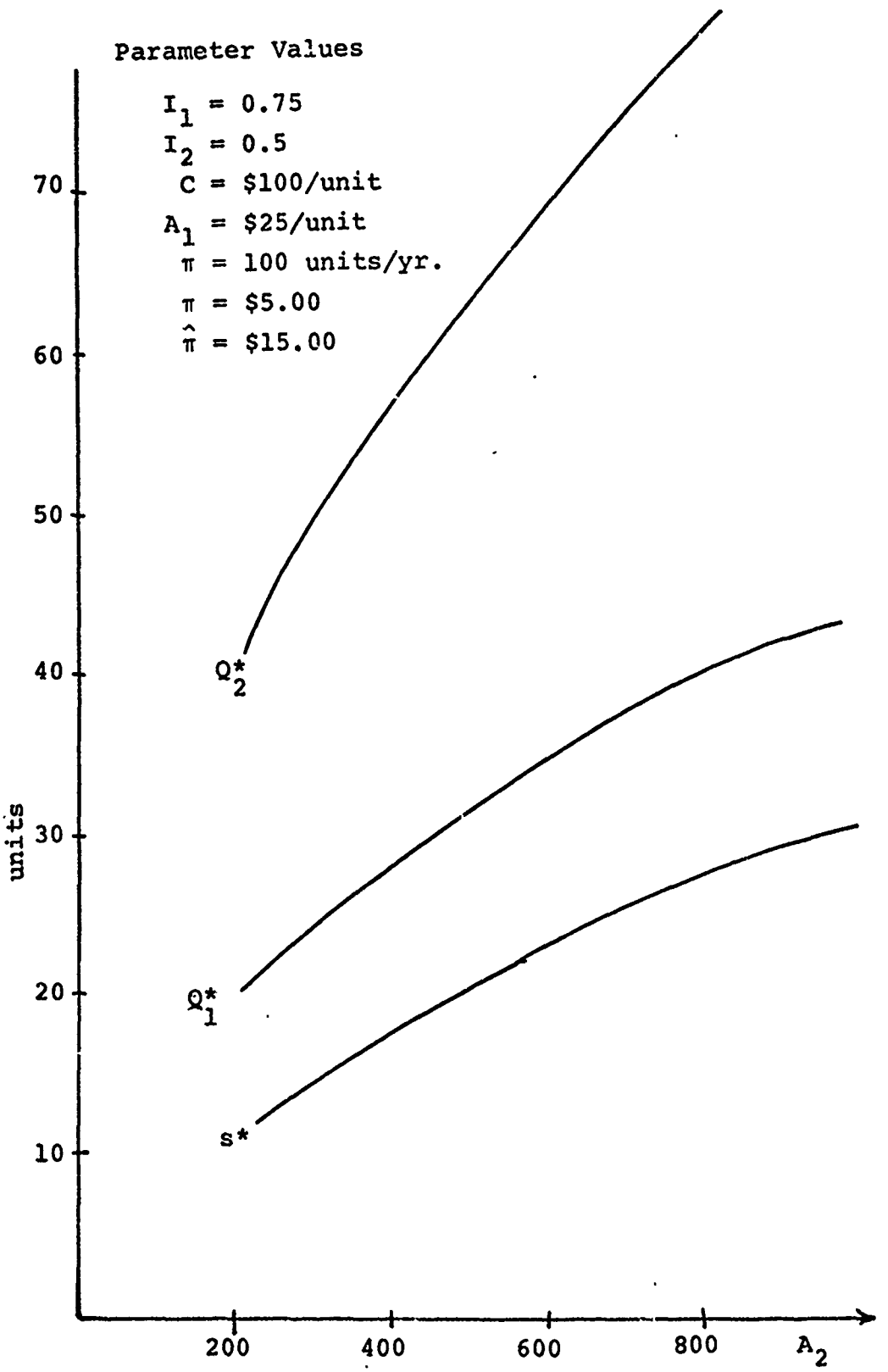

Figure 19. Q_1^*, Q_2^* and s* : I_2 Backorder Model with Low Backorder Costs.

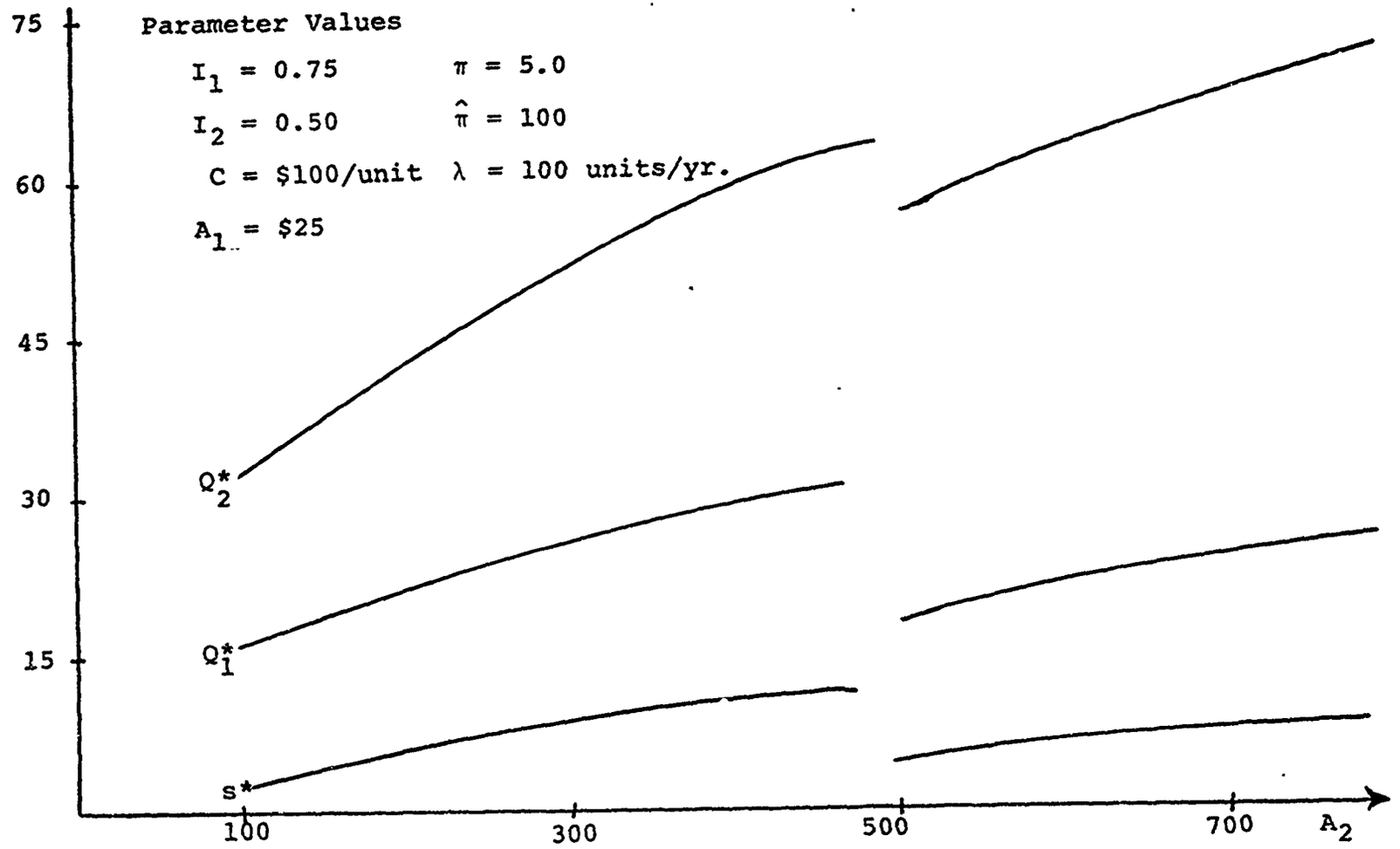

Figure 20. Q_1^*, Q_2^* and s* : I_2 Backorder Model with High Backorder Costs.

LIST OF REFERENCES

1. Hadley, G., and Whitin, T. M., Analysis of Inventory Systems, Prentice-Hall, Inc., 1963.