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SYSTEMS OF QUASILINEAR EQUATIONS AND THEIR APPLICATIONS TO GAS DYNAMICS

By

B. L. Rozhdestvenskiy and N. N. Yanenko

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The fundamentals of the theory of difference schemes are set forth and a variety of numerical solution methods for gas dynamics problems employed in practical calculations are set forth.

A theory of the generalized solution is outlined forsystems of quasilinear equations of the hyperbolic type.

The monograph contains the reuslts of recent work on these problem areas.

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CONTENTS

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INTRODUCTION

Theoretical physics employs a variety of models in describing the behavior of a continuous medium (gas, liquid, or solid); in most cases the models lead to nonlinear differential equations with partial derivatives. This is not fortuitous. Actually, the interaction of two gas molecules depends on their velocities. For this reason, the coefficients of differential equations of a continuous medium describing the averaged pattern of molecular interaction depend not on the time and point in space, but solely on the state of the medium at the given point, i.e., the equations are nonlinear.

The mechanics of a continuous medium is a principal, but not the only field of practical use of systems of nonlinear differential equations in partial derivatives. In describing most real physical processes, we arrive at nonlinear equations, and only substantial additional assumptions on the smallness of the amplitudes of the field waves or the amplitudes of fluctuations in the medium, amplitudes of deviations from the equilibrium state, and so on lead to nonlinear equations, which are studied more profoundly. Chapter Four of this book presents examples of problems in physics, chemistry, and mathematics that are associated with nonlinear equations.

Study of general properties of nonlinear equations and methods of their solution is a fast-growing field of modern mathematics.

Given the wealth of interesting facts and the diversity of original and ingenious methods of investigation and solution of nonlinear equations, this field of mathematics has until now not had as solid theoretical foundation as the theory of linear equations. This is because, first of all, the principle of superpositioning of solutions is not applicable to nonlinear differential equations, so that the set of solutions is not linear.

Among hyperbolic systems of nonlinear equations with partial derivatives, the simplest are the systems of quasilinear equations. Systems with two independent variables have been most thoroughly studied; these systems describe, in particular, the nonsteady one-dimensional and supersonic two-dimensional steady flows of compressive gases and liquids. But even for these systems, at present time there is not a complete enough theory; there are no general theorems of the existence and uniqueness of solutions to problems with initial data (Cauchy's problem).

This situation is explained by the fact that the solution to Cauchy's problem as a whole for hyperbolic systems of nonlinear equations is associated with the marked complexity both of the formulation of the problem as well as methods of solving it. And almost all the principal difficulties arising here appear already for the situation of two independent variables, and we can expect that solutions to multidimensional equations in gas dynamics locally have generally the same features as solutions to one-dimensional equations.

So the study of hyperbolic systems of nonlinear equations with two independent variables represents a wholly necessary and thus far still unsurmounted stage in the exploration of more general nonlinear equations.

From these considerations, the authors decided to limit themselves generally to the theory of hyperbolic systems with two independent variables and to study one-dimensional nonsteady flows of compressible liquids and gases. Therefore, as a rule, we will consider one of the independent variables to be time and denote it by the letter t.

Let us clarify at this point the present status of the problem of the solvability of Cauchy's problem for hyperbolic systems of quasilinear equations and the difficulties arising in attempts to construct the solution to this problem overall. The fundamental method in solving hyperbolic systems of quasilinear equations is the method of characteristics, presented in detail in Chapter One. It is used to prove the existence, uniqueness, and continuous dependence on initial data of the classical solution to Cauchy's problem. These results are highly satisfactory in the sense that the classical solution is constructed throughout the domain of variables e and x, where they exist. We note that the domain of existence of the classical solution, generally speaking, is bounded, since solutions to nonlinear equations, in contrast to linear equations, exhibit the property of unbounded increase in the value of the derivatives, which is called the gradient catastrophe.

The significance of this property is that even at as smooth initial values as desired, the first derivatives of the solution remain bounded, generally speaking, only within a finite time interval. At some $t_0 > 0$, they become unbounded, and when $t > t_0$ no classical solution to the formulated Cuachy's problem exists any longer.

From the viewpoint of gas dynamics this corresponds to the formation of a shock wave (a compression jump) from a compression wave. Thus, if we wish to define the solution to Cauchy's problem for any $t \geq 0$, i.e., overall (and this is precisely how the problem stands, for example, in gas dynamics), then we must first of all give a definition of the solution, since the solution to a system of equations in the usual sense -- a classical solution, does not exist when $t > t_0$, as we stated above.

In most physical problems and, in particular, in gas dynamics the determination of the generalized solution is dictated by the way in which the problem is formulated. Thus, for example, in gas dynamics the basic physical laws from which we derive all consequences are the laws of conservation of mass, momentum, and energy. These laws of conservation are in the nature of integral relations, and they are applicable not only to smooth (differentiable) flows. Rather, differential equations of gas dynamics are derived from these laws of conservation on the assumption of the smoothness of flow.

Thus, we define the generalized solution of gas dynamics equations as a flow (possibly even with discontinuous parameters) satisfying the main laws of conservation: of mass, of momentum, and of energy. To this we add the requirement of thermodynamics on the increase in the entropy of each system closed in the thermal sense. The view is widely held, thus far not refuted by a single example, that a determinate solution exists, is unique, and satisfies all rational requirements.

Here a most essential requirement is that of thermodynamics dealing with the rise in entropy, which shows the possible direction of the process of rapid change in the gas state. This requirement does not figure in an examination of classical solutions to equations in gas dynamics for a gas deprived of viscosity and thermal conductivity, since in smooth flows the entropy of the system is retained by virtue of the same fundamental laws of conservation.

Another approach to generalized(discontinuous) flows of an ideal gas deprived of viscosity and thermal conductivity is also well known in gas dynamics. Since gas without dissipation is an idealization of gas subject to dissipative processes, naturally we can consider its discontinuous flow as the "limiting flow" of a viscous thermally conductive gas as the coefficients of viscosity and thermal conductivity tend to zero. Here it is assumed that viscous flows are always described by classical solutions of differential equations and that the limit as the dissipative coefficients approach zero does exist and is unique in a rational sense. And actually, thus far this assumption has not been overthrown by a single example, though exact proofs have been obtained thus far only for the very particular case of a stationary shock wave.

Here we must bear in mind that in many cases real gases exhibit sufficiently small dissipation so that they can be "approximated" by nondissipative gases. However, the occurrence of dissipative processes, even though limited in extent, leads to an increment in the system's entropy. Thus, the requirement of entropy increase in the discontinuous flow of an ideal gas is associated with the representation of this flow as the "limiting" flow of a viscous thermally conductive gas.

Let us note that from the mathematical point of view the requirement of an entropy increase is a requirement insuring the uniqueness of the generalized solution and its stability with respect to perturbations.

Though this formulation of the problem of the flow of compressible gases was known more than a century ago and even Riemann investigated the simplest discontinuous flows, there has been relatively limited progress in investigating general properties of generalized solutions of equations in gas dynamics. Thus, and we have already mentioned this earlier, up till now there have been no satisfactory existence and uniqueness theorems.

On the other hand, the demands of practice stemming from the urgent need for practical investigation of discontinuous flows, and also the new computational possibilities associated with the emergence of high speed computers has led to a situation in which, in spite of our inadequate information about the general properties of discontinuous flows, different numerical algorithms have been devised and employed for satisfactory calculation of flows containing shock waves. It must be noted that most of the hypotheses, about which we referred to earlier, in developing the numerical algorithms have been regarded as reliable.

Since the direct and rigorous sustantiation of various assumptions on generalized solutions in gas dynamics is a difficult problem, the natural desire arose to test our views even though with model equations and systems of equations which to some extent simulate equations of gas dynamics.

A consequence of this desire was the emergence in recent decades of the so-called theory of generalized solutions of systems of quasilinear equations, or, more concisely, the theory of systems of quasilinear equations (this is usually what is referred to as systems of hyperbolic type). This theory sets out to introduce on analogy with gas dynamics the concept of the generalized solution as an "arbitrary" system of quasilinear equations in partial derivatives of the hyperbolic type, to demonstrate its existence, uniqueness, and continuous dependence on initial problem data, and to study the properties of these solutions. At least formally this theory is more general than one-dimensional gas dynamics and includes the latter as a particular case.

The theory has attracted any mathematicians and several results of Soviet and foreign scientists have aroused expectations of its further development.

Beginning with this view of the advancement of the theory of generalized (discontinuous) solutions of systems of quasilinear equations, the authors confine themselves to the case of only two independent variables and cover the following fundamental problem areas in the book:

1. Methods of constructing classical solutions to systems of quasilinear equations; proofs of existence and uniqueness theorems, and the continuous functions of classical solutions; conditions of forming discontinuities in solutions of arbitrary systems of quasilinear equations. These problem areas are taken up in Chapter one of the book. Here are presented results obtained for classical solutions of systems of quasilinear equations in recent years.

2. Classical and generalized solutions to equations of gas dynamics for one-dimensional nonsteady flows. This problem is taken up in chapter two of the book. The authors deem it advisable to examine in detail several problems in gas dynamics discussed in many reference works. Presented are the fundamentals of thermodynamics, the derivation of equations of gas dynamics for different symmetries of one-dimensional flow, Hugoniot's conditions, general properties of flows, the theory of the shock transition, and self-modeling and analytic solutions of gas dynamics. Including these traditional problem areas of gas dynamics in the book made it possible to deal with, from a unified point of view, several mathematical problems that arise in gas dynamics; moreover, most of the numerical

methods in gas dynamics are actually based on this material. Covered in greater detail than elsewhere is the fundamental problem of the theory of discontinuous solutions as equations of gas dynamics, as well as general systems of quasilinear solutions of the hyperbolic type -- problem of the collapse of an arbitrary discontinuity, and also the interaction of shock waves with each other, with traveling waves, and with the contact boundary.

3. Chapter Three in the book deals with difference methods of solving gas dynamics equations. These methods have now become the principal means of investigating problems in gas dynamics, therefore progress in studying discontinuous flows is closely linked with difference methods.

In this book we were obligated to present the fundamental concepts of the theory of difference methods. Unfortunately, most statements in this theory apply only to the case of linear equations.

The present status of the validation of difference methods used in the numerical solution of problems in gas dynamics, briefly stated, amounts to the following. Classical solutions (smooth flows) can be calculated with practically arbitrary accuracy. The main methods -- the numerical method of characteristics -- is adequately substantiated for classical solutions. At the same time, numerical methods used in calculating discontinuous flows strictly speaking have not been substantiated and in most cases a number of hypotheses on solution behavior, on the approximation of some solutions by others, and so on are used. Most often simply equations for which the behavior of the discontinuous solution is well known are employed to verify any particular assumptions. It is not fortuitous that in this chapter in most cases each scheme is checked with one of the simplest quasilinear equations whose solution can be explicitly written.

This principle in substantiating difference methods shows that progress in this field is closely bound up with progress in investigating general properties of the generalized solutions of systems of quasilinear equations and, in particular, solutions of gas dynamics equations. On the other hand, difference methods yield experimental material and most strongly stimulate advances in the theory of generalized solutions.

4. Chapter Four deals with the theory of generalized solutions of hyperbolic systems of quasilinear equations and contains the main results attained in this field in recent years. Here the chief success must be regarded as the construction of a theory of the generalized solution of a single quasilinear equation, which can be assumed to be almost consummated. The existence, uniqueness, and continuous dependence of a generalized solution on initial data are proven for this equation; the equivalence of definitions of generalized solution from the viewpoint of the law of conservation, on the one hand, and as a limit of "viscid solutions," on the other, is shown.

At the same time, just as in gas dynamics, the study of generalized solutions of systems of equations encounter great difficulties, and here thus far only very scanty results have been forthcoming. The main problem, which is now undergoing comprehensive investigation, is the problem of the disintegration of an arbitrary discontinuity. By means of this simplest problem, we can study the structure of the generalized solution and even construct generalized solutions for the case of a system of two equations, by relying on the former solution.

Chapter four represents the main result obtained for a single quasilinear equation; covered in this chapter is the problem of disintegration of a discontinuity for an arbitrary hyperbolic system of quasilinear equations; also presented are some results appertaining to more general cases. This chapter concludes with a description of several problems of different fields of science associated with the theory of systems of quasilinear equations and, in particular, discontinuous solutions of such equations.

The book is divided into chapters, sections, and subsections. The numbering of formulas is self-contained in each subsection. Therefore in designating formulas, along with formula number the subsection number and the section number are added, so that formula (2.7.18) stands for formula (18) in subsection 7 of section 2 of a given chapter. Only when the reference is not made outside the confines of a given subsection is the formula number alone indicated.

In writing the book, we try to treat as fully as possible the entire range of problems associated with classical and generalized solutions of gas dynamics equations and more general quasilinear systems. Still, our personal points of view, undoubtedly, have affected the choice of material.

In writing the book, the authors consulted with different teams of Soviet mathematicians. Among these we can cite the collectives headed by M. V. Keldysh, A. N. Tikhonov and A. A. Samarskiy, and I. M. Gel'fand. Our opinions and points of view were inevitably affected by consultation with friends and colleagues in work; several results were made known to us by oral communication with them.

For a number of years each of us has given special courses to students on the subject areas of this book. As a result of working on the book, a number of new results, presented here for the first time, were obtained.

Summing up, it must be clear that mathematical theory of discontinuous solutions of systems of quasilinear equations and, in particular, equations of gas dynamics though containi[illegible] many remarkable results and containments, is still far from its culmination [illegible]pe that our book will afford the reader a grasp of modern methods of solutio[illegible] and investigation of systems of quasilinear equations and at the same time spur him to further investigation in this highly interesting and rapidly growing field of applied mathematics.

This book grows out of many long years of work during which we always enjoy the cooperation of many of our friends and colleagues at work as well as many of our students. To all we express our heartfelt gratitude.

We are also indebted to A. N. Tikhonov whose advice we were continually favored with.

The assistance of L. V. Ovsyanikov was especially valuable for us, since he not only reviewed the manuscript of the entire book and made a number of valuable comments, but also placed our disposal materials which we used in writing section 13 of Chapter One.

A. A. Samarskiy read the manuscript of Chapter Three of this book and made several valuable comments.

We were given a great deal of help by N. N. Kuznetsov, who read the entire manuscript, made several valuable observations, and as an editor of this book did much to promote its improvement.

We express our deep sense of appreciation to all of these.

CHAPTER ONE FUNDAMENTALS OF THE THEORY OF HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS WITH TWO INDEPENDENT VARIABLES

Section I. Basic Definitions

In this book we will limit ourselves to considering differential equations for functions dependent only on two independent variables.

The system of relations

$$\mathcal{F}_i\left(x, t, u_1, u_2, \ldots, u_n, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}, \frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_n}{\partial t}\right)=0 \quad (1)$$
$$(i=1, 2, \ldots, m),$$

relating values of the unknown functions $u_1(x, t)$, $u_2(x, t)$, ... $u_n(x, t)$ and their first derivatives $\frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}, \frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_n}{\partial t}$, is called a first-order system of differential equations with respect to the function $u_1, \ldots, u_n$. System (1) is referred to as determinate for the case when $m = n$. We will limit ourselves to considering only this case.

Introducing the vectors

$$u=\{u_1, \ldots, u_n\}, \quad \frac{\partial u}{\partial x}=\left\{\frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}\right\},$$
$$\frac{\partial u}{\partial t}=\left\{\frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_n}{\partial t}\right\},$$

we can write system (1) more concisely:

$$\mathcal{F}_i\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)=0 \quad (i=1, 2, \ldots, n). \quad (2)$$

The functions $u_i = v_i(x, t)$, exhibiting continuous first derivatives and satisfying the equations of system (2), are called the solution of this system of equations.

If the system of nonlinear differential equations(2) can be represented in a form that is solvable with respect to the derivatives of the functions $u_1, \ldots, u_n$ with respect to any derivative (for example, t):

$$\frac{\partial u_i}{\partial t} = \varphi_i\left(x, t, u, \frac{\partial u}{\partial x}\right) \quad (i = 1, 2, \ldots, n), \tag{3}$$

then this form of system (2) will be called the normal form. System (3) is called a Cauchy-Kovalevski type system. We note that when system (2) is reduced to normal form the transformation of variable x, t is admitted.

System (2) is called a system of quasilinear equations if the functions $\mathcal{F}_i$ are linear with respect to the variables $\partial u/\partial x$, $\partial u/\partial t$; if however functions $\mathcal{F}_i$ are linear over a set of variables u, $\partial u/\partial x$, $\partial u/\partial t$, then system (2) is called linear.

A first-order system of quasilinear equations can be written as

$$\sum_{j=1}^{n} a_{ij}\frac{\partial u_j}{\partial t} + b_{ij}\frac{\partial u_j}{\partial x} = c_i \quad (i = 1, 2, \ldots, n), \tag{4}$$

where the coefficients a_{ij}, b_{ij}, c_i depend on x, t, u. If the coefficients a_{ij}, b_{ij} do not depend on u, then system (4) is called semilinear (if in this case c_i is linearly dependent on u, then it is linear). We can somewhat simplify the notation of system (4), if we bring into our consideration the above-defined vectors u, $\partial u/\partial x$, $\partial u/\partial t$, the vector $c = \{c_1, \ldots, c_n\}$ and the matrix

$$A = \begin{Vmatrix} a_{11} & \ldots & a_{1n} \\ a_{n1} & \ldots & a_{nn} \end{Vmatrix} = ((a_{ij})), \quad B = ((b_{ij})).$$

When using matrical notations it is assumed that the symbol Au and uA denote vectors whose components can be computed by the rule:

$$(Au)_k = \sum_{j=1}^{n} a_{kj}u_j, \quad (uA)_k = \sum_{j=1}^{n} a_{jk}u_j = (A'u)_k \quad (k = 1, 2, \ldots, n), \tag{5}$$

where A' is a transposed vector.

If matrix A is symmetrical, then $A' = A$, and $Au = uA$. The scalar derivative of vectors u, v is given by the formula

$$uv = (u, v) = \sum_{i=1}^{n} u_i v_i.$$

Therefore for formulas (5) it follows that $v(Au) = (vA) = vAu$. We will denote by the norm $\| u \|$ of the vector u the quantity

$$\| u \| = \sqrt{(u, u)} = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

We will refer to as the norm of matrix A that smallest number $\| A \|$ which for any vector u satisfies the inequalities $\| Au \| \leqslant \| A \| \, \| u \|$. It is not difficult to see that $\| A \| = \sqrt{\lambda}$, where λ is the largest eigenvalue of matrix AA' (or $A'A$, which amounts to the same thing). Since $\lambda \leqslant \operatorname{Sp} AA'$, then

$$\| A \| \leqslant \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

Let us recall several more definitions from linear algebra. The vector $l = \{ l_1, \ldots, l_n \}$ and the number ξ are called, respectively, the left eigenvector and the eigenvalue of matrix A if

$$lA = \xi l, \quad \| l \| \neq 0. \qquad (6)$$

Similarly, vector r is called the right eigenvector of matrix A if

$$Ar = \xi r, \quad \| r \| \neq 0. \qquad (7)$$

By formulas (5) and (6), the eigenvalues ξ of matrix A is a root of the characteristic equation

$$\operatorname{Det} ((a_{ij} - \xi \delta_{ij})) = 0 \qquad (8)$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$). To each eigenvalue ξ of matrix A corresponds a linear space of left eigenvectors l and right eigenvector r. The measure of these spaces is $n - \beta$, where β is the length of the matrix

$$A - \xi E = ((a_{ij} - \xi \delta_{ij})). \qquad (9)$$

The matrix rank (9), as we know, is not smaller than $n - \alpha$, where α is the multiplicity of the root ξ of equation (8).

Let us assume that the eigenvalues ξ of matrix A are real. Let us number them in increasing order, i.e., we will assume that

$$\xi_1 \leq \xi_2 \leq \dots \leq \xi_n. \qquad (10)$$

The equality sign in (10) is admitted owing to the possibility of multiple roots of equation (8), and each multiple root of ξ is repeated in (10) as many times as its multiplicity.

If for any eigenvalue ξ of matrix A of multiplicity α, the matrix rank (9) is $n - \alpha$, then the eigenvectors, both the left l and the right r corresponding to all eigenvalues, form the basis in space E_n of the vectors $u = \{u_1, \dots, u_n\}$.

Thus, in this case we will assume that there exist eigenvectors l^1, $l^2, \dots, l^n$ forming the basis in space E_n, i.e., satisfying the condition

$$\det \Lambda = \det((l_i^j)) = \det \begin{vmatrix} l_1^1 & \dots & l_n^1 \\ \dots & \dots & \dots \\ l_1^n & \dots & l_n^n \end{vmatrix} \neq 0. \qquad (11)$$

The index of the left eigenvector l^k in this case corresponds to the number of eigenvalue ξ_k; the latter are ordered by means of inequalities (10).

If $\xi_k \neq \xi_j$, then l^k and r^j are orthogonal. In fact, suppose

$$l^k A = \xi_k l^k, \quad A r^j = \xi_j r^j. \qquad (12)$$

Multiplying scalarly the first of the equalities (12) by r^j, the second by l^k, and subtracting, we get

$$(\xi_k - \xi_j) l^k r^j = (l^k A) r^j - l^k (A r^j) = l^k (A r^j) - l^k (A r^j) = 0. \qquad (13)$$

Since $\xi_k \neq \xi_j$, from this follows the orthogonality of l^k and r^j. For the case when all eigenvalues of matrix A are simple, the equality sign in inequalities (10) is canceled out and the left and right eigenvectors form a biorthogonal system, i.e.,

$$l^k r^j = \sum_{a=1}^{n} l_a^k r_a^j = 0 \qquad \text{when} \qquad k \neq j. \tag{14}$$

If matrix A is symmetric then we can assume that $r^k = l^k$. We require that the left eigenvectors l^k of matrix A satisfies the norm on condition that:

$$\| l^k \| = 1 \qquad (k = 1, \ldots, n). \tag{15}$$

Then, if for any eigenvalue $\xi = \xi_k$, the matrix rank (9) is $n - \alpha$, the eigenvectors l^k form a normed basis in E_n and naturally satisfy condition (11); if in this case matrix A is symmetric, then the basis $\{l^k\}$ can be chosen as orthonormed.

Matrix A is referred to as positively defined if all the eigenvalues are positive; it is nonsingular if $\xi = 0$ is not its eigenvalue, and singular, otherwise.

Limiting ourselves to this brief recapitulation of linear algebra, let us write system (4) as

$$A\, \partial u/\partial t + B\, \partial u/\partial x = c. \tag{16}$$

For the case when matrix A is nonsingular, system (16) reduces to the normal form (3) and can be, after transformations, written as

$$\partial u/\partial t + A_1\, \partial u/\partial x = b, \tag{17}$$

where $A_1 = A_1(x, t, u)$, $b = b(x, t, u)$ are certain new matrices and a new vector, respectively. Below we will limit ourselves to studying system (16) which can be reduced to normal form (17).

Above we made an assumption on $A(x, t, u)$ of the system of equation (16). However, A depends on u, i.e., on the assumption which is as a rule unknown to us. Therefore let us stipulate in which sense we make the assumptions on the coefficients of systems (16) and (17).

1) Either we will assume that solution $u = u(x, t)$ of systems (16) and (17) is given as a function of the variable x, t, then the realization of any restriction imposed on matrices A, B, and A_1, and on vectors c, b, is verified forthwith.

2) Or else these restrictions are satisfied by identity (for any values $u = \{u_1, \ldots, u_n\}$) in some singly-connected domain of space (x, t, u) in which

the system of quasilinear equations and its solutions will be considered.

In this chapter we will impose the restrictions mainly in the second sense.

Section II. Characteristic Directions of a System of Quasilinear Equations

1. Derivative relative to direction. Assume f(x, t) is a differentiable function of its variables. At some point (x_0, t_0) we will consider the expression

$$A\frac{\partial f}{\partial t}+B\frac{\partial f}{\partial x}, \qquad (1)$$

assuming that A, B are not simultaneously equal to zero.

For any A and B that are continuous in some neighborhood of the point (x_0, t_0) we can find a smooth curve Γ running through this point and such that when it is suitably parametrized equation (1) is proportional to the derivative of the function f(x, t) at the curve Γ with respect to the parameter τ.

Actually, suppose the curve Γ is given by the equations:

$$\Gamma: x=x(\tau),\quad t=t(\tau),\quad x(\tau_0)=x_0,\quad t(\tau_0)=t_0. \qquad (2)$$

Then at the curve Γ the function f(x, t) is a function of one variable τ: $f(x(\tau), t(\tau)) = F(\tau)$. Let us reply that expression (1) be proportional to $F'(\tau)$ whatever the function f.

This will be done if

$$dx/d\tau = \alpha B,\quad dt/d\tau = \alpha A, \qquad (3)$$

where α is any derivative function τ. Clearly, the essential condition uniquely defining the direction of curve Γ at the point (x_0, t_0) is the equation

$$dx/B = dt/A, \qquad (4)$$

and formulas (3) define the corresponding parametrization.

We will call the derivative $F'(\tau)$ for the natural parametrization of Γ when $\alpha = \frac{1}{\sqrt{A^2+B^2}}$ the derivative of function f with respect to direction Γ. In this case the parameter τ is the length of the arc of the curve Γ. When $\alpha = 1$, expression (1) will be called the derivative of function f with respect to parameter τ in the direction of the curve Γ. This simple concept finds important applications in the theory of equations with partial derivatives.

Let us consider the simplest differential equation

$$A(x, t)\frac{\partial u}{\partial t} + B(x, t)\frac{\partial u}{\partial x} = 0, \tag{5}$$

assuming that functions A and B are continuously differentiable. Equations

$$\frac{dt}{d\tau} = A(x, t), \quad \frac{dx}{d\tau} = B(x, t) \tag{6}$$

or equation (4) defines the single-parametric family of curves Γ. The parameter τ is defined along each of these curves uniquely if along some (arbitrarily chosen) curve γ_0 intersection of the curve Γ we set $\tau = \tau_0$ (Figure 1.1).

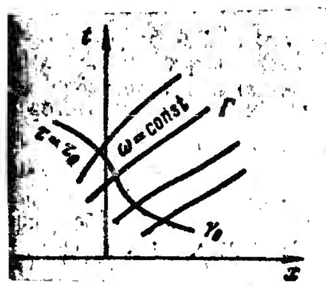

Figure 1.1

Let us bring into correspondence to each curve Γ the value of a certain parameter ω (for example, the length of the arc of the curve γ_0 measured from an arbitrary point on it to the point at which intersects with the given curve Γ). Then to each point (x, t) there will correspond one and only one number of pair τ, ω.

We can therefore assume the function u(x, t) to be the function of the variables τ, ω; the equations of the lines Γ are of the form $\omega =$ const, while equation(5), according to the foregoing, is written as

$$\frac{\partial u(\tau, \omega)}{\partial \tau} = 0. \tag{7}$$

Hence it follows that $u = F(\omega)$ is the general solution of equation (5) and the function u(x, t) is constant along Γ curves τ, ω.

The directions of the curves Γ defined by the vector $\{B, A\}$ are called the characteristic directions of equation (5), and the curves Γ are the characteristics.

Let us note that the form (7) of equation (5) no longer assumes the existence of the derivatives $\partial u/\partial x$, $\partial u/\partial t$: equation (7) is satisfied by an arbitrary function $F(\omega)$, in particular, even a discontinuous function. Here the function $u = F(\omega)$ can be interpreted as the solution of equation (5) in the generalized sense.

2. Hyperbolic systems of quasilinear equations. Let us consider the system of quasilinear equations

$$\partial u/\partial t + A\,\partial u/\partial x = b. \tag{1}$$

Multiplying it by the vector l, we get the scalar equation

$$l\frac{\partial u}{\partial t} + lA\frac{\partial u}{\partial x} = lb. \tag{2}$$

If l is the left eigenvector of matrix A, then equation (2) can be written as

$$l\left(\frac{\partial u}{\partial t} + \xi\frac{\partial u}{\partial x}\right) = lb, \tag{3}$$

where ξ is the corresponding eigenvalue of matrix A.

In equation (3) all components u_i of vector u are differentiated in the same direction. Actually, by writing equation (3) in components, we get

$$\sum_{i=1}^{n} l_i\left(\frac{\partial u_i}{\partial t} + \xi\frac{\partial u_i}{\partial x}\right) = \sum_{i=1}^{n} l_i b_i.$$

denoting by

$$\left(\frac{du_i}{dt}\right) = \frac{\partial u_i}{\partial t} + \xi\frac{\partial u_i}{\partial x}$$

the derivative of function $u_i(x\ t)$ with respect to the variable t in the direction $dx/dt = \xi$, we see that equation (3) contains a linear combination of the derivative (du_i/dt). The equation $dx/dt = \xi$ defines the direction of differentiation in equality (3), called the characteristic direction of the system of equations (1), which is common to all functions $u_i(x, t)$.

We will refer to the quasilinear equations (1) as hyperbolic in some singly-connected domain D of the space of variables x, t, u, if the following conditions are satisfied at each point of this domain:

1) all eigenvalues $\xi = \xi_k(x, t, u)$ of matrix $A = A(x, t, u)$ are real; and

2) there exists the basis $\{l^1(x, t, u), \ldots, l^n(x, t, u)\}$ in the space E_n, composed of left eigenvectors of matrix A governed by the norming condition, i.e., there exist normed eigenvectors $l^1, l^2, \ldots, l^n$ satisfying the condition (1.11).

Let us note that a system (1) is semilinear, then the eigenvalue ξ_k and the left eigenvectors l^k do not depend on u. Therefore the condition of hyperbolicity for the semilinear systems is defined in some domain of variable (x, t) for arbitrary u (in a cylindrical domain).

As a part of the definition of hyperbolicity, let us note that often conditions 1) and 2) are supplemented further by the requirement of a determinate smoothness of eigenvectors l^k and eigenvalues ξ_k. Thus, for example, in the book [1] system (1) is called hyperbolic if conditions 1) and 2) are satisfied, and moreover, $\xi_k(x, t, u)$ and $l^k(x, t, u)$ exhibit the same smoothness as the elements of matrix A(x, t, u).

In the following we will of course have to resort to assumptions on the smoothness of ξ_k and l^k. We will do this to the extent that it is necessary. Let us note in this regard that a given smoothness of l^k and ξ_k does not always stem from the assumption of the same smoothness for the matrix A.

Let us show this in the following example of a system of two quasilinear equations:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial t} + a(u_1, u_2)\frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} = 0.$$

Matrix A in this case is of the form

$$A = A(u) = \begin{Vmatrix} u_1 & 0 \\ a(u_1, u_2) & u_2 \end{Vmatrix},$$

and its eigenvalues ξ_1, ξ_2 are determined from the equation

$$(\xi - u_1)(\xi - u_2) = 0,$$

from whence $\xi_1 = u_1$, $\xi_2 = u_2$. The eigenvectors l^1, l^2 are defined from the equations

$$0 \cdot l_1^1 + a(u_1, u_2) l_2^1 = 0, \quad 0 \cdot l_1^1 + (u_2 - u_1) l_2^1 = 0, \qquad (4)$$
$$(u_1 - u_2) l_1^2 + a(u_1, u_2) l_2^2 = 0, \quad 0 \cdot l_1^2 + 0 \cdot l_2^2 = 0. \qquad (5)$$

taking the norming condition into account, let us now define l^1 and l^2 in the domain $u_1 \neq u_2$:

$$\left. \begin{array}{l} l^1 = \{1;\ 0\}, \\ l^2 = \left\{ \dfrac{\frac{a(u_1, u_2)}{u_1 - u_2}}{\sqrt{1 + \left(\frac{a}{u_2 - u_1}\right)^2}},\ \dfrac{1}{\sqrt{1 + \left(\frac{a}{u_2 - u_1}\right)^2}} \right\}. \end{array} \right\} \qquad (6)$$

Let us consider the straight line $u_1 = u_2$ on the plane of variables (u_1, u_2). If $a(u_1, u_1) \neq 0$, then from (5) we find $l^2 = \{1;\ 0\}$, and the hyperbolicity condition 2) is violated on the straight line $u_1 = u_2$. Let us consider in more detail the case $a(u_1, u_1) = 0$. Here equations (4) and (5) become identical at the straight line $u_1 = u_2$ and we can select as l^1 and l^2 two arbitrary noncollinear unit factors and, therefore, the system is hyperbolic.

But if we have required that the vectors l^1, l^2 exhibit a certain smoothness, then the requirement of the same smoothness for the function $a(u_1, u_2)$ would not guarantee this. Suppose, for example, that we require the vectors l^1, l^2 to be continuous at the straight line $u_1 = u_2$.

Obviously, the only vector l^1 that is continuous at the straight line $u_1 = u_2$ would be the vector $l^1(u_1, u_2) = \{1; 0\}$; as for the vector l^2, its continuity necessitates, for example, the continuity of the function $a(u_1, u_2)(u_2 - u_1)^{-1}$. For example, if

$$a(u_1, u_2) = |u_1 - u_2|^{\alpha} b(u_1, u_2) \quad (b(u_1, u_1) \neq 0),$$

then when $\alpha > 1$ the unknown continuous vector l^2 is given by formula (6)(when $u_1 \neq u_2$) by the formula $l^2(u_1, u_1) = \{0; 1\}$. When $\alpha < 1$ no continuous function l^2 exists.

It is not difficult to verify that in this example, to ensure the continuity of m-th derivatives of l^2 we must require that the function $a(u_1, u_2)$ have m continuous derivatives, and also (m + 1)-th derivatives at the straight line $u_1 = u_2$.

The situation becomes very much simpler in a highly important particular case of hyperbolic systems, which we will call hyperbolic in the narrow sense.

We will refer to the system of equations (1) as hyperbolic in the narrow sense in a singly-connected domain D of variables (x, t, u) if at each point of this domain the eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$ of matrix A are real and distinct.

In this case the eigenvalues can be ordered, and we will assume that everywhere in D the following inequalities are satisfied:

$$\xi_1(x, t, u) < \xi_2(x, t, u) < \ldots < \xi_n(x, t, u).$$

Then, as indicated in section I, the eigenvectors $l^k(x, t, u)$ are linearly independent. It is easy to see that in this case $l^k(x, t, u)$, $\xi_k(x, t, u)$ possess the same degree of smoothness as the elements of the matrix A(x, t, u).

And so, system (1) hyperbolic in domain D, by multiplying it by the left eigenvector l^k, is reduced to the form

$$l^k(x, t, u)\left[\frac{\partial u}{\partial t} + \xi_k(x, t, u)\frac{\partial u}{\partial x}\right] = f_k(x, t, u) \quad (k = 1, \ldots, n), \tag{7}$$

where

$$f_k = l^k b = \sum_{\alpha=1}^{n} l^k_{\alpha} b_{\alpha}.$$

From condition 2) of the definition of hyperbolic system (1) follows the equivalence of system (7) of initial condition (1).

We will call equation (7) the characteristic form of the system of equation (1).

Expanding the notation of this system in components thusly:

$$\sum_{\alpha=1}^{n} l_{\alpha}^{k}(x, t, u)\left[\frac{\partial u_{\alpha}}{\partial t}+\xi_{k}(x, t, u)\frac{\partial u_{\alpha}}{\partial x}\right]=f_{k}(x, t, u) \quad (k=1, \ldots, n).$$

Sometimes we will write it in the following form:

$$l^{k}(x, t, u)\left(\frac{du}{dt}\right)_{k}=f_{k}(x, t, u) \quad (k=1, \ldots, n),$$

where the symbol $(df/dt)_k$ denotes the quantity

$$\left(\frac{df}{dt}\right)_{k}=\frac{\partial f}{\partial t}+\xi_{k}\frac{\partial f}{\partial x}.$$

3. Hyperbolic system of nonlinear equations. Let us consider a system of nonlinear equations written in the normal form:

$$\frac{\partial v_i}{\partial t}+\varphi_{i}\left(x, t, v, \frac{\partial v}{\partial x}\right)=0 \quad (i=1, \ldots, n). \tag{1}$$

Setting

$$\partial v_i/\partial x = \omega_i, \tag{2}$$

let us write system (1) in the form

$$\frac{\partial v_i}{\partial t}+\varphi_{i}(x, t, v, \omega)=0 \quad (i=1, 2, \ldots, n). \tag{3}$$

Suppose $\varphi_k(x, t, v, \omega) \in C_2$. Let $A(x, t, v, \omega)$ refer to the square matrix of all n:

$$A=\left(\left(\frac{\partial \varphi_{k}(x, t, v, \omega)}{\partial \omega_{i}}\right)\right)=\left(\left(\frac{\partial \varphi}{\partial \omega}\right)\right). \tag{4}$$

We will call the system of nonlinear equations (1) hyperbolic in some domain of variation of variables x, t, v, ω, if at each point of this domain the eigenvalues $\xi = \xi_k(x, t, v, \omega)$ and the left eigenvectors $l^k(x, t, v, \omega)$ of matrix A satisfy requirements 1) and 2) of subsection 2.

The hyperbolic system of nonlinear equations (1) reduces to a hyperbolic system of quasilinear equations. Differentiating each of the equations (1) with respect to the variable x and taking the symbols (2) into account, we get

$$[\text{illegible}] + \frac{\partial \varphi_i}{\partial v_\alpha}\omega_\alpha + (\varphi_i)'_x = 0 \qquad (5)$$

In formula (5) summation*) is carried out with respect to the Greek subscript α within the limits from 1 to n. Below, for simplicity of notation, we will often employ this convention.

Combining equations (3) and (5), we get a system of 2n equations

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= -\varphi(x, t, v, \omega), \\ [\text{illegible}] + A(x, t, v, \omega)\frac{\partial \omega}{\partial x} &= \bar{f}(x, t, v, \omega), \end{aligned}\right\} \qquad (6)$$

where

$$\left.\begin{aligned} &\bar{f}_i = -[\varphi_i(x, t, v, \omega)]'_x - \frac{\partial \varphi_i}{\partial v_\alpha}\omega_\alpha, \quad v = (v_1, \ldots, v_n), \\ &\omega = (\omega_1, \ldots, \omega_n), \quad \varphi = (\varphi_1, \ldots, \varphi_n), \quad \bar{f} = (\bar{f}_1, \ldots, \bar{f}_n), \end{aligned}\right\} \qquad (7)$$

which we can consider as a system of quasilinear equations with respect to 2n unknowns

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*) To avoid confusion, let us stress that the summation is carried out only with respect to Greek subscript. For example, $l^{k}_{\alpha}\xi_k \frac{\partial u_\alpha}{\partial x} = \sum_{\alpha=1} l^{k}_{\alpha}\xi_k \frac{\partial u_\alpha}{\partial x}$, and summation is now carried out with respect to the Latin subscript k.

$$u = [illegible] = \left\{v_1, \dots, v_n; \frac{\partial v_1}{\partial x}, \dots, \frac{\partial v_n}{\partial x}\right\}$$

Let us show that the system of 2n quasilinear equations (6) is hyperbolic. Multiplying the second group of squation (6) by the left eigenvector $l^k(x, t, v, \omega)$ of matrix $A(x, t, v, \omega)$, we get

$$l^k\left[\frac{\partial \omega}{\partial t} + \xi_k \frac{\partial \omega}{\partial x}\right] = f_k \quad [illegible] \quad (k = 1, \dots, n) \tag{8}$$

$$f_k = l^k f \tag{9}$$

Thus, system (6) is reduced to the form

$$l^k\left[\frac{\partial \omega}{\partial t} + \xi_k \frac{\partial \omega}{\partial x}\right] = f_k, \quad \frac{\partial v_k}{\partial t} = \varphi_k(x, t, v, \omega) \quad (k = 1, \dots, n), \tag{10}$$

from whence comes its hyperbolicity.

If $\varphi_k = \varphi_k(x, t, \omega)$, i.e., $\partial \varphi_k / \partial v_i \equiv 0$, then the first group of equations (10) can be considered independently as a hyperbolic system of n quasilinear equations with respect to n unknowns $\omega_1, \dots, \omega_n$.

Of course, we cannot refer to the equivalence of system (10) and system (1).

First of all, it is not any solution v_k, ω_k of system (10) that yields the solution $v_k(x, t)$ to system (1). Actually, the solution v_k, ω_k of system (10) does not necessarily, generally speaking, satisfy equations (2).

As we show in subsection 3 of section IX, satisfying conditions (2) reduces to satisfying them at the straight line $t = 0$. Thus, solutions to system (10) reduce to the solution $v(x, t)$ to system (1) only if conditions (2) are satisfied. On the other hand, the nonequivalence of system (10) and (1) is manifested also in that the solution to system (10) satisfying conditions (2) requires that $v(x, t) \in C_2$. At the same time the definition of the solution $v(x, t)$ to system (1) requires only its continuous differentiability.

Therefore the equivalence of system (10) to system (1) cannot hold only for solutions $v(x, t)$ to system (1) of class C_2.

Section III. Riemann Invariants

1. Invariants of a semilinear system of equations. In each equation of the characteristic system (2.2.7) functions $u_i(x, t)$ are differentiated in the same direction. In several cases further simplification of the characteristic system is possible: by change of variables we can succeed in differentiating only one function of the variable x, t, u in each of the equations.

Let us first consider the case of the semilinear system. Then equation (2.2.7) can be written as

$$\left(\frac{dr_k}{dt}\right)_k = \frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = g_k \quad (k = 1, \ldots, n), \tag{1}$$

where

$$r_k = l^k_\alpha u_\alpha, \quad g_k = f_k + u_\alpha \left(\frac{dl^k_\alpha}{dt}\right)_k = f_k + u_\alpha \left[\frac{\partial l^k_\alpha}{\partial t} + \xi_k \frac{\partial l^k_\alpha}{\partial x}\right]. \tag{2}$$

Since for the hyperbolic system $\mathrm{Det}((l^k_i)) \neq 0$, then

$$\frac{\partial(r_1, \ldots, r_n)}{\partial(u_1, \ldots, u_n)} = \mathrm{Det}\,\Lambda \neq 0 \tag{3}$$

and the variables $r_k(x, t, u)$ can be taken as new unknown functions. Let us express from equations (2) $u_1, \ldots, u_n$ in terms of $r_1, \ldots, r_n$:

$$u_k = \lambda^k_\alpha r_\alpha = \lambda^k r, \tag{4}$$

where we let r stand for the vector $\{r_1, \ldots, r_n\}$, and λ^k_α are the coefficients of the matrix Λ^{-1} that is the inverse of matrix Λ:

$$\Lambda = ((l^k_i(x, t))), \quad \Lambda^{-1} = ((\lambda^k_i(x, t))).$$

Substituting formulas (4) in the right sides of system (2), we arrive at the system of equations

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = g_k(x, t, r) \quad (k = 1, \ldots, n), \tag{5}$$

which we will call a system written in invariants.

Let us illustrate the concept of invariants with the example of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (a = \text{const}).$$

It reduces to the hyperbolic system

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 0,$$

whose characteristic form is

$$\left(\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x}\right) - a\left(\frac{\partial v}{\partial t} - a\frac{\partial v}{\partial x}\right) = 0,$$

$$\left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right) + a\left(\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x}\right) = 0.$$

Therefore, the invariants defined by formulas (2) are as follows:

$$r_1 = u - av, \quad r_2 = u + av.$$

(Let us note that here we have used the nonnormed vectors l^1, l^2.)

The system written in the invariants:

$$\frac{\partial r_1}{\partial t} - a\frac{\partial r_1}{\partial x} = 0, \quad \frac{\partial r_2}{\partial t} + a\frac{\partial r_2}{\partial x} = 0,$$

shows that the invariant r_1 is constant along the straight lines $x + at =$ const, and r_2 — along the line $x - at =$ const; therefore

$$r_1 = f(x + at), \quad r_2 = g(x - at),$$

where f and g are arbitrary functions.

Returning to the function u, we get a known general solution of the wave equation:

$$u = \frac{f(x + at) + g(x - at)}{2}.$$

In the case of a system of quasilinear equations, the vectors l^k depend on x, t, u. Let us consider the differential forms

$$\omega_k(x, t, u, du) = l^k(x, t, u)\,du = l_c^k(x, t, u)\,du_a. \qquad (6)$$

Suppose that each of these forms, considered for fixed values of variables x, t has an integrating cofactor $\mu_k = \mu_k(x, t, u)$ so that for any $k = 1, \ldots, n$, we have

$$\mu_k(x, t, u)\,\omega_k(x, t, u, du) = \mu_k l_a^k\,du_a = \frac{\partial r_k(x, t, u)}{\partial u_a}\,du_a \qquad (7)$$

(Let us recall that the summation is carried out only with respect to the Greek subscripts; the number k in formula (7) is fixed). Equations (2.2.7) after multiplication by μ_k become

$$\left(\frac{dr_k(x, t, u)}{dt}\right)_k = \frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = \bar{g}_k(x, t, u) \qquad (k = 1, \ldots, n), \quad (8)$$

and

$$\bar{g}_k = \mu_k f_k + r'_{kt} + \xi_k r'_{kx}. \qquad (9)$$

In formula (9) the variables r'_{kx}, r'_{kt} are partial derivatives of the functions $r_k(x, t, u)$, respectively, with respect to x and t for fixed values of the variables $u = \{u_1, \ldots, u_n\}$.

Now using independence of the functions $r_k(x, t, u)$, let us express the variable u in terms of them, after which we get from (8) the following system of linear equations:

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = g_k(x, t, r) \qquad (k = 1, \ldots, n). \qquad (10)$$

The quantities r_k are called invariants (Riemann invariants), and the system (10) is called a system in invariants. Riemann first introduced the concept of invariants in his classical work [2]. If systems (2.2.1) and (2.2.7) are homogeneous and do not depend explicitly on x, $t(A = A(u), f \equiv 0)$, then equation (10) is also homogeneous:

$$\frac{\partial r_k}{\partial t}+\xi_k(r)\frac{\partial r_k}{\partial x}=0 \qquad (k=1, \ldots, n), \tag{11}$$

i.e., the functions $r_k(x, t)$ are constant along the integral curves of the equation

$$\frac{dx}{dt}=\xi_k(r(x, t)), \tag{12}$$

which are called characteristics of system of equation (11).

2. Systems of two and three quasilinear equations. We know that not every differential form $\omega_k(u, du)$ has an integrating cofactor. An exception is the case $n = 2$ when this cofactor always exists. In this case the Riemann invariants can be defined thusly. Suppose the equations

$$\omega_k(x^0, t^0, u, du)=0 \qquad (k=1, 2)$$

have integrals

$$\Phi_k(x^0, t^0, u)=\text{const} \qquad (k=1, 2).$$

Then, obviously, the following functions can be taken as Riemann invariants:

$$r_k = \Phi_k(x, t, u).$$

Now let us consider the case $n = 3$. We know (cf [3, 4]) that in this case the arbitrary differential form

$$\omega_k(x, t, u, du) = l^k(x, t, u)\, du \tag{1}$$

(x, t fixed) can be represented one of the following forms

a) dU, b) $V\, dU$, c) $dU + V\, dW$,

where U, V, W are functions of x, t, u. The cases a), b), and c) follow one after the other in order of generality.

If the forms ω_k refer, when $k = 1, 2, 3$, to types a) and b), then this means that integrating cofactor μ_k is present for each form ω_k, i.e., the possibility of reducing the system of quasilinear equations to invariants*).

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*) We know (cf for example [5]) that the form $l^k\, du$ belongs to type a) if $\operatorname{rot} l^k = 0$, and this form belongs to type b), if $l^k \operatorname{rot} l^k = 0$, where these conditions are essential and sufficient (the operation rot is taken with respect to the variables u_1, u_2, u_3).

In the general case the forms ω_k belong to the type c) and the characteristic system reduces the form .

$$\left(\frac{dU_k}{dt}\right)_k + V_k\left(\frac{dW_k}{dt}\right)_k = f_k \quad (k = 1, 2, 3), \tag{2}$$

where U_k, W_k, V_k are functions of the variables u_1, u_2, u_3, x, t. Suppose $V_3 = 0$; then equations (2.2.7) can be written in the form

$$[illegible] = f_k \quad (k = 1, 2), \tag{3}$$

$$\frac{\partial U_3}{\partial t} + \lambda_3 \frac{\partial U_3}{\partial x} = k_3, \tag{4}$$

where $U_3 = U_3(x, t, u)$.

If the forms $\omega_k(x, t, u, du)$ (k = 1, 2) are considered at the surface $U_3 = U_3(x, t, u) =$ const, $x = x_0$, $t = t_0$, then they have an integrating multiplier. Hence it follows that

$$\mu_k(x, t, u)\,\omega_k(x, t, u, du) = d\bar{U}_k(x, t, u) - \frac{\partial U_k}{\partial U_3}dU_3 - \frac{\partial U_k}{\partial t}dt - \frac{\partial U_k}{\partial x}dx \quad (k = 1, 2),$$

where it is assumed that

$$\bar{U}_k(x, t, u) = U_k(x, t, u_1, u_2, U_3(u_1, u_2, u_3, x, t)).$$

The system of quasilinear equations in this case can be written in the following characteristic form:

$$\left.\begin{aligned} \left(\frac{dU_k}{dt}\right)_k + \left(\eta_k - \frac{\partial U_k}{\partial U_3}\right)\left(\frac{dU_3}{dt}\right)_k = \bar{g}_k \quad (k = 1, 2), \\ \left(\frac{dU_3}{dt}\right)_3 = \bar{g}_3, \end{aligned}\right\} \tag{5}$$

where η_k, $\bar{g}_k$ are functions of the variables $U_k(x, t, u)$, x, t.

Section IV. Transformations of Systems of Quasilinear Equations

1. Transformation of systems with respect to solution. By the transformation of dependent and independent variables

$$x' = x'(x, t), \quad t' = t'(x, t), \quad v = v(x, t, u), \tag{1}$$

which have a reciprocal, i.e., such that

$$\frac{\partial[x', t']}{\partial[x, t]} \neq 0, \quad \frac{\partial[v_1, v_2, \dots, v_n]}{\partial[u_1, u_2, \dots, u_n]} \neq 0,$$

the hyperbolic system of quasilinear equations is converted into some new hyperbolic system of quasilinear equations. The characteristic directions of the hyperbolic system are the invariants of transformation (1). This means that if the direction $dx/dt = \xi_k$ is characteristic for the original syste[m], then after transformation (1) the direction

$$\frac{dx'}{dt'} = \xi_k' = \frac{\frac{\partial x'}{\partial t} + \frac{\partial x'}{\partial x}\xi_k}{\frac{\partial t'}{\partial t} + \frac{\partial t'}{\partial x}\xi_k}$$

will also be characteristic.

Let us consider the transformation of independent variables used in gasdynamics, which we will call the transformation of independent variables with respect to the solution. Suppose the new variables x', t' are associated with the old x, t by the formulas

$$\left.\begin{aligned} dx' &= \varphi_1(x, t, u)\,dx - \psi_1(x, t, u)\,dt, \\ dt' &= \varphi_2(x, t, u)\,dx - \psi_2(x, t, u)\,dt. \end{aligned}\right\} \tag{2}$$

For the line x' = const, t' = const to form a regular net for any solutions u = u(x, t) of the initial system, i.e., for one and only one point x', t' to correspond to each point x, t, it is sufficient that in the singly-connected domain of variable x, t under consideration, the following conditions are satisfied:

$$\frac{\partial \varphi_i(x, t, u)}{\partial t} + \frac{\partial \psi_i(x, t, u)}{\partial x} = 0 \quad (i = 1, 2), \tag{3}$$

$$\Delta = \varphi_1\psi_2 - \varphi_2\psi_1 \neq 0. \tag{4}$$

In inequality (3) $u = u(x, t)$ is the arbitrary solution of the initial system; in the differentiation, the dependence u on x, t must be taken into account.

Since inequality (3) must obtain for any solution $u = u(x, t)$ of the initial system, they must themselves be its corollaries. Let us assume that this does occur and that (4) is satisfied. Then from (2) follow the differentiation formulas

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \varphi_1 + \frac{\partial u}{\partial t'} \varphi_2, \quad \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x'} \psi_1 - \frac{\partial u}{\partial t'} \psi_2 \tag{5}$$

By formulas (5), the derivatives $\partial u/\partial t$, $\partial u/\partial x$ are linearly expressed by $\partial u/\partial t'$, $\partial u/\partial x'$, and after substitution in the initial system, obviously, we again will obtain a system of quasilinear equations.

Transformation (2) is one that is more general compared to the ordinary transformation of independent variables (1). For its applicability, however (in the case when φ_i and ψ_i depend on u) it is necessary that the system of quasilinear equations have as corollaries special equations. As we will see in section V, it is not any system of quasilinear equations that has even one equation of the type (3) as the corollary.

Let us present an example of the conversion of independent variables with respect to solution. The system of equations of gas dyanmics (cf Chapter Two, Section II).

$$\left.\begin{array}{c} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0, \quad p = p(\rho, S), \\ \dfrac{\partial S}{\partial t} + u\dfrac{\partial S}{\partial x} = 0 \end{array}\right\} \tag{6}$$

contains three equations with three unknowns ρ, u, S. Let us consider the transformation

$$dx' = \rho\, dx - \rho u\, dt, \quad dt' = dt. \tag{7}$$

In this case $\varphi_1 = \rho$, $\psi_1 = \rho u$, $\varphi_2 = 0$, $\psi_2 = -1$; conditions (3) are satisfied by virtue of the first equation in (6), and condition (4) leads to the requirement $\rho > 0$. Thus, if $\rho > 0$, then

$$\frac{\partial}{\partial x} = \rho \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \rho u \frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - u \frac{\partial}{\partial x},$$

and system (6) changes, after transformation (7), into the new system:

$$\frac{\partial}{\partial t'}\left(\frac{1}{\rho}\right) - \frac{\partial u}{\partial x'} = 0, \quad \frac{\partial u}{\partial t'} + \frac{\partial p}{\partial x'} = 0, \quad \frac{\partial S}{\partial t'} = 0. \tag{8}$$

in gas dynamics the variables x, t are called Eulerian variables, and the variables $q = x'$, $t' = t$ are Lagrangian variables.

2. Hodograph transformation. For a homogenous system of quasilinear equations whose coefficients do not explicitly depend on x, t in the case $n = 2$:

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \quad u = \{u_1; u_2\}. \tag{1}$$

Let us interchange the roles of dependent and independent variables, i.e., we can assume that $x = x(u_1, u_2)$, $t = t(u_1, u_2)$. Simple calculations lead to the result

$$\frac{\partial u_1}{\partial x} = \Delta \frac{\partial t}{\partial u_2}, \quad \frac{\partial u_2}{\partial x} = -\Delta \frac{\partial t}{\partial u_1}, \quad \frac{\partial u_1}{\partial t} = -\Delta \frac{\partial x}{\partial u_2}, \quad \frac{\partial u_2}{\partial t} = \Delta \frac{\partial x}{\partial u_1}, \tag{2}$$

where

$$\Delta = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial t} \end{vmatrix} = \frac{\partial [u_1; u_2]}{\partial [x; t]}. \tag{3}$$

If $\Delta \neq 0$, then by substituting formulas (2) into (1), we arrive at a linear system of two equations:

$$\left.\begin{aligned} \frac{\partial x}{\partial u_2} - a_{11}(u) \frac{\partial t}{\partial u_2} + a_{12}(u) \frac{\partial t}{\partial u_1} = 0, \\ \frac{\partial x}{\partial u_1} - a_{21}(u) \frac{\partial t}{\partial u_2} + a_{22}(u) \frac{\partial t}{\partial u_1} = 0, \end{aligned}\right\} \tag{4}$$

where $a_{ij}(u)$ are elements of matrix A(u).

This transformation of variables is called the hodograph transformation; it is used in gas dynamics.

3. Extended system. Let us write a system of quasilinear equations of the hyperbolic type in the following characteristic form:

$$l^k(x, t, u)\left[\frac{\partial u}{\partial t} + \xi_k(x, t, u)\frac{\partial u}{\partial x}\right] = \\ = l^k_\alpha\left[\frac{\partial u_\alpha}{\partial t} + \xi_k\frac{\partial u_\alpha}{\partial x}\right] = f_k(x, t, u) \quad (k = 1, \ldots, n). \tag{1}$$

In many investigations, along with system (1) it is useful to consider a system of equations in which the unknowns are also derivatives of the solution u(x, t). This system is obtained by differentiating (1) and is its differential corollary. We will call system (1) and its differential corollaries an extended system.

Let us denote

$$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial t} = q \quad \left(\frac{\partial u_i}{\partial x} = p_i, \ \frac{\partial u_i}{\partial t} = q_i\right); \tag{2}$$

then system (1) can be written as

$$l^k(q + \xi_k p) = l^k_\alpha(q_\alpha + \xi_k p_\alpha) = f_k \quad (k = 1, \ldots, n). \tag{3}$$

Differentiating each equation (3) with respect to the variables t, x, we get

$$l^k\left(\frac{\partial q}{\partial t} + \xi_k\frac{\partial p}{\partial t}\right) = \bar{\mathcal{G}}_k, \quad l^k\left(\frac{\partial q}{\partial x} + \xi_k\frac{\partial p}{\partial x}\right) = \bar{\mathcal{F}}_k, \tag{4}$$

where

$$\bar{\mathcal{G}}_k = \frac{\partial f_k}{\partial t} + \frac{\partial f_k}{\partial u_\alpha} q_\alpha - (q_\alpha + \xi_k p_\alpha)\left(\frac{\partial l_\alpha^k}{\partial t} + \frac{\partial l_\alpha^k}{\partial u_\beta} q_\beta\right) - l_\alpha^k\left(\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial u_\beta} q_\beta\right) p_\alpha,$$
$$\bar{\mathcal{F}}_k = \frac{\partial f_k}{\partial x} + \frac{\partial f_k}{\partial u_\alpha} p_\alpha - (q_\alpha + \xi_k p_\alpha)\left(\frac{\partial l_\alpha^k}{\partial x} + \frac{\partial l_\alpha^k}{\partial u_\beta} p_\beta\right) - l_\alpha^k\left(\frac{\partial \xi_k}{\partial x} + \frac{\partial \xi_k}{\partial u_\beta} p_\beta\right) p_\alpha \qquad (5)$$

From (2) follow, as conditions of integrability the equations

$$\partial q/\partial x = \partial p/\partial t, \qquad (6)$$

and, therefore, equations (4) can be rewritten as

$$l^k\left(\frac{\partial q}{\partial t} + \xi_k \frac{\partial q}{\partial x}\right) = \bar{\mathcal{G}}_k, \quad l^k\left(\frac{\partial p}{\partial t} + \xi_k \frac{\partial p}{\partial x}\right) = \bar{\mathcal{F}}_k, \qquad (7)$$

where $\bar{\mathcal{G}}_k$, $\bar{\mathcal{F}}_k$ are, by (5), functions of x, t, u, p, q.

We will call equations (2) and (7) an extended system. The extended system can be written in a different representation. The equations

$$\frac{\partial u}{\partial t} = q, \quad l^k\left(\frac{\partial q}{\partial t} + \xi_k \frac{\partial q}{\partial x}\right) = \bar{\mathcal{G}}_k \qquad (8)$$

constitute an extended system of 2n equations if the variables p appearing in $\bar{\mathcal{G}}_k$ are canceled out by means of equations (3). Here it is assumed that $\xi_k \neq 0$*).

The extended system was introduced in this form by R. Courant and P. Lax in article [6].

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*) The requirement $\xi_k \neq 0$ is not essential. By change of variables we can achieve the result that $\xi_k \neq 0$ for all k = 1, ..., n.

If we consider an extended system in symmetric form (7), then it is equivalent to the initial system in the class $u(x, t) \in C_2$.

Let us dwell on a remarkable feature of the extended system. As was shown, a hyperbolic semilinear system can be reduced to invariants. This, generally speaking, does not obtain for systems of quasilinear equations. However, an extended system of any hyperbolic system of quasilinear equations already possess this property, i.e., is reducible to invariants.

Actually, denoting

$$\mathcal{P}_k = l^k p = l^k_\alpha p_\alpha, \quad \mathcal{Q}_k = l^k q \tag{9}$$

and converting in equations (7) to the variables $\mathcal{P}_k$, $\mathcal{Q}_k$, we get

$$\left.\begin{aligned} \frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} &= \mathcal{F}_k(x, t, u, \mathcal{P}, \mathcal{Q}), \\ \frac{\partial \mathcal{Q}_k}{\partial t} + \xi_k \frac{\partial \mathcal{Q}_k}{\partial x} &= \mathcal{G}_k(x, t, u, \mathcal{P}, \mathcal{Q}), \end{aligned}\right\} \tag{10}$$

where

$$\left.\begin{aligned} \mathcal{F}_k &= F_k + p_\alpha\left[\frac{\partial l^k_\alpha}{\partial t} + \xi_k \frac{\partial l^k_\alpha}{\partial x} + \frac{\partial l^k_\alpha}{\partial u_\beta}(q_\beta + \xi_k p_\beta)\right], \\ \mathcal{G}_k &= G_k + q_\alpha\left[\frac{\partial l^k_\alpha}{\partial t} + \xi_k \frac{\partial l^k_\alpha}{\partial x} + \frac{\partial l^k_\alpha}{\partial u_\beta}(q_\beta + \xi_k p_\beta)\right]. \end{aligned}\right\} \tag{11}$$

Since $\mathrm{Det}((\;{}^{k}_{\alpha}\;)) \neq 0$, then the variables p, q are uniquely expressed by $\mathcal{P}$, $\mathcal{Q}$, and can be canceled out of $\mathcal{F}_k$, $\mathcal{G}_k$.

Adding to equations (10) the equation

$$\frac{\partial u}{\partial x} = p = \lambda^k_\alpha \mathcal{P}_\alpha, \quad \frac{\partial u}{\partial t} = q = \lambda^k_\alpha \mathcal{Q}_\alpha, \tag{12}$$

we get a system of 4n quasilinear equations written in invariants.

We can reduce the number of equations down to 2n, if, for example, we add to the first group of equations (10) the first group of equations (2), i.e., consider a system of 2n equations in the invariants:

$$\frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} = \mathcal{F}_k, \quad \frac{\partial u}{\partial x} = p = \lambda^k_\alpha \mathcal{P}_\alpha \tag{13}$$

and assume that in the functions $\mathcal{F}_k$ the variables q are canceled out by means of (3), and p — by means of (12).

However, the second group of equations (13) is unsuitable for investigation. We will transform it. From equations (3) we have

$$q_k = \lambda^k_\alpha f_\alpha - \lambda^k_\alpha l^\alpha_\beta \xi_\alpha p_\beta = \lambda^k_\alpha f_\alpha - \lambda^k_\alpha \xi_\alpha \mathcal{P}_\alpha. \tag{14}$$

Therefore instead of system (13) we can consider the system

$$\frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} = \mathcal{F}_k, \quad \frac{\partial u_k}{\partial t} = q_k = \lambda^k_\alpha f_\alpha - \lambda^k_\alpha \xi_\alpha \mathcal{P}_\alpha, \tag{15}$$

which is also written in variants, and the function $\mathcal{F}_k$ are functions of x, t, u, $\mathcal{P}$.

Let us write extended system (15) in its final form:

$$\frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} = \mathcal{F}_k(x, t, u, \mathcal{P}), \tag{16}$$

$$\frac{\partial u_k}{\partial t} = F_k(x, t, u, \mathcal{P}). \tag{17}$$

From formulas (15), (11), (5), and (3) it follows that

$$\mathcal{F}_k = \mathcal{F}^k(x, t, u) + \mathcal{F}^k_\alpha(x, t, u)\mathcal{P}_\alpha + \mathcal{F}^k_{\alpha\beta}(x, t, u)\mathcal{P}_\alpha \mathcal{P}_\beta, \tag{18}$$

$$F_k = F^k(x, t, u) + F^k_\alpha(x, t, u)\mathcal{P}_\alpha. \tag{19}$$

where $\mathcal{F}^k$, $\mathcal{F}^k$, $\mathcal{F}^k_{\alpha\beta}$, F^k, F^k_α are certain functions dependent only on x, t, u. The formulas for these variables are quite cumbersome and we will not write them out here. Let us however note that F^k, F^k_α are expressed in terms of the coefficients of the initial system, and $\mathcal{F}^k$, $\mathcal{F}^k_\alpha$, $\mathcal{F}^k_{\alpha\beta}$ in terms of the coefficients and their first derivatives with respect to the variables x, t, u.

The extended system (16) and (17) will be used by us in the following in estimating the growth of the solution to the system of quasilinear equations and its derivatives (cf section VIII).

By section II, a hyperbolic system of n nonlinear equations reduces to a system of 2n quasilinear equations. The extended system for arbitrary hyperbolic system of quasilinear equations reduces, in turn, to equations in invariants. Therefore a hyperbolic system of n nonlinear equations reduces to a system of not more than 4n quasilinear equations in invariants by means of forming an extended system.

Section V. Conservative Systems of Quasilinear Equations

1. Definitions. If the equation

$$\frac{\partial\varphi(x, t, u)}{\partial t} + \frac{\partial\psi(x, t, u)}{\partial x} = F(x, t, u) \qquad (1)$$

is a corollary of the system of quasilinear equations

$$\partial u/\partial t + A\partial u/\partial x = b \qquad (2)$$

for any solutions to system (2), then we call it the law of conservation of system (2).

Suppose system (2) has m laws of conservation (1) corresponding to the functions $\varphi_1, \ldots, \varphi_m; \psi_1, \ldots, \psi_m$. These laws of conservation will be called independent in the domain D if the functions 1, $\varphi_1(x_0, t_0, u), \ldots, \varphi_m(x_0, t_0, u)$ are linearly independent for all x_0, t_0 of the domain D under consideration.

If $\varphi = \varphi(x, t)$, then by the definition of equality (1) is not independent law of conservation.

If system (2) has n independent laws of conservation satisfying the condition

$$\frac{\partial[\varphi_1, \ldots, \varphi_n]}{\partial[u_1, \ldots, u_n]} \neq 0,$$

then we call it conservative, otherwise — nonconservative.

Thus, the conservative system (2) can be reduced to the form

$$\frac{\partial \varphi(x, t, u)}{\partial t} + \frac{\partial \psi(x, t, u)}{\partial x} = F(x, t, u), \qquad (3)$$

where we understand φ, ψ, and F to refer to vectors with n components. Let us note that a system of the type (3) is often called "divergent"; sometimes this term applies only to the case F = 0. We obtain equations that will serve for defining all laws of conservation of system (2), i.e., functions of φ and ψ. To do this, let us multiply system (2) by the vector $\alpha = \alpha(x, t, u) = \{\alpha_1, \ldots, \alpha_n\}$ and require that this result be of the form (1).

Let us arrive at equations

$$\alpha = \frac{\partial \varphi}{\partial u}, \quad \alpha A = \frac{\partial \psi}{\partial u}, \quad F = \alpha b - \varphi'_t - \psi'_x,$$

where

$$\frac{\partial \varphi}{\partial u} = \left\{\frac{\partial \varphi}{\partial u_1}, \ldots, \frac{\partial \varphi}{\partial u_n}\right\}, \quad \frac{\partial \psi}{\partial u} = \left\{\frac{\partial \psi}{\partial u_1}, \ldots, \frac{\partial \psi}{\partial u_n}\right\}.$$

Canceling the vector α from these equations, we get the system

$$\frac{\partial \varphi}{\partial u} A(x, t, u) = \frac{\partial \psi}{\partial u}, \qquad (4)$$

in which only two unknown functions $\varphi(x, t, u)$ and $\psi(x, t, u)$ appear. The system (4) consists of n equations and is described in components as follows:

$$\frac{\partial \psi}{\partial u_i} = \sum_{j=1}^{n} a_{ji}(x, t, u) \frac{\partial \varphi}{\partial u_j} \quad (i = 1, 2, \ldots, n). \qquad (5)$$

The variables x and t appear in the coefficients of this system as parameters.

A set of linearly independent solutions to system (5) defines the system of independent laws of conservation of system (2).

If system (2) is linear or semilinear, it is conservative. Naturally, in this case A = A(x, t) and system (5) has n independent solutions:

$$\varphi_k(x, t, u) = u_k, \quad \psi_k(x, t, u) = a_{k\alpha}(x, t) u_\alpha \quad (k = 1, \ldots, n).$$

When $n \leqslant 2$, system (5) is either indeterminate or determinate and has an infinite number of solutions. When $n \geqslant 3$, system (5) is over determined and cannot in general have a single solution φ, ψ, which would depend essentially on u. The proof of this assertion can be obtained for the example of the system

$$\frac{\partial u_1}{\partial t}+u_2\frac{\partial u_1}{\partial x}=0,\quad \frac{\partial u_2}{\partial t}+u_3\frac{\partial u_2}{\partial x}=0,\quad \frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x}=0,$$

for which it is easy to establish that system (5) does not have nontrival solutions (cf [7]).

2. Laws of conservation of gas dynamics. As an example (cf [8]), let us consider a system of equations of gas dynamics in Lagrangian coordinates (chapter two, section II), which we will write as

$$\frac{\partial V}{\partial t}-\frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial t}+\frac{\partial p(V,\varepsilon)}{\partial x}=0,\quad \frac{\partial}{\partial t}\left(\varepsilon+\frac{u^2}{2}\right)+\frac{\partial pu}{\partial x}=0. \tag{1}$$

Let us pose the problem of finding all laws of conservation of this system of equations (obviously, system (1) has already been written in the form of laws of conservation and is therefore conservative). Representing the system as

$$\frac{\partial V}{\partial t}-\frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial t}+p'_V\frac{\partial V}{\partial x}+p'_\varepsilon\frac{\partial \varepsilon}{\partial x}=0,\quad \frac{\partial \varepsilon}{\partial t}+p\frac{\partial u}{\partial x}=0,$$

we will write for it the system of equations (5.1.5) with respect to $\varphi=\varphi(V, u, \varepsilon)$, $\psi=\psi(V, u, \varepsilon)$:

$$\frac{\partial \psi}{\partial V}=p'_V(V,\varepsilon)\frac{\partial \varphi}{\partial u}, \tag{2}$$

$$\frac{\partial \psi}{\partial u}=-\frac{\partial \varphi}{\partial V}+p(V,\varepsilon)\frac{\partial \varphi}{\partial \varepsilon}, \tag{3}$$

$$\frac{\partial \psi}{\partial \varepsilon}=p'_\varepsilon(V,\varepsilon)\frac{\partial \varphi}{\partial u}. \tag{4}$$

Suppose $p=p(V,\varepsilon)$ is a doubly-continuously differentiable functions of its variable; we will assume that $\varphi(V, u, \varepsilon)$, $\psi(V, u, \varepsilon)$ is also doubly-continuously differentiable. Combining equalities (2) and (4), we obtain

$$p'_\varepsilon(V,\varepsilon)\frac{\partial \psi}{\partial V}-p'_V(V,\varepsilon)\frac{\partial \psi}{\partial \varepsilon}=0,$$

which denotes a functional dependence with fixed change u of the variables ψ, p, i.e.,

$$\psi=\psi(p, u). \tag{5}$$

Similarly, writing out the conditions for consistency of equalities (2) and (4), we get

$$p'_{\varepsilon}\frac{\partial}{\partial V}\left(\frac{\partial\varphi}{\partial u}\right)=p'_V\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\varphi}{\partial u}\right),$$

i.e.

$$\partial\varphi/\partial u = F(p, u) \qquad (6)$$

where F(p, u) is an arbitrary function.

Substituting (5) into (3), we find

$$p\frac{\partial\varphi}{\partial\varepsilon}-\frac{\partial\varphi}{\partial V}=\Phi(p, u), \qquad (7)$$

where Φ (p, u) is an arbitrary function.

Integrating (6), we get

$$\varphi(V, u, \varepsilon)=F_1(p, u)+F_2(\varepsilon, V). \qquad (8)$$

where as before F_1, F_2 are certain arbitrary functions.

Substituting this expression into equality (7), we give the latter the following form:

$$p\frac{\partial F_2(\varepsilon, V)}{\partial\varepsilon}-\frac{\partial F_2(\varepsilon, V)}{\partial V}=\Phi(p, u)-\frac{\partial F_1(p, u)}{\partial p}\left[pp'_{\varepsilon}-p'_V\right].$$

Since here the left side does not depend on the variable u, then

$$\Phi'_u(p, u)=\frac{\partial^2 F_1(p, u)}{\partial p\,\partial u}\left[pp'_{\varepsilon}-p'_V\right]. \qquad (9)$$

We will limit ourselves to a consideration of the case when the variable pp' is functionally independent (as a function of the variables V, ε) with the function p(V, ε):

$$p'_V\frac{\partial}{\partial\varepsilon}\left[pp'_{\varepsilon}-p'_V\right]+p'_{\varepsilon}\frac{\partial}{\partial V}\left[pp'_{\varepsilon}-p'_V\right]. \qquad (10)$$

Therefore, in the right side of equality (9) appears a function of p, u, V, ε that is not reducible to a function of p, u. Therefore equality (9) is possible if and only if

$$\frac{\partial^2 F_1(p, u)}{\partial p\,\partial u}=0, \qquad \Phi'_u(p, u)=0.$$

Thus, assuming that condition (10) is satisfied, $\Phi(p, u) = \Phi(p)$ and, by (8)

$$\varphi(V, u, \varepsilon) = F_1(u) + F_2(\varepsilon, V) \qquad (11)$$

From equalities (2), (4), and (11) follows

$$\psi(V, u, \varepsilon) = p\frac{dF_1(u)}{du} + F_3(u). \qquad (12)$$

Therefore, substituting formulas (11) and (12) into equality (3), we get

$$p\frac{d^2F_1(u)}{du^2} + \frac{dF_3(u)}{du} = p\frac{\partial F_2}{\partial \varepsilon} - \frac{\partial F_2}{\partial V} \qquad (13)$$

Since $F_2 = F_2(\varepsilon, V)$, then the right side in (13) does not depend on u; therefore

$$\frac{d^2F_1(u)}{du^2} = C_1, \quad \frac{dF_3(u)}{du} = C_2$$

and

$$F_1(u) = C_1\frac{u^2}{2} + C_2 u, \quad F_3 = C_3 u,$$

since the constants are imaginary. Finally, equation (13) becomes

$$p(V, \varepsilon)\frac{\partial F_2}{\partial \varepsilon} - \frac{\partial F_2}{\partial V} = C_1 p(V, \varepsilon) + C_3,$$

i.e., $F_2(\varepsilon, V)$ satisfies the linear equation in partial derivatives, assuming

$$F_2(\varepsilon, V) = C_1\varepsilon - C_3 V + \bar{F}_2(\varepsilon, V),$$

we get for $\bar{F}_2(\varepsilon, V)$ the homogeneous equation

$$p(V, \varepsilon)\frac{\partial \bar{F}_2}{\partial \varepsilon} - \frac{\partial \bar{F}_2}{\partial V} = 0. \qquad (14)$$

As for the variables V, ε, let us consider the ordinary differential equation

$$d\varepsilon + p(V, \varepsilon)\,dV = 0. \qquad (15)$$

Suppose $S = S(V, \varepsilon) = \text{const}$ is the general integral of this equation, i.e.,

$$S'_\varepsilon(V, \varepsilon)\,p(V, \varepsilon) - S'_V(V, \varepsilon) = 0.$$

The function $S(V, \varepsilon)$ is called entropy in thermodynamics. Since equation (15) determines the characteristics of equation (14), then (cf for example [5]) the general solution to equation (14) is

$$F_2(V, s) = f(S(V, s)),$$

where f is an arbitrary function of one variable. And thus, we obtain the general solution to equations (2) - (4) for φ:

$$\varphi = C_1\left(s + \frac{u^2}{2}\right) + C_2 u - C_3 V + f(S(V, s)), \tag{16}$$

after which we can easily find ψ:

$$\psi = C_1 p u + C_2 p + C_3 u + 0 \cdot f(S(V, s)). \tag{17}$$

Formulas (16) and (17) give us a grasp of all laws of conservation of this system of equation for the case when the inequality (10) is satisfied. It is easy to note that they contain laws of conservation (1), and also a new law of conservation

$$\frac{\partial f(S(V, s))}{\partial t} = 0,$$

which in gas dynamics is called the law of conservation of entropy.

From our proof it follows that when condition (10) is satisfied the equations of gas dynamics (1) do not have any other laws of conservation except for the known laws of conservation of mass, momentum, energy, and entropy.

By way of yet another example, let us consider the system

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial u_i} + \frac{\partial}{\partial x}\frac{\partial \mathscr{L}^1}{\partial u_i} = 0 \qquad (i = 1, \ldots, n), \tag{18}$$

where $\mathscr{L} = \mathscr{L}(u_1, \ldots, u_n)$, $\mathscr{L}' = \mathscr{L}'(u_1, \ldots, u_n)$ are scalar functions (cf [9]). System (18) is hyperbolic if the matrix

$$\mathscr{L}_{uu} = \left(\left(\frac{\partial^2 \mathscr{L}}{\partial u_i \partial u_j}\right)\right)$$

is sign-determined.

The system of equations (18), obviously, is conservative. It is easily seen that it has yet another law of conservation, independent of the laws of conservation of (18) if $\mathscr{L}_{uu}$ is a variable matrix:

$$\varphi = u_\alpha \mathscr{L}_{u_\alpha} - \mathscr{L}, \qquad \psi = u_\alpha \mathscr{L}^1_{u_\alpha} - \mathscr{L}^1.$$

It is interesting to note that equations of gas dynamics, as well as certain other systems of equations in mathematical physics are reducible to the form (18).

3. Potential of the solution of a conservative system of quasilinear equations. Let us consider a conservative system of n quasilinear equations

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial x} + \text{[illegible]}$$

By the definition of conservativeness

$$\frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(u_1, \ldots, u_n)} \text{[illegible]}$$

Therefore, as new dependent variable we can select the variable $v_i = \varphi_i(x, t, u)$ and consider only the conservative systems of the special form:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, u)}{\partial x} = F(x, t, u). \qquad (1)$$

Suppose we know the solution $u(x, t) \in C_1$ to system (1). Let us find the vector $\mathcal{F}(x, t)$ such that

$$\frac{\partial \mathcal{F}(x, t)}{\partial x} = F(x, t, u(x, t)).$$

Obviously, the vector $\mathcal{F}(x, t)$ is nonuniquely determined; for determinateness, we must set

$$\mathcal{F}(x, t) = \int_{x_0(t)}^{x} F(\xi, t, u(\xi, t))\, d\xi. \qquad (2)$$

where $x = x_0(t)$ is a smooth curve uniquely projectible onto the axis $x = 0$. The system (1) can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[\varphi(x, t, u) - \mathcal{F}(x, t)] = 0.$$

Integrating this equation of the domain $\mathcal{G}_C$ bonded by the contour C, we conclude that the contour integral

$$\oint_C u\, dx - [\varphi(x, t, u) - \mathcal{F}(x, t)]\, dt = 0$$

tends to zero for any piecewise-smooth closed contour C. Therefore the curvilinear integral

$$\Phi(x, t) = \int_{(x_0, t_0)}^{(x, t)} u\, dx - [\varphi - \mathcal{F}]\, dt \qquad (3)$$

does not depend on the path of integration and defines the vector $\Phi(x, t) \in C_2$, if $\varphi \in C_1$, $F \in C$.

From (3) follow the formulas

$$\frac{\partial \Phi}{\partial x} = u, \quad \frac{\partial \Phi}{\partial t} = -\varphi(x, t, u) + \mathcal{F}(x, t).$$

Canceling out u and using formula (2), we find

$$\frac{\partial \Phi}{\partial t} + \varphi\left(x, t, \frac{\partial \Phi}{\partial x}\right) = \int F\left(\xi, t, \frac{\partial \Phi(t, \xi)}{\partial \xi}\right) d\xi \qquad (4)$$

Now the system of nonlinear integro-differential equations (4) can be considered independently of (1). If we know the solution $\Phi(x, t) \in C_2$ to system (4), then $u = \partial\Phi/\partial x \in C_1$ is the solution of system (1). Reducing system (1) to system (4), we can consider, in this way, less smooth solutions u(x, t) of system (1) as derivatives of solutions Φ (x, t) of system (4) that have greater smoothness.

For this reason, this approach find use in the examination of generalized (for example, discontinuous) solutions of systems of quasilinear equations.

We will call the vector Φ (x, t) the potential of the solution u(x, t) of system of equations (1) (cf [10]).

Let us note several particular cases. If $F \equiv 0$, then system (4) becomes a nonlinear system of the Cauchy-Kovalevski type. Reducing system (1) to system (4) in this case must be compared with the opposite procedure -- reducing a nonlinear system to a system of quasilinear equations (section II). Therefore this processes of increasing solution smoothness can be used even in the following.

Section VI. Formulation of Cauchy's Problem for a Hyperbolic System of Quasilinear Equations

1. Formulation of the problem. For a hyperbolic system of quasilinear equations

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = b, \qquad (1)$$

which we will also write in the characteristic form

$$l^k \left[\frac{\partial u}{\partial t} + \xi_k \frac{\partial u}{\partial x}\right] = f_k \qquad (k = 1, \ldots, n),$$

let us consider the following problem:

In some vicinity of the arc $a \leq \tau \leq b$ of curve $\mathcal{L}$

$$x = x(\tau), \quad t = t(\tau)$$

find the solution u(x, t) of system (1) that takes on assigned values on $\mathcal{L}$,

$$u(x(\tau), t(\tau)) = u^0(\tau), \quad a \leqslant \tau \leqslant b. \qquad (2)$$

conditions (2) are called initial, the vector-function u^0 is the initial function, and curve $\mathcal{L}$ is the initial curve.

The problem (1) and (2) is called the problem with initial data, or Cauchy's problem.

Cauchy's problem for equation (1) is interpreted geometrically as a problem of constructing in the space of n + 2 dimensions of variables (x, t, u) a two-dimensional integral surface u = u(x, t) passing through the given curve

$$x = x(\tau), \qquad t = t(\tau), \qquad u = u^0(\tau),$$

which we will also called the initial curve.

To render the formulation of Cauchy's problem more precise, we must indicate:

a) the smoothness of the matrix A(x, t, u), vector b(x, t, u) (or l^k, ξ_k, f_k), the initial curve, and the function $u^0(\tau)$ (we will call these variables the initial data of Cauchy's problem);

b) the domain G of the variables x, t in which we seek the solution to Cauchy's problem.

These problems will be examined in the following subsections in the construction of the solution of Cauchy's problem.

Let us note that, by definition, the solution u(x, t) of system (1) is continuously differentiable ($u \in C_1$). If u(x, t) exhibits less smoothness, but in some sense satisfies system (1), then the function u(x, t) is called the generalized solution of system (1).

In this chapter we construct the solution $u(x, t) \in C_1$ for a hyperbolic system of quasilinear equations; a generalized solution $u(x, t) \in C$ will be constructed for linear and semilinear systems. Following K. O. Friedrichs, we will call the latter the solution to Cauchy's problem in the broad sense.

2. Solvability of Cauchy's problem. Characteristics. Suppose $x(\tau)$, $t(\tau)$, $u^0(\tau) \in C_1$, $l^k(x, t, u)$, ξ_k, $f_k \in C$ and some vector-function $u(x, t) \in C_1$ takes on the value $u^0(\tau)$ on the curve $\mathscr{L}$ and suppose its derivatives p, q satisfy the equations of system (6.1.1) on the line $\mathscr{L}$.

Let us formulate the problem of whether the derivatives p, q based on these data can be determined (on the line $\mathscr{L}$), i.e., the problem of whether the function $u(x, t) \in C_1$ satisfying these requirements exists.

On the line $\mathscr{L}$ we have the equality

$$u(x(\tau), t(\tau)) = u^0(\tau), \quad l^k[q + \xi_k p] = f_k, \tag{1}$$

where l^k, ξ_k, f_k are obviously known functions of the variable τ on $\mathscr{L}$.

Differentiating $u(x(\tau), t(\tau)) = u^0(\tau)$ with respect to τ, we get

$$t'(\tau) q + x'(\tau) p = \frac{du^0(\tau)}{d\tau} = \varphi(\tau), \tag{2}$$

and, therefore, $\varphi(\tau) \in C$.

Equations (1) and (2) form a system of 2n equations for the determination (on the line $\mathscr{L}$) of the derivatives p, q. Since the matrix $((l^k_\alpha))$ is non-singular, then by canceling out the vector q from equations (1) and (2), we get

$$[t'(\tau)\xi_k - x'(\tau)]\, l^k p = [t'(\tau)\xi_k - x'(\tau)]\, l^k_\alpha p_\alpha = \bar{f}_k, \tag{3}$$

where $\bar{f}_k$ is a continuous function of the variable τ. The determinant $D(\tau)$ of system (3) can be easily computed:

$$D(\tau) = [\mathrm{Det}\,((l^k_\alpha))] \prod_{k=1}^{n} [t'(\tau)\xi_k - x'(\tau)] = \mathrm{Det}\,\Lambda \prod_{k=1}^{n} [t'(\tau)\xi_k - x'(\tau)].$$

It differs from zero if for all $k = 1, \ldots, n$

$$\left.\frac{dx}{dt}\right|_{\mathscr{L}} = \frac{x'(\tau)}{t'(\tau)} \neq \xi_k = \xi_k(x(\tau), t(\tau), u^0(\tau)). \tag{4}$$

We assume that $|x'(\tau)| + |t'(\tau)| \neq 0$. If $t'(\tau) = 0$, then plainly $D(\tau) \neq 0$, since ξ_k are bounded.

Thus, if conditions (4) are satisfied, then the system of equation (3) has the unique solution $p = p(\tau)$ and, therefore, the derivatives p, q of the function u(x, t) are uniquely defined on the line $\mathcal{L}$ from conditions (1).

Now let us assume that $u(x, t) \in C_2$; $x(\tau)$, $t(\tau)$, $u^0(\tau) \in C_2$, l^k, ξ_k, $f_k \in C_1$. If $u = u^0$ on $\mathcal{L}$, the first derivatives of u(x, t) satisfy on $\mathcal{L}$ equations (6.1.1), and the second derivatives of u(x, t) satisfy on $\mathcal{L}$ the differential corollaries of system (6.1.1) (i.e., equations obtained by formal differentiation of system (6.1.1) with respect to variable x, t), then providing that conditions (4) are satisfied, on $\mathcal{L}$ the second derivatives of u(x, t) are also uniquely defined.

Similarly, if conditions (4) are satisfied, we can define on $\mathcal{L}$ derivatives of any order m of the function u(x, t) if conditions (1) are satisfied then, moreover, all derivatives of u up to order m inclusively satisfy on $\mathcal{L}$ all differential corollaries of equations (6.1.1) up to order m inclusively. Of course, the input data must be sufficiently smooth.

Let us note that, as we can easily appreciate, in these arguments it is sufficient to speak not of all differential corollaries, but only of those that are obtained by differentiating equations (6.1.1) in any fixed direction not coincident with the direction of the curve $\mathcal{L}$ (so-called exit direction). For example, the direction of the normal is such a direction.

This procedure of determining derivatives can be extended as far as one wishes if the initial data are analytic, and it permits constructing an analytic solution for the problems (6.1.1) and (6.1.2) for such data. This fact is the analytic basis of the well-known Cauchy-Kovalevski method.

If conditions (4) are satisfied for all $k = 1, \ldots, n$ on curve $\mathcal{L}$, we will call Cauchy's problem normal.

The curve $\mathcal{L}$ assigned in this space of n + 2 variables x, t, u by the equations

$$x = x(\tau), \qquad t = t(\tau), \qquad u = u^0(\tau), \qquad (5)$$

is called the characteristic of number k_o of system (5.1.1) if the following equalities satisfy at this curve:

$$\left.\frac{dx}{dt}\right|_{k_0} = \frac{x'(\tau)}{t'(\tau)} = \xi_{k_0}(x(\tau), t(\tau), u^0(\tau)). \qquad (6)$$

For the case whence several characteristic values ξ_k coincide on $\mathcal{L}$, the curve $\mathcal{L}$ can be a characteristic of several numbers k at the same time.

Sometimes we will also refer to the projection of the curve (5) onto the plane of variables (x, t) as a characteristic, bearing in mind, however, that equality (6) has been satisfied for it.

Suppose the curve $\mathcal{L}$ under Cauchy's problem conditions is the characteristic of number k_o. The left side of the equation of system (3) corresponding to $k = k_o$ then tends to zero. If the right side of this equation $\bar{f}_k(\tau)$ does not tend identically to zero, then the system of equations (3) has no solutions at all in general. Therefore there does not exist a function $u(x, t) \in C_1$ which on $\mathcal{L}$ takes assigned values $u^0(\tau)$ that would satisfy on $\mathcal{L}$ system (6.1.1). Nor does there exist the solution $u(x, t) \in C_1$ to Cauchy's problem (6.1.1) and (6.1.2).

Thus, if the initial curve is a characteristic, then the initial conditions (6.1.2) and the system (6.1.1), generally speaking, contradict each other, and Cauchy's problem is nonsolvable*).

For Cauchy's problem to be physically meaningful also in this case, we must stipulate that $\bar{f}_{k_o}(\tau) \equiv 0$.

Thus, if the initial curve has a characteristic of number k_o, then the initial data cannot be assigned arbitrarily; they must satisfy the condition

$$\bar{f}_{k_0}(\tau) = t'(\tau) f_{k_0}(x(\tau), t(\tau), u^0(\tau)) - l^{k_0}(x(\tau), t(\tau), u^0(\tau)) \frac{du^0(\tau)}{d\tau} \equiv 0. \qquad (7)$$

*) From this examination follows another definition for the characteristic, as a curve $\mathcal{L}$ at which a linear combination of equations of the system under consideration contains only interior derivatives, i.e., derivatives with respect to the parameter τ in the direction of the curve $\mathcal{L}$.

which is called the condition of solvability.

Suppose the initial curve $\mathscr{L}$ is a characteristic of number k_0 and suppose the solvability condition (7) has been met. Then system (3) is consistent, but has infinitely many solutions. Therefore, the solution u(x, t) to Cauchy's problem is not determined uniquely by the initial condition (6.1.2) and there exist infinitely many solutions to system (6.1.1) satisfying the very same initial conditions. Thus, we arrive at the general definition of the characteristic of system (6.1.1):

A characteristic curve is a curve $\mathscr{L}$ for which Cauchy's problem is either nonsolvable or solvable, but not uniquely.

For the unique determination of the solution u(x, t) for the case when the curve $\mathscr{L}$ is a characteristic and when the solvability conditions have been met, several additional conditions can be imposed. Examples of these problems are taken up in section XI.

Related to Cauchy's problem is the problem of the extension of the solution u(x, t) through curve $\mathscr{L}$. Suppose the solution $u(x, t) \in C_1$ is known along one side of curve $\mathscr{L}$ and it is required to extend it to the other side. This problem reduces to Cauchy's problem with the curve $\mathscr{L}$ as the initial curve.

If $\mathscr{L}$ is not a characteristic, then this Cauchy's problem is normal and the problem of extending the solution is uniquely solved. In this case, from the condition of the continuity of the extension follows the noncontinuity of all derivatives of u(x, t) which exist on the line $\mathscr{L}$, in particular, there follows $u(x, t) \in C_1$. If however the curve $\mathscr{L}$ is a characteristic, then the corresponding Cauchy's problem is nevertheless solvable, for the solvability condition (7) has obviously met (since u(x, t) is the solution on one side of $\mathscr{L}$). However, it is solved nonuniquely.

Let us consider, for example, a continuous extension. As we have seen, the valuesof u(x, t) at the curve $\mathscr{L}$ do not uniquely define its first derivatives p, q; therefore the continuous extension of the solution with discontinuity of the first derivative at the characteristic $\mathscr{L}$ is possible by an infinite set of ways. If however the continuity at $\mathscr{L}$ of the first derivatives is required, then derivatives of higher order can experience discontinuity, so that in this case the extension is defined nonuniquely.

And so, a characteristic is a line through which a solution is extended nonuniquely.

The problem of extending the solution u(x, t) through the characteristic $\mathcal{L}$ is uniquely solved only for the case of analytic solutions, just as, obviously, is the case for any problem of the analytic extension of a function.

Now we will concentrate on the normal Cauchy's problem. By transformation of the variables $x' = x'(x, t)$, $t' = t'(x, t)$ converting the curve $\mathcal{L}$ to the segment of axis $t' = 0$, the general problem reduces to a special Cauchy's problem with initial conditions assigned at the axis $t = 0$: find the solution u(x, t) to system (6.1.1) satisfying the initial conditions

$$u(x, 0) = u^{0}(x), \qquad a \leq x \leq b. \tag{8}$$

From the boundedness of the variables ξ_k it follows that this Cauchy's problem is normal. We will solve Cauchy's problem (6.1.1) and (8) only in the half-plane $t \geqslant 0$. The solution u(x, t) is constructed in the half-plane $t \leqslant 0$ by analogy, when necessary.

3. Domain of dependence and domain of determinacy. The concept of correctness of Cauchy's problem. Suppose we know the solution u(x, t) to the system (6.1.1), taking on the initial value (6.2.8). Let us draw the characteristic $x = x_k(x^0, t^0, \tau)$, given by the equation

$$\frac{dx_k}{d\tau} = \xi_k(x_k, \tau, u(x_k, \tau)) \qquad (k = 1, 2, \ldots, n),$$

until they intercept the axis $t = 0$ through the point $\mathcal{M} = (x^0, t^0)$ of the half-plane $t > 0$. Suppose they intercept the axis $t = 0$ at several points, the farthest of which are denoted by a' and b' $(a' < b')$ (Figure 1.2). The segment of the initial axis $t = 0$ $a' \leq x \leq b'$ is called the domain of the dependence of the solution u at the point $\mathcal{M}$.

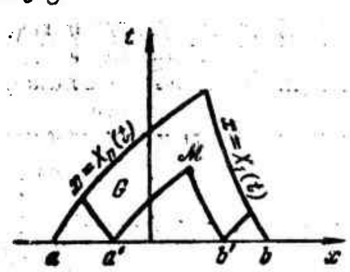

Figure 1.2

The domain of determinacy G of the solution to Cauchy's problem is the name given to the region of the half-plane $t \geqslant 0$ consisting of all points (x, t) for which the domain of dependence $a' \leqslant x \leqslant b'$ belongs to the initial segment $[a, b]$, i.e., $[a', b'] \subseteq [a, b]$.

Finally, the domain of the influence of segment $a' \leqslant x \leqslant b'$ of the initial axis refers to the domain G' of the half-plane $t \geqslant 0$ consisting of all points (x, t) whose domain of dependence has a nonempty intersection with the segment $[a', b']$.

Since the characteristic of the system can be found only simultaneously with the solution $u(x, t)$, determining the domain of determinacy is difficult. The situation becomes much simpler for the case of a semilinear system, when $\xi_k = \xi_k(x, t)$. Here the domain of determinacy G is given by the conditions

$$G: \quad t \geqslant 0, \qquad X_n(t) \leqslant x \leqslant X_1(t),$$

where $X_1(t)$, $X_n(t)$ denote the solutions to the differential equations

$$\frac{dX_n(t)}{dt} = \max_{k=1,\dots,n} \{\xi_k(X_n(t), t)\}, \quad \frac{dX_1(t)}{dt} = \min_{k=1,\dots,n} \{\xi_k(X_1(t), t)\},$$

which take, on where $t = 0$, the values $X_n(0) = a$, $X_1(0) = b$.

For the case of a system of quasilinear equations, the a priori determination of G is difficult. However, if we know that $\|u(x, t)\| \leqslant U$, then the following assertion is valid:

$$\bar{G} \subseteq G,$$

where

$$\bar{G}: t \geqslant 0, \quad \bar{X}_n(t) \leqslant x \leqslant \bar{X}_1(t)\},$$

$$\frac{d\bar{X}_n}{dt} = \max_{k=1,\dots,n} \max_{\|u\|<U} \{\xi_k(\bar{X}_n(t), t, u)\}, \quad \bar{X}_n(0) = a,$$

$$\frac{d\bar{X}_1}{dt} = \min_{k=1,\dots,n} \min_{\|u\|<U} \{\xi_k(\bar{X}_1(t), t, u)\}, \quad \bar{X}_1(0) = b.$$

Cauchy's problem is called correct or correctly formulated if its solution $u(x, t)$ exists, is unique, and depends continuously on initial data. Of course, the question of the metric in which the continuous function obtains depends on the classes of the solutions and initial data under consideration and is solved in each of these classes in a different way. When proving the existence theorems for the solution to Cauchy's problem, it will be stated in which sense the continuous dependence of solutions on initial data obtains.

Let us explain these concepts with the example of a linear system with constant coefficients:

$$\partial u/\partial t + A\partial u/\partial x = 0. \qquad (1)$$

The invariants

$$r_k = l_k u = l^k_a u_a \qquad (2)$$

satisfy the equations

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = 0 \quad (k = 1, \ldots, n),$$

i.e., system (1) is decomposible into n independent equations. The characteristics of system (1) are straight lines:

$$x = x_k = x^0_k + \xi_k t.$$

Therefore the domain of determinacy of the solution to Cauchy's problem for system (1) is the triangle

$$G: \quad t \geqslant 0, \quad a + \xi_n t \leqslant x \leqslant b + \xi_1 t,$$

and the domain of dependence of the solution at the point $\mathcal{M}$ (x, t) is the segment [a', b'] of the axis t = 0, where $a' = x - \xi_n t$, $b' = x - \xi_1 t$ (Figure 1.3).

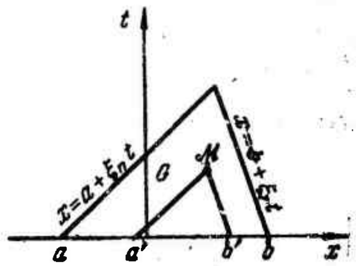

Figure 1.3

The functions $r_k(x, t)$ are easily defined:

$$r_k(x, t) = f_k(x - \xi_k t),$$

where f_k are arbitrary functions.

If the initial conditions $u(x, 0) = u^0(x)$, are posed, then by (2)

$$r_k(x, 0) = l^k u^0(x) = r^0_k(x).$$

From whence

$$r_k(x, t) = r^0_k(x - \xi_k t) = l^k u^0(x - \xi_k t) = l^k_\alpha u^0_\alpha(x - \xi_k t).$$

Returning by formulas (2) to the functions u, we get

$$u_k(x, t) = \lambda^k_\alpha l^\alpha_\beta u^0_\beta(x - \xi_\alpha t).$$

From this formula there directly follows the continuous dependence on initial data — matrix A and initial functions $u^0(x)$ — of the solution to Cauchy's problem of system (1) with constant coefficients.

4. Method of characteristics and review of results. Sections VII and VIII will set forth in detail the application of the method of characteristics to proving the principal theorems on the solvability of Cauchy's problem. Here we will only briefly describe the concept of the method of characteristics in order that the reader not interested in details will grasp the method of characteristics and the results achieved without having to read the proofs of the related theorem.

Suppose that for the system

$$l^k\left[\frac{\partial u}{\partial t}+\xi_k(x,t,u)\frac{\partial u}{\partial x}\right]=f_k(x,t,u) \qquad (1)$$

the initial conditions $u(x, 0) = u^0(x)$ (2)

are formulated. For sake of simplicity, assume that $a = -\infty$, $b = \infty$; the quantities l^k, ξ_k, f_k, u^0 are assumed to be sufficiently smooth functions of their variables.

Suppose that in the strip $0 \leq t \leq t_0$ the smooth solution $u(x, t)$ to problem (1), (2) is known. The functions

$$\bar{l}^k=l^k(x,t,u(x,t)),\quad \bar{\xi}_k=\xi_k(x,t,u(x,t)),\quad \bar{f}_k=f_k(x,t,u(x,t))$$

can then be regarded as functions of the variables x, t, and system (1) can be considered as a system of linear equations

$$\bar{l}^k\left[\frac{\partial u}{\partial t}+\bar{\xi}_k\frac{\partial u}{\partial x}\right]=\bar{f}_k$$

and it can be written in the invariants:

$$\frac{\partial r_k}{\partial t}+\bar{\xi}_k\frac{\partial r_k}{\partial x}=\bar{g}_k(x,t). \qquad (3)$$

Here the functions $\bar{g}_k(x, t)$ are expressed in terms of $u(x, t)$, f_k, l^k, ξ_k and the first derivatives of $\bar{l}^k$.

Each of the equation (3) can be integrated. Actually, if we let $x = x_k(x^o, t^o, \tau)$ stand for the solution to the problem

[redacted]

then the expression $\frac{\partial}{\partial \tau} + \bar{\xi}_k \frac{\partial}{\partial x}$ is the differentiation operator with respect to the variable τ in the direction of the characteristic $x = x_k(x^o, t^o, \tau)$; therefore

[redacted] (4)

where r_k^o are equal, in accordance with (3.1.2),

[redacted]

In view of the hyperbolicity of system (1), matrix $((\bar{l}^k_\alpha))$ is nonsingular, therefore $u(x, t)$ can be determined from (4):

[redacted]

(cf formula (3.1.4)). However, the solution $u(x, t)$ is unknown to us and, therefore, so are the quantities $\bar{l}^k$, $\bar{\xi}_k$, $\bar{f}_k$. So, with the exception of the most simple cases, construction of the solution $u(x, t)$ does not reduce to this uncomplicated procedure, but requires the application of the method of successive approximations.

Suppose that in the strip $0 \leq t \leq t_o$ an approximate value $\overset{(s)}{u}(x, t)$ of the solution to Cauchy's problem (1), (2) is known. Then we can determine the quantities $\overset{(s)}{\bar{l}^k}$, $\overset{(s)}{\bar{\xi}_k}$, $\overset{(s)}{\bar{f}_k}$ and find the next approximation of $\overset{(s+1)}{u}(x, t)$ by the above-indicated method.

Thus, the approximation $\overset{(s+1)}{u}(x, t)$ can be regarded as the results of applying to $\overset{(s)}{u}(x, t)$ a certain operator T: $\overset{(s+1)}{u}(x, t) = T\overset{(s)}{u}(x, t)$.
This operator is nonlinear and contains the operation of differentiation with respect to x, t and integration along the characteristics. The solution $u(x, t)$ to problem (1), (2) under this approach satisfies the equation $u(x, t) = Tu(x,t)$, which obviously symbolically describes Cauchy's problem (1), (2). To prove the convergence of successive approximations $\{\overset{(s)}{u}(x, t)\}$, we must first of all establish their uniform boundness in some strip $0 \leq t \leq t_o$. Then, the proof of convergence reduces to establishing the complete continuity of operator T (i.e., the fact that it maps any bonded set into a compact set). Finally, it is

shown that the limit possesses the required smoothness and is a solution to a problem. Ordinarily this last stage involves investigating the sequence of derivatives $\partial u^{(s)}/\partial x$. These problems are studied in detail in the next two sections.

The first results on the existence and uniqueness of solutions to Cauchy's problem were obtained by the Cauchy-Kovalevskaya method for systems of the Cauchy-Kovalevskaya equations on the assumption that the initial data of the Cauchy's problem were analytic. These burdensome restrictions detract from the value of the results, since Cauchy's problem for hyperbolic equations is best considered with minimum requirements on the smoothness of initial data.

In 1927 H. Levy (cf [11]) showed that essentially the solution to a hyperbolic system of linear equations with two independent variables reduces to the solution of Cauchy's problem for systems of ordinary differential equations. This work laid the foundation of the classical method of characteristics*).

We will briefly present here the result obtained recently on the question of the solvability of Cauchy's problem for hyperbolic systems of equations with two independent variables.

In 1948 K. O. Fridrichs (cf [12]) considered the problem of the existence and uniqueness of the solution to the problem (1), (2) for systems of linear, semilinear, and quasilinear equations. For the linear system K. C. Fridrichs proposed continuous differentiability of $l^k(x, t)$, Lipshits-continuity of $\xi_k(x, t)$, and the continuity of $f_k(x, t, u) = f^k(x, t) + f^k_\alpha(x, t)\, u_\alpha$. Given these conditions, he established the existence of a solution in the broad sense on the assumption only of continuity of $u^o(x)$ (the concept of the solution in the broad sense will be taken up in section VII). For a system of quasilinear equations Fridrichs required that l^k_α, $\xi_k \in C_2$, $f_k \in C_1$, $u^o(x) \in C_2$. Here the existence of the solution $u(x, t) \in C_2$ was proven.

R. Courant and P. Lax (cf [6]) used the concept of an extended system in invariants. In spite of the elegance of the proof, in this work more rigid assumptions on the smoothness of the initial data are made. Thus, for example,

*) The method of characteristics was developed earlier for a single equation.

the existence of third derivatives is required for the initial functions $u^0(x)$.

In papers by A. Douglas (cf [13]) and P. Hartman and A. Wintner (cf [14]), published in 1952, the solution to the problem (1), (2) is constructed on the assumption of continuous differentiability of initial data. A. Douglas first constructed the solution for smoother initial data; these initial data are considered by means of passage to the limit. F. Hartman and A. Wintner constructed the solution under these assumptions directly by the method of characteristics. Lemma 2, presented in the next section, plays a key role in their construction.

Thus, the minimum requirement on initial data under which the existence and uniqueness of the solution $u(x, t) \in C_1$ have been proven at the present time is the requirements of the continuous differentiability of these data. It must further be required that, as shown by the simplest examples, no solution $u(x, t) \in C_1$ exists if the initial data are not differentiable.

Let us note that several existence theorems can be gotten by specialization of more general theorems to the case of two independent variables. Thus, for example, papers by I. G. Petrovskiy (cf [15]), and S. A. Khristianovich (cf [16]) presented general results, from which derive, in particular, existence theorems for the case of interest to us. Let us however note that here the requirements imposed on the initial data are naturally overstated.

Finally, we make several remarks on the presentation of these questions discussed in this book. Cauchy's problem for the linear system is studied on the basis of the paper [12] by K. O. Fridrichs; the existence theorem is proven with these same assumptions.

The method of characteristics as propounded by F. Hartman and A. Wintner is adopted has the basis for systems of quasilinear equations. However, our presentation differs in several key respects. We cite a number of them. Proof of the boundness of successive approximations and their derivatives is usually extremely cumbersome in the method of characteristics. Gross estimates using the "majorant" which we employ essentially considerably simplify and make more general these estimates by bypassing the necessity of arithmetic computation. Another point of distinction in our presentation is in the proof of uniform convergence in the domain of determinacy G not only of the successive approximations, but also of the sequence of their first derivatives.

5. Two lemmas.

Lemma 1. Suppose that the vector-function $u(t) = \{u_1, \ldots, u_n\}$ continuous on the segment $0 \leqslant t \leqslant t_0$ satisfies the inequality

$$\|u(t)\| \leqslant U_0 + \int_0^t [A(\tau) + B(\tau) \max_{0 \leqslant \xi \leqslant \tau} \|u(\xi)\|] d\tau, \tag{1}$$

and suppose that when $0 \leqslant t \leqslant t_0$ $|A(t)| \leqslant A$, $|B(t)| \leqslant B$, $U_0 \geqslant 0$. Then when $0 \leqslant t \leqslant t_0$ the following estimate obtains:

$$\|u(t)\| \leqslant \max_{0 \leqslant \tau \leqslant t} \|u(\tau)\| \leqslant U_0 e^{Bt} + \frac{A}{B}(e^{Bt} - 1). \tag{2}$$

When $B = 0$, formula (2) is transformed into the obvious inequality:

$$\|u(t)\| \leqslant \max_{0 \leqslant \tau \leqslant t} \|u(\tau)\| \leqslant U_0 + At.$$

Proof. Let $U(t)$ stand for the quantity $\max_{0 \leqslant \tau \leqslant t} \|u(\tau)\|$. Suppose $t \geqslant 0$ is any number from the segment $[0, t_0]$ and $U(t) = \|u(t')\|$ $(0 \leqslant t' \leqslant t)$. Writing the inequality (1) for the point t', we get

$$\|u(t')\| = U(t) \leqslant U_0 + \int_0^{t'} [A + BU(\tau)] d\tau \leqslant$$
$$\leqslant U_0 + \int_0^t [A + BU(\tau)] d\tau = U_0 + At + B\int_0^t U(\tau) d\tau.$$

Applying this estimate many times, we will have

$$U(t) \leqslant U_0\left[1 + Bt + \ldots + \frac{(Bt)^s}{s!}\right] + $$
$$+ A\left[t + \frac{Bt^2}{2!} + \ldots + \frac{(Bt)^{s+1}}{B(s+1)!}\right] + U(t_0)\frac{(Bt)^{s+1}}{(s+1)!},$$

from whence follows inequality (2).

Corollary. Suppose the continuous vector $u(x, t) \in C_1$ satisfies the inequality

$$\left|\frac{du}{dt}\right| \leqslant \|f(t)\| + |f^k_\alpha u_\alpha(t)|.$$

Then estimate (2) holds, where $U_0 = \|u(0)\|$, $A = \max_t \|f(t)\|$, $B = \max_t \|((f^k_\alpha(t))\|$.

Lemma 2. Suppose $u(x, t, \tau)$, $v(x, t, \tau) \in C_1$. Then the function

$$I(x, t) = \int_0^t u(x, t, \tau) \frac{\partial v(x, t, \tau)}{\partial \tau} d\tau$$

is continuously differentiable ($I(x, t) \in C_1$).

Proof. Let us compute, for example, the derivative $\partial I/\partial x$. We have:

$$\frac{1}{\Delta x}[I(x+\Delta x, t) - I(x, t)] =$$
$$= \frac{1}{\Delta x}\int_0^t \left\{u(x+\Delta x, t, \tau)\frac{\partial v(x+\Delta x, t, \tau)}{\partial \tau} - u(x, t, \tau)\frac{\partial v(x, t, \tau)}{\partial \tau}\right\} d\tau =$$
$$= \int_0^t \frac{u(x+\Delta x, t, \tau) - u(x, t, \tau)}{\Delta x} \cdot \frac{\partial v(x+\Delta x, t, \tau)}{\partial \tau} d\tau +$$
$$+ \frac{1}{\Delta x}\int_0^t u(x, t, \tau)\frac{\partial}{\partial \tau}[v(x+\Delta x, t, \tau) - v(x, t, \tau)] d\tau.$$

Performing integration by parts in the last integral and passing to the limit as $\Delta x \to 0$, we get

$$\frac{\partial I(x, t)}{\partial x} = u(x, t, t) v_x(x, t, t) - u(x, t, 0) v_x(x, t, 0) +$$
$$+ \int_0^t [u_x(x, t, \tau) v_\tau(x, t, \tau) - u_\tau(x, t, \tau) v_x(x, t, \tau)] d\tau \tag{3}$$

Formula (3) proves lemma 2 and simultaneously gives us a rule for computing derivatives of the function $I(x, t)$.

Section VII. Cauchy's Problem for Linear and Semilinear Systems

1. Existence and uniqueness of the solution of Cauchy's problem in the broad sense. Let us consider the semilinear system

$$\frac{\partial u}{\partial t} + A(x, t)\frac{\partial u}{\partial x} = b(x, t, u), \tag{1}$$

and suppose that the system

$$\frac{\partial r_k}{\partial t} + \xi_k(x, t)\frac{\partial r_k}{\partial x} = g_k(x, t, r) \tag{2}$$

is the notation of system (1) in invariants.

Suppose that for some segment [a, b] of axis $t = 0$ the initial conditions

$$u(x, 0) = u^0(x). \qquad (3)$$

are assigned for system (1). Note that the segment [a, b] can be unbounded.

Denoting $r_k^0(x) = l^k(x, 0)\, u^0(x)$, we get the initial conditions

$$r(x, 0) = r^0(x) \qquad (4)$$

for system (2).

Let us assume that the functions A, ξ_k, $l^k \in C_1$ in G (recall that G refers to the domain of determinacy of our problem), f_k, and $\partial f_k/\partial u_\alpha \in C_0$ in the domain $\bar{G} \times \{u \mid -\infty < \|u\| < \infty\}$, $u^0(x) \in C_0$ on [a, b]. Then g_k, $\partial g_k/\partial r_\alpha \in C_0$ in the domain $G \times \{r \mid -\infty \|r\| < \infty\}$, $r^0(x) \in C_0$ on [a, b].

The functions $r(x, t)$ continues in G are called the solution to Cauchy's problem (2), (4) in the broad sense if $r(x, 0) = r^0(x)$ and if each of the functions $r_k(x, t)$ is continuously differentiable with respect to the variable t along the corresponding characteristic $x = x_k(\xi, \tau, t)$, where

$$\frac{d}{dt} r_k(x_k(\xi, \tau, t), t) = g_k(x_k, t, r(x_k, t)). \qquad (5)$$

The vector $u(x, t)$ obtained from the vector $r(x, t)$ by formulas (3.1.4) will be called the solution in the broad sense of Cauchy's problem (1), (3).

The uniqueness theorem of the solution in the broad sense. Suppose that two solutions in the broad sense $u(x, t)$ and $\bar{u}(x, t)$ to problem (1), (3) exist in G. To these correspond two solutions in the broad sense $r(x, t)$, $\bar{r}(x, t)$ of problem (2), (4).

We introduce the difference $v(x, t) = r(x, t) - \bar{r}(x, t)$ $(v(x, 0) = 0)$ (6)
Subtracting from equation (5) written for the function $r(x, t)$ this same equation, written for $\bar{r}(x, t)$, we get

$$\frac{dv_k(x_k(\xi, \tau, t), t)}{dt} = g_\alpha^k(x_k, t)\, v_\alpha(x_k, t). \qquad (7)$$

where $g_\alpha^k(x, t)$ denotes the quantities

$$g_\alpha^k(x, t) = \int_0^1 \frac{\partial g_k}{\partial r_\alpha}(x, t, r(x, t) - \lambda v(x, t))\, d\lambda.$$

By the definition of a solution in the broad sense, the functions r, $\bar{r}$, v are continuous in G. Therefore the functions g^k_α are continuous in G and are bounded in any strip $0 \leq t \leq t_0$.

Integrating equation (7) with respect to t from 0 to τ, allowing for condition (6), we get

$$v_k(\xi, \tau) = \int_0^\tau g^k_\alpha(x_k, t) v_\alpha(x_k, t)\,dt \quad (k = 1, \ldots, n).$$

In view of the boundedness of matrix $((g^k_\alpha))$, hereafter applying lemma 1 from section VI it follows that everywhere in G $\|v(x, t)\| = 0$ and, therefore, $r(x, t) \equiv \bar{r}(x, t)$, $u(x, t) \equiv \bar{u}(x, t)$. The theorem is proven.

Of course, from this theorem naturally follows that the classical, i.e., continuously differentiable, solution to Cauchy's problem is also unique.

Existence theorem of the solution in the broad sense for a linear system. We will construct the solution to the problem (2), (4) for a linear system by the method of successive approximations.

Suppose $g_k(x, t, r) = g^k(x, t) + g^k_\alpha(x, t)\, r_\alpha$. By formula (5), the solution r_k satisfies the equation

$$r_k(x, t) = r^0_k(x_k(x, t, 0)) + \int_0^t g^k(x_k(x, t, \tau), \tau)\,d\tau + \int_0^t g^k_\alpha(x_k(x, t, \tau), \tau)\, r_\alpha(x_k(x, t, \tau), \tau)\,d\tau \quad (8)$$

Applying the method of successive approximations, we set

$$\overset{(s+1)}{r}_k(x, t) = \overset{(0)}{r}_k(x, t) + \int_0^t g^k_\alpha(x_k(x, t, \tau), \tau)\, \overset{(s)}{r}_\alpha(x_k(x, t, \tau), \tau)\,d\tau \quad (9)$$

$$(s = 0, 1, \ldots),$$

where

$$\overset{(0)}{r}_k(x, t) = r^0_k(x_k(x, t, 0)) + \int_0^t g^k(x_k(x, t, \tau), \tau)\,d\tau.$$

Hence it is clear that $\overset{(s+1)}{r}_k(x, t)$ are defined and continuous in G and have a continuous derivative in t in the corresponding characteristic direction. Let us prove the uniform convergence of the sequence $\{\overset{(s)}{r}(x, t)\}$.

From formula (9) follows

$$\left\| \overset{(s+1)}{r}(x,t) - \overset{(s)}{r}(x,t) \right\| < B \int_0^t \sup_{G_\tau} \left\| \overset{(s)}{r} - \overset{(s-1)}{r} \right\| d\tau \qquad (10)$$

where G_τ denotes the intersection of G with the strip $0 \leq t \leq \tau$, and B is the constant bounding the norm of matrix $((g^k_\alpha))$ in the domain G.

Denoting

$$\overset{(s)}{v}(t) = \sup_{G_t} \left\| \overset{(s+1)}{r_k}(\xi,\eta) - \overset{(s)}{r_k}(\xi,\eta) \right\|$$

let us writing (10) as

$$\overset{(s)}{v}(t) < B \int_0^t \overset{(s-1)}{v}(\tau)\, d\tau$$

Then

$$\overset{(s)}{v}(t) < \frac{(Bt)^s}{s!} \overset{(0)}{v}(t) \to 0 \qquad \text{when } s \to \infty$$

and, therefore, the sequence $\{\overset{(s)}{r}(x, t)\}$ uniformly converges to the continuous function $r(x,t)$ if the domain G is finite with respect to variable t. If however the domain G is infinite in t, then the sequence $\{\overset{(s)}{r}(x, t)\}$ converges uniformly in any of its finite (with respect to t) subdomains G_{t_0}. Passing in the equality (9) to the limit as $s \to \infty$, we get the result that $r(x, t)$ satisfies equations (8). Since $x_k(x, t, \tau) = x_k(x_0, t_0, \tau)$, if the point (x_0, t_0) lies on this characteristic, then equalities (8) can be rewritten as

$$r_k(x_k(x_0, t_0, t), t) = r_k^0(x_k(x_0, t_0, 0)) + \int_0^t g^k(x_k(x_0, t_0, \tau), \tau)\, d\tau +$$
$$+ \int_0^t g^k_\alpha(x_k(x_0, t_0, \tau), \tau)\, r_\alpha(x_k(x_0, t_0, \tau), \tau)\, d\tau \qquad (11)$$

From the continuity of the integrands follows the continuous differentiability of the right side of (11) with respect to the variable t. Therefore, $r_k(x, t)$ is continuously differentiable along the characteristic $x = x_k$; here the equality (5) is satisfied. Thus, $r_k(x, t)$ is the solution to the problem (2), (4) in the broad sense. By formulas (3.1.4) the solution $u(x, t)$ to the problem (1), (3) can be obtained.

For the case of a semilinear system, successive approximations are assigned analogously to (9):

$$\overset{(s+1)}{r_k}(x, t) = \overset{(0)}{r_k}(x, t) + \int_0^t g_k(x_k(x, t, \tau), \tau)\,\overset{(s)}{r}(x_k(x, t, \tau), \tau)\,d\tau,$$

and converge uniformly in the subdomains G_{t_0} of the domain G in which they are uniformly bounded.

Thus, the construction of the solution in the broad sense for the semilinear system differs from the linear case only by the fact that the domain G_{t_0} in which the solution of the semilinear system remains bounded appears as the domain of convergence of the successive approximations. We will delay the discussion of the boundedness of the solution to section VIII.

2. Existence of the classical solution to Cauchy's problem for the linear system. Suppose the system (7.1.1) is linear, i.e., $b(x, t, u) = b_1(x, t) + B(x, t)u$. Then in the system (7.1.2) $g_k(x, t, r) = g^k(x, t) + g^k_\alpha(x, t)\, r_\alpha$, where $g^k(x, t) = l^k b_1$, and

$$g^k_\alpha(x, t) = \gamma^k_\alpha(x, t) + \left[\frac{\partial l^k_\beta}{\partial t} + \xi_k \frac{\partial l^k_\beta}{\partial x}\right] \lambda^\beta_\alpha(x, t), \tag{1}$$

where $\gamma^k_\alpha = l^k_\beta\, B_{\beta\delta}\, \lambda^\delta_\alpha$ (cf formulas (3.1.2)).

We assume that l^k, ξ_k, $B_{\alpha\beta} \in C_1$ in G. Obviously λ^α_β, $\gamma^k_\alpha \in C_1$ in G. Suppose also that $u^0(x) \in C_1$, $r^0(x) \in C_1$.

Let us show that the solution constructed above is under these assumptions continuously differentiable in G and, therefore, yields the solution to the problem (7.1.2), (7.1.4) in the ordinary sense.

In the case (1) formula (7.1.9) becomes

$$\overset{(s+1)}{r_k}(x, t) = \overset{(0)}{r_k}(x, t) + \int_0^t \gamma^k_\alpha(x_k(x, t, \tau), \tau)\,\overset{(s)}{r_\alpha}(x_k(x, t, \tau), \tau)\,d\tau +$$
$$+ \int_0^t \left[\frac{d}{d\tau} l^k_\beta (x_k(x, t, \tau), \tau)\right] \lambda^\beta_\alpha(x_k(x, t, \tau), \tau)\,\overset{(s)}{r_\alpha}(x_k(x, t, \tau), \tau)\,d\tau. \tag{2}$$

Obviously, from these assumptions there follows the continuous differentiability in G of the first two members of the right side of (2) if in G the approximation $\overset{(s)}{r}(x, t)$ is continuously differentiable. As for the last member of formula (2), it is also continuously differentiable in G if $\overset{(s)}{r}(x, t) \in C_1$. This follows from lemma 2 of section VI.

Differentiating (2) with respect to variable x (the last term in formula (6.5.3)), we get

$$\frac{\partial \overset{(s+1)}{r_k}}{\partial x} = \frac{\partial \overset{(0)}{r_k}}{\partial x} + \int_0^t \frac{\partial x_k(x, t, \tau)}{\partial x}\left[\frac{\partial \gamma_\alpha^k}{\partial x}\overset{(s)}{r_\alpha} + \gamma_\alpha^k \frac{\partial \overset{(s)}{r_\alpha}}{\partial x}\right] d\tau +$$

$$+ \left[\frac{d}{dx} l_\beta^k(x_k(x, t, \tau), \tau)\right]_{\tau=t} \lambda_\alpha^\beta(x, t)\overset{(s)}{r_\alpha}(x, t) -$$

$$- \left[\frac{d}{dx} l_\beta^k(x_k(x, t, \tau), \tau)\right]_{\tau=0} \lambda_\alpha^\beta(x_k(x, t, 0), 0)\overset{(s)}{r_\alpha}(x_k(x, t, 0), 0) +$$

$$+ \int_0^t \left\{\frac{\partial}{\partial x}\left[\lambda_\alpha^\beta(x_k(x, t, \tau), \tau)\overset{(s)}{r_\alpha}(x_k(x, t, \tau), \tau)\right] \times\right.$$

$$\times \left[\frac{\partial}{\partial \tau} l_\beta^k(x_k(x, t, \tau), \tau)\right] - \frac{\partial}{\partial \tau}\left[\lambda_\alpha^\beta(x_k(x, t, \tau), \tau)\overset{(s)}{r_\alpha}(x_k(x, t, \tau), \tau)\right] \times$$

$$\left. \times \left[\frac{\partial}{\partial x} l_\beta^k(x_k(x, t, \tau), \tau)\right]\right\} d\tau.$$

Hence it follows that

$$\left\| \frac{\partial \overset{(s+1)}{r}}{\partial x} - \frac{\partial \overset{(s)}{r}}{\partial x} \right\| \leqslant A^1(x, t) \left\| \overset{(s)}{r}(x, t) - \overset{(s-1)}{r}(x, t) \right\| +$$

$$+ \int_0^t A^2(x, t, \tau) \max_{G_\tau} \left\| \overset{(s)}{r} - \overset{(s-1)}{r} \right\| d\tau +$$

$$+ \int_0^t A^3(x, t, \tau) \max_{G_\tau} \left\| \frac{\partial \overset{(s)}{r}}{\partial x} - \frac{\partial \overset{(s-1)}{r}}{\partial x} \right\| d\tau,$$

and the functions A^1, A^2, A^3 are continuous and bounded in G.

Since the sequence $\{\overset{(s)}{r}(x, t)\}$ converges in G and the quantities A^1, A^2, A^3 are bounded in G, then the uniform convergence in G of the sequence $\{\partial \overset{(s)}{r_k} / \partial x\}$ is proven on analogy to the preceding. This means that the above-constructed solution in the general sense is continuously differentiable in the variable x.

The continous differentiability of $r(x, t)$ with respect to t is similarly proven. Incidently, this already follows from the continuity in G of the combination of derivatives

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = g_k(x, t, r).$$

Thus, when the formulated conditions are satisfied, the solution $r(x, t)$ in the broad sense is also the solutions to Cauchy's problem (7.1.2) and (7.1.4) in the ordinary sense. Passing by formulas (3.1.4) from r to u, we conclude that the resulting function u is a solution to the system (7.1.1) in the ordinary sense.

The following assertion is quite analogously proven for the semilinear system: if in the domain G_{t_0} the solution $u(x, t)$ of the problem (7.1.1), (7.1.3) is bounded and ξ_k, $l^k \in C_1$; $f_k(x, t, u) \in C_1$, $u_0(x) \in C_1$, the above-constructed solution in the broad sense is continuously differentiable, i.e., $u(x, t) \in C_1$.

By direct differentiation of systems of equations (7.1.2) we see that on these assumptions, the derivatives p, q satisfy in the broad sense the equations of the extended system.

3. Some properties of solutions of linear and semilinear systems. The solution to Cauchy's problem in the broad sense for linear and semilinear systems is uniformly continuous in G if the domain G is finite; if G is an unbounded region, uniform continuity obtains in any finite region of G. Here, for the case of a semilinear system the uniform continuity of the solution in the broad sense obtains only in the domain G_{t_0} of the boundedness of the solution. These properties are easily derived from formula (7.1.8) defining the solution $r(x, t)$.

In the case of the classical solution ($u(x, t) \in C_1$), the derivatives p, q satisfy in the broad sense equations of the extended system (cf subsection 3 of section IV). Noting that in the case of a semilinear system the extended system is linear with respect to the derivative p, q, we conclude that the derivatives of a solution to a semilinear system remains bounded in the domain G_{t_0} in which the solution itself remains bounded.

Suppose now $u(x, t)$ and $\bar{u}(x, t)$ are two solutions in the broad sense to Cauchy's problem, whose initial data we denote by l^k, ξ_k, f_k, $u^0(x)$ and $\bar{l}^k$,

$\bar{\xi}_k$, $\bar{f}_k$, $\bar{u}^o(x)$. We will assume that the initial data satisfy the conditions formulated in the proof of the existence theorem of the solution in the broad sense.

It is easy to see that when ξ_k, $\bar{\xi}_k \in C_1$ $\bar{x}_k(x, t, \tau) \longrightarrow x_k(x, t, \tau)$, if $\bar{\xi}_k \longrightarrow \xi_k$; then when $\bar{\xi}_k \longrightarrow \xi_k$, $\bar{G} \longrightarrow G$. From formula (7.1.6) it follows that if $\bar{\xi}_k \longrightarrow \xi_k$, $\bar{l}_k \longrightarrow l_k$, and so on, then $\bar{r}_k(x, t) \longrightarrow r_k(x, t)$, $\bar{u}_k(x, t)$ ---- $u_k(x, t)$. Thus, the solution to Cauchy's problem in the broad sense depends continuously (in the norm C) on the initial data of this problem. Thus, Cauchy's problem is correct in this formulation.

If the initial data are continuously differentiable, then as we have seen, the solution has continuous derivatives and is classical. Of course, the above statements on continuous dependence apply also to these solutions.

If, further, not only the initial data, but also the derivatives of the initial data uniformly approach each other, then not only $\bar{u} \longrightarrow u$, but also

$$\frac{\partial \bar{u}}{\partial x} \longrightarrow \frac{\partial u}{\partial x}, \quad \frac{\partial \bar{u}}{\partial t} \longrightarrow \frac{\partial u}{\partial t}.$$

This follows from the fact that the derivatives of the solution satisfy, in the broad sense, the equations of the extended system which are linear with respect to the derivatives. Similar conclusions can be made also about derivatives of higher order if the requirements on the smoothness of the initial data are made suitably more rigorous.

The construction of the solution to Cauchy's problem reduces to constructing a representation that transforms the initial function into the solution to Cauchy's problem at the time instant t. Let us consider, for sake of specificity, the linear system with right side equal to zero (system (7.1.1) with $b = Bu$). Then $u = Su^o$, where the operator S is linear. Obviously, the domain of definition of this operator is a set of continuously differentiable functions. We will consider S as the mapping $C \longrightarrow C$. Then the operator S is defined on an everywhere-dense set (which is a set of continuously differentiable functions in C) and is bounded on the set. According to the familiar theorem of functional analysis, S permits continuous expansion with norm preserved to the entire space C. S* is the result of this expansion. Then $u = S^*u^o$, $u^o \in C$ is the solution to Cauchy's problem in the broad sense whose existence we have proven to be independent.

Hence it follows that to construct the generalized (in the sense of extensions of operators) solution, it is sufficient to approximate the initial function $u^0_\delta(x)$ by an element of everywhere-dense set $u^0_\delta(x)$ (i.e., "to smooth" the initial function) to construct the smooth solution $u_\delta = Su^0_\delta$, and to pass to the limit as $\delta \longrightarrow 0$, using the corresponding metric.

Similar considerations apply also for the semilinear case. It is complicated only by the fact that the operator S assigning the solution to Cauchy's problem: $u = S(u^0)$, is now nonlinear. It is determinate, bounded, and continuous, as the operator from C to C on a set of continuously differentiable functions satisfying the equality

$$\|u^0\| < U_{t_0}, \qquad U_{t_0} = \text{const} \tag{1}$$

($0 \leq t < t_0$ is the strip in which the solution is considered). Quite analogously to the foregoing case, the operator S can be extended to the continuous bounded operator defined on the entire set of elements C satisfying inequality (1). The result of this extension, just as in the linear case, yields the solution in the broad sense.

We stress that generalized solutions of linear and semilinear equations are, thus, limits of the classical solutions in a given metric.

Section VIII. Cauchy's Problem for Systems of Quasilinear Equations

1. Growth estimate of a solution and its derivatives. Majorant system. For the system of quaslinear equation $l^k = l^k(x, t, u)$; $\xi_k = \xi_k(x, t, u)$. In this case the construction of the solution of Cauchy's problem is complicated compared with the linear system. Let us indicate several points of distinction for this case;

(1) for a system of quasilinear equation we can no longer introduce the concept of a solution in the broad sense in view of the absence of invariants;

(2) the domain G of the determinacy of the solution to Cauchy's problem is defined simultaneously with the solution u(x, t) and, generally speaking, cannot be prespecified; and

(3) the solution u (x, t) and its derivatives do not remain bounded.

Therefore, first of all we establish the pre-estimates of the solution and its derivatives and indicate the domain $\tilde{G} \subseteq G$ of variables x, t, in which the solution and its derivatives remain clearly bounded.

Let us introduce the domain $G_0(U)$ of the space (x, t, u) as given by the conditions:

$$G_0(U) = \{a_0 \leq x \leq b_0,\ 0 \leq t \leq T_0;\ \|u\| \leq U\},$$
$$a_0 < a,\quad b_0 > b.$$

Suppose $l^k(x, t, u),\ \xi_k(x, t, u),\ f_k(x, t, u) \in C_1(G_0(U))$ for any $U > 0$; $u^0(x) \in C_1(a, b)$.

By subsection 3 of section IV, the extended system for a hyperbolic system of quasilinear equations is of the form

$$\frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} = \mathcal{F}_k; \qquad \frac{\partial u_k}{\partial t} = F_k, \tag{1}$$

where
$$\mathcal{F}_k = \mathcal{F}^k(x, t, u) + \mathcal{F}^k_\alpha(x, t, u)\mathcal{P}_\alpha + \mathcal{F}^k_{\alpha\beta}(x, t, u)\mathcal{P}_\alpha\mathcal{P}_\beta,$$
$$F_k = F^k(x, t, u) + F^k_\alpha(x, t, u)\mathcal{P}_\alpha,$$
$$\mathcal{P}_k = l^k_\alpha(x, t, u)\, p_\alpha.$$

The quantities $\mathcal{F}^k$, $\mathcal{F}^k_\alpha$, $\mathcal{F}^k_{\alpha\beta}$; F^k, F^k_α are expressed by l^k_α, ξ_k, f_k, and their first derivatives. Therefore these quantities are continuous in the domain $G_0(U)$ for any $U > 0$. Let us introduce the following notation:

$$\mathcal{F}_0(U) = \max_{G_0(U)} \|\mathcal{F}(x, t, u)\|, \quad \mathcal{F} = \{\mathcal{F}^k\},$$
$$\mathcal{F}_1(U) = \max_{G_0(U)} \|\mathcal{F}^k_\alpha(x, t, u)\|,$$
$$\mathcal{F}_2(U) = \max_{G_0(U)} \max_{\beta_0 = 1, \ldots, n} \|\mathcal{F}^k_{\alpha\beta_0}(x, t, u)\|,$$
$$F_0(U) = \max_{G_0(U)} \|F(x, t, u)\|,$$
$$F_1(U) = \max_{G_0(U)} \|F^k_\alpha(x, t, u)\|$$

and let us consider a system of two ordinary differential equations:

$$\frac{d\mathcal{P}}{dt}=\mathcal{F}_0(U)+\mathcal{F}_1(U)\mathcal{P}+\mathcal{F}_2(U)\mathcal{P}^2, \tag{2}$$

$$\frac{dU}{dt}=F_0(U)+F_1(U)\mathcal{P}, \tag{3}$$

which we will call the majorant system. Let U_0, $\mathcal{P}_0$ stand for the quantities

$$U_0=\max_{a<x<b}\|u^0(x)\|, \quad \mathcal{P}_0=\max_{a<x<b}\left\|l_\alpha^k(x,\ 0,\ u^0(x))\frac{du_\alpha^0(x)}{dx}\right\|.$$

For the system of equations (2), (3), we assign the initial conditions:

$$\mathcal{P}(0)=\mathcal{P}_0, \qquad U(0)=U_0. \tag{4}$$

For a comparison of equations (1), (2), and (3) it follows that if

$$\|u(x,\ t)\|\leqslant U(t), \quad \|\mathcal{P}(x,\ t)\|\leqslant\mathcal{P}(t), \tag{5}$$

then

$$\left|\frac{\partial u(x,\ t)}{\partial t}\right|\leqslant\frac{dU(t)}{dt}, \quad \left\|\left\{\frac{\partial\mathcal{P}_k(x,\ t)}{\partial t}+\xi_k\frac{\partial\mathcal{P}_k(x,\ t)}{\partial x}\right\}\right\|\leqslant\frac{d\mathcal{P}(t)}{dt}.$$

Since from (4) follows the satisfaction of condition (5) when $t = 0$, then for any $t \geqslant 0$ $\|u(x,\ t)\| \leqslant U(t)$, $\|\mathcal{P}(x,\ t)\| \leqslant \mathcal{P}(t)$. Thus, the functions $U(t)$, $\mathcal{P}(t)$ majorize the growth of the solution $u(x,\ t)$ and its first derivatives.

Suppose that when $0 \leqslant t \leqslant t_0$ the solution $U(t)$, $\mathcal{P}(t)$ of the majorant system satisfying the initial conditions (4) remain bounded. Then clearly when $0 \leqslant t \leqslant t_0$ the solution $u(x,\ t)$ and its derivatives $p(x,\ t)$ remain bounded. Therefore we can determine the domain $\overline{G} \subseteq G$ by specifying it as follows:

$$\overline{G}=\{0\leqslant t\leqslant t_0,\quad \overline{X}_1(t)\leqslant x\leqslant\overline{X}_2(t)\}.$$

$$\frac{d\overline{X}_1}{dt}=\max_{k=1,\ \dots,\ n}\ \max_{\|u\|<U(t)}\{\xi_k(\overline{X}_1(t),\ t,\ u)\}, \quad \overline{X}_1(0)=a,$$

$$\frac{d\overline{X}_2}{dt}=\min_{k=1,\ \dots,\ n}\ \min_{\|u\|<U(t)}\{\xi_k(\overline{X}_2(t),\ t,\ u)\}, \quad \overline{X}_2(0)=b.$$

We will construct the solution to Cauchy's problem for a system of quasi-linear equations in the domain $\overline{G} \subseteq G$ (Figure 1.4).

Let us write out the majorant system for several examples.

(1) Linear system. Suppose $l_\alpha^k = l_\alpha^k(x, t)$, $\xi_k = \xi_k(x, t)$, $f_k = f^k(x, t) + f_\alpha^k(x, t) u_\alpha$. Then

$$\mathcal{F}_0(U) = \mathcal{F}_0 + \bar{\mathcal{F}}_0 U, \quad \mathcal{F}_1(U) = \mathcal{F}_1, \quad \mathcal{F}_2(U) = 0,$$
$$F_0(U) = F_0 + \bar{F}_0 \cdot U, \quad F_1(U) = F_1,$$

where $\mathcal{F}_0, \bar{\mathcal{F}}_0, \mathcal{F}_1; F_0, \bar{F}_0, F_1$ are constants that depend only on the domain G of variables x, t. The majorant system for all the systems of linear equations is also linear. This means that the solution to the majorant system, and together with it the solution u(x, t) and its derivatives p (x, t) remain bounded in any finite region of the half-plane $t \geq 0$. The quantity t_0 in this case is arbitrary, and the domain $\bar{G}$ coincides with the domain of determinacy G.

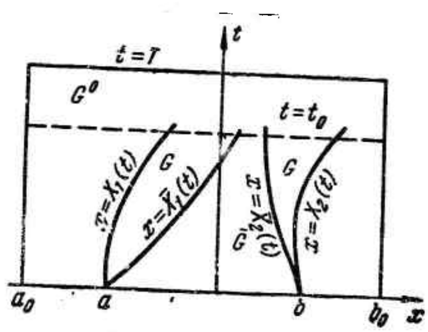

Figure 1.4

(2) Semilinear system. Suppose $l_\alpha^k = l_\alpha^k(x, t)$, $\xi_k = \xi_k(x, t)$, $f_k = f_k(x, t, u)$. Then $\mathcal{F}_2(U) = 0$. The majorant system (2), (3) takes on the form

$$\frac{d\mathcal{P}}{dt} = \mathcal{F}_0(U) + \mathcal{F}_1(U)\mathcal{P}, \quad \frac{dU}{dt} = F_0(U) + F_1(U)\mathcal{P}.$$

Hence it follows that $\mathcal{P}(t)$ is bounded if U(t) is bounded. This fact expresses the general property of solutions of semilinear system: the derivatives of the solution remain bounded as long as the solution itself is bounded.

2. Theorems of the uniqueness and existence of a solution. On the assumptions made in subsection 1, let us consider the Cauchy's problem

$$l^k(x, t, u)\left[\frac{\partial u}{\partial t} + \xi_k(x, t, u)\frac{\partial u}{\partial x}\right] = f_k(x, t, u), \qquad (1)$$

$$u(x, 0) = u^0(x). \qquad (2)$$

First of all let us prove the uniqueness theorem. Suppose that in the domain G exist the two solutions $u(x, t)$ and $\bar{u}(x, t)$ to Cauchy's problem (1), (2). Then the difference $v(x, t) = u - \bar{u}$ satisfies, in domain G, the system of linear equations

$$\bar{l}^k_\alpha\left[\frac{\partial v_\alpha}{\partial t} + \bar{\xi}_k\frac{\partial v_\alpha}{\partial x}\right] = \bar{f}^k_\alpha v_\alpha \quad (k = 1, \ldots, n)$$

and the zero initial conditions $v(x, 0) \equiv 0$. Here we introduce the notations:

$$\left.\begin{aligned}
&\bar{l}^k_\alpha = l^k_\alpha(x, t, \bar{u}(x, t)), \quad \bar{\xi}_k = \xi_k(x, t, \bar{u}(x, t)),\\
&\bar{f}^k_\alpha = \int_0^1 \Big\{\frac{\partial f_k}{\partial u_\alpha}(x, t, \bar{u} + \lambda v(x, t)) -\\
&\qquad - \frac{\partial u_\beta(x, t)}{\partial t}\frac{\partial l^k_\beta}{\partial u_\alpha}(x, t, \bar{u} + \lambda v(x, t)) -\\
&\qquad\qquad - \frac{\partial u_\beta(x, t)}{\partial x}\frac{\partial}{\partial u_\alpha}(l^k_\beta \cdot \xi_k)(x, t, \bar{u} + \lambda v)\Big\}\, d\lambda.
\end{aligned}\right\} \qquad (3)$$

The quantities $\bar{l}^k_\alpha$, $\bar{\xi}_k$ are obviously continuously differentiable in G, and $\bar{f}^k_\alpha$ are continuous.

By the uniqueness theorem of the solution to Cauchy's problem for the system of linear equations established in section VII, we obtain the result that in G $v(x, t) \equiv 0$, i.e., $u(x, t) \equiv \bar{u}(x, t)$. The uniqueness theorem stands proven.

To prove the existence theorem let us employ the method of successive approximation. Suppose we have constructed the approximation $\overset{(s)}{u}(x, t) \in C_1$. Define $\overset{(s+1)}{u}(x, t)$ as a solution to Cauchy's problem for the linear system

$$l^k(x, t, \overset{(s)}{u}(x, t))\left[\frac{\partial \overset{(s+1)}{u}}{\partial t} + \xi_k(x, t, \overset{(s)}{u}(x, t))\frac{\partial \overset{(s+1)}{u}}{\partial x}\right] =$$
$$= f_k(x, t, \overset{(s)}{u}(x, t)), \qquad (4)$$

taking on the initial values (2):

$$\overset{(s+1)}{u}_k(x, 0) = \varphi^0_k(x) \qquad (5)$$

From the existence theorem of the solution to the linear system of equations established in section VII it follows that in the domain of determinacy $\overset{(s+1)}{G}$ for Cauchy's problem (4), (5) there exists the solution $\overset{(s+1)}{u}(x, t) \in C_1$, so that all successive approximations are defined and continuously differentiable in the domain $\overset{(s)}{G}$.

The first stage of our proof will be proving the existence of some domain $\tilde{G}$ belonging to all domains $\overset{(s)}{G}$, and also the domain $\bar{G}$, and such that successive approximations and their first derivatives are uniformly bounded in this domain.

Denote

$$\overset{(s)}{\mathscr{P}}_k = l^k_\alpha(x, t, \overset{(s-1)}{u}) \frac{\partial \overset{(s)}{u}_\alpha}{\partial x}$$

and let us write the extended system for the linear system (4). It is of the form

$$\left.\begin{aligned}
&\frac{\partial \overset{(s+1)}{\mathscr{P}}_k}{\partial t} + \xi_k(x, t, \overset{(s)}{u}) \frac{\partial \overset{(s+1)}{\mathscr{P}}_k}{\partial x} = \mathscr{F}^k(x, t, \overset{(s)}{u}) + \\
&\quad + \Phi^k_\alpha(x, t, \overset{(s)}{u}, \overset{(s-1)}{u}) \overset{(s)}{\mathscr{P}}_\alpha + \Psi^k_\alpha(x, t, \overset{(s)}{u}, \overset{(s-1)}{u}) \overset{(s+1)}{\mathscr{P}}_\alpha + \\
&\qquad + \mathscr{F}^k_{\alpha\beta}(x, t, \overset{(s)}{u}, \overset{(s-1)}{u}) \overset{(s)}{\mathscr{P}}_\alpha \overset{(s+1)}{\mathscr{P}}_\beta, \\
&\frac{\partial \overset{(s+1)}{u}_k}{\partial t} = F^k(x, t, \overset{(s)}{u}) + F^k_\alpha(x, t, \overset{(s)}{u}) \overset{(s)}{\mathscr{P}}_\alpha.
\end{aligned}\right\} \qquad (6)$$

We will not here write out explicit expressions for the functions appearing in the system, since they are obtained quite analogously to formulas (4.3.16) - (4.3.19). Let us merely note that they are associated with the functions appearing in the system (8.1.1) by the following obvious formulas:

$$\left.\begin{aligned}
&\mathscr{F}^k_\alpha(x, t, u) = \Phi^k_\alpha(x, t, u, u) + \Psi^k_\alpha(x, t, u, u), \\
&\mathscr{F}^k_{\alpha\beta}(x, t, u) = \mathscr{F}^k_{\alpha\beta}(x, t, u, u).
\end{aligned}\right\} \qquad (7)$$

Along with system (6) let us consider the system of ordinary differential equations

$$\left.\begin{aligned} \frac{d\tilde{P}}{dt} &= \mathcal{F}_0(\tilde{U}) + \Phi_1(\tilde{U})\tilde{P} + \Phi_2(\tilde{U})\tilde{P}^2, \\ \frac{d\tilde{U}}{dt} &= F_0(\tilde{U}) + F_1(\tilde{U})\tilde{P}, \end{aligned}\right\} \quad (8)$$

where the functions $\mathcal{F}_0$, F_0, F_1 are defined in subsection 1, and

$$\Phi_1(\tilde{U}) = \Phi'(\tilde{U}) + \Psi(\tilde{U}),$$

$$\Phi'(\tilde{U}) = \max_{\substack{|u| \leq \tilde{U} \\ |v| \leq \tilde{U}}} |\Phi_k^i(x, t, u, v)|,$$

$$\Psi(\tilde{U}) = \max_{\substack{|u| \leq \tilde{U} \\ |v| \leq \tilde{U}}} |\Psi_k^i(x, t, u, v)|,$$

$$\Phi_2(\tilde{U}) = \max_{\substack{|u| < \tilde{U} \\ |v| < \tilde{U}}} \max_{1 < h < n} |\mathcal{F}_{kh}^i(x, t, u, v)|.$$

For system (8) we assign the initial conditions: $\tilde{P}(0) = \mathcal{P}_0$, $\tilde{U}(0) = U_0$ (compare with conditions (1.4)).

Since from (7) it follows that $\mathcal{F}_1(U) \leq \Phi_1(U)$, $\mathcal{F}_2(U) \leq \Phi_2(U)$, the solution to system (8) majorizes the solution to majorant system (8.1.3) - (8.1.3): $U(t) \leq \tilde{U}(t)$, $\mathcal{P}(t) \leq \tilde{P}(t)$. So if the domain $\tilde{G}$ is constructed according to function $\tilde{U}(t)$ just as domain $\bar{G}$ was constructed in subsection 1 according to function $U(t)$, then $\tilde{G} \subseteq \bar{G}$.

Now let us assume that all the successive approximations $\overset{(k)}{u}$ satisfy the inequalities $k = 1, 2, \cdots, s$,

$$\|\overset{(k)}{u}\| < \tilde{U}(t), \quad \|\overset{(k)}{\mathcal{P}}\| < \tilde{P}(t). \quad (9)$$

Denoting

$$U_{s+1}(t) = \sup_x \|\overset{(s+1)}{u}\|, \quad \mathcal{P}_{s+1}(t) = \sup_x \|\overset{(s+1)}{\mathcal{P}}\|,$$

from system (6) we have

$$\frac{d\mathcal{P}_{s+1}}{dt} < \mathcal{F}_0(\tilde{U}) + \Phi'(\tilde{U})\tilde{P} + \Psi(\tilde{U})\mathcal{P}_{s+1} + \Phi_2(U)\tilde{P}\mathcal{P}_{s+1},$$

$$\frac{dU_{s+1}}{dt} < F_0(\tilde{U}) + F_1(\tilde{U})\tilde{P},$$

such that, obviously, $U_{s+1}(t) \leq \tilde{U}(t)$, $\mathcal{P}_{s+1}(t) \leq P(t)$. Since the initial approximation can be chosen so that (9) is satisfied, then we have thus proven that all successive approximations satisfy equalities (9). Hence follows the existence of the domain $\tilde{G}$ belonging to $\bar{G}$, as well as to all domains $\overset{(s)}{G}$ in which inequalities (9) are satisfied.

The second stage of our proof will be to demonstrate the uniform convergence in the domain $\tilde{G}$ of the sequence $\{\overset{(s)}{u}\}$.

Suppose

$$\overset{(s)}{r}_k = P_k(x, t, \overset{(s-1)}{u})(\overset{(s)}{u} - \overset{(s-1)}{u})$$

Then from (4) we get

$$\frac{\partial \overset{(s+1)}{r}_k}{\partial t} + \xi_k(x, t, \overset{(s)}{u}) \frac{\partial \overset{(s+1)}{r}_k}{\partial x} = \overset{(s)}{g}{}^k_\alpha \overset{(s+1)}{r}_\alpha + \overset{(s)}{f}{}^k_\alpha \overset{(s)}{r}_\alpha$$

where everywhere in $\tilde{G}$, by virtue of (9), $|\overset{(s)}{f}{}^k_\alpha| < B$, $|\overset{(s)}{g}{}^k_\alpha| < B$.

Since the domain of determinacy of the system contains the domain $\tilde{G}$, then by integrating along the characteristics, we obtain for each point in $\tilde{G}$

$$|\overset{(s+1)}{r}_k| < \int_0^t |\overset{(s)}{f}{}^k_\alpha \overset{(s)}{r}_\alpha + \overset{(s)}{g}{}^k_\alpha \overset{(s+1)}{r}_\alpha| \, d\tau,$$

such that

$$R_{s+1}(t) \leqslant B \int_0^t [R_s(\tau) + R_{s+1}(\tau)] \, d\tau,$$

where

$$R_s(t) = \max_{(x, \tau) \in G, \ \tau < t} \|\overset{(s)}{r}\|.$$

Employing lemma 1 of section VI, we get

$$R_{s+1}(t) \leqslant C \int_0^t R_s(\tau) \, d\tau,$$

or

$$R_{s+1}(t) \leqslant \text{const} \frac{(Ct)^s}{s!},$$

which then proves uniform convergence in $\tilde{G}$ of the sequence $\{\overset{(s)}{u}\}$.

Now let finally advance to the last step of the proof. Let us show that derivatives of successive approximations uniformly converge in the domain $\tilde{G}$. Obviously, this is tantamount to proving uniform convergence of the sequence $\{\overset{(s)}{\mathcal{P}}\}$.

We first prove the equicontinuity of this sequence in x. In other words, we show that there exists a function $M(\delta)$, $M(\delta) \longrightarrow 0$ when $\delta \longrightarrow 0$, such that everywhere in $\tilde{G}$ and for all s

$$\|\overset{(s)}{\mathcal{F}}(x'', t) - \overset{(s)}{\mathcal{F}}(x', t)\| < M(\delta) \qquad \text{where} \quad |x'' - x'| \leq \delta.$$

Above it was shown that the sequence $\{\overset{(s)}{\mathcal{F}}\}$ is uniformly bounded. Hence it follows that the functions $\overset{(s)}{u}(x, t)$, and together with them the functions

$$\xi_k(x, t, \overset{(s)}{u}(x, t)), \ \mathcal{F}^k(x, t, \overset{(s)}{u}), \ \Phi^k_\alpha(x, t, \overset{(s)}{u}, \overset{(s-1)}{u}),$$

$$\Psi^k_\alpha(x, t, \overset{(s)}{u}, \overset{(s-1)}{u}), \ \mathcal{F}^k_{\alpha\beta}(x, t, \overset{(s)}{u}, \overset{(s-1)}{u})$$

(cf systems (6)) are equicontinuous in $\tilde{G}$. Moreover, from the equicontinuity of the functions $\xi_k(x, t, \overset{(s)}{u})$ there follows the analogous property of the functions $\overset{(s)}{x}_k(x, t, \tau)$ giving the characteristics of system (6).

Therefore, denoting

$$M_s(t, \delta) = \max_{k < s} \sup_{\substack{|x'' - x'| < \delta \\ \tau < t}} \|\overset{(k)}{\mathcal{F}}(x'', \tau) - \overset{(k)}{\mathcal{F}}(x', \tau)\|$$

and integrating equations (6) along the characteristics, we get

$$M_{s+1}(t, \delta) < (t+1) N(\delta) + C \int_0^t M_{s+1}(\tau, \delta)\, d\tau,$$

where the function $N(\delta)$ is such that $N(\delta) \longrightarrow 0$ when $\delta \longrightarrow 0$. Using lemma 1 from section VI, we conclude that the required function $M(\delta)$ exists such that the sequence $\{\overset{(s)}{\mathcal{F}}\}$ is equicontinuous.

Since, by the familiar Arzela's lemma, from any uniformly bounded and equicontinuous sequence we can separate a convergent (uniformly) subsequence, then some sequence $\{\overset{(s_k)}{\mathcal{F}_\alpha}\}$, therefore also $\{\overset{(s_k)}{p_\alpha}\}$ is uniformly convergent in $\tilde{G}$ to the continuous function p_α. By the familiar theorem of analysis, this means that the function $u_\alpha = \lim_{s \to \infty} \overset{(s)}{u}_\alpha$ is continuously differentiable in $\tilde{G}$ and $\partial u_\alpha / \partial x = p_\alpha$.

Hence it follows that the family $\{\overset{(s)}{p}_\alpha\}$ has only one limit point, and therefore the sequence $\{\overset{(s)}{p}_\alpha\}$ is not only compact, but convergent. Thus, the proof of the convergence of the sequence $\overset{(s)}{p}$ (consequently, also of $\overset{(s)}{q}$) is complete.

Passing in the system (4) to the limit, we can conclude that u with a solution to problem (1), (2). The existence theorem is proven.

3. Certain properties of solutions to Cauchy's problem for systems of quasilinear equations. Suppose $u(x, t)$ and $\bar{u}(x, t)$ are two solutions to Cauchy's problem for a system of quasilinear equations, corresponding to initial data $l^k_\alpha, \xi_k, f_k, u^0; \bar{l}^k_\alpha, \bar{\xi}_k, \bar{f}_k, \bar{u}^0$. We will assume that the initial data of these two Cauchy's problems satisfy the conditions in subsection 2, i.e., are continuously differentiable.

In the intersection G of the domains of determinacy of solutions $u(x, t)$ and $\bar{u}(x, t)$, the difference $v(x, t) = u(x, t) - \bar{u}(x, t)$ satisfies the system of linear equations

$$\bar{l}^k\left[\frac{\partial v}{\partial t}+\bar{\xi}_k\frac{\partial v}{\partial x}\right]=\gamma^k_\alpha v_\alpha+\Delta f_k,$$

where γ^k_α, Δf_k are bounded and continuous in G and Δf_k tends to zero when $\bar{l}^k_\alpha, \bar{\xi}_k, \bar{f}_k \to l^k_\alpha, \xi_k, f_k$.

As we have seen in section VII, solutions of systems of linear equations depend continuously on initial data of the Cauchy's problem. Therefore it follows from this that in G $v(x, t) \Rightarrow 0$ when $\bar{l}^k_\alpha \Rightarrow l^k_\alpha$, $\bar{\xi}_k \Rightarrow \xi_k$, $\bar{f}_k \Rightarrow f_k$ and $v(x, 0) = u^0(x) - u^0(x) \Rightarrow 0$.

Here, however, we must make a clarification. It is essential that the initial data of each Cauchy's problem have bounded derivatives.

In contrast to the case of a semilinear system, the strip $0 \leq t \leq t_0$ in which the solution $u(x, t)$ (and its derivatives) remains bounded depends on the derivatives of the initial functions and $t_0 \longrightarrow 0$ if $\|du^0/dx\| \longrightarrow \infty$. Therefore the continuous dependence of solutions to Cauchy's problem for a system of quasilinear equations obtains only in the case of initial data with uniformly bounded derivatives.

If as before we symbolically write out the procedure for the solution of Cauchy's problem in the form of the equality $u(x, t) = Su^0(x)$, the nonlinear operator S defines the solution $u(x, t)$ only in the domain $\bar{G}$ of the half-plane $t \geqslant 0$. The width of the strip $0 \leq t \leq t_0$ in which the domain $\bar{G}$ is enclosed depends on the derivatives du^0/dx and tends to zero as $\|du^0/dx\| \longrightarrow \infty$.

Therefore the operator S, in contrast to the case of the linear system does not admit of extensions to the class of continuous initial functions $u^0(x)$. For this reason the generalized solution to a system of quasilinear equations cannot be determined formally by extension of the space of possible solutions. The concept of the classical solution to a system of quasilinear equations must, thus, be introduced independently of the concept of the solution of this system. Generalized solutions will be studied in detail in chapter four.

Let us note a limited extension of the class of initial data for which the existence of the solution to Cauchy's problem follows from the foregoing.

The function $\xi_k(x, t, u)$ is called Lipschitz-continuous in the domain $G_0(U)$ with respect to the totality of the variables x, t, u if there exists a constant $C > 0$ such that

$$|\xi_k(\bar{x}, \bar{t}, \bar{u}) - \xi_k(x, t, u)| \leqslant C\{|\bar{x} - x| + |\bar{t} - t| + \|\bar{u} - u\|\}$$

for any $(\bar{x}, \bar{t}, \bar{u}), (x, t, u) \in G_0(U)$.

If we consider the class of Lipschitz-continuous initial data characterized by the Lipschitz constant K, it can be regarded as the closure of the class of initial data with first derivative uniformly bounded by the same constant K. Therefore the solution to Cauchy's problem with Lipschitz-continuous initial data can be treated as the limit of the classical solutions $u(x, t) \in C_1$ since the latter form a family with uniformly bounded first derivatives.

Of course, this limit no longer is the solution to Cauchy's problem in the ordinary sense since it does not possess continuous first derivatives. However, it is the Lipschitz continuous function of the variables x, t and exhibits derivatives almost everywhere in the domain $\bar{G}$. These derivatives almost everywhere in $\bar{G}$ satisfy the system of quasilinear equations. The class of Lipschitz-continuous solutions u(x,t) of Cauchy's problem is an example of the formal extension of the operator S defined in the class C_1 to the class of Lipschitz-continuous input data.

Section IX. Cauchy's Problem for a Single Equation

1. One quasilinear equation. The results of section VIII unreservedly

apply to the case of a single quasilinear equation. However they are too general to apply to this case in which there are important simplifying details.

Therefore let us consider in greater detail the Cauchy's problem for a single quasilinear equation

$$\frac{\partial u}{\partial t} + \xi(x, t, u)\frac{\partial u}{\partial x} = f(x, t, u) \qquad (1)$$

with initial condition $u(x, 0) = u_0(x), \quad a \leq x \leq b \qquad (2)$

Integration of equation (1) leads to the solution of the system of two ordinary differential equations

$$\frac{dX}{dt} = \xi(X, t, U), \quad \frac{dU}{dt} = f(X, t, U), \qquad (3)$$

which are called the characteristic system of equation (1). Each solution $x = X(t)$, $u = U(t)$ gives the characteristics in the space of variables x, t, u.

It is assumed that the functions ξ, f are continuously differentiable. Then one and only one characteristic passes through any point (x_0, t_0, u_0).

Cauchy's problem (1), (2) can be geometrically interpreted as the problem of constructing the integral surface of equation (1) passing through a given initial curve: $t = 0$, $u = u_0(x)$. Since here we wish to obtain a unique differentiable function $u(x, t)$ of variables x, t, this surface naturally must be uniquely projected onto the plane $u = 0$ of variable x, t.

Since the solution u is uniquely determined along each characteristic passing through the point (x_0, t_0, u_0), this problem amounts to constructing a surface consisting of characteristics join through the given initial curve and uniquely projectible onto the plane $u = 0$.

Let $X = X(t, x_0, u_0)$, $U = U(t, x_0, u_0)$ stand for the solution to the characteristic system (3) satisfying the initial conditions

$$X(0, x_0, u_0) = x_0, \quad U(0, x_0, u_0) = u_0. \qquad (4)$$

Then the solution $u(x, t)$ to Cauchy's problem (1), (2) is yielded by the formula

$$u(X(t, x_0, u_0(x_0)), t) = U(t, x_0, u_0(x_0)). \quad (5)$$

Formula (5) implicitly defines the function u(x, t), which in the case of $u_0(x) \in C_1$ is continuously differentiable at all points x, t in which the equation

$$x = X(t, x_0, u_0(x_0)) \quad (6)$$

is uniquely solvable with respect to parameter x_0 and in which the right side of equation (5) remains bounded.

Suppose that at these points

$$x_0 = X^{-1}(t, x) = x_0(x, t) \quad (7)$$

is the result of solving equation (6) with respect to x_0. Then from formula (5) we obtain an explicit formula for the solution u(x, t) of the problem (1), (2):

$$u(x, t) = U(t, x_0(x, t), u_0(x_0(x, t))). \quad (8)$$

Let us explain graphically the construction of this solution to Cauchy's problem (1), (2). We draw through any point $x_0 \in [a, b]$ the characteristic (4) on the plane u = 0 (plane (x, t)), setting $u_0 = u_0(x_0)$ (Figure 1.5). We will also call this projection (6) a characteristic.

A continuously differentiable function $U(t, x_0, u_0(x_0))$ of variable t, which then yields the solution u(x, t) at line $x = X(t, x_0, u_0(x_0))$, is assigned at the characteristic (6).

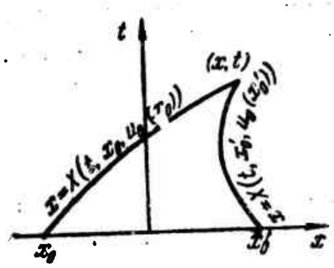

Figure 1.5

It may be that at several points (x, t) as t > 0 to more lines $x = X(t, x_0, u_0(x_0))$ corresponding to different values of the parameter x_0 (Figure 1.5) can intercept each other. At these points equation (6) with respect to

x_0 has more than one solution and formulas (7) and (8) define some multivalued function of variables x, t. In this case no continuous solution u(x, t) to the problem (1), (2) exists.

Let us show that when $0 \leq t \leq t_0$ and for sufficiently small $t_0 > 0$, a unique characteristic (6) passes through any point $(x, t) \in G$, i.e., equation (6) has a unique solution with respect to x_0.

To explain the possibility of the unique solvability of equation (6) with respect to x_0, it is sufficient to show that

$$\dot{X} = \frac{\partial X(t, x_0, u_0(x_0))}{\partial x_0} > 0, \tag{9}$$

since $X(0, x_0, u_0(x_0)) \equiv x_0$ and

$$\dot{X}(0, x_0, u_0(x_0)) = 1. \tag{10}$$

Denoting

$$\dot{U} = \frac{\partial U(t, x_0, u_0(x_0))}{\partial x_0} \quad (\dot{U}(0, x_0, u_0(x_0)) \equiv u_0'(x_0)) \tag{11}$$

and differentiating equations (3), we get

$$\frac{d\dot{X}}{dt} = \xi_u' \dot{U} + \xi_x' \dot{X}, \quad \frac{d\dot{U}}{dt} = f_u' \dot{U} + f_x' \dot{X}, \tag{12}$$

where for brevity we omit the argument for all quantities. And so, $\dot{X}$ and $\dot{U}$ satisfy the system of ordinary linear differential equations (12) and the initial conditions (10) and (11). Hence it is clear that for sufficiently small t_0 inequality (9) will obtain.

Thus, there exists a $t_0 > 0$ such that when $0 \leq t \leq t_0$ formulas (7) and (8) define the function $u(x, t) \in C_1$ satisfying equation (1) and initial condition (2).

Cauchy's problem (1), (2) presupposes, as we have already pointed out, the existence of a unique function $u(x, t) \in C_1$ of variables x, t satisfy equation (1) and initial condition (2). At the same time the more general problem of determining the integral surface S passing through the initial curve does not at all assume that this surface is uniquely projectible onto the plane of variable x, t, and can have, and as a rule does have a solution in a larger domain of variable x, t and does the Cauchy's problem (1), (2).

We will, for example, seek the equation of the surface S in the form

$$(x, t, u) = 0. \qquad (13)$$

Any characteristic (4) of equation (1) must lie on the surface S, therefore

$$\varphi (X, t, U) = 0. \qquad (14)$$

Differentiating (14) with respect to variable t and taking (3) into account, we get the equation

$$\frac{\partial\varphi}{\partial t} + \xi(x, t, u)\frac{\partial\varphi}{\partial x} + f(x, t, u)\frac{\partial\varphi}{\partial u} = 0, \qquad (15)$$

which is a first-order linear differential equation for the function dependent on three independent variables (x, t, u).

The surface S is defined by equations (13) and (14) uniquely for any x_0, t at which X, U are finite and is a smooth surface ($\varphi \in C_1$) if $u_0(x) \in C_1$, ξ, f $\in C_1$.

From equation (13) the function $u(x, t) \in C_1$ is defined, yielding the solution to the problem (1), (2) only in the domain of x, t values in which equation (6) is uniquely solvable with respect to x_0.

Thus, the difference in the formulation of Cauchy's problem (1), (2) and the problem of defining the surface S is that in the first case we seek the integral surface u = u(x, t) uniquely projectible onto the point u = 0; in the second case this surface can be arbitrary.

For sufficiently small t_0 values in the case $u_0(x) \in C_1$ above these Cauchy's problems are equivalent; overall (i.e., for any $t > 0$), the geometrical formulation of Cauchy's problem is the more general and admits for the solution if and only if problem (1), (2) is nonsolvable.

If we assume that the function u(x, t) describes any physical quantity in the space of variable x, t, then naturally this quantity must be a unique function of x, t. Therefore the physics problems reduced to Cauchy's problem (1), (2) require the definition of the unique function u = u(x, t). As we have seen, Cauchy's problem (1), (2) is solvable in this formulation in the class of continuous solutions $u(x, t) \in C$ only in a sufficiently small strip $0 \leq t \leq t_0$.

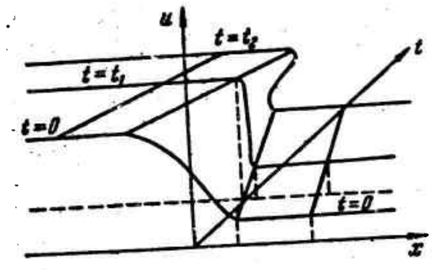

Figure 1.6

Figure 1.6 shows the typical appearance of the surface S. From this figure it is clear that $\left|\frac{\partial u(x, t)}{\partial x}\right| \longrightarrow \infty$ at those points (x, t) close to which the surface S is nonuniquely projected onto the plane of variables (x, t).

Let us explain the foregoing with the example of the simplest quasilinear equation

$$\partial u/\partial t + u\,\partial u/\partial x = 0, \qquad (16)$$

for which we pose the initial condition

$$u(x, 0) = u_0(x) = \begin{cases} u^- = \alpha a + \beta & \text{when } x \leq a \\ \alpha x + \beta & \text{when } a \leq x \leq b, \\ u^+ = \alpha b + \beta & \text{when } x \geq b. \end{cases} \qquad (17)$$

The initial function $u_0(x)$ is continuous when $-\infty < x < \infty$; the derivative $u'_0(x)$ suffers a discontinuity at the points $x = a$, $x = b$. Let us construct the solution to problem (16), (17) satisfying equation (16) in the broad sense at the points at which the derivatives $\partial u/\partial t, \partial u/\partial x$ did not exist.

The characteristic system (3) of equation (16) has the solution

$$X(t, x_0, u_0) = x_0 + u_0 t, \quad U(t, x_0, u_0) = u_0, \qquad (18)$$

which remains bounded for any values t, x_0, u_0. Suppose $\alpha \geq 0$. Projections of the characteristics (18) onto the plane $u = 0$ are of the form shown in Figure 1.7. In this case through each point (x, t) of the half-plane $t \geq 0$

passes the unique characteristic $x = X(t, x_0, u_0(x_0))$, i.e., equation (6) has a unique solution with respect to x_0. The function $u(x, t)$ is constant along the characteristics (6), therefore in the zone I, i.e., when $x \leqslant a + u^- t$, $u(x, t) = u^- = \alpha a + \beta$, in zone III, when $x \geqslant b + u^+ t$, $u(x, t) = u^+ = \alpha b + \beta$. In zone II, when $a + u^- t \leqslant x \leqslant b + u^+ t$, equation (6) can be solved with respect to

$$x_0 = \frac{x - \beta t}{1 + \alpha t}.$$

By formula (8), let us define the solution $u(x, t)$ in zone II:

$$u(x, t) = u_0(x_0(x, t)) = \alpha \cdot \frac{x - \beta t}{1 + \alpha t} + \beta = \frac{\alpha x + \beta}{1 + \alpha t}.$$

Thus, the solution to Cauchy's problem (16), (17) when $\alpha \geqslant 0$ is given by the formula

$$u(x, t) = \begin{cases} u^- = \alpha a + \beta & \text{when} \quad x \leqslant a + u^- t, \\ \dfrac{\alpha x + \beta}{1 + \alpha t} & \text{when} \quad a + u^- t \leqslant x \leqslant b + u^+ t, \\ u^+ = \alpha b + \beta & \text{when} \quad x \geqslant b + u^+ t. \end{cases} \qquad (19)$$

Solution (19) is continuously differentiable when $t \geqslant 0$ everywhere, except for the line $x = a + u^- t$, $x = b + u^+ t$, where the first derivatives suffer a discontinuity.

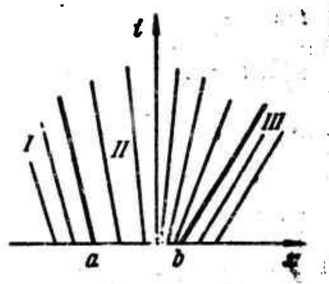

Figure 1.7

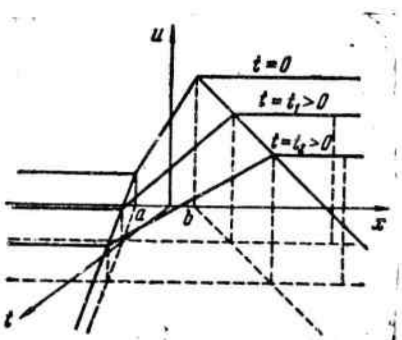

Figure 1.8

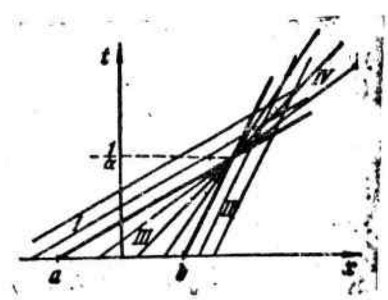

Figure 1.9

In the space of variable x, t, u, solution (19) defines the integral surface S shown in Figure 1.8. This surface is uniquely projected onto the plane $u = 0$ when $t \geqslant 0$.

For the case $\alpha < 0$ $u^- > u^+$, the pattern of characteristics in projections onto the plane (x, t) is one of the forms shown in Figure 1.9. All characteristics (6) when $a \leqslant x_0 \leqslant b$ converge at the point $x = (a + b) - \beta/\alpha$, $t = -1/\alpha > 0$. In zone I, $u(x, t) = u^-$; in zone II $u(x, t) = u^+$. In zone III $u(x, t) = (\alpha x + \beta)/(1 + \alpha t)$, since $\alpha < 0$, then this formula does not define the solution u(x, t) when $t = -1/\alpha$. Finally, in zone IV, the function $U(t, x_0(x, t), u_0(x_0(x, t)))$ is three-valued and takes on the following three values: $u_I(x, t) = u^-$, $u_{II}(x, t) = (\alpha x + \beta)/(1 + \alpha t)$, $u_{III}(x, t) = u^+$. Thus, for the case $\alpha < 0$ the continuous solution u(x, t) of Cauchy's problem (16), (17) exists only when $t < -1/\alpha$, and the integral surface S is determinate for all $t \geqslant 0$ (Figure 1.10); however, when $t \geqslant -1/\alpha$ it is not projected uniquely onto the plane $u = 0$.

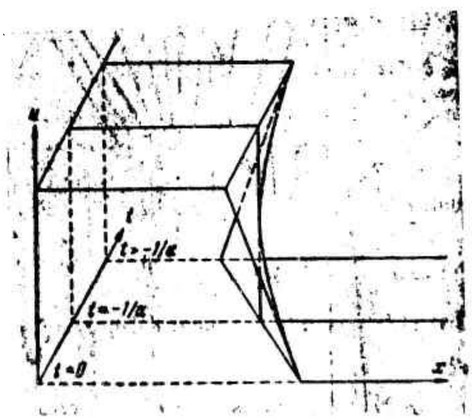

Figure 1.10

2. One nonlinear equation. Cauchy's problem for a nonlinear equation

$$\frac{\partial v}{\partial t} + \varphi(x, t, v, \omega) = 0, \quad \omega = \frac{\partial v}{\partial x}, \quad v(x, 0) = v_0(x) \qquad (1)$$

for the case $\varphi \in C_2$, $v_0(x) \in C_2$ is reduced after differentiation with respect to x of equation (1) to Cauchy's problem for a system of two quasilinear equations

$$\left.\begin{aligned} \frac{\partial \omega}{\partial t}+\varphi'_{\omega}(x,\ t,\ v,\ \omega)\frac{\partial \omega}{\partial x} &= -\varphi'_{v}\omega-\varphi'_{x}, \\ \frac{\partial v}{\partial t} &= -\varphi(x,\ t,\ v,\ \omega) \end{aligned}\right\} \tag{2}$$

with the initial conditions: $v(x,\ 0) = v_0(x)$, $\omega(x,\ 0) = v'_0(x)$. Transforming the second equation of system (2), let us write it as

$$\frac{\partial \omega}{\partial t}+\varphi'_{\omega}\frac{\partial \omega}{\partial x} = -\varphi'_{v}\omega-\varphi'_{x}, \tag{3}$$

$$\frac{\partial v}{\partial t}+\varphi'_{\omega}\frac{\partial v}{\partial x} = -\varphi+\varphi'_{\omega}\omega. \tag{4}$$

Equation (4) is usually called the strip condition.

We can readily see that if we know the solution to the characteristics system for equation (1):

$$\left.\begin{aligned} \frac{dX}{dt} &= \varphi'_{\omega}(X,\ t,\ V,\ \Omega), \\ \frac{dV}{dt} &= \Omega\varphi'_{\omega}(X,\ t,\ V,\ \Omega)-\varphi(X,\ t,\ V,\ \Omega), \\ \frac{d\Omega}{dt} &= -\Omega\varphi'_{v}(X,\ t,\ V,\ \Omega)-\varphi'_{x}(X,\ t,\ V,\ \Omega), \end{aligned}\right\} \tag{5}$$

where

$$X=X(t,\ x_0,\ v_0,\ \omega_0),\quad V=V(t,\ x_0,\ v_0,\ \omega_0),\quad \Omega=\Omega(t,\ x_0,\ v_0,\ \omega_0),$$

satisfying the initial conditions:

$$X(0,\ x_0,\ v_0,\ \omega_0)=x_0,\quad V(0,\ x_0,\ v_0,\ \omega_0)=v_0,\quad \Omega(0,\ x_0,\ v_0,\ \omega_0)=\omega_0, \tag{6}$$

then the solution $v(x,\ t) \in C_2$ to Cauchy's problem (1) is given by the formula

$$v\left(X\left(t,\ x_0,\ v_0(x_0),\ \frac{dv_0(x_0)}{dx_0}\right),\ t\right)=V\left(t,\ x_0,\ v_0(x_0),\ \frac{dv_0(x_0)}{dx_0}\right), \tag{7}$$

which parametrically defines the function $v(x,\ t)$. If $x_0 = x_0(x,\ t)$ is the result of the unique solution of the equation

$$x = X\left(t, x_0, v_0(x_0), \frac{dv_0(x_0)}{dx_0}\right) \qquad (8)$$

with respect to the parameter x_0, then from (7) follows the explicit formula for the solution v(x, t) to Cauchy's problem (1):

$$v(x, t) = V\left(t, x_0(x, t), v_0(x_0(x, t)), v_0'(x_0(x, t))\right) \qquad (9)$$

Formula (9) defines the unique function $v(x, t) \in C_2$ only at first point (x, t) at which equation (8) is uniquely solvable with respect to x_0.

The solution to Cauchy's problem (1) was obtained by us only for the case $v_0(x) \in C_2$. Simple examples show that if $v_0'(x)$ is only continuous, then generally speaking there does not exist the solution $v(x, t) \in C_1$ to problem (1). Formulation of Cauchy's problem (1) for the case of initial functions $v_0(x) \in C_1$ is in need of refinement.

3. Hyperbolic system of nonlinear equation. For the case of the Cauchy's problem for a hyperbolic system of nonlinear equations

$$\frac{\partial v}{\partial t} + \varphi(x, t, v, \omega) = 0, \qquad (1)$$

$$v(x, 0) = v^{(0)}(x), \qquad (2)$$

$$\left(\omega = \frac{\partial v}{\partial x} = \left\{\frac{\partial v_1}{\partial x}, \ldots, \frac{\partial v_n}{\partial x}\right\}\right) \qquad (3)$$

we will assume that $\varphi \in C_2$, $v^0 \in C_2$. Then, by subsection 3 of section II, the functions v, ω are the solution to the system

$$\frac{\partial v}{\partial t} = -\varphi(x, t, v, \omega), \quad \frac{\partial \omega}{\partial t} + A(x, t, v, \omega)\frac{\partial \omega}{\partial x} = f \qquad (4)$$

(cf section III, subsection 2) where the metric $A = \frac{\partial \varphi}{\partial \omega} = \left(\left(\frac{\partial \varphi_i}{\partial \omega_j}\right)\right)$. If we compose the initial conditions

$$v(x, 0) = v^0(x), \quad \omega(x, 0) = \omega^0(x) = \frac{dv^0(x)}{dx}, \qquad (5)$$

for system (4), and the question of whether the function v(x, t) defined as a result of solving Cauchy's problem (4), (5) is the solution to the initial problem (1), (2) as a problem of satisfying equality (3) for any $(x, t) \in G$.

Differentiating the first group of equations (4) with respect to x, subtracting from the result the second group of equations (4), and considering that $\bar{f} = -\varphi'_x - \frac{\partial \varphi}{\partial v}\omega$ (cf formula (2.3.7)), we find

$$\frac{\partial}{\partial t}\left(\frac{\partial v_i}{\partial x} - \omega_i\right) = -\frac{\partial \varphi_i}{\partial v_a}\left(\frac{\partial v_a}{\partial x} - \omega_a\right) \quad (i = 1, \ldots, n) \tag{6}$$

and by (5),

$$\frac{\partial v_i}{\partial x}(x, 0) = \omega_i(x, 0). \tag{7}$$

Based on lemma 1 (subsection 5 of section VI), from (6) and (7) it follows that

$$\frac{\partial v_i}{\partial x}(x, t) \equiv \omega_i(x, t), \tag{8}$$

i.e., equality (3) is satisfied by identity.

Since the existence of the solution to the system of quasilinear equations was proven for us only for the class C_1, the hyperbolic systems of nonlinear equations the constructed solutions belong to class C_2.

Section X. Behavior of Derivatives of the Solution to a System of Quasilinear Equations

1. Weak discontinuity. Transport equation. Sections VII and VIII outlined the construction of the solution to Cauchy's problem for a system of quasilinear equations possessing continuous first derivatives. In considering the Lipschitz-continuous initial data, we arrive at a certain generalization of classical solutions -- to Lipschitz-continuous solutions to Cauchy's problem possessing first derivatives almost everywhere in the domain of definition.

Consider a more particular case of generalized solutions u(x, t) of a system of quasilinear equations -- the class of continuous functions u(x, t) exhibiting piecewise-continuous first derivatives. Let us assume that the vector-function u(x, t) is continuous and exhibits first derivatives that are continuous everywhere except for certain piecewise-differentiable lines on which the first derivatives p, q suffer first-order discontinuity; let us assume that exterior to the lines of discontinuity of the first derivatives of u(x, t) the following system of quasilinear equations

$$l_a^k\left[\frac{\partial u_a}{\partial t}+\xi_k\frac{\partial u_a}{\partial x}\right]=f_k. \qquad (1)$$

is satisfied. Suppose x = x(t) is the equation of the line of discontinuity of the first derivatives of the function u(x, t); let us denote

$$p_k^{\pm}=p_k^{\pm}(x(t),\ t)=p_k(x(t)\pm 0,\ t)=\frac{\partial u_k}{\partial x}(x(t)\pm 0,\ t),$$

$$q_k^{\pm}=q_k^{\pm}(x(t),\ t)=q_k(x(t)\pm 0,\ t)=\frac{\partial u_k}{\partial t}(x(t)\pm 0,\ t).$$

If $p^+\neq p^-$, but the solution $u(x, t)$ is continuous on the line x = x(t), then this feature of the solution is called a weak discontinuity, and line x = x(t) is called a line of weak discontinuity.

From the condition of continuity of u(x, t) at the weak discontinuity line x = x(t), it follows that $x'(t)p^- + q^- = x'(t)p^+ + q^+$, i.e.

$$[p]x'(t) = -[q], \qquad (2)$$

where $[p] = p^+ - p^-$, $[q] = q^+ - q^-$. In the assumptions made, the function u(x, t) to the left and right of the line x = x(t) satisfies system (1); therefore points of this point

$$l^k(q^+ +\xi_k p^+)=f_k, \quad l^k(q^- +\xi_k p^-)=f_k \qquad (3)$$

(the quantities l^k, ξ_k, f_k are continuous at the line x = x(t)).

Subtracting the second group from the first group of the equations (3), we get

$$l_a^k(q_a^+ - q_a^-)+\xi_k(\mathcal{P}_k^+ - \mathcal{P}_k^-)=0,$$

where

$$\mathcal{P}_k^{\pm}=l_a^k p_a^{\pm} \qquad (k=1,\ \ldots,\ n).$$

Canceling out $(q^+ - q^-)$ by means of (2), we get

$$[\xi_k - x'(t)]\,[\mathcal{P}_k]=[\xi_k - x'(t)]\,[\mathcal{P}_k^+ - \mathcal{P}_k^-]=0 \qquad (4)$$
$$(k=1,\ \ldots,\ n).$$

We denote

$$\mathcal{P}_k^+ - \mathcal{P}_k^- = [\mathcal{P}_k] = \eta_k.$$

If $x'(t) \neq \xi_k$ for all $k = 1, \ldots, n$ then $\mathcal{P}_k^+ = \mathcal{P}_k^-$. In view of the linear independence of the eigenvectors l^k, in this case $p_i^- = p_i^+$ for all $i = 1, \ldots, n$, i.e., the discontinuity of the derivatives is absent.

Therefore, $x'(t) = \xi_s(x, t), t, u(x(t), t))$. (5)

This equation denotes that the line of weak discontinuity $x = x(t)$ is a characteristic of system (1). This conclusion naturally is in agreement with the definition of the characteristic Cauchy line through which the solution to system (1) is extended nonuniquely (cf section VI, subsection 2).

Suppose that over the segment of the line of weak discontinuity $x = x(t)$ under consideration equality (5) is set aside, and suppose $x = x(t)$ is the m-tuple characteristic of system (1), i.e., equality (5) is solid when $s = j, j + 1, \ldots, j + m - 1$.

Then from (4) and (5) it follows that

$$\eta_k = \mathcal{P}_k^+ - \mathcal{P}_k^- = 0 \quad \text{when} \quad k < j \quad \text{and} \quad k \geqslant j + m.$$

Let us derive the equations better satisfied by η_k characterizing the value of the weak discontinuity. Since the solution $u(x, t)$ to the left and right of the weak discontinuity line $x = x(t)$ is a classical solution of system (1), the quantity $\mathcal{P}_k^{\pm}$ satisfy at the line $x = x(t)$ the equations of the extended system (section IV, subsection 3), in the broad sense, written for the charac-teristic $x = x(t)$:

$$\left(\frac{d}{dt}\mathcal{P}_k^+\right)_k = \frac{\partial \mathcal{P}_k^+}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k^+}{\partial x} = \mathcal{F}^k + \mathcal{F}^k_\alpha \mathcal{P}_\alpha^+ + \mathcal{F}^k_{\alpha\beta}\mathcal{P}_\alpha^+\mathcal{P}_\beta^+, \quad (6)$$

$$\left(\frac{d}{dt}\mathcal{P}_k^-\right)_k = \frac{\partial \mathcal{P}_k^-}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k^-}{\partial x} = \mathcal{F}^k + \mathcal{F}^k_\alpha \mathcal{P}_\alpha^- + \mathcal{F}^k_{\alpha\beta}\mathcal{P}_\alpha^-\mathcal{P}_\beta^- \quad (7)$$

$$(k = j, j+1, \ldots, j+m-1).$$

The coefficients of the equations (6) and (7) are continuous at the line $x = x(t)$, therefore we do not furnish their sign $\pm$.

Subtracting (7) from (6), we get

$$\left(\frac{d\eta_k}{dt}\right)_k = \frac{\partial \eta_k}{\partial t} + \lambda_k \frac{\partial \eta_k}{\partial x} = \mathcal{T}^k_\alpha \eta_\alpha + \mathcal{T}^k_{\alpha\beta}\left[\varphi^+_\alpha \varphi^+_\beta - \varphi^-_\alpha \varphi^-_\beta\right] \quad (8)$$

$$(k = j, j+1, \ldots, j+m-1).$$

Since

$$\varphi^+_\alpha \varphi^+_\beta - \varphi^-_\alpha \varphi^-_\beta = \varphi^+_\alpha \varphi^+_\beta - \varphi^+_\alpha \varphi^-_\beta + \varphi^+_\alpha \varphi^-_\beta - \varphi^-_\alpha \varphi^-_\beta = \varphi^+_\alpha \eta_\beta + \varphi^-_\beta \eta_\alpha$$

and $\eta_\alpha = 0$ when $\alpha \neq j, j+1, \ldots, j+m-1$, then equation (8) can be written in the form

$$\left(\frac{d\eta_k}{dt}\right)_k = \sum_{\alpha} \mathcal{T}'^k_\alpha \eta_\alpha \quad (k = j, j+1, \ldots, j+m-1), \quad (9)$$

where

$$\mathcal{T}'^k_\alpha = \mathcal{T}^k_\alpha + \sum_{\beta=1}^{n}\left(\mathcal{T}^k_{\alpha\beta}\varphi^+_\beta + \mathcal{T}^k_{\beta\alpha}\varphi^-_\beta\right).$$

From the linearity of the system of ordinary differential equations (9) there follows an important conclusion: the weak discontinuity of the solution to a hyperbolic system of quasilinear equations, extended along the characteristic, can neither arise or disappear if the solution and its first derivative remain bounded.

For the case of a system that is hyperbolic in the narrow sense (section II, subsection 2), the characteristic $x = x(t)$ is simple. Therefore system (9) is converted to a single ordinary differential equation.

Equations (9) are called transport equations for the weak discontinuity. Noting that

$$\varphi^+_\alpha \varphi^+_\beta - \varphi^-_\alpha \varphi^-_\beta = \eta_\alpha \eta_\beta + \varphi^-_\alpha \eta_\beta + \varphi^-_\beta \eta_\alpha,$$

the system (9) can take on the following form:

$$\left(\frac{d\eta_k}{dt}\right)_k = \mathcal{T}^k_\alpha \eta_\alpha + \mathcal{T}^k_{\alpha\beta}\left[\eta_\alpha \eta_\beta + \varphi^-_\alpha \eta_\beta + \varphi^-_\beta \eta_\alpha\right] \quad (10)$$

$$(k = j, j+1, \ldots, j+m-1).$$

System (10) is nonlinear. From it we can conclude that the value of the weak discontinuity η can become infinity in a finite time. Actually, for example, for systems that are hyperbolic in the narrow sense, system (10) is converted into a single equation of the Ricatti or Bernoulli type. However, the values of η_k tend to infinity only simultaneously with $\mathcal{T}^-_k$, $\mathcal{T}^+_k$. Therefore this

$$\left(\frac{d\eta_k}{dt}\right)_k = [illegible] + \mathcal{T}^-_{k\alpha}\eta_\alpha + \mathcal{T}^+_{k\alpha\beta}\left[\mathcal{P}^+_\alpha\mathcal{P}^+_\beta - \mathcal{P}^-_\alpha\mathcal{P}^-_\beta\right] \quad (k = j,\ j+1,\ \ldots,\ j+m-1) \tag{8}$$

Since

$$\mathcal{P}^+_\alpha\mathcal{P}^+_\beta - \mathcal{P}^-_\alpha\mathcal{P}^-_\beta = \eta_\alpha\eta_\beta + \mathcal{P}^-_\alpha\eta_\beta + \mathcal{P}^-_\beta\eta_\alpha$$

and $\eta_\alpha = 0$ when $\alpha \neq j, j+1, \ldots, j+m-1$, then equation (8) can be written in the form

$$\left(\frac{d\eta_k}{dt}\right)_k = \sum_{\alpha} \mathcal{R}_{k\alpha}\eta_\alpha \quad (k = j,\ j+1,\ \ldots,\ j+m-1) \tag{9}$$

where

$$\mathcal{R}_{k\alpha} = \mathcal{T}^-_{k\alpha} + \sum_{\beta=1}^{n}\left(\mathcal{T}^+_{k\alpha\beta}\mathcal{P}^+_\beta + \mathcal{T}^+_{k\beta\alpha}\mathcal{P}^-_\beta\right).$$

From the linearity of the system of ordinary differential equations (9) there follows an important conclusion: the weak discontinuity of the solution to a hyperbolic system of quasilinear equations, extended along the characteristic, can neither arise or disappear if the solution and its first derivative remain bounded.

For the case of a system that is hyperbolic in the narrow sense (section II, subsection 2), the characteristic $x = x(t)$ is simple. Therefore system (9) is converted to a single ordinary differential equation.

Equations (9) are called transport equations for the weak discontinuity. Noting that

$$\mathcal{P}^+_\alpha\mathcal{P}^+_\beta - \mathcal{P}^-_\alpha\mathcal{P}^-_\beta = \eta_\alpha\eta_\beta + \mathcal{P}^-_\alpha\eta_\beta + \mathcal{P}^-_\beta\eta_\alpha,$$

the system (9) can take on the following form:

$$\left(\frac{d\eta_k}{dt}\right)_k = \mathcal{T}^-_{k\alpha}\eta_\alpha + \mathcal{T}^+_{k\alpha\beta}\left[\eta_\alpha\eta_\beta + \mathcal{P}^-_\alpha\eta_\beta + \mathcal{P}^-_\beta\eta_\alpha\right] \quad (k = j,\ j+1,\ \ldots,\ j+m-1). \tag{10}$$

System (10) is nonlinear. From it we can conclude that the value of the weak discontinuity η can become infinity in a finite time. Actually, for example, for systems that are hyperbolic in the narrow sense, system (10) is converted into a single equation of the Ricatti or Bernoulli type. However, the values of η_k tend to infinity only simultaneously with $\mathcal{T}^-_k$, $\mathcal{T}^+_k$. Therefore this effect is not specific for the weak discontinuity of a solution, but is the consequence of the general property of the unbounded increase in derivatives of the solution to a hyperbolic system of quasilinear equations.

For a system of two quasilinear equations that is hyperbolic in the narrow sense, the transport equation in form (10) was obtained by J. Nitsche (cf [17]).

We have established that for a system that is hyperbolic in the narrow sense the weak discontinuity differs from zero at all points of the characteristic. Therefore, the weak discontinuity of a solution to Cauchy's problem occurs only when the initial functions exhibit discontinuity of the first derivatives.

The arbitrary discontinuity of derivatives of initial functions breaks down into weak discontinuities, which extend, generally speaking, over all characteristics exiting from the point of discontinuity of the derivatives of the initial functions satisfying conditions (9) at each characteristic. Sometimes this effect is called the breakdown of the arbitrarily weak discontinuity.

2. Unboundedness of derivatives. Gradient catastrophe. By subsection 1 of section VIII, the graph of solution u(x, t) and its first derivatives with increase in t is estimated by means of a solution to the majorant system (8.1.2), (8.1.3). This system is a nonlinear system of two ordinary differential equations and from it they directly follow the fact that for sufficiently large $t > 0$ the quantities $\mathcal{P}(t)$, $U(t)$ simultaneously tend to infinity. Thus, the growth estimate of the solution and its derivatives by means of the solution to the majorant system leads to the conclusion that for an arbitrary hyperbolic system of quasilinear equations, the solution u(x, t) and its derivatives p(x, t) tend to infinity with growth of t, generally speaking.

This conclusion applies to an arbitrary system of quasilinear equations. However, particular classes of systems of quasilinear equations are also of interest, for example, systems whose solutions remain bounded for any values of the variable t.

This property is exhibited, for example, by systems of linear equations, and also by systems leading to invariants, i.e., those representable in the form

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = f_k(x, t, r) \quad (k = 1, \ldots, n) \tag{1}$$

if here f_k does not grow too rapidly with growth in r, for example, if for any x, t, u

$$\left| \frac{\partial f_k}{\partial r_l} \right| < C \quad (k, l = 1, \ldots, n)$$

It is not difficult to note that the solution r(x, t) to systems of this type remain bounded for any t value, however, their derivatives nevertheless increase unboundedly up to absolute value $\xi_k(x, t, r)$ depends essentially on $r = \{r_1, \ldots, r_n\}$.

The effect of the formation of unbounded derivatives when the solution to a system of quasilinear equations is bounded is called the gradient catastrophe.

Let us explain this with a simple example. Consider the homogeneous system of two quasilinear equations whose coefficients do not depend explicitly on x, t it leads to invariants and can be written as

$$\frac{\partial r_k}{\partial t} + \xi_k(r) \frac{\partial r_k}{\partial x} = 0 \quad (k = 1, 2) \tag{2}$$

Let us assume that $\partial \xi_k / \partial r_k > 0$ and let us consider for the system (2) Cauchy's problem with the initial conditions formulated for the entire axis t = 0:

$$r_1(x, 0) = r_1^0(x), \quad r_2(x, 0) = r_2^0(x) = r_2^0 = \text{const.} \tag{3}$$

Suppose $r_1^0(x) \in C_1, \; |r_1^0{}'(x)| < M.$ The solution to Cauchy's problem (2), (3) reduce to Cauchy's problem for a single quasilinear equation:

$$\frac{\partial r_1}{\partial t} + \xi_1(r_1, r_2^0) \frac{\partial r_1}{\partial x} = 0, \quad r_1(x, 0) = r_1^0(x).$$

By section IX, the solution $r_1(x, t)$ of this problem is yielded by the formula $r_1(x, t) = r_1^0\,(x - \xi_1(r_1(x, t), r_2^0) \cdot t)$.

Let us compute the derivative $\partial r_1 / \partial x$:

$$\frac{\partial r_1}{\partial x} = \frac{\frac{dr_1^0(x_0)}{dx_0}}{1 + \frac{\partial \xi_1}{\partial r_1}(r_1(x, t), r_2^0)\frac{dr_1^0(x_0)}{dx_0}\cdot t},$$

where $x_0 = x - \xi_1(r_1, r_2^0)\cdot t$. Hence it follows that when $\frac{dr_1^0(x_0)}{dx_0} < 0$,

the derivative $\partial r_1/\partial x$ monotonically decreases with increase in t at the characteristic $x = x_0 + \xi_1(r_1, r_2^0)\cdot t$ and when

$$t = t_m = \frac{1}{\frac{d}{dx_0}\xi_1(r_1^0(x_0), r_2^0)} > 0,$$

becomes unbounded.

Thus, if $\frac{\partial \xi_k}{\partial r_k} \neq 0$, then as a rule the derivatives of the solution r(x, t) to system (2) increase unboundedly (with respect to module) with increase in the variable t.

3. Strongly and weakly nonlinear systems of quasilinear equations. We will call a system of quasilinear equations (10.2.1) weakly nonlinear in some domain of space of variable x, t, r if in this domain

$$\frac{\partial \xi_k(x, t, r)}{\partial r_k} \equiv 0 \quad (k = 1, 2, \ldots, n), \tag{1}$$

otherwise we will call the system (10.2.1) strongly nonlinear.

By definition, a weakly nonlinear system of two quasilinear equations is written in the form

$$\left.\begin{aligned} \frac{\partial r_1}{\partial t} + \xi_1(x, t, r_2)\frac{\partial r_1}{\partial x} = f_1(x, t, r_1, r_2), \\ \frac{\partial r_2}{\partial t} + \xi_2(x, t, r_1)\frac{\partial r_2}{\partial x} = f_2(x, t, r_1, r_2). \end{aligned}\right\} \tag{2}$$

Note that if $\frac{\partial \xi_1}{\partial r_2} \neq 0,\ \frac{\partial \xi_2}{\partial r_1} \neq 0,$ then the system (2) is reducible to the form

$$\frac{\partial r_1}{\partial t} + r_2\frac{\partial r_1}{\partial x} = f_1, \quad \frac{\partial r_2}{\partial t} + r_1\frac{\partial r_2}{\partial x} = f_2.$$

We now will show that the derivatives of the solution of weakly nonlinear system (2) remain bounded for any t values if the solution r(x, t) itself remains bounded.

Theorem. Suppose the solution r(x, t) to system (2) is bounded*) when $0 \leq t \leq T$:

$$|r_k(x, t)| \leqslant R, \tag{3}$$

but system (2) is hyperbolic in the narrow sense**), i.e.,

$$\xi_2(x, t, r_1(x, t)) - \xi_1(x, t, r_2(x, t)) \geqslant \varepsilon > 0 \quad \text{where } 0 \leqslant t \leqslant T, \tag{4}$$

and the functions ξ_k, $f_k \in C_1$. Then the derivatives $\partial r_k/\partial x$, $\partial r_k/\partial t$ are bounded when $0 \leqslant t \leqslant T$ if they are bounded when $t = 0$.

Proof. The solution r(x, t) will be considered a solution to Cauchy's problem for system (2) with the initial conditions

$$r_k(x, 0) = r_k^0(x) \quad (-\infty < x < \infty).$$

By the conditions of the theorem,

$$|r_k^0(x)| \leqslant R, \quad \left|\frac{dr_k^0}{dx}(x)\right| \leqslant P_0.$$

Suppose $x = x_1(t, x_0)$ is the equation of the characteristic of system (2) passing through the point $x = x_0$ of axis $t = 0$. We rewrite the first equation of system (2) in the form

$$\left(\frac{dr_1}{dt}\right)_1 = \frac{\partial r_1}{\partial t} + \xi_1(x, t, r_2(x, t))\frac{\partial r_1}{\partial x} = f_1(x, t, r_1, r_2).$$

If we assume the function $r_2(x, t)$ to be known, then the definition of $r_1(x, t)$ reduces to the solution to Cauchy's problem for a system of two ordinary differential equations

$$\frac{dx_1}{dt} = \xi_1(x_1, t, r_2(x_1, t)), \tag{5}$$

$$\frac{d\bar{r}_1}{dt} = f_1(x_1, t, \bar{r}_1, r_2(x_1, t)) \quad (\bar{r}_1 = \bar{r}_1(t, x_0)) \tag{6}$$

*) As we have noted, condition (3) will be satisfied automatically if $|\partial f_k/\partial r_i| \leqslant C$.

**) It is sufficient for us that condition (4) be satisfied for the given solution r(x, t). Of course, system (2) can be hyperbolic in the narrow sense by identity, i.e., for any r_1, r_2.

with the initial data

$$x_1(0, x_0)=x_0, \quad \bar{r}_1(0, x_0)=r_1^0(x_0). \tag{7}$$

If $\bar{r}_1(t, x_0)$ is the solution to problem (5) - (7), then the formula

$$r_1(x_1(t, x_0), t)=\bar{r}_1(t, x_0) \tag{8}$$

defines the solution $r_1(x, t)$.

Let us note

$$\dot{x}_1(t, x_0)=\frac{\partial}{\partial x_0} x_1(t, x_0)$$

and let us differentiate equation (5) with respect to the parameter x_0. We get

$$\frac{d\dot{x}_1}{dt}=\left\{\frac{\partial}{\partial x}[\xi_1(x, t, r_2(x, t))]\right\}_{x=x_1(t, x_0)} \dot{x}_1(t, x_0). \tag{9}$$

Since

$$\left(\frac{d\xi_1(x, t, r_2)}{dt}\right)_2=\frac{\partial\xi_1}{\partial t}+\xi_2\frac{\partial\xi_1}{\partial x}=$$

$$=\frac{\partial\xi_1}{\partial r_2}(x, t, r_2)\left[\frac{\partial r_2}{\partial t}+\xi_2\frac{\partial r_2}{\partial x}\right]+\xi_{1t}'+\xi_2\xi_{1x}',$$

then

$$\frac{\partial\xi_1}{\partial t}+\xi_2\frac{\partial\xi_1}{\partial x}=\bar{f}_2(x, t, r), \tag{10}$$

where

$$\bar{f}_2=\frac{\partial\xi_1}{\partial r_2}f_2+\xi_{1t}'+\xi_2\xi_{1x}'.$$

Subtracting from (10) the inequality

$$\frac{\partial\xi_1}{\partial t}+\xi_1\frac{\partial\xi_1}{\partial x}=\left(\frac{d\xi_1}{dt}\right)_1,$$

we find

$$\frac{\partial\xi_1}{\partial x}=\frac{\bar{f}_2-\left(\frac{d\xi_1}{dt}\right)_1}{\xi_2-\xi_1}. \tag{11}$$

On analogy with the foregoing, we get

$$\frac{\partial\xi_2}{\partial t}+\xi_1\frac{\partial\xi_2}{\partial x}=\left(\frac{d\xi_2}{dt}\right)_1=\bar{f}_1.$$

Let us transform equality (11) by means of the identity transformations:

$$[illegible] = \frac{\bar f_2-\bar f_1}{\xi_2-\xi_1} + \frac{\bar f_1-\left(\frac{d\xi_1}{dt}\right)_1}{\xi_2-\xi_1} = \frac{\bar f_2-\bar f_1}{\xi_2-\xi_1} + \frac{\left(\frac{d\xi_2}{dt}\right)_1-\left(\frac{d\xi_1}{dt}\right)_1}{\xi_2-\xi_1} =$$
$$= \frac{\bar f_2-\bar f_1}{\xi_2-\xi_1} + \left(\frac{d}{dt}\ln[\xi_2(x, t, r_1)-\xi_1(x, t, r_2)]\right)_1.$$

Substituting this equality into equation (9), we transform the latter to become

$$\frac{d}{dt}\ln\left[\frac{\dot{x}_1(t, x_0)}{\xi_2(x, t, r_1(x, t))-\xi_1(x, t, r_2(x, t))}\right]_{x=x_1(t, x_0)} = \left[\frac{\bar f_2-\bar f_1}{\xi_2-\xi_1}\right]_{x=x_1(t, x_0)} \tag{12}$$

From the initial condition (7), we have $\dot{x}_1(0, x_0) = 1$; therefore, integrating equation (12) from 0 to t, we get

$$\frac{\dot{x}_1(t, x_0)[\xi_2(x_0, 0, r_1^0(x_0))-\xi_1(x_0, 0, r_2^0(x_0))]}{[\xi_2(x, t, r_1(x, t))-\xi_1(x, t, r_2(x, t))]_{x=x_1(t, x_0)}} = \exp\int_0^t \frac{\bar f_2-\bar f_1}{\xi_2-\xi_1}\,dt. \tag{13}$$

From the assumption (3) on the boundedness of this solution under continuity of the functions f_k, $\bar f_k$, ξ_k, it follows that there exists same number $M > 0$ such that

$$|\bar f_k| < \frac{M}{2}, \quad |\xi_2-\xi_1| < M, \quad |\bar f_2-\bar f_1| \leq M.$$

Moreover, by condition (4) of the theorem, we have

$$\xi_2-\xi_1 \geq \varepsilon > 0.$$

Therefore from formula (13) we get the estimate

$$\frac{\varepsilon}{M}e^{-\frac{Mt}{\varepsilon}} < \dot{x}_1(t, x_0) < \frac{M}{\varepsilon}e^{\frac{Mt}{\varepsilon}}, \tag{14}$$

which shows that the field of characteristics of the first family $x = x_1(t, x_0)$ is differentiable with respect to x_0 for all t from the interval $[0, T]$.

Hence it follows that the characteristics $x = x_1(t, x_0)$ do not intersect each other when $0 \leq t \leq T$. Now it is easy to obtain the proof of our theorem. Denoting

$$\dot{r}_1 = \dot{r}_1(t, x_0) = \frac{\partial r_1(t, x_0)}{\partial x_0}$$

and differentiating equation (6) with respect to parameter x_0, we get

$$\frac{\partial \dot{r}_1}{\partial t} = \frac{\partial f_1}{\partial r_1}\dot{r}_1 + \dot{x}_1\left[\frac{\partial f_1}{\partial r_2}\cdot\frac{\partial r_2}{\partial x}\Big|_{x=x_1} + f'_{1x}\right]. \qquad (15)$$

Let us note

$$P(t) = \sup_{\substack{0<\tau<t \\ -\infty<x<\infty}} \max_{k=1,2}\left|\frac{\partial r_k(x,\tau)}{\partial x}\right|$$

and we will assume that the constant M is so large that for all $0 \leqslant t \leqslant T$, $-\infty < x < \infty$

$$\left|\frac{\partial f_i}{\partial r_j}\right| < M, \quad |f'_{ix}| < M \qquad (i, j = 1, 2). \qquad (16)$$

Integrating equation (15), we get

$$\dot{r}_1(t, x_0) = \dot{r}_1(0, x_0)\exp\left\{\int_0^t \frac{\partial f_1}{\partial r_1}d\tau\right\} + \int_0^t\left[\frac{\partial f_1}{\partial r_2}\cdot\frac{\partial r_2}{\partial x} + f'_{1x}\right]\dot{x}_1(\tau, x_0)\exp\left\{\int_\tau^t \frac{\partial f_1}{\partial r_1}d\zeta\right\}d\tau.$$

Substituting here the estimates (14) and (16) and using initial condition (7), we get

$$|\dot{r}_1(t, x_0)| < P_0 e^{Mt} + \frac{M^2}{\varepsilon}e^{\frac{2M\varepsilon+M}{\varepsilon}t}\int_0^t [P(\tau)+1]\,d\tau. \qquad (17)$$

From formula (8) we have

$$\frac{\partial r_1(x,t)}{\partial x} = \frac{\dot{r}_1(t, x_0)}{\dot{x}_1(t, x_0)},$$

such that from estimates (17) and (14) there follows

$$\left|\frac{\partial r_1(x,t)}{\partial x}\right| < \frac{M}{\varepsilon}e^{Mt+\frac{Mt}{\varepsilon}}P_0 + \frac{M^3}{\varepsilon^2}e^{\frac{2Mt(\varepsilon+1)}{\varepsilon}}\int_0^t [P(\tau)+1]\,d\tau.$$

It is easy to observe that the estimate is analogously obtained for the quantity $\partial r_2/\partial x$; therefore

$$\left|\frac{\partial r_k}{\partial x}\right| \leqslant P_0 \frac{M}{\varepsilon} e^{Mt+\frac{Mt}{\varepsilon}} + \frac{M^2}{\varepsilon^2} e^{\frac{2Mt(\varepsilon+1)}{\varepsilon}} \int_0^t [P(\tau)+1]\, d\tau \qquad (k = 1, 2)$$

From these inequalities there derives the estimate

$$P(t) \leqslant AP_0 + B \int_0^t [P(\tau)+1]\, d\tau,$$

where

$$P(0) = P_0, \quad A = \frac{M}{\varepsilon} e^{Mt\left(1+\frac{1}{\varepsilon}\right)}, \quad B = MA^2,$$

is valid when $0 \leqslant t \leqslant T$.

Now applying lemma 1 from section VI to the resulting inequality, we get $P(t) \leqslant [AP_0 + Bt]e^{Bt}$, from whence follows the boundedness of the derivatives $\partial r_k(x, t)/\partial x$ over the entire interval $0 \leqslant t \leqslant T$. The theorems stand proven.

In view of the arbitrary selection of T, the derivatives $\partial r_k/\partial x$ of a solution of weakly nonlinear system (2) remain bounded in any strip with respect to the variable t in which theorem conditions (3) and (4) are satisfied.

From the proven theorem there follows:

Corollary. Cauchy's problem for a weakly nonlinear system of two quasilinear equations, hyperbolic in the narrow sense, is solvable in the domain of determinacy G if the solution $r(x, t)$ remains bounded in it*).

Let us explain this corollary in more detail. For an arbitrary system of quasilinear equations, the derivatives become unbounded even when the solution itself is bounded. If we consider Cauchy's problem with initial conditions assigned, for example, for the entire initial axis $t = 0$, then for a strongly nonlinear solution the derivatives tend to infinity for a finite value $t_0 > 0$ and when $t > t_0$ no solution (classical) to this Cauchy's problem exists.

For a weakly nonlinear system that is hyperbolic in the narrow sense, whose solution remains bounded (for example, when $|\partial f_k/\partial r_i| \leqslant C$ $(i, k = 1, 2)$),

*) This very same property of weakly nonlinear systems was recently proven [32] for the arbitrary system (10.2.1) when $f_k \equiv 0$.

the derivatives remain bounded for all $t > 0$. Therefore the solution to Cauchy's problem can be constructed in any finite strip $0 \leqslant t \leqslant T$ by the procedure outlined in section VIII. Thus, for a weakly nonlinear system there exists a solution to Cauchy's problem as a whole, i.e., for any finite values of the variable t. This circumstance brings weakly nonlinear systems closer to linear systems.

On the other hand, this shows that any feature of initial data, being smooth when $t = 0$, no longer is reproducible when $t > 0$. Therefore the generalized solutions of a weakly nonlinear system that is hyperbolic in the narrow sense can be considered as the limits to smooth solutions at once for the entire half-plane $t \geqslant 0$, much as occurs for linear equations.

Let us consider by way of example the weakly nonlinear system of two equations *)

$$\frac{\partial r_k}{\partial t} + \xi_k(r_{3-k}) \frac{\partial r_k}{\partial x} = 0 \qquad (k = 1, 2). \qquad (18)$$

Here $f_1 = f_2 = \bar{f}_1 = \bar{f}_2 = 0$; therefore formula (13) converts to the equality

$$\dot{x}_k(t, x_0^k) = \frac{\xi_2(r_1(x, t)) - \xi_1(r_2(x, t))}{\xi_2(r_1^0(x_0^k)) - \xi_1(r_2^0(x_0^k))} \qquad (k = 1, 2),$$

and for the derivative $\partial r_k / \partial x$ we have

$$\frac{\partial r_k(x, t)}{\partial x} = \frac{dr_k^0(x_0^k)}{dx_0^k} \cdot \frac{\xi_2(r_1^0(x_0^k)) - \xi_1(r_2^0(x_0^k))}{\xi_2(r_1(x, t)) - \xi_1(r_2(x, t))}. \qquad (19)$$

Hence follows the more exact estimate of derivatives of the solution to system (18):

$$\left| \frac{\partial r_k(x, t)}{\partial x} \right| \leqslant P_0 \frac{M}{\varepsilon}.$$

Let us note an interesting consequence of formula (19): if $\xi_2(r_1(x, t)) = \xi_1(r_2(x, t))$, then $\partial r_k(x, t) / \partial x = \infty$.

Remark. The definition of weakly nonlinear systems of quasilinear equations was introduced only for systems leading to invariants. This is possible in the general case only when $n \leqslant 2$ (cf section III). The theorem on the boundedness of derivatives was all the more so proven only for $n = 2$.

*) The general integral of system (18) was obtained in the paper [33].

The question of separating of the class of systems which do not lead to unboundedness of derivatives for the case $n \geqslant 3$ remains open. Possibly, derivatives of the solution to a system that is hyperbolic in the narrow sense remain bounded for the case when the following conditions are satisfied:

$$r_\alpha^k(x, t, u)\frac{\partial \xi_k(x, t, u)}{\partial u_\alpha} \equiv 0 \qquad (\alpha, k = 1, \ldots, n) \tag{20}$$

(cf [18]). We can easily see that the conditions (20) and the conditions (1) coincide in the event that invariants exist.

If conditions (20) are satisfied, then it is easy to see that the coefficients $\mathcal{F}^k_{\alpha\alpha}$ of the extended system (4.3.16) - (4.3.19) identically tend to zero. In combination with the requirement of hyperbolicity in the narrow sense, this possibly enables us to prove the boundedness of derivatives of the solutions of such systems as a consequence of the boundedness of the solution itself.

Section XI. Remarks on the Mixed Problem

1. Formulation of the mixed problem for a linear system. Let us consider the typical mixed problem:

Find the solution u(x, t) to a hyperbolic system of linear equations that takes on, when t = 0, the assigned values

$$u(x, 0) = u^0(x), \qquad a \leqslant x \leqslant b \tag{1}$$

and that satisfies certain boundary conditions

$$c_\alpha^l(x, t)u_\alpha(x, t)\big|_{\Gamma_1} = c^l u\big|_{\Gamma_1} = c_l(x, t) \qquad (1 \leqslant l \leqslant n_1), \tag{2}$$

$$d_\alpha^l(x, t)u_\alpha(x, t)\big|_{\Gamma_2} = d^l u\big|_{\Gamma_2} = d_l(x, t) \qquad (1 \leqslant l \leqslant n_2), \tag{3}$$

which are specified for certain lines Γ_1, Γ_2, exiting, respectively, from endpoints x = a and x = b of the interval [a, b] of the axis t = 0 (Figure 1.11).

We will assume that the curves Γ_1, Γ_2 have a continuously variable tangent and $l^k_\alpha(x, t)$, $\xi_k(x, t)$, $f^k(x, t)$, $f^k_\alpha(x, t) \in C_1$ in the domain

$$l_\alpha^k(x, t),\ \xi_k(x, t),\ f^k(x, t),\ f_\alpha^k(x, t) \in C_1$$

bounded by the curves Γ_1, Γ_2 and by the axis $t = 0$; c^i_α, $c_1 \in C_1$ at the curve Γ_1, d^i_α, $d_i \in C_1$ at the curve Γ_2.

Suppose the conditions for the agreement of initial conditions (1) and boundary conditions (2) and (3) are satisfied:

$$\left.\begin{aligned} c^l_\alpha(a, 0)\, u^0_\alpha(a) &= c_l(a, 0) \qquad (1 \leqslant l \leqslant n_1), \\ d^l_\alpha(b, 0)\, u^0_\alpha(b) &= d_l(b, 0) \qquad (1 \leqslant l \leqslant n_2). \end{aligned}\right\} \tag{4}$$

If conditions (4) are not satisfied, then the solution u(x, t) of the mixed problem is discontinuous, and it must then be regarded as a generalized solution.

Let us decompose the vectors c^i and d^i into the vectors $l^k(x, t)$:

$$c^l(x, t) = \mu^l_\alpha(x, t)\, l^\alpha(x, t), \qquad d^l(x, t) = \nu^l_\alpha(x, t)\, l^\alpha(x, t)$$

Then the boundary conditions (2) and (3) will be rewritten as

$$\mu^l_\alpha l^\alpha_\beta u_\beta \big|_{\Gamma_1} = c_l(x, t) \qquad (l = 1, \ldots, n_1),$$
$$\nu^l_\alpha l^\alpha_\beta u_\beta \big|_{\Gamma} = d_l(x, t) \qquad (l = 1, \ldots, n_2),$$

or, in invariants,

$$\mu^l_\alpha r_\alpha \big|_{\Gamma_1} = c_l(x, t) \qquad (l = 1, \ldots, n_1), \tag{5}$$
$$\nu^l_\alpha r_\alpha \big|_{\Gamma_2} = d_l(x, t) \qquad (l = 1, \ldots, n_2). \tag{6}$$

Suppose conditions (5) and (6) are consistent and are linearly independent, i.e., the rank of the matrix $((\mu^i_\alpha))$ is n_1; and the rank $((\nu^i_\alpha))$ is n_2.

Suppose G^0 is the domain of determinacy of the solution to Cauchy's problem with initial condition (1).

Obviously, the curves Γ_1 and Γ_2 must lie outside the domain G^0, since the solution to the linear system of equations is uniquely determined in the domain G^0 by the initial condition (1) and, in general, does not satisfy in it the conditions (5) and (6).

Let us consider the case when the curve $x = X_1(t)$ intersects Γ_1 at the point D, and the curve $x = X_n(t)$ (cf section VI, subsection 3) intersects

Γ_2 at the point E.

The solution u(x, t) is uniquely defined in the domain G^0 and can be constructed in this domain by the method of successive approximations (cf section VII). Therefore it is sufficient to consider the problem of constructing u(x, t) in the domain ACD; the solution is similarly constructed in the domain BCE.

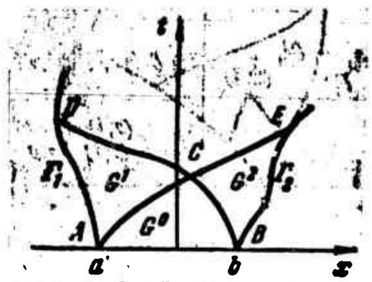

Figure 1.11

To explain the conditions for the solvability of the mixed problem, it is essential to know which families of the characteristics $x = x_k(\xi, 0, t)$ exiting from the points of the interval [a, b] of the initial axis ($a \leq \xi \leq b$) intersects the arcs AD and BE of the curve Γ_1 and Γ_2.

Suppose that at $k = k_1, k_2, \ldots, k_p$, the characteristics $x = x_k(\xi, 0, t)$ for $\xi \in [a, b]$ intersect the segment AD by the curve Γ_1, and when $k = \bar{k}_1, \bar{k}_2, \ldots, \bar{k}_s$ intersect the segment BE by the curve Γ_2.

Let us number the variables $\xi_k(x, t)$, setting $k_1 = 1$, $k_2 = 2, \ldots, k_p = p$.

Denoting $u^0(x, t)$, $r^0(x, t)$ as the solution to Cauchy's problem with initial condition (1) in the domain G^0, we advance to the next problem in the domain ACD(G^1):

Find the solution r(x, t) the linear system

$$\frac{\partial r_k}{\partial t} + \xi_k \frac{\partial r_k}{\partial x} = g^k(x, t) + g^k_\alpha(x, t) r_\alpha, \tag{7}$$

satisfying the conditions

$$r_k(x, t)|_{AC} = r_k^0(x, t)|_{AC} \quad (k = 1, \ldots, n), \tag{8}$$

at the line AC and the conditions (5) at the line AD.

By the definition of the domain G^0, its frontier consists of the segments of the characteristics of system (7).

Thus, the definition of the solution in the domain G^1 reduced to defining the solution of the system (7) taking on assigned values at the characteristic AC and satisfying conditions (5) at the line AD (Figure 1.12). In view of the existence of the solution to system (7) in the domain G^0, the values $r^0(x, t)$ at the line AC satisfy the solvability conditions (section VI, subsection 2). Thus, here we first encounter the problem when the initial values are assigned at the characteristic. The problem with data at the characteristic is usually called Goursat's problem.

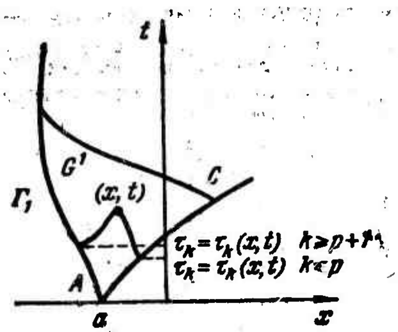

Figure 1.12

Let us consider an arbitrary point (x, t) in the domain G^1 (Figure 1.12) and draw through it all characteristics $x = x_k(x, t, \tau)$ of this system (7). Under our assumption, characteristics $x = x_k(x, t, \tau)$ intersect the curve AC when $1 \leq k \leq p$ at several points $(x_k(x, t, \tau_k), \tau_k)$, and here $\tau_k = \tau_k(x, t) \leq t$.

Similarly, the characteristics $x = x_k$ intersect the curve AD at the points $(x_k(x, t, \tau_k), \tau_k)$ when $k \geq p + 1$ and $\tau_k = \tau_k(x, t) \leq t$.

We will call boundary conditions (5) correct if:

(1) the number n_1 of boundary conditions (5) equals the number $q = n - p$ of characteristics $x = x_k(x, t, \tau)$ descending from arbitrary point (x, t) G^1 on line AD:

(2) conditions (5) can be uniquely solved with respect to the quantities $r_k(x, t)$ when $k \geqslant p + 1$.

Suppose these conditions are satisfied, i.e., $n_1 = q = n - p$ and when $(x, t) \in AD$

$$\operatorname{Det}\begin{vmatrix} \mu^1_{p+1} & \cdots & \mu^1_{p+q=n} \\ \cdots & \cdots & \cdots \\ \mu^q_{p+1} & \cdots & \mu^q_n \end{vmatrix} \neq 0.$$

Then conditions (5) can be rewritten as

$$r_i(x, t)\big|_{AD} = \bar{c}_i(x, t) + \sum_{\alpha=1}^{p} \bar{\mu}^i_\alpha(x, t)\, r_\alpha(x, t)\big|_{AD}$$
$$(i = p+1,\ p+2, \ldots,\ p+q = n).$$

In the following we will omit the bar over c_i, μ^i_α. Solution $r(x, t)$ satisfies in G^1 the equations

$$r_k(x, t) = r^0_k(x_k(x, t, \tau_k), \tau_k) +$$
$$+ \int_{\tau_k(x, t)}^{t} [g^k(x_k(x, t, \tau), \tau) + g^k_\alpha(x_k(x, t, \tau), \tau)\, r_\alpha(x_k(x, t, \tau), \tau)]\, d\tau$$
$$(k \leqslant p),$$
$$r_k(x, t) = c_k(x_k(x, t, \tau_k), \tau_k) +$$
$$+ \sum_{\alpha=1}^{p} \mu^k_\alpha(x_k(x, t, \tau_k), \tau_k)\, r_\alpha(x_k(x, t, \tau_k), \tau_k) +$$
$$+ \int_{\tau_k(x, t)}^{t} [g^k(x_k(x, t, \tau), \tau) + g^k_\alpha(x_k(x, t, \tau), \tau)\, r_\alpha(x_k(x, t, \tau), \tau)]\, d\tau$$
$$(k \geqslant p + 1).$$

We will seek the solution $r(x, t)$ by the method of successive approximations:

$$\overset{(s+1)}{r_k}(x, t) = r_k^0(x_k(x, t, \tau_k), \tau_k) + \int_{\tau_k(x,t)}^{t} \{g^k(x, \tau) + g_a^k(x, \tau)\overset{(s)}{r_a}(x, \tau)\}_{x=x_k(x,t,\tau)} d\tau \qquad (k \leqslant p),$$

$$\overset{(s+1)}{r_k}(x, t) = \Big[c_k(\zeta, \eta) + \sum_{a=1}^{p} \mu_a^k(\zeta, \eta)\overset{(s)}{r_a}(\zeta, \eta)\Big]_{\substack{\zeta = x_k(x, t, \tau_k) \\ \eta = \tau_k}} + \int_{\tau_k(x,t)}^{t} \{g^k(x, \tau) + g_a^k(x, \tau)\overset{(s)}{r_a}(x, \tau)\}_{x=x_k(x,t,\tau)} d\tau \qquad (k \geqslant p+1),$$

selecting the suitable initial approximation exhibiting continuous first derivatives in the domain G^1.

If we denote

$$\overset{(s)}{V}(t) = \max_{G_t^1} \max_{k=1,\dots,n} |\overset{(s)}{r_k}(\zeta, \tau) - \overset{(s-1)}{r_k}(\zeta, \tau)|,$$

then on analogy with section VII we obtain the estimate

$$\overset{(s+1)}{V}(t) < B\int_0^t \overset{(s)}{V}(\tau)d\tau + pn\mu\int_0^t \overset{(s)}{V}(\tau)d\tau = (B + pn\mu)\int_0^t \overset{(s)}{V}(\tau)d\tau.$$

Here we assume that in G^1

$$B \geqslant k|g_a^k(x, t)|, \quad \mu > |\mu_a^k(x, t)|\,|_{\Gamma_1}.$$

From the resulting estimate follows the uniform convergence in the domain G^1 of the sequence $\{\overset{(s)}{r}(x, t)\}$ to the solution $r(x, t)$ of the mixed problem.

The solution r (x, t) (u(x, t)) constructed in the domain G^1 exhibits in it all properties of the solution of Cauchy's problem enumerated in section VII; it is continuously differentiable and depends continuously on the initial data of the mixed problem, as well as depending on curve Γ_1, if it satisfies properties (1) and (2).

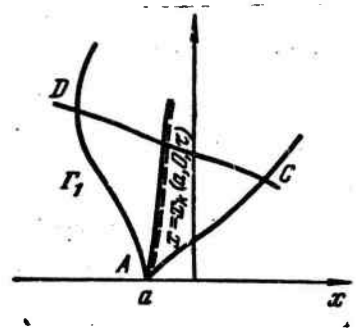

Figure 1.13

Note, however, that the line AC, generally speaking, is the discontinuity line of derivatives of the solution to the mixed problem. For the solution to the mixed problem to have continuous first derivatives in the domains $G^0 + G^1$, it is necessary that the initial and boundary conditions satisfy the conditions of congruence for the derivatives. These conditions will be obtained below for the more general case.

Above, for sake of simplicity it was assumed that for a fixed k all characteristics of the k-th family $x = x_k(x, t, \tau)$ passing through any point $(x, t) \in G^1$ intersects, when $\tau < t$, either only the line AC or only the line AD.

It may be, however, that this is not the case.

Suppose, for example, that through the point A passes the characteristic $x = x_k(a, 0, \tau)$ partitioning domain G^1 into two parts (Figure 1.13). In this case the solution is constructed on the analogy with the preceding, with obvious changes.

Note that this characteristic will also be a discontinuity line of the first derivatives.

In general, derivatives of the solution $r(x, t)$ suffer discontinuity at the characteristics exhibiting from the point A if and only if at this point the conditions of consistency of the derivatives are not satisfied.

2. Correctness of boundary conditions for a system of quasilinear equations. For a hyperbolic system of quasilinear equations we imposed the initial conditions $u(x, 0) = u^0(x)$ (1)

and the boundary conditions

$$c_i(x, t, u)|_{\Gamma_1} = 0 \quad (i = 1, 2, \ldots, n_1), \quad (2)$$

$$d_i(x, t, u)|_{\Gamma_2} = 0 \quad (i = 1, \ldots, n_2). \quad (3)$$

We will assume that the coefficients of the system satisfy their requirements that were imposed in section VIII when proving the existence theorem of the solution to Cauchy's problem.

Suppose $u^0(x) \in C_1$, and the curves Γ_1, Γ_2 are certain curves with a continuous tangent line in the half-plane $t \geqslant 0$ and passing, respectively, through the points (a, 0), (b, 0), and $c_i(x, t, u)$, $d_i(x, t, u)$ are continuously differentiable functions of their arguments.

We will state that the consistency conditions (conditions for continuity of a solution) are satisfied that the point (a, 0) if

$$c_i(a, 0, u^0(a)) = 0 \quad (i = 1, \ldots, n_1).$$

Let us establish the consistency conditions for the derivatives at the point (a, 0). Suppose $x = X(t)$ is the equation of the curve Γ_1. Let us assume the existence of the solution $u(x, t) \in C_1$ of a system of quasilinear equations satisfying conditions (1) and (2).

Differentiating the boundary conditions (2) with respect to the variable t at the line $x = X(t)$ we get

$$\frac{\partial c_i}{\partial u_\alpha}(q_\alpha + X'(t)\, p_\alpha) + \frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} X'(t) = 0 \quad (i = 1, \ldots, n_1).$$

The derivatives $p_\alpha = \partial u_\alpha / \partial x$ at $t = 0$ are defined from initial conditions (1), therefore from the system of equations

$$l^k_\alpha [q_\alpha + \xi_k p_\alpha] = f_k$$

we can define the derivatives $q_\alpha = \partial u_\alpha / \partial t$ at the initial axis $t = 0$:

$$q_k(x, 0) = -\lambda_\alpha^k \xi_\alpha l_\beta^\alpha p_\beta(x, 0) + \lambda_\alpha^k f_\alpha = q_k^0(x),$$
$$p_k(x, 0) = p_k^0(x) = \frac{du_k^0(x)}{dx}.$$

In these formulas the quantities λ_α^k, ξ_α, l_β^α, f_α are known functions of the variable x, for example: $\lambda_\alpha^k = \lambda_\alpha^k(x, 0, u^0(x))$ and so on.

We will state that at the point A(a, 0) of curve Γ_1 the conditions for consistency of the derivatives are met if

$$\frac{\partial c_i}{\partial u_\alpha}[q_\alpha(a) + X'(0) p_\alpha(a)] + c_{it} + X_i'(0) c_{ix} = 0 \quad (i = 1, \ldots, n_1);$$

here the functions $\frac{\partial c_i}{\partial u_\alpha}$, $\frac{\partial c_i}{\partial t}$, $\frac{\partial c_i}{\partial x}$ are taken at the point (a, 0), for example, $\frac{\partial c_i}{\partial u_\alpha} = \frac{\partial c_i}{\partial u_\alpha}(a; 0, u^0(a))$.

Assuming as before the existence of the solution $u(x, t) \in C_1$ of the mixed problem, let us establish requirements that must be satisfied by the boundary conditions. In general, the conclusion that the curves Γ_1, Γ_2 must lie exterior to the domain of determinancy G^0 of the solution to Cauchy's problem for essentially nonlinear equations $(\partial \xi_k / \partial u_j \neq 0)$ is invalid. Correctly formulated mixed problems exist when the curves Γ_1 and Γ_2 are in the domain G^0. An example of this problem is the problem of the piston in gas dynamics (cf Chapter Two, Section III). However, the solutions of these problems are discontinuous. Confining ourselves to a consideration of classical solutions, we now exclude this case, assuming that Γ_1 and Γ_2 lie outside G^0. Let us denote

$$\bar{l}_a^k(x, t) = l_a^k(x, t, u(x, t)),$$
$$\bar{\xi}_k = \xi_k(x, t, u(x, t))$$

and so on and we will consider our problem as a mixed problem for the linear system

$$\bar{l}_a^k\left[\frac{\partial u_a}{\partial t} + \bar{\xi}_k \frac{\partial u_a}{\partial x}\right] = \bar{f}_k$$

given the initial and boundary conditions (1) - (3).

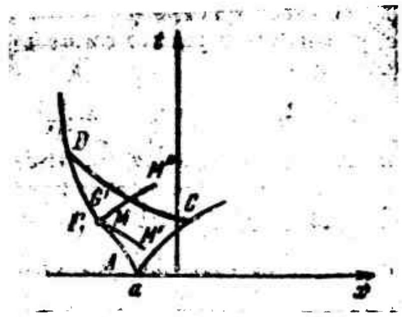

Figure 1.14

The functions $\bar{r}_k = \bar{l}^k_\alpha u_\alpha$ satisfy the linear system of equations

$$\frac{\partial \bar{r}_k}{\partial t} + \bar{\xi}_k \frac{\partial \bar{r}_k}{\partial x} = \bar{g}^k(x, t) + \bar{g}^k_\alpha(x, t)\bar{r}_\alpha.$$

Let us examine domain G^1 (Figure 1.14) and let point $(X(t), t)$ lie on the line Γ_1.

We will call the characteristic $x = x_k(X(t), t, \tau)$ the arriving characteristic at the point $(X(t), t)$ of line Γ_1 if it lies in the domain G^1 when $\tau \leqslant t$ and an exit characteristic if it lies in the domain G^1 when $\tau \geqslant t$. In Figure 1.14 MM' is the arriving characteristic, and MM" is the exit characteristic.

Suppose that at each point of Γ_1 the characteristics $x = x_k$ when $k = 1, 2, \ldots, p$ are arriving, and when $k = p + 1, \ldots, n$ are exit characteristics.

As for the case of the linear system, we require that

(1) the number of conditions (2) equals $q = n - p$, and

(2) equations (2) are uniquely solved with respect to the quantities $\bar{r}_{p+1}, \bar{r}_{p+2}, \ldots, \bar{r}_n$ for known $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_p$.

If (2) is rewritten as

$$c_l(x, t, \bar{\lambda}^k_\alpha \bar{r}_\alpha) = 0 \qquad (l = 1, 2, \ldots, n_1 = q),$$

then condition (2) will be satisfied if

$$\mathrm{Det}\left(\left(\frac{\partial c_l}{\partial u_\alpha}\lambda^\alpha_k\right)\right) \neq 0 \qquad (l = 1, \ldots, q; \quad k = p + 1, \ldots, n = p + q)$$
$$(x, t) \in \Gamma_2.$$

We note, however, that it proves very difficult to verify the correctness of the boundary conditions for a system of nonlinear equations, since conditions (1) and (2) depend on the solution u(x, t), which is unknown to us. Nevertheless, in solving the mixed problem we can proceed as follows.

Over a sufficiently small segment of the curve Γ_1 adjoining the point A, the solution u(x, t) (if it exists) will be sufficiently close to the value $u^0(a)$. This makes it possible to verify the conditions for the correctness of the formulation of the mixed problem for sufficiently small values of the variable t. If they are satisfied, then we solve the problem for this small interval and we will consider the values of the solution u(x, t) at the endpoint of the interval as new initial values. In this way we can construct the solution to the mixed problem in the entire domain of the variable x, t, where it exists.

Let us consider an example. For a system of two quasilinear equations

$$\frac{\partial r_1}{\partial t} + \xi_1(r_1, r_2)\frac{\partial r_1}{\partial x} = 0, \quad \frac{\partial r_2}{\partial t} + \xi_2(r_1, r_2)\frac{\partial r_2}{\partial x} = 0$$

the initial conditions

$$r_i(x, 0) = r_i^0 = \text{const}, \quad |x| < a$$

and the boundary conditions

$$c(t, r_1(-a, t), r_2(-a, t)) = 0, \quad d(t, r_1(a, t), r_2(a, t)) = 0$$

are formulated. Suppose $\xi_1 < 0$, $\xi_2 > 0$. The consistency conditions are satisfied if

$$c(0, r_1^0, r_2^0) = 0, \quad d(0, r_1^0, r_2^0) = 0. \tag{4}$$

When this condition is met, the solution r(x, t) is continuous in some neighborhood of the axis t = 0. If in addition to (4) the conditions for the consistency of the derivatives

$$\frac{\partial c}{\partial t}(0, r_1^0, r_2^0) = 0,$$
$$\frac{\partial d}{\partial t}(0, r_1^0, r_2^0) = 0,$$

are satisfied, then the solution r(x, t) has continuous derivatives.

The solution r (x, t) is constant in the domain G^0 (Figure 1.15): $r_i(x, t) = r_i^0$. In the domain G^1, $r_1(x, t) = r_1^0$. The boundary conditions are correct if

$$\frac{\partial c(t, r_1^0, r_2)}{\partial r_2} \neq 0, \quad \frac{\partial d(t, r_1, r_2^0)}{\partial r_1} \neq 0.$$

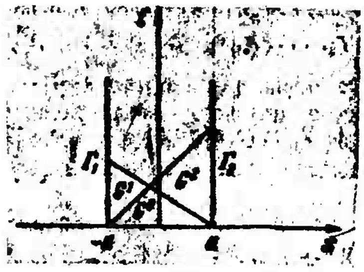

Figure 1.15

Section XII. Analytic Methods of Separating Solutions to Systems of Differential Equations With Two Independent Variables

1. Investigation of the compatibility of several overdetermined systems. Analytic methods of seeking solutions to systems of quasilinear equations in many cases lead to overdetermined systems, i.e., systems in which the number of equations exceed the number of unknowns. Here analysis of the compatibility of the overdetermined system is called for.

At the present time the most universal method of analyzing the compatibility of systems of equations is Cartan's method of external forms (cf [3, 4, 21, 22]). In a number of examples we will present a simple method of investigating compatibility which precedes Cartan's method and is sufficient for our purposes.

Example 1. Let us consider first of all the system of equations

$$\frac{\partial u_i}{\partial x_j} = f_{ij}(x_1, x_2, u_1, \dots, u_n) \quad (i = 1, \dots, n;\ j = 1, 2) \tag{1}$$

for n unknown functions $u_1, u_2, \dots, u_n$ of two independent variables x_1 and x_2. This system can also be written in differentials:

$$du_i = f_{i\alpha}\, dx_\alpha \quad (i = 1, \dots, n;\ \alpha = 1, 2). \tag{2}$$

Compiling the conditions for the integrability of equations (1), i.e., equating the mixed derivatives:

$$\frac{\partial^2 u_i}{\partial x_1 \partial x_2} = \frac{\partial^2 u_i}{\partial x_2 \partial x_1} \quad (i = 1, \dots, n), \tag{3}$$

we find

$$\frac{df_{i1}}{dx_2} = \frac{\partial f_{i1}}{\partial x_2} + \frac{\partial f_{i1}}{\partial u_\alpha} f_{\alpha 2} = \frac{df_{i2}}{dx_1} = \frac{\partial f_{i2}}{\partial x_1} + \frac{\partial f_{i2}}{\partial u_\alpha} f_{\alpha 1} \tag{4}$$
$$(i = 1, \dots, n;\ \alpha = 1, \dots, n).$$

If relations (4) are satisfied by identity, system (1) is called wholly integrable. In this case solution (1) can be defined by using the following algorithm. Let us assign at the point M_0: $x_j = x_{j0}$ ($j = 1,2$) of the value $u_i = u_{io}$ ($i = 1, \ldots, n$) and let us consider a certain curve $x_j = x_j(t)$ passing through the point M_0 and through the arbitrary point $M(x_1, x_2)$. Let us further consider the system of ordinary differential equations

$$\frac{du_i}{dt} = f_{i\alpha}[x_1(t), x_2(t), u_1, \ldots, u_n] \frac{dx_\alpha}{dt} \qquad (5)$$
$$(\alpha = 1, 2).$$

For known conditions imposed on the function $f_{i\alpha}(x, u)$ and at the curve of integration $x_j = x_j(t)$, system (5) has a unique solution which takes on the value $u_i = u_{io}$ at the point M_0 and is defined everywhere in the domain G containing the point M. Thus, $u(x_1, x_2)$ can be defined at each point of the domain. Let us show that for the case of a wholly integrable system the value u(x) at the point M(x) does not depend on the choice of the curve $x_j = x_j(t)$. Suppose $\mathcal{L}_1$, $\mathcal{L}_2$ are two curves that have common endpoints M_0, M, and are bounded together with a certain domain $G_0 \subset G$ (Figure 1.16). Then

$$\oint_{\mathcal{L}_1+\mathcal{L}_2} du_i = \oint_{\mathcal{L}_1+\mathcal{L}_2} f_{i\alpha}\, dx_\alpha. \qquad (6)$$

By the Gauss-Ostrogradskiy theorem,

$$\oint_{\mathcal{L}_1+\mathcal{L}_2} f_{i\alpha}\, dx_\alpha = \iint_{G_0} \left(\frac{df_{i2}}{dx_1} - \frac{df_{i1}}{dx_2}\right) dx_1\, dx_2. \qquad (7)$$

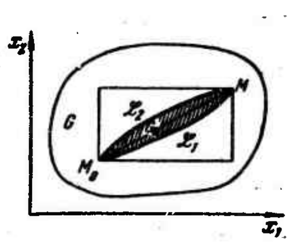

Figure 1.16

By conditions (4), integral (7) is equal to 0, which means the independence of the value u(x) from the selection of the curve of integration. In practice, it is more convenient to adopt as the path of integration the strict raise at M_0M or a broken line whose segments are parallel to the axes x_1, x_2.

If conditions (4) are not satisfied identically, they constitute a system of finite relations between $u_1, \ldots, u_n, x_1, x_2$, which makes it possible to cancel out several of the quantities u_i and to reduce system (1) to an analogous system with a smaller number of unknown functions. Extending the analogy further, we arrive either at an inconsistent system, or at a wholly integrable system. In the case of compatibility, we obtain a set of solutions dependent on arbitrary constants (a class of solutions with arbitrary constant).

Similarly, an analysis is made of compatibility for the system

$$\frac{\partial u_i}{\partial x_j}=f_{ij}(x_1, \ldots, x_m, u_1, \ldots, u_n) \quad (i=1, \ldots, n;\ j=1, \ldots, m), \qquad (8)$$

in which u_i depends on m arguments. Conditions of total integrability are of the form

$$\frac{df_{ij}}{dx_k}=\frac{\partial f_{ij}}{\partial x_k}+\frac{\partial f_{ij}}{\partial u_\alpha}f_{\alpha k}=\frac{df_{ik}}{dx_j}=\frac{\partial f_{ik}}{\partial x_j}+\frac{\partial f_{ik}}{\partial u_\alpha}f_{\alpha j} \qquad (9)$$
$$(i,\ \alpha=1, \ldots, n;\ j,\ k=1, \ldots, m).$$

and for system (8) we have no more than the arbitrary constant in the solutions.

Example 2. Now let us consider the linear homogeneous system with a single unknown scalar function of $u(x_1, \ldots, x_m)$:

$$L_i u=a_{i\alpha}\frac{\partial u}{\partial x_\alpha}=0 \qquad (i=1, \ldots, p;\ \alpha=1, \ldots, m). \qquad (10)$$

where coefficients $a_{i\alpha}$ are sufficiently smooth functions of $x_1, \ldots, x_m$. In this case the compatibility algorithm is known and reduces to the successive formation of so-called Poisson's brackets (cf [4, 23, 241]).

Let us form the commutant

$$[L_iL_j]=L_iL_j-L_jL_i. \qquad (11)$$

for the linear operators $L_i = a_{i\alpha}\frac{\partial}{\partial x_\alpha}$, $L_j = a_{j\alpha}\frac{\partial}{\partial x_\alpha}$. It is not difficult to see that the operator $[L_iL_j]$ is a first-order linear differential operator:

$$[L_iL_j]=b_{ij\alpha}\frac{\partial}{\partial x_\alpha}, \quad b_{ijk}=a_{i\beta}\frac{\partial a_{jk}}{\partial x_\beta}-a_{j\beta}\frac{\partial a_{ik}}{\partial x_\beta}. \qquad (12)$$

Operator $[L_iL_j]$ is called the Poisson's bracket. If $u(x_1, \ldots, x_m)$ is solution (10), then it satisfies, as a consequence, also the first-order linear homogeneous equation:

$$[L_iL_j]u=0 \quad (i,\ j=1, \ldots, p). \qquad (13)$$

Adjoining system (13) to system (10), we obtain an extended first-order system of linear homogeneous equations and can apply to it the algorithm for the formation of Poisson's brackets. This cycle of operations will be called an extension. After a finite number of extensions we arrive at a linear system containing all the preceding equations, for which the adding on of Poisson's brackets does not yield new equations, i.e., the commutants of the differential operators of the system are linear combinations of these operators. The systems are called complete. Thus, by definition, system (10) is called complete if

$$[L_i, L_j] = C^q_{ij} L_q \quad (i, j, q = 1, \ldots, p), \quad C^q_{ij} = C^q_{ij}(x_1, \ldots, x_m) \qquad (14)$$

For a complete system (10) equations (13) are no longer differentiable, but algebraic consequences.

Assuming the equations of the complete system (10) to be linearly independent, we see that two cases are possible:

(a) $p = m$, then system (10) admits only of the trivial solution $u = \text{const}$; and

(b) $p < m$.

We can ensure (cf [23, 24]) that in the second case the system is reduced by change of variables to a single linear homogeneous equation for one unknown function v of $m - p$ arguments $y_1, \ldots, y_{m-p}$, and thus the solution depends on a single arbitrary function of $m - p$ arguments.

Thus, the final conditions for compatibility consists in estimating the rank of the matrix of the complete system.

The distinguishing feature of the investigation of the compatibility of a linear system with a single function is the simplicity and homogeneity of the operations employed.

This is related to the fact that the conditions for the compatibility of linear equations are again linear equations, i.e., the extended system has the same structure as the initial.

Example 3. Let us consider the overdetermined system of two nonlinear equations, one of which is the Monge-Ampere equation

$$b\begin{vmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} & \frac{\partial^2 u}{\partial x_2^2} \end{vmatrix} + a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + a = 0 \quad (\alpha, \beta = 1, 2), \tag{15}$$

and the other is the first-order equation

$$\varphi\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = 0. \tag{16}$$

Here b, $a_{\alpha\beta}$, a are functions of $x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$, $a_{\alpha\beta} = a_{\beta\alpha}$.

We now consider conditions under which the overdetermined system (15), (16) admits of a family of solution dependent on a single arbitrary function of one argument.

Setting

$$du = p_\alpha dx_\alpha, \quad dp_i = p_{i\alpha} dx_\alpha \quad (i, \alpha = 1, 2), \tag{17}$$

let us write equations (15) and (16) in the form of finite relations in terms of $x_1, x_2, u, p_1, p_2, p_{11}, p_{12} = p_{21}, p_{22}$:

$$b\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} + a_{\alpha\beta} p_{\alpha\beta} + a = 0 \quad (\alpha, \beta = 1, 2), \tag{18}$$

$$\varphi(x_1, x_2, u, p_1, p_2) = 0. \tag{19}$$

Equation (18) for fixed x_1, x_2, u, p_1, p_2 defines a three-dimensional space of components p_{11}, p_{12}, p_{22}, a second-order surface (quadric) that has, generally speaking, two families of rectilinear generatrices.

As we know (cf [23]), these families are defined by the equation

$$\left.\begin{aligned} bp_{11} - b\mu p_{12} + a_{22} - \mu\lambda_1 = 0, \\ bp_{12} - b\mu p_{22} + \lambda_2 - \mu a_{11} = 0 \end{aligned}\right\} \tag{20}$$

and, therefore

$$\left.\begin{aligned} bp_{11} - b\mu p_{12} + a_{22} - \mu\lambda_2 = 0, \\ bp_{12} - b\mu p_{22} + \lambda_1 - \mu a_{11} = 0, \end{aligned}\right\} \tag{21}$$

where λ_1, λ_2 are the roots of the characteristic equation

$$\lambda^2 + 2a_{12}\lambda + a_{11}a_{22} - a_{12}^2 = 0 \tag{22}$$

Equations (20), and correspondingly (21), for fixed μ define a specific generatrix, and the value p_{12} is a point on this generatrix. Changing independently μ and p_{12}, we obtain the initial quadric (18).

Differentiating relation (19) with respect to x_1 and x_2, we get

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial p_1} p_{11} + \frac{\partial\varphi}{\partial p_2} p_{12} + \frac{\partial\varphi}{\partial x_1} + \frac{\partial\varphi}{\partial u} p_1 = 0, \\ \frac{\partial\varphi}{\partial p_1} p_{12} + \frac{\partial\varphi}{\partial p_2} p_{22} + \frac{\partial\varphi}{\partial x_2} + \frac{\partial\varphi}{\partial u} p_2 = 0. \end{aligned}\right\} \tag{23}$$

For system (15), (16) to be compatible and to admit of a one-functional arbitrary constant, the linear algebraic system (18), (23) must admit of an infinite number of solutions. Actually, otherwise p_{11}, p_{12}, p_{22} would be determined from conditions (18) and (23) as functions of x_1, x_2, u, p_1, p_2, the system of equations in total differential (17) with the closed and would admit only of an arbitrary constant (cf example 1). The requirement of an infinite number of solutions to system (18), (23) signifies that the straight line (23) is one of the generatrices (18). Assuming that the straight line (23) belongs to the family (20) or (21), we see that $\frac{\partial\varphi}{\partial x_i}, \frac{\partial\varphi}{\partial p_i}, \frac{\partial\varphi}{\partial u}$ must satisfy the equations

$$\frac{\frac{\partial\varphi}{\partial p_1}}{b} = \frac{\frac{\partial\varphi}{\partial p_2}}{-b\mu} = \frac{\frac{\partial\varphi}{\partial x_1} + p_1\frac{\partial\varphi}{\partial u}}{a_{22} - \mu\lambda_1}, \quad \frac{\frac{\partial\varphi}{\partial p_1}}{b} = \frac{\frac{\partial\varphi}{\partial p_2}}{-b\mu} = \frac{\frac{\partial\varphi}{\partial x_2} + p_2\frac{\partial\varphi}{\partial u}}{\lambda_2 - \mu a_{11}} \tag{24}$$

or else, correspondingly,

$$\frac{\frac{\partial\varphi}{\partial p_1}}{b} = \frac{\frac{\partial\varphi}{\partial p_2}}{-b\mu} = \frac{\frac{\partial\varphi}{\partial x_1} + p_1\frac{\partial\varphi}{\partial u}}{a_{22} - \mu\lambda_2}, \quad \frac{\frac{\partial\varphi}{\partial p_1}}{b} = \frac{\frac{\partial\varphi}{\partial p_2}}{-b\mu} = \frac{\frac{\partial\varphi}{\partial x_2} + p_2\frac{\partial\varphi}{\partial u}}{\lambda_1 - \mu a_{11}}. \tag{25}$$

Canceling μ from equations (24), we advance to a system of equations for $\varphi(x_1, x_2, u, p_1, p_2)$:

$$\left.\begin{aligned} b\left(\frac{\partial\varphi}{\partial x_1}+p_1\frac{\partial\varphi}{\partial u}\right)-a_{22}\frac{\partial\varphi}{\partial p_1}-\lambda_1\frac{\partial\varphi}{\partial p_2}=0,\\ b\left(\frac{\partial\varphi}{\partial x_2}+p_2\frac{\partial\varphi}{\partial u}\right)-\lambda_2\frac{\partial\varphi}{\partial p_1}-a_{11}\frac{\partial\varphi}{\partial p_2}=0.\end{aligned}\right\} \quad (26)$$

Canceling out μ from equation (25), we thus obtain the equations

$$\left.\begin{aligned} b\left(\frac{\partial\varphi}{\partial x_1}+p_1\frac{\partial\varphi}{\partial u}\right)-a_{22}\frac{\partial\varphi}{\partial p_1}-\lambda_2\frac{\partial\varphi}{\partial p_2}=0,\\ b\left(\frac{\partial\varphi}{\partial x_2}+p_2\frac{\partial\varphi}{\partial u}\right)-\lambda_1\frac{\partial\varphi}{\partial p_1}-a_{11}\frac{\partial\varphi}{\partial p_2}=0.\end{aligned}\right\} \quad (27)$$

Let us show that equations (26), and therefore (27), are also sufficient for the compatibility of equations (15) and (16) with a one-functional arbitrariness. Suppose φ is solution (26), and $u(x_1, x_2)$ is the solution to equation (16); let us show that $u(x_1, x_2)$ is the solution to (15). By differentiating (16), we get equation (23). Equations (26) signify that we can introduce the parameter μ so as to satisfy equations (24). Equations (24) signify that the straight line (23) lies in the quadrant (18), that is, $u(x_1, x_2)$ satisfies equation (15). The assertion stands proven.

And so, equations (26), and therefore (27), are necessary and sufficient conditions for the compatibility of system (15), (16) with one-functional arbitrariness. Thus, the problem of determining the compatibility of system (15), (16) reduces to the familiar and simple algorithm for investigating the compatibility of the linear homogeneous system with a single unknown function, which we treated in example (2).

Martin [27], Ludford [28], and Yu. S. Zav'yalov [29] made an analysis of the compatibility of system (15), (16) on the special assumption when

$$b=1, \quad a_{\alpha\beta}=0, \quad a=f^2(x_1, x_2). \quad (28)$$

In this case system (26) takes from the form

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial x_1}+p_1\frac{\partial\varphi}{\partial u}+f\frac{\partial\varphi}{\partial p_2}=0,\\ \frac{\partial\varphi}{\partial x_2}+p_2\frac{\partial\varphi}{\partial u}-f\frac{\partial\varphi}{\partial p_1}=0.\end{aligned}\right\} \tag{29}$$

Let us denote

$$x_1=x_1,\quad x_2=x_2,\quad x_3=u,\quad x_4=p_1,\quad x_5=p_2. \tag{30}$$

Twice extending system (29) by means of the formation of Poisson's brackets, we arrive at the system

$$L_i=a_{i\alpha}\frac{\partial\varphi}{\partial x_\alpha}=0 \qquad (i,\ \alpha=1,\ \ldots,\ 5), \tag{31}$$

where the matrix a_{ij} is of the form

$$\left\|\begin{matrix}1 & 0 & x_4 & 0 & f\\ 0 & 1 & x_5 & -f & 0\\ 0 & 0 & -2f & f_1 & f_2\\ 0 & 0 & -3f_1 & f_{11} & f_{21}\\ 0 & 0 & -3f_2 & f_{12} & f_{22}\end{matrix}\right\|. \tag{32}$$

For the presence of functional arbitrary choice, it is necessary that the rank of the matrix of the system obtained by adjoining Poisson's brackets does not exceed 4. Denoting by Δ_i (i = 1, ..., 5) the algebraic complements to matrix (32), equation (31) can be rewritten as

$$\frac{\partial\varphi}{\partial x_i}=\Delta_i\omega, \tag{33}$$

where ω is some function of x_1, x_2. From (33) follows the representation

$$d\varphi=\omega(\Delta_\alpha dx_\alpha) \qquad (\alpha=1,\ \ldots,\ 5), \tag{34}$$

which means that ω is the integrating cofactor of the differential form $\Delta_\alpha dx_\alpha$. From equation (33) we get equations for ω and the compatibility conditions for system (31).

Without delving into details, and referring the reader to the works [27 - 29], where a complete analysis was made, we only point to the end result: for a system (31) to be compatible and to admit of a solution dependent on an arbitrary function of one argument, it is necessary and sufficient that there be the possibility of representing $f(x_1, x_2)$ in one of two forms:

$$f = F(\alpha_1 x_1 + \alpha_2 x_2), \tag{35}$$

$$f = \frac{1}{(x_1+\alpha_1)(x_2+\alpha_2)} F\left(\frac{x_1+\alpha_1}{x_2+\alpha_2}\right) = \left(\frac{1}{x_2+\alpha_2}\right)^2 F\left(\frac{x_1+\alpha_1}{x_2+\alpha_2}\right), \tag{36}$$

where F is the arbitrary function of a single argument, α_1, α_2 are arbitrary constants.

Different expressions are obtained for the function φ, depending on the function f.

If f is represented in the form corresponding to (36):

then

$$f = \frac{1}{(x_2+\alpha_2)^2} \sqrt{F'\left(\frac{x_1+\alpha_1}{x_2+\alpha_2}\right)}, \tag{37}$$

$$\varphi = x_3 - x_4(x_1+\alpha_1) - x_5(x_2+\alpha_2) \pm g\left(\frac{x_1+\alpha_1}{x_2+\alpha_2}\right), \tag{38}$$

where $g(\theta)$ is associated with $F(\theta)$ by the relation $g' = \sqrt{F'}$ (39)
and the sign $\pm$ denotes the different possibilities of selection of the roots of the characteristic equation.

If f is represented in the form corresponding to (35):

then

$$f = \sqrt{F'(\alpha_1 x_1 + \alpha_2 x_2)}, \tag{40}$$

$$\varphi = \alpha_2 x_4 - \alpha_1 x_5 \pm g(\alpha_1 x_1 + \alpha_2 x_2), \tag{41}$$

where g as before is associated with F by the relation (39).

Finally, for the case when $f = 0$, $\varphi = \varphi(x_4, x_5)$. (42)

These results were used by Martin [27], Ludford [28], and Yu. S. Zav'yalov [29] to obtain generalized Riemann waves (cf chapter 2, section IX, subsection 3).

2. Solutions with degenerate hodograph of systems of quasilinear equations. The main task in the analytic theory of differential equations with partial derivatives is to obtain particular solutions and to construct solutions of a broader class by relying on them. Particular solutions are obtained for the most part by means of contraction of the space of the equation, i.e., by reducing the number of active variables.

Thus, for example, Fourier's method allows us to proceed from an equation with partial derivatives to ordinary equations and thus to obtain particular solutions containing the additional (passive) parameters. Then, the general integrals obtained by superpositioning particular solutions contains arbitrary functions of passive parameters. However, Fourier's method is applicable only for an extremely narrow class of linear equations. For the case of nonlinear equations the method of contracting the space of equations is also used. It allows us to obtain particular solutions, but the superposition principle becomes inapplicable, and obtaining a broad class of solutions that contain arbitrary functions of passive parameters becomes greatly complicated.

The familiar method of envelopes permitting converting from solutions containing arbitrary parameters (total integral) to solutions containing arbitrary functions (general integral) becomes, generally speaking, unsuitable for systems of equations with several functions.

If $u_i(x, t, a)$ $(i = 1, \ldots, n)$ (1)

is a solution to an arbitrary nonlinear system

$$\left\{ F_i\left(x, t, u_1, \ldots, u_n, \frac{\partial u_1}{\partial t}, \ldots, \frac{\partial u_n}{\partial t}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_n}{\partial x}\right) = 0 \quad (i = 1, \ldots, n), \right. \qquad (2)$$

then the envelope does not always correspond to it. This fact is pure geometrical in origin.

For the case $n = 1$ the space (u_1, x, t) of equation (2) is three-dimensional and infinitely close two-dimensional integral manifolds (1) corresponding to values of the parameters a, a + da intersect along the line (characteristic). A one-parametric family of characteristics forms an envelope surface, which is an integral manifold (2). If $n = 2$, the space (u_1, u_2, x, t) of equation (2) becomes four-dimensional and infinitely close two-dimensional manifolds (1) intersect, generally speaking, not along the line but at a point.

Thus, for a one-parametric family of solutions of system (2), the aggregation of manifolds of intersection yields now not a two-dimensional integral manifold, but only some line.

Let us consider by way of illustration nonlinear equations for the potential of a conservative system of equations.

For the homogeneous conservative system

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u_1, \ldots, u_n)}{\partial x} = 0 \qquad (i = 1, \ldots, n) \tag{3}$$

Let us introduce the potentials (cf section V, subsection 3), which are associated with $u_1, \ldots, u_n$ by the relations

$$u_i = \frac{\partial \Phi_i}{\partial x}, \quad \varphi_i = -\frac{\partial \Phi_i}{\partial t}. \tag{4}$$

For Φ_i we obtain the equation

$$\frac{\partial \Phi_i}{\partial t} + \varphi_i\left(\frac{\partial \Phi_1}{\partial x}, \ldots, \frac{\partial \Phi_n}{\partial x}\right) = 0 \qquad (i = 1, \ldots, n). \tag{5}$$

It is not difficult to see that we have a 2n-parametric family of solutions to system (5): $\Phi_i = a_i x + b_i t + c_i \qquad (i = 1, \ldots, n) \qquad (6)$
where a_i, b_i, c_i are constants, and a_i and b_i are associated by the relation

$$b_i + \varphi_i(a_1, \ldots, a_n) = 0. \tag{7}$$

In the $(n + 2)$-dimensional space $\{\Phi_1, \ldots, \Phi_n, t, x\}$ of system (5), surfaces given by the totality of equations (6), with fixed parameters a_i and c_i, are two-dimensional planes. Suppose the functions

$$a_i = a_i(\tau), \quad c_i = c_i(\tau), \quad b_i = -\varphi_i(a_1(\tau), \ldots, a_n(\tau)) \tag{8}$$

separate from family (6) some one-parametric family. We require that the corresponding planes intersect along a line (characteristic); the set of intersection points is determined from conditions (6) and from the additional equations

$$0 = \dot{a}_i x + \dot{b}_i t + \dot{c}_i, \tag{9}$$

where the dot over the letter denotes differentiation with respect to τ.

For a characteristic (the general line of intersection of the planes (6) and (9)) to exist, it is necessary and sufficient that system (9) be of rank 1, i.e., that the condition

$$\frac{\dot{b}_i(\tau)}{\dot{a}_i(\tau)} = \mu(\tau), \quad \frac{\dot{c}_i(\tau)}{\dot{a}_i(\tau)} = \nu(\tau). \tag{10}$$

be satisfied. From this we get equations for determination of a_i and μ:

$$[a_{i\alpha} + \mu(\tau)\delta_{i\alpha}]\,\dot{a}_\alpha = 0, \tag{11}$$

$$\mathrm{Det}((a_{ij} + \mu\delta_{ij})) = 0, \tag{12}$$

where

$$a_{ij} = \frac{\partial\varphi_i(a_1, \ldots, a_n)}{\partial a_j}. \tag{13}$$

Equations (11) and (12) define $a_i(\tau)$ with accuracy up to one arbitrary function of parameter τ. From equations (10), $c_i(\tau)$ is also defined with an accuracy up to one arbitrary function of parameter τ. Since parameter τ is undetermined, we obtain a family of solutions depending essentially only on a single arbitrary function of one parameter, i.e., this family is not a general integral.

Let us show that the resulting family is a family of so-called simple waves.

Definition. A simple wave of a system of equations

$$\frac{\partial u_i}{\partial t} + a_{i\alpha}(u_1, \ldots, u_n)\frac{\partial u_\alpha}{\partial x} = 0 \quad (i, \alpha = 1, \ldots, n) \tag{14}$$

is the name given to the solution $u_i = u_i(x, t)$ satisfying the condition:

$$\text{rank} \begin{Vmatrix} \frac{\partial u_1}{\partial t} & \frac{\partial u_2}{\partial t} & \cdots & \frac{\partial u_n}{\partial t} \\ \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \cdots & \frac{\partial u_n}{\partial x} \end{Vmatrix} = 1. \tag{15}$$

Equation (15) signifies that functions $u_i(x, t)$ can be represented as

$$u_i(x, t) = u_i(\tau), \quad \tau = \tau(x, t), \tag{16}$$

where $\tau(x, t)$ is some parametric function. Substituting (16) and (14), we find

$$\dot{u}_i \frac{\partial \tau}{\partial t} + a_{i\alpha}\dot{u}_\alpha \frac{\partial \tau}{\partial x} = 0. \tag{17}$$

For solution $u_i(x, t) = u_i[\tau(x, t)]$ to be nontrivial, the equations

$$\frac{a_{i\alpha}\dot{u}_\alpha}{\dot{u}_i} = \xi = -\frac{\frac{\partial \tau}{\partial t}}{\frac{\partial \tau}{\partial x}}. \tag{18}$$

must be satisfied. Therefore for this is necessary and sufficient that

$$\text{Det}((a_{i\alpha} - \xi\delta_{i\alpha})) = 0, \tag{19}$$

then ξ is the eigenvalue of matrix $\| a_{i\alpha} \|$, and the vector $u = \{\dot{u}\}$ is its right eigenvector.

From the algebraic relations (18) and (19) we find ξ and u, and $\tau(x,t)$ is defined as the solution of the equation

$$\frac{\partial \tau}{\partial t} + \xi \frac{\partial \tau}{\partial x} = 0. \tag{20}$$

If system (14) is hyperbolic, then there exist n eigenvalues $\xi_1, \ldots, \xi_n$ and travelling waves corresponding to them, which we will denote with the same number as the characteristic. It is not difficult to see that in the k-th simple wave the characteristics of the k-th family are straight lines.

If system (14) is conservative, i.e., if the following conditions are satisfied:

$$a_{ij}(u_1, \ldots, u_n) = \frac{\partial \varphi_i(u_1, \ldots, u_n)}{\partial u_j}, \tag{21}$$

where $\varphi_i(u_1, \ldots, u_n)$ are certain functions, then we can proceed to equation (5) in potentials. We will show that the envelope of family (6), (8) is a simple wave. Taking (6), (7), and (9) into account, we have

$$u_i = \frac{\partial \Phi_i}{\partial x} = (\dot{a}_i x + \dot{b}_i t + \dot{c}_i)\frac{\partial \tau}{\partial x} + a_i = a_i(\tau). \tag{22}$$

Thus, at the envelope surface of family (6), (8) functions $u_i(x, t)$ depend on a single parameter, and by definition solution $u_i = u_i(x, t)$ is a simple wave. The statement stands proven.

A special case of the simple wave is the centered wave, when straight lines of the characteristic of the k-th family intersect at the same point x_0, t_0. Then we can choose the inclination of the characteristic

$$\tau = \frac{x - x_0}{t - t_0}. \tag{23}$$

as the parametric function. Equation (20) retains its meaning, and the relation

$$\xi_n = \tau. \tag{24}$$

is valid.

Let us note one interesting property of simple waves. Relation (16) signifies that the (n - 1)-th functional relation exists in the k-th wave:

$$r_i^k(u_1, \ldots, u_n) = c_i \qquad (i = 1, \ldots, n;\ i \neq k). \tag{25}$$

Let us consider the one-parametric family of k-th centered waves in which the constants c_i ($i \neq k$) from (25) are fixed, and x_0, t_0 are associated by the function $x_0 = \varphi(t_0)$. Then we have a one-parametric family of integral surfaces of equation (20), which has an envelope. This envelope is a simple wave, but no longer centered. Therefore, simple waves are centered envelopes.

Let us now consider simple waves of inhomogeneous systems.

For the inhomogeneous system

$$\frac{\partial u_i}{\partial t}+a_{i\alpha}(u_1, \ldots, u_n)\frac{\partial u_\alpha}{\partial x}=g_i(u_1, \ldots, u_n) \quad (i=1, \ldots, n), \tag{26}$$

assumption (16) leads to the equation

$$\frac{\partial \tau}{\partial t}+f_i\frac{\partial \tau}{\partial x}=F_i \quad (i=1, \ldots, n), \tag{27}$$

where

$$f_i=\frac{a_{i\alpha}\dot{u}_\alpha}{\dot{u}_i}, \quad F_i=\frac{g_i}{\dot{u}_i} \quad (i, \alpha=1, \ldots, n). \tag{28}$$

For simple waves with arbitrary functional to exist, it is necessary and sufficient that the rank of the matrix

$$\begin{Vmatrix} 1 & f_1 & F_1 \\ 1 & f_2 & F_2 \\ \cdot & \cdot & \cdot \\ 1 & f_n & F_n \end{Vmatrix} \tag{29}$$

equal 1. In the general cases condition is not satisfied. For waves to exist with arbitrary constant it is necessary that the rank of matrix (29) be 2.

From this condition, n - 2 quantities $\dot{u}_i (i = 1, \ldots, n)$ are determined for every two of them, for example, $\dot{u}_3, \ldots, \dot{u}_n$ -- at a spacing of $\dot{u}_1, \dot{u}_2$.

Then, considering these functions, we arrive at subsystem (27) when i = 1, 2, where f_i, F_i are now the functions only of u_1, u_2. Solving (27) with respect to $\frac{\partial \tau}{\partial t}, \frac{\partial \tau}{\partial x}$, we find

$$\frac{\partial \tau}{\partial x}=\Phi_1=\frac{F_2-F_1}{f_2-f_1}, \quad \frac{\partial \tau}{\partial t}=\Phi_2=\frac{F_1f_2-F_2f_1}{f_2-f_1}. \tag{30}$$

Setting up the conditions for integrability of equations (30), we find

$$\dot{\Phi}_1\Phi_2=\dot{\Phi}_2\Phi_1. \tag{31}$$

Hence follows the integral $\Phi_2 / \Phi_1 = C$. (32)

From equations (30), bearing (32) in mind, we find

$$\frac{\partial \tau}{\partial t} - C\frac{\partial \tau}{\partial x} = 0. \quad (33)$$

The simplest solution for τ is of the form

$$\tau = \xi = x + Ct. \quad (34)$$

Thus, u_1, u_2 are functions of parameter ξ and the following system of ordinary differential equations is satisfied:

$$\frac{dr_i}{d\xi} = \frac{g_i(r_1, r_2)}{f_i(r_1, r_2) + C} \quad (i = 1, 2). \quad (35)$$

Let us further point to a class of equations that admit of simple waves with arbitrary functional. For the system

$$\frac{\partial r_i}{\partial t} + f_i(r_1, r_2)\frac{\partial r_i}{\partial x} = \frac{g_i(r_1, r_2)}{x^{\alpha}} \quad (i = 1, 2) \quad (36)$$

the parametric function $\tau(x, t)$ satisfies the equations

$$\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x} = \frac{g_i}{x^{\alpha} r_i'}. \quad (37)$$

From whence we find

$$\frac{\partial \tau}{\partial t} = \frac{\Phi_1}{x^{\alpha}}, \quad \frac{\partial \tau}{\partial x} = \frac{\Phi_2}{x^{\alpha}}, \quad (38)$$

where

$$\Phi_1 = \frac{\frac{g_1 f_2}{r_1'} - \frac{g_2 f_1}{r_2'}}{f_2 - f_1}, \quad \Phi_2 = \frac{\frac{g_2}{r_2'} - \frac{g_1}{r_1'}}{f_2 - f_1}. \quad (39)$$

The conditions for the integrability of equalities (38) yield

$$\frac{\Phi_2 \Phi_1'}{x^{2\alpha}} - \alpha \frac{\Phi_1}{x^{\alpha+1}} = \frac{\Phi_1 \Phi_2'}{x^{2\alpha}}. \quad (40)$$

Hence it follows that when $\alpha = 1$ system (36) admits of a simple wave with an arbitrary constant.

Let us indicate the classes of equations leading to equations admitting of simple waves.

Suppose the system

$$\frac{\partial r_i}{\partial t}+f_i(r_1, r_2)\frac{\partial r_i}{\partial x}=\frac{g_i(r_1, r_2)}{x^{\alpha}} \quad (i=1, 2) \tag{41}$$

has the coefficients $f_i(r_1, r_2)$, $g_i(r_1, r_2)$, exhibiting the property of homogeneity, such that the relations

$$\left.\begin{aligned} f_i(\theta r_1, \theta r_2)=\theta^{\beta_1} f_i(r_1, r_2), \\ g_i(\theta r_1, \theta r_2)=\theta^{\beta_2} g_i(r_1, r_2). \end{aligned}\right\} \tag{42}$$

are valid. By the substitution $r_i = x^{\gamma} R_i$ (43)
system (41) is reduced to the form

$$\frac{\partial R_i}{\partial t}+x^{\gamma\beta_1} f_i(R_1, R_2)\frac{\partial R_i}{\partial x}=x^{\gamma\beta_2-\gamma-\alpha} g_i(R_1, R_2)-\gamma x^{\gamma\beta_1-1} f_i R_i \quad (i=1, 2). \tag{44}$$

If

$$\gamma[(\beta_2-1)-\beta_1]=\alpha-1, \tag{45}$$

then after the substitution

$$z=\frac{1}{-\gamma\beta_1+1} x^{-\gamma\beta_1+1} \tag{46}$$

system (4,) takes on the form

$$\frac{\partial R_i}{\partial t}+f_i(R_1, R_2)\frac{\partial R_i}{\partial z}=\frac{1}{(1-\gamma\beta_1)z}[g_i(R_1, R_2)-\gamma f_i R_i]. \tag{47}$$

i.e., admits of a simple wave.

By the substitution $r_i = (x/t)^{\gamma} R_i$ (48)
system (41) is converted into the form

$$\frac{\partial R_i}{\partial t}+\left(\frac{x}{t}\right)^{\gamma\beta_1} f_i(R_1, R_2)\frac{\partial R_i}{\partial x}= =\gamma\frac{R_i}{t}-\gamma\left(\frac{x}{t}\right)^{\gamma\beta_1-1}\frac{R_i}{t} f_i(R_1, R_2)+\left(\frac{x}{t}\right)^{\gamma\beta_2-\gamma-\alpha}\frac{g_i(R_1, R_2)}{t^{\alpha}}. \tag{49}$$

If $\quad \nu\beta_1 = 1, \quad \alpha = 1, \quad \nu\beta_2 - \nu - \alpha = 0,$ (50)
then system (49) becomes

$$\frac{\partial R_i}{\partial \tau} + f_i(R_1, R_2)\frac{\partial R_i}{\partial z} = H_i(R_1, R_2), \qquad (51)$$

where

$$H_i = \nu R_i [1 - f_i(R_1, R_2)] + g_i(R_1, R_2), \quad \tau = \ln t, \quad z = \ln x, \qquad (52)$$

i.e., also admits of a simple wave.

The possibility of the substitutions given above is related to the group properties of the equations (cf section XIII on this subject). In particular equation (51) is invariant with respect to the similarity transformation $z \to kz$, $\tau \to k\tau$.

Relation (16) indicates that in the case of a simple wave the two-dimensional integral surface $u_i = u_i(x, t)$ maps on to the plane of hodograph purely $\{u_1, \ldots, u_n\}$ in the form of the line l: $u_i = u_i(\tau)$ $(i = 1, \ldots, n)$; therefore we can now speak of simple waves as solutions with degenerate hodograph.

If the system is homogeneous, then to the line l there corresponds a family of solutions with a one-functional arbitrary choice.

In the general case the two-dimensional integral surface $u_i = u_i(x, t)$ $(i = 1, 2, \ldots, n)$ maps onto a two-dimensional plane in the hodograph space. It can be shown, as a rule, that to the surface S there corresponds a family of solutions with not more than an arbitrary constant and only in exceptional cases does surface S map onto a family of solutions with a functional arbitrary choice.

For sake of simplicity let us limit ourselves to the case of a system with three unknowns

$$\frac{\partial u_i}{\partial t} + a_{i\alpha}\frac{\partial u_\alpha}{\partial x} = 0 \qquad (i, \alpha = 1, 2, 3). \qquad (53)$$

To each solution $u_i = u_i(x, t)$ of system (53) that is not a simple wave there corresponds a wholly determinate surface S in the space $\{u_1, u_2, u_3\}$, which we will give, for specificity, by the equation

$$u_3 = \varphi(u_1, u_2). \tag{54}$$

After the substitution of (54) into (53), we arrive at the over-determined system

$$\left.\begin{aligned} &\frac{\partial u_i}{\partial t} + b_{i\alpha}\frac{\partial u_\alpha}{\partial x} = 0 \quad (i,\ \alpha = 1,\ 2), \\ &\frac{\partial \varphi}{\partial u_\alpha}\frac{\partial u_\alpha}{\partial t} + b_{3\alpha}\frac{\partial u_\alpha}{\partial x} = 0 \quad (\alpha = 1,\ 2), \end{aligned}\right\} \tag{55}$$

where

$$b_{i\alpha} = a_{i\alpha} + a_{i3}\frac{\partial \varphi}{\partial u_\alpha} \quad (i = 1,\ 2,\ 3;\ \alpha = 1,\ 2). \tag{56}$$

The condition for the consistency of system (55) leads to a third-order quasilinear equations for the function $\varphi(u_1, u_2)$. By defining surface (54), we can restore the solution by quadrature.

We will not slight the operations, referring the reader to work [25] and to section IX of chapter two.

3. Solutions characterized by the differential relation. The analysis made in the preceding subsections shows that solutions with a degenerate hodograph (simple waves) do not always have a functional arbitrary choice. This means that to obtain solutions with an arbitrary functional choice or with an arbitrary constant choice with a large number of constants, classes of solutions must be separated in a more general fashion.

Functional relations in the space of the initial equation must be replaced by functional relations in the space of the extended system (cf section IV, subsection 3). Functional relations containing not only the unknown and independent variables as such, but also derivatives, will be called differential relations (cf [26]).

The highest order of derivatives appearing in a differential relation will be called the order of the relation.

Let us clarify these concepts with the example of a system of two inhomogeneous equations written in invariants:

$$\left.\begin{aligned}\frac{\partial r_1}{\partial x_2}+f_1(r_1, r_2, x_1, x_2)\frac{\partial r_1}{\partial x_1}=g_1(r_1, r_2, x_1, x_2),\\ \frac{\partial r_2}{\partial x_2}+f_2(r_1, r_2, x_1, x_2)\frac{\partial r_2}{\partial x_1}=g_2(r_1, r_2, x_1, x_2).\end{aligned}\right\} \tag{1}$$

We will seek the solution $r_i = r_i(x_1, x_2)$ of system (1) satisfying the first-order differential relation:

$$F\left(x_1, x_2, r_1, r_2, \frac{\partial r_1}{\partial x_1}, \frac{\partial r_2}{\partial x_1}, \frac{\partial r_1}{\partial x_2}, \frac{\partial r_2}{\partial x_2}\right)=0. \tag{2}$$

Clearly, using equations (1) the dependence of F on $\frac{\partial r_1}{\partial x_2}$, $\frac{\partial r_2}{\partial x_2}$ can be canceled out and relation (2) can be rewritten as

$$\Phi\left(x_1, x_2, r_1, r_2, \frac{\partial r_1}{\partial x_1}, \frac{\partial r_2}{\partial x_1}\right)=0. \tag{3}$$

Here Φ is a thus far unknown function, but a fixed function in our entire treatment.

Let us find the conditions under which a family of solutions satisfying the fixed relation (3) as a one-functional arbitrary choice.

Let us consider the first extension of equations (1) and relation (3). We set

$$\frac{\partial r_i}{\partial x_1}=p_i, \quad \frac{\partial r_i}{\partial x_2}=q_i, \quad \frac{\partial p_i}{\partial x_1}=s_i, \quad \frac{\partial p_i}{\partial x_2}=t_i \qquad (i=1, 2). \tag{4}$$

In equations (1) take on tne form of the finite relations

$$q_i+f_ip_i=g_i \qquad (i=1, 2). \tag{5}$$

using (4) and (5), we find

$$dr_i=p_i\,dx_1+q_i\,dx_2=p_i\,dx_1+(g_i-f_ip_i)\,dx_2. \tag{6}$$

The conditions for the integrability of relation (6) lead to the equations

$$\left.\begin{aligned}\frac{\partial p_1}{\partial x_2}+f_1\frac{\partial p_1}{\partial x_1}=t_1+f_1s_1=F_1,\\ \frac{\partial p_2}{\partial x_2}+f_2\frac{\partial p_2}{\partial x_1}=t_2+f_2s_2=F_2,\end{aligned}\right\} \tag{7}$$

where

$$F_i = \frac{\delta g_i}{\delta x_1} - p_i \frac{\delta f_i}{\delta x_1}, \tag{8}$$

and the symbol $\delta/\delta x_i$ denotes differentiation with respect to x_i, taking into account the dependence of r_1, r_2 (but not p_1, p_2) on x_1, x_2:

мости r_1, r_2 (но не p_1, p_2) от x_1, x_2:

$$\left.\begin{aligned}
\frac{\delta}{\delta x_1} &= \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial r_1} + p_2 \frac{\partial}{\partial r_2}, \\
\frac{\delta}{\delta x_2} &= \frac{\partial}{\partial x_2} + q_1 \frac{\partial}{\partial r_1} + q_2 \frac{\partial}{\partial r_2} = \\
&= \frac{\partial}{\partial x_2} + [g_1 - f_1 p_1] \frac{\partial}{\partial r_1} + [g_2 - f_2 p_2] \frac{\partial}{\partial r_2} = \\
&= \frac{\partial}{\partial x_2} + g_1 \frac{\partial}{\partial r_1} + g_2 \frac{\partial}{\partial r_2} - f_1 p_1 \frac{\partial}{\partial r_1} - f_2 p_2 \frac{\partial}{\partial r_2}.
\end{aligned}\right\} \tag{9}$$

Differentiating relation (3) we find

$$\left.\begin{aligned}
\frac{\partial \Phi}{\partial p_\alpha} s_\alpha + \frac{\delta \Phi}{\delta x_1} &= 0 \quad (\alpha = 1, 2), \\
\frac{\partial \Phi}{\partial p_\alpha} t_\alpha + \frac{\delta \Phi}{\delta x_2} &= 0 \quad (\alpha = 1, 2).
\end{aligned}\right\} \tag{10}$$

In the four-dimensional space of derivatives s_1, s_2, t_1, t_2, each of the equations (7) and (10) constitutes a three-dimensional plane. For the functional arbitrary choice it is necessary that these planes have a common straight line, i.e., that the rank of the matrix

$$\begin{Vmatrix}
f_1 & 1 & 0 & 0 & -F_1 \\
0 & 0 & f_2 & 1 & -F_2 \\
\frac{\partial \Phi}{\partial p_1} & 0 & \frac{\partial \Phi}{\partial p_2} & 0 & \frac{\delta \Phi}{\delta x_1} \\
0 & \frac{\partial \Phi}{\partial p_1} & 0 & \frac{\partial \Phi}{\partial p_2} & \frac{\delta \Phi}{\delta x_2}
\end{Vmatrix} \tag{11}$$

be equal to three. Actually, if these planes intersect at a point, then this signifies that from equations (7) and (10) the derivatives $s_i = \partial p_i/\partial x_1$, $t_i = \partial p_i/\partial x_2$ are determined in terms of x_1, x_2, r_1, r_2, p_1, p_2, and we arrive at the system of equations with an arbitrary constant choice that we investigated in subsection 1.

Hence, as the first corollary, when $f_2 \neq f_1$ we get

$$\frac{\partial \Phi}{\partial p_1} \cdot \frac{\partial \Phi}{\partial p_2} = 0 \qquad (12)$$

Let us assume for sake of definiteness that $\partial \Phi / \partial p_2 = 0$. Then, as the second consequence we have

$$\frac{\partial \Phi}{\partial x_2} + f_1 \frac{\partial \Phi}{\partial x_1} + p_1 \frac{\partial \Phi}{\partial r_1} = 0, \qquad (13)$$

Taking (8) and (9) into account, relation (13) can be transformed to

$$A + p_2 B = 0 \qquad (14)$$

where

$$A = f_1 \frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial x_2} + g_1 \frac{\partial \Phi}{\partial r_1} + g_2 \frac{\partial \Phi}{\partial r_2} + \left[\frac{\partial g_1}{\partial x_1} + \left(\frac{\partial g_1}{\partial r_1} - \frac{\partial f_1}{\partial x_1} \right) p_1 - \frac{\partial f_1}{\partial r_1} p_1^2 \right] \frac{\partial \Phi}{\partial p_1}, \qquad (15)$$

$$B = (f_1 - f_2) \frac{\partial \Phi}{\partial r_2} + \left(\frac{\partial g_1}{\partial r_2} - \frac{\partial f_1}{\partial r_2} p_1 \right) \frac{\partial \Phi}{\partial p_1}. \qquad (16)$$

If $B \neq 0$, then relation (14) is a new differential relation, from which we can explicitly define p_2; in this case we arrive at the arbitrary constant choice. Therefore, for a functional arbitrary choice the linear equation

$$A = 0, \qquad B = 0. \qquad (17)$$

must be satisfied.

Thus, the following statement is valid:

For differential relation (3) to admit of an arbitrary functional choice it is necessary that Φ satisfy one of the two systems:

$$\left.\begin{aligned} & \frac{\partial \Phi}{\partial p_2} = 0, \\ & f_1 \frac{\partial \Phi}{\partial x_1} + \frac{\partial \Phi}{\partial x_2} + g_1 \frac{\partial \Phi}{\partial r_1} + g_2 \frac{\partial \Phi}{\partial r_2} + \\ & \qquad + \left[\frac{\partial g_1}{\partial x_1} + \left(\frac{\partial g_1}{\partial r_1} - \frac{\partial f_1}{\partial x_1} \right) p_1 - \frac{\partial f_1}{\partial r_1} p_1^2 \right] \frac{\partial \Phi}{\partial p_1} = 0, \\ & (f_1 - f_2) \frac{\partial \Phi}{\partial r_2} + \left(\frac{\partial g_1}{\partial r_2} - \frac{\partial f_1}{\partial r_2} p_1 \right) \frac{\partial \Phi}{\partial p_1} = 0 \end{aligned}\right\} \qquad (18)$$

or

$$\left.\begin{aligned} &\frac{\partial\Phi}{\partial p_1}=0, \\ &f_2\frac{\partial\Phi}{\partial x_1}+\frac{\partial\Phi}{\partial x_2}+g_1\frac{\partial\Phi}{\partial r_1}+g_2\frac{\partial\Phi}{\partial r_2}+ \\ &\qquad+\left[\frac{\partial g_2}{\partial x_1}+\left(\frac{\partial g_2}{\partial r_2}-\frac{\partial f_2}{\partial x_1}\right)p_2-\frac{\partial f_2}{\partial r_2}p_2^2\right]\frac{\partial\Phi}{\partial p_2}=0, \\ &(f_2-f_1)\frac{\partial\Phi}{\partial r_1}+\left(\frac{\partial g_2}{\partial r_1}-\frac{\partial f_2}{\partial r_1}p_2\right)\frac{\partial\Phi}{\partial p_2}=0. \end{aligned}\right\} \quad (19)$$

Differential relation (3) corresponding to the condition $\partial\Phi/\partial p_2 = 0$ will be referred to as the first-order relation, and relation (3) corresponding to the condition $\partial\Phi/\partial p_1 = 0$ — as the second-order relation.

Again we have reduced the problem of investigating consistency to a standard problem for a system of linear homogeneous equations.

The study of systems (18) and (19) is carried out with the aid of the familiar algorithm for the formation of Poisson's brackets. Since each of the systems is considered, essentially, in a five-dimensional space: the second and third equations of (18) -- in the space $(x_1, x_2, r_1, r_2, p_1)$, the second and third equations of (19) -- correspondingly in the space $(x_1, x_2, r_1, r_2, p_2)$, then a sufficient condition for the existence of a relation with a one-functional arbitrary choice is the condition $r = 4$. (20)
where r is the rank of the complete Jacobian system formed by adjoining Poisson's brackets to equations(18), and, respectively, to equations (19).

The algorithm for defining the differential relation with a one-functional arbitrary choice can be transferred almost without modification for the case of a relation of arbitrary order. Without carrying out the operations, we state only the final result. On analogy with the preceding, the differential relation

$$\Phi(x_1, x_2, r_i^0, r_i^1, \ldots, r_i^{k+1})=0, \qquad (21)$$

where the following is assumed:

$$r_i^0=r_i, \quad r_i^s=\frac{\partial r_i^{s-1}}{\partial x_1} \quad (s=1, \ldots, k+1),$$

can be of two orders: a first-order differential relation when $\frac{\partial \Phi}{\partial r_2^{k+1}} = 0$, and a second-order differential relation, when $\frac{\partial \Phi}{\partial r_1^{k+1}} = 0$.

For the case of a first-order relation function Φ satisfies the system

$$\left.\begin{aligned} & f_1 \frac{\delta \Phi}{\delta x_1} + \frac{\delta \Phi}{\delta x_2} + g_\alpha^k \frac{\partial \Phi}{\partial r_\alpha^k} + \left[\frac{\delta g_1^k}{\delta x_1} + r_1^{k+1} \left(\frac{\partial g_1^k}{\partial r_1^k} - \frac{d f_1}{d x_1} \right) \right] \frac{\partial \Phi}{\partial r_1^{k+1}} = 0 \\ & \qquad (\alpha = 1, 2), \\ & (f_1 - f_2) \frac{\partial \Phi}{\partial r_\alpha^k} + \frac{\partial g_1^k}{\partial r_\alpha^k} \frac{\partial \Phi}{\partial r_1^{k+1}} = 0, \quad (\alpha = 1, 2) \end{aligned}\right\} \quad (22)$$

where

$$\left.\begin{aligned} & \frac{\delta}{\delta x_1} = \frac{\partial}{\partial x_1} + r_\alpha^\beta \frac{\partial}{\partial r_\alpha^{\beta-1}}, \quad \frac{\delta}{\delta x_2} = \frac{\partial}{\partial x_2} + [g_\alpha^{\beta-1} - f_\alpha r_\alpha^\beta] \frac{\partial}{\partial r_\alpha^{\beta-1}}, \\ & \frac{d}{d x_1} = \frac{\partial}{\partial x_1} + r_\alpha^1 \frac{\partial}{\partial r_\alpha^0} \quad (\alpha = 1, 2;\ \beta = 1, \ldots, k+1). \end{aligned}\right\} \quad (23)$$

and the quantities g_i^s $(i = 1, 2;\ s = 1, \ldots, k)$ are defined from the recursion relations

$$g_i^s = \frac{d g_i^{s-1}}{d x_1} - \frac{d f_i}{d x_1} r_i^s, \quad s \geq 1, \quad g_i^0 = g_i. \qquad (24)$$

Here d/dx_1 is the total derivative with respect to x_1, taking account of all arguments on which functions g_1^{s-1}, f_1 depend.

The space of system (22) is a space of the valuables x_1, x_2, r_1^s, r_2^s, r_1^{k+1} $(s = 0, \ldots, k)$. Since it has the dimension $2(k + 2) + 1$, for the relation to admit of a one-functional arbitrary choice it is necessary and sufficient that the rank of the complete system corresponding to system (22) be equal to $2(k + 2)$.

Remark. Differential relations for a given system of differential equations can contain not only derivatives, but also potentials.

Let us consider the conservative system

$$\frac{\partial u_i}{\partial x_2} + \frac{\partial \varphi_i(u_1, \ldots, u_n)}{\partial x_1} = \frac{\partial u_i}{\partial x_2} + a_{i\alpha} \frac{\partial u_\alpha}{\partial x_1} = 0, \quad a_{i\alpha} = \frac{\partial \varphi_i}{\partial u_\alpha}. \qquad (25)$$

For it the potentials

$$u_i = \frac{\partial \Phi_i}{\partial x_1}, \quad \varphi_i(u_1, \ldots, u_n) = -\frac{\partial \Phi_i}{\partial x_2}, \tag{26}$$

can be introduced, satisfying the conditions

$$\frac{\partial \Phi}{\partial x_2} + \varphi_i\left(\frac{\partial \Phi_1}{\partial x_1}, \ldots, \frac{\partial \Phi_n}{\partial x_1}\right) = 0. \tag{27}$$

If we take as the starting point not system (25), but the system in potential (27), differential relations must also contain the variables $\Phi_1, \ldots, \Phi_n$. Thus, the first-order differential relation for equation (27) is of the form

$$\Psi\left(x_1, x_2, \Phi_1, \ldots, \Phi_n, \frac{\partial \Phi_1}{\partial x_1}, \ldots, \frac{\partial \Phi_n}{\partial x_1}\right) = \Psi(x_1, x_2, \Phi_1, \ldots, \Phi_n, u_1, \ldots, u_n) = 0. \tag{28}$$

We similarly discuss relations of this type in subsection 3 of section IX, chapter two.

In concluding this subsection, let us deal with the application of the concept of differential relation to several problems in the theory of linear equations. We limit ourselves to Darboux's equation:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f(x_1, x_2) u, \tag{29}$$

which plays a major role in hydrodynamics.

Setting $\partial u / \partial x_i = r_i \quad (i = 1, 2),$ (30)

we write (29) in the form of the system

$$\frac{\partial r_1}{\partial x_2} = fu, \quad \frac{\partial r_2}{\partial x_1} = fu. \tag{31}$$

The extended system for equations (30) and (31) is of the form

$$\frac{\partial r_1^s}{\partial x_2} = g_1^s, \quad \frac{\partial r_2^s}{\partial x_1} = g_2^s, \tag{32}$$

where

$$g_i^s = \frac{\partial^s}{\partial x_i^s}(fu) = \binom{s}{\alpha} \frac{\partial^{s-\alpha} f}{\partial x_i^{s-\alpha}} r_i^{\alpha-1} \quad (i = 1, 2;\ \alpha = 0, \ldots, s), \tag{33}$$

$$r_i^s = \frac{\partial r_i^{s-1}}{\partial x_i} \quad (r_i^0 = r_i,\ r_i^{-1} = u). \tag{34}$$

Since part of the higher-order derivatives is determined from the conditions of the extended system, a relation of the (k + 1)-th order can be sought for in the form $\varphi(x_1, x_2, u, r_i^0, \ldots, r_i^k) = 0 \quad (i = 1 \text{ or } 1 = 2) \qquad (35)$
Without going into details of analysis of the consistency of (35) with the equations of the extended system, let us give the final result.

For functional arbitrary choice the condition

$$\frac{\partial\varphi}{\partial r_1^k}\frac{\partial\varphi}{\partial r_2^k} = 0, \tag{36}$$

is necessary, so that relation (35) will be of the first or second order, depending on whether condition $\frac{\partial\varphi}{\partial r_2^K} = 0$ or $\frac{\partial\varphi}{\partial r_1^K} = 0$ is satisfied. Let us assume for concreteness that $\frac{\partial\varphi}{\partial r_2^K} = 0$ (first-order relation). The following theorem is valid:

Theorem. For the first-order relation (35) to admit of functional arbitrary choice, it is necessary and sufficient that φ not depend on u, r_2^s (s = 0, ..., k) and that it satisfies the linear homogeneous system

$$\left(\frac{\partial^a f}{\partial x_1^a}\right)\frac{\partial\varphi}{\partial r_1^a} = 0 \quad (a = 0, 1, \ldots, k). \tag{37}$$

$$\frac{\partial\varphi}{\partial x_2} + P_a\frac{\partial\varphi}{\partial r_1^a} = 0 \quad (a = 0, 1, \ldots, k), \tag{38}$$

where

$$P_0 = 0, \quad P_s = \binom{s}{a}\frac{\partial^{s-a} f}{\partial x_1^{s-a}} r_i^{a-1} = \binom{s}{a+1}\frac{\partial^{s-a-1} f}{\partial x_1^{s-a-1}} r_i^a \ (s = 1, \ldots, k). \tag{39}$$

An analogous statement is valid also for the second-order relation. Conditions for the consistency of system (37), (38) lead to conditions on the function f.

The conditions for the existence of first-order relation*)

$$\varphi(x_1, x_2, r_1) = 0 \tag{40}$$

is of the form $\qquad f = 0, \qquad (41)$

i.e., the Laplace invariant tends to zero, and Darboux's equation converts to an equation of oscillations.

---

*) The order of the relation is established for equation (29) with respect to the function u.

For a second-order relation $\varphi(x_1, x_2, r_1, r_1^1) = 0$ (42)
to exist, it is necessary and sufficient that

$$f = \frac{\partial^2 \ln f}{\partial x_1 \partial x_2}. \tag{43}$$

This means that by the Laplace transformation (cf [30]) we can reduce equation (29) to the form

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0. \tag{44}$$

These criteria point to the intimate connection between Laplace transformation and the differential relation method.

Obviously, the following assertion is valid: if equation (29) admits of a differential relation of (k + 1)-th order, then by the k-th Laplace transformation it can be reduced to the form (44). Since the Laplace transformation theory is group-oriented (cf [30]), this also points to the close connection between the concept of differential relation and the group properties of differential equations.

The effective construction of Riemann's function is possible for hyperbolic equations admitting of a differential relation (cf [4]). We will briefly summarize Riemann's method, confining ourselves for simplicity to Darby's equation, which is self-adjoint.

If $u(x_1, x_2)$, $v(x_1, x_2)$ are solutions to equation (29), for any domain G bounded by the curve C, the relation

$$\int_G \int (vLu - uLv)\, dx_1\, dx_2 = \oint_C X_1\, dx_1 + X_2 dx_2 = 0, \tag{45}$$

is valid, where

$$L = \frac{\partial^2}{\partial x_1 \partial x_2} - f, \tag{46}$$

$$X_1 = \frac{1}{2}\left(u \frac{\partial v}{\partial x_1} - v \frac{\partial u}{\partial x_1}\right), \quad X_2 = \frac{1}{2}\left(v \frac{\partial u}{\partial x_2} - u \frac{\partial v}{\partial x_2}\right), \tag{47}$$

and the integral along the contour C is taken counterclockwise.

Riemann's function $R(\xi_1, \xi_2; x_1, x_2)$ is defined as function $v(\xi_1, \xi_2; x_1, x_2)$, which is -- with respect to x_1, x_2 -- the solution to equation (29) and satisfies the additional conditions:

$$\begin{aligned} R=1, \quad x_1=\xi_1, \quad & x_2 \text{ -- arbitrary} \\ R=1, \quad x_2=\xi_2, \quad & x_1 \text{ -- arbitrary} \end{aligned} \qquad (48)$$

In other words, $R(\xi_1, \xi_2; x_1, x_2)$ is the solution of (29), which tends to 1 at the characteristics PM and MN.

Suppose $u(x_1, x_2)$ is the solution of (29) for which $u, \partial u/\partial x_1, \partial u/\partial x_2$ are given at the line PN, i.e., for which Cauchy's problem is posed at the line PN.

Employing the identity (45) for the functions u, v = R for the domain G and the contour C = PMNP (Figure 1.17), we get after uncomplicated transformations

$$u(M)=\frac{u(P)+u(N)}{2}-\frac{1}{2}\left[\int_N^P\left(u\frac{\partial R}{\partial x_1}-R\frac{\partial u}{\partial x_1}\right)dx_1+\right. \\ \left.+\left(R\frac{\partial u}{\partial x_2}-u\frac{\partial R}{\partial x_2}\right)dx_2\right] \qquad (49)$$

Since $\partial u/\partial x_1$, u, R, $\partial R/\partial x_2$ are given at the curve PN, formula (49) yields the solution to Cauchy's problem for equation (29).

The explicit representation of Riemann's function for equation (29) is possible only in particular cases. Thus, it is possible for equation (29) admits of differential equations.

The case of a first-order relation is obvious. Then $f \equiv 0$, R = 1, and formula (49) is the familiar D'Alembert's formula.

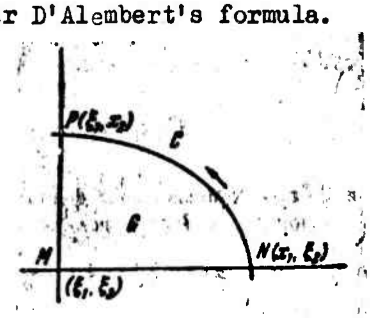

Figure 1.17

Let us consider the case when equation (29) admits of a second-order relation of the first and second kinds and when condition (43) is satisfied.

We will seek Riemann's function in the class of solutions admitting of a differential relation of the second order. The equations for the definition of the differential relation and the corresponding class of solutions are of the form:–

$$\frac{\partial u}{\partial x_1} = r_1, \quad \frac{\partial u}{\partial x_2} = r_2, \tag{50}$$

$$\frac{\partial r_1}{\partial x_2} = fu, \quad \frac{\partial r_2}{\partial x_1} = fu, \tag{51}$$

$$\frac{\partial r_1}{\partial x_1} = \Psi(x_1, x_2, r_1). \tag{52}$$

Equations for the consistency of the first of the equations (51) and (52) leads to a system for Ψ:

$$\frac{\partial \Psi}{\partial x_2} = f r_1, \tag{53}$$

$$\frac{\partial \Psi}{\partial r_1} = \frac{\partial \ln f}{\partial x_1}, \tag{54}$$

which is consistent given the condition

$$f = \frac{\partial^2 \ln f}{\partial x_1 \partial x_2}. \tag{55}$$

Assuming condition (55), expression

$$d\Psi = f r_1\, dx_2 + \frac{\partial \ln f}{\partial x_1} dr_1 \tag{56}$$

is the complete differential, and for ψ we obtain the expression

$$\Psi = r_1 \frac{\partial \ln f}{\partial x_1} + F(x_1), \tag{57}$$

where $f(x_1)$ is an arbitrary function.

Taking (52) into account, for r_1 we get the equation

$$\frac{\partial r_1}{\partial x_1} - \frac{\partial \ln f}{\partial x_1} r_1 = F(x_1), \tag{58}$$

whose integral is of the form $r_1(x_1, x_2) = C(x_1, x_2)\, f(x_1, x_2)$, (59)
where

$$[illegible] \qquad (60)$$

and $g(x_2)$ is an arbitrary function.

Let us define $F(x_1)$, starting from the condition

$$u(x_1, \xi_2) - \xi_2 + \frac{\partial u}{\partial x_1}(x_1, \xi_2) - r_1(x_1, \xi_2) = 0. \qquad (61)$$

From (61) it follows that $C(x_1, \xi_2) = 0$. (62)

This is possible only in the case when $F(x_1) = 0$, $g(\xi_2) = 0$, (63)

so that $r_1(x_1, x_2)$ can be represented as

$$r_1(x_1, x_2) = g(x_2) f(x_1, x_2). \qquad (64)$$

Taking the first of the equations (50) into account, we find

$$u(x_1, x_2) = g(x_2) \int_{\xi_1}^{x_1} f(x_1, x_2)\, dx_1 + K(x_2). \qquad (65)$$

Since $u(x_1, \xi_2) = 1$, $g(\xi_2) = 0$, then $K(\xi_2) = 1$. (66)

Bearing in mind that $u(\xi_1, x_2) = 1$, we find $K(x_2) = 1$ (67)

and for $u(x_1, x_2)$ we obtain the representation

$$u(x_1, x_2) = g(x_2) \int_{\xi_1}^{x_1} f(x_1, x_2)\, dx_1 + 1. \qquad (68)$$

Satisfying the first of the equations (51), we derive an equation for $g(x_2)$:

$$\frac{dg}{dx_2} + g \frac{\partial \ln f}{\partial x_2} = g \int_{\xi_1}^{x_1} f(x_1, x_2)\, dx_1 + 1. \qquad (69)$$

Taking condition (55) into account, equation (69) can be rewritten as

$$g'(x_2) + l(\xi_1, x_2)\, g(x_2) = 1, \qquad (70)$$

where

$$l(x_1, x_2) = \frac{\partial \ln f}{\partial x_2}, \quad f = \frac{\partial l}{\partial x_1}. \qquad (71)$$

From this we get an expression for $g(x_2)$:

$$g(x_2) = g(\xi_1, \xi_2, x_2) = \frac{\int_{\xi_2}^{x_2} f(\xi_1, x_2)\, dx_2}{f(\xi_1, x_2)} \tag{72}$$

and for Riemann's function:

$$s(x_1, x_2) = R(\xi_1, \xi_2; x_1, x_2) = \frac{\int_{\xi_2}^{x_2} f(\xi_1, x_2)\, dx_2 \int_{\xi_1}^{x_1} f(x_1, x_2)\, dx_1}{f(\xi_1, x_2)} + 1. \tag{73}$$

Noting that function f, which is the solution to equation (55), is of the form (cf [30])

$$f = 2\frac{\alpha_1'(x_1)\,\alpha_2'(x_2)}{[\alpha_1(x_1) + \alpha_2(x_2)]}. \tag{74}$$

we finally obtain

$$R(\xi_1, \xi_2; x_1, x_2) = 2\frac{[\alpha_2(x_2) - \alpha_2(\xi_2)]\,[\alpha_1(x_1) - \alpha_1(\xi_1)]}{[\alpha_1(\xi_1) + \alpha_2(\xi_2)]\,[\alpha_1(x_1) + \alpha_2(x_2)]} + 1. \tag{75}$$

Section XIII. Group Properties of Differential Equations

The task of seeking an multiplying solution is closely bounded up with the group properties of differential equations. The fact of simple waves is enveloping center waves, and the latter of self-modeling, i.e., invariant relative to the homothetic transformation $x^1 \longrightarrow kx^1$, $x^2 \longrightarrow kx^2$ in the plane x^1, x^2, is group in origin.

The possibility of reducing quasilinear equations to a form suitable for obtaining simple waves can be found after analyzing their group properties.

The long-known self-modeling solutions of one-dimensional gas dynamics essentially were derived by a group analysis, whose specific form is dimension.

Group analysis enables us to construct regular algorithms to find particular solutions without involving additional considerations, based only on the given system of differential equations.

1. One-parametric Lie group. Consider in the n-dimensional space X_n of variables $\{x^1, \ldots, x^n\}$ a system of ordinary differential equations

$$\frac{dx^i}{dt} = \xi^i(x^1, \ldots, x^n). \tag{1}$$

for which Cauchy's problem $x^i(t_o) = x^i_o$. (2)
can be formulated. Given sufficiently smooth functions $\xi^i(x^1, \ldots, x^n)$ and on the condition that $\sum (\xi^i)^2 > 0$, problem (1), (2) has a unique solution

$$x^i(t) = f^i(x^1_0, \ldots, x^n_0, t). \tag{3}$$

which is the sufficient number of times of the differentiable function of the initial values x^i_o and parameter t in some domain of variation of parameters x^i_o and t.

Problem (1), (2) can be given in the following geometric interpretation: the parameter t is time, the curve $x^i(t) = f^i(x^1_o, \ldots, x^n_o, t)$ represent the trajectories of some steady flow, $x^1_o, \ldots, x^n_o$ are Lagrangian coordinates, and $x^1, \ldots, x^n$ are Euler coordinates.

Solution (3) to problem (1), (2) as the property of invariancy relative to displacement in terms of parameter t: if in the problem (1), (2) we preserve the initial x^i_c values and replace t_o with $t_o + \tau$, then it will have the solution

$$x^i_1(t) = f^i(x^1_0, \ldots, x^n_0, t - \tau). \tag{4}$$

Formula (3) can be symbolically represented as

$$x(t) \Rightarrow S(t, t_0)\, x(t_0). \tag{5}$$

where $S(t, t_o)$ is an operator converting $x(t_o)$ to $x(t)$. Due to the invariancy of the solution relative to displacement in t, operator $S(t, t_o)$ has the property

$$S(t, t_0) = S(t - t_0, 0) = S(t - t_0). \tag{6}$$

Therefore, after solving Cauchy's problem with the initial time instants $t = t_o$, t_1, t_2, we arrive at the property of composition

$$S(t_1+t_2)=S(t_2)S(t_1)=S(t_1)S(t_2). \qquad (7)$$

To this we must add the property of continuous fitting of the solution to the initial data $S(\tau) \longrightarrow E \quad (\tau \longrightarrow 0)$ (8)
and the property of inversibility $S(\tau)\,S(-\tau) = E,$ (9)
where E is the identity operator.

The totality of operators (transforms) S(t) exhibiting the properties (7) - (9) forms, by definition, a one-parametric continuous group (Lie group).

We will call $\xi = \{\xi^1, \ldots, \xi^n\}$ the direction vector of the one-parametric Lie group and refer to the group with the direction vector ξ by the symbol $G_1(\xi)$.

2. Invariance of the group. The scalar function $F(x) = F(x^1, \ldots, x^n)$ is called the invariant of the group $G_1(\xi)$ if $F(Sx) \equiv F(x)$ (1)
for any transformation $S \in G_1(\xi)$.

Let us introduce the concept of the Lie derivative of the function F(x). For x^i the Lie derivative, by definition, is the quantity

$$\frac{\delta x^i}{\delta t}=\xi^i \quad (i=1, \ldots, n). \qquad (2)$$

The Lie derivative of the function F(x) is defined by the rule of differentiation of a complex function:

$$\frac{\delta F(x)}{\delta t}=\frac{\partial F}{\partial x^\alpha}\frac{\delta x^\alpha}{\delta t}=\xi^\alpha\frac{\partial F}{\partial x^\alpha} \quad (\alpha=1, \ldots, n). \qquad (3)$$

The Lie derivative $\delta F(x)/\delta t$ is none other than the derivative of the function F(x) along the projectory relative to parameter t.

The differential operator

$$L(\xi)=\xi^\alpha\frac{\partial}{\partial x^\alpha} \quad (\alpha=1, \ldots, n) \qquad (4)$$

will be called the infinitesimal operator (in the following, simply the operator) of the group $G_1(\xi)$ and we will state that the operator $L(\xi)$ generates the group of finite transformations $G_1(\xi)$.

Thus, the Lie derivative of the function F(x) is none other than the results of applying the operator $L(\xi)$ to F(x). It [?] not difficult to see that F(x) is the invariant of $G_1(\xi)$ if and only if the Lie derivative of the function F(x) is equal to zero:

$$\frac{dF}{dt} = L(\xi)F = 0.$$

The expression

$$\delta F = \left(\xi^\alpha \frac{\partial F}{\partial x^\alpha}\right)\delta t$$

will be called the Lie differential of the function F. In particular, $\delta x^i = \xi^i \delta t$ is the Lie differential of the function x^i.

We will understand the differential dx^i to refer to the expression

$$dx^i = \frac{\partial f^i}{\partial x_0^\alpha} dx_0^\alpha \quad (\alpha = 1, \ldots, n). \qquad (7)$$

computed for a fixed t.

Similarly, the differential of the function F(x) is defined:

$$dF = \frac{\partial F}{\partial x^\alpha} dx^\alpha = \frac{\partial F}{\partial x^\alpha}\frac{\partial f^\alpha}{\partial x_0^\beta} dx_0^\beta \quad (\alpha, \beta = 1, \ldots, n). \qquad (8)$$

It is not dificult to see that the operators δ and d are permutable, so that the relation

$$\delta dx^i = d\delta x^i, \qquad (9)$$

is valid. The manifold Φ given by the equation

$$F^l(x^1, \ldots, x^n) = C^l \quad (l = 1, \ldots, q), \qquad (10)$$

is called the invariant manifold relative to $G_1(\xi)$ if the transformations $S \in G_1$ translate the points of the manifold again to points of the manifold.

For the invariancy Φ, it is necessary and sufficient that the vector at the points Φ touch Φ, i.e., that the condition

$$\xi^\alpha \frac{\partial F^l}{\partial x^\alpha} = LF^l = 0 \quad (l = 1, \ldots, q; \alpha = 1, \ldots, n). \qquad (11)$$

be satisfied that the manifold (10). If an arbitrary point on the manifold

Φ is not invariant relative to G_1, while at the same time the manifold Φ is invariant, then we say that the group G_1 induces the continuous group of transformations at the manifold Φ. Let us assign Φ by the parametric equations

$$x^i = f^i(v^1, \ldots, v^p) \quad (p = n - q; \ i = 1, \ldots, n). \tag{12}$$

Then the group $G_1(\xi)$ induces in the space $v = v^1, \ldots, v^p$ the group $g_1(\eta)$, where the direction vector $\eta = \{\eta^1, \ldots, \eta^p\}$ is related with the direction vector $\xi = \{\xi^1, \ldots, \xi^n\}$ by the relation

$$\xi^i = \frac{\delta x^i}{\delta t} = \frac{\partial f^i}{\partial v^\beta} \cdot \frac{\delta v^\beta}{\delta t} = \frac{\partial f^i}{\partial v^\beta} \eta^\beta \quad (i = 1, \ldots, n; \ \beta = 1, \ldots, p). \tag{13}$$

which derive from equality (12).

Suppose

$$\Omega(x, dx) = A_\alpha(x^1, \ldots, x^n) dx^\alpha \tag{14}$$

is a linear differential form. Then the Lie differential of the form Ω is, by definition, the expression

$$\delta\Omega = \delta(A_\alpha dx^\alpha) = \delta A_\alpha dx^\alpha + A_\alpha \delta dx^\alpha. \tag{15}$$

employing the commutativity of operators d and δ, we have

$$\delta\Omega = \delta A_\alpha dx^\alpha + A_\alpha d\delta x^\alpha = \left[\frac{\partial A_\alpha}{\partial x^\beta} \xi^\beta dx^\alpha + A_\alpha d\xi^\alpha\right] \delta t \quad (\alpha, \beta = 1, \ldots, n). \tag{16}$$

The Lie derivative of form Ω, by definition, is the expression

$$\Lambda(x, \xi, dx, d\xi) = \frac{\partial A_\alpha}{\partial x^\beta} \xi^\beta dx^\alpha + A_\alpha d\xi^\alpha, \tag{17}$$

which itself is a linear differential form.

We will call the form Ω invariant relative to $G_1(\xi)$ if its Lie derivative, i.e., the form $\Lambda(x, \xi, dx, d\xi)$, is equal to zero.

Suppose the group $G_1(\xi)$ leaves the form

$$\Omega^i = A^i_\alpha(x)\,dx^\alpha \quad (i = 1, \ldots, r;\ \alpha = 1, \ldots, n) \tag{18}$$

invariant, as well as the manifold Φ given either by equations (10) or by parametric equations (12).

The forms Ω^i convert on Φ to the forms

$$\omega^i = a^i_\beta\,dv^\beta, \quad a^i_\beta = A^i_\alpha \frac{\partial x^\alpha}{\partial v^\beta} \quad (i = 1, \ldots, r;\ \alpha = 1, \ldots, n;\ \beta = 1, \ldots, p), \tag{19}$$

which remain invariant relative to the group $g_1(\eta)$ induced by the group $G_1(\xi)$ on Φ.

By definition, the Lie derivatives of the forms ω^i are equal to zero, i.e., the equalities

$$\Lambda^i(v, \eta; dv, d\eta) = \frac{\partial a^i_\beta}{\partial v^\gamma}\eta^\gamma dv^\beta + a^i_\beta\,d\eta^\beta = 0 \quad (i = 1, \ldots, r;\ \beta, \gamma = 1, \ldots, p). \tag{20}$$

are satisfied. Suppose $G_1(\xi_1)$, $G_2(\xi_2)$, . . . , $G_p(\xi_p)$ are one-parametric groups of transformations in the space X_n. We will seek the combined invariants of these groups. If $F(x) = F(x^1, \ldots, x^n)$ is the invariant of $G_1(\xi_1)$, $G_2(\xi_2)$, . . . , $G_p(\xi_p)$, then the relations

$$\left.\begin{array}{l} L_1F = \xi^\alpha_1 \dfrac{\partial F}{\partial x^\alpha} = 0, \\ L_2F = \xi^\alpha_2 \dfrac{\partial F}{\partial x^\alpha} = 0, \\ \cdots\cdots\cdots\cdots \\ L_pF = \xi^\alpha_p \dfrac{\partial F}{\partial x^\alpha} = 0 \\ (\alpha = 1, \ldots, n). \end{array}\right\} \tag{21}$$

must obtain. Suppose system (21) is complete, i.e., the operators $[L_i, L_j]$ are linear combinations of $L_i (i = 1, 2, \ldots, p)$:

$$[L_i, L_j] = c^{\alpha}_{ij} L_{\alpha} \quad (i, j, \alpha = 1, \ldots, p).$$

If the coefficients c^{α}_{ij} are constant and satisfy the conditions

$$c^{\alpha}_{ij} = -c^{\alpha}_{ji}, \quad c^{\alpha}_{ij} c^{l}_{\alpha k} + c^{\alpha}_{ki} c^{l}_{\alpha j} + c^{\alpha}_{jk} c^{l}_{\alpha i} = 0,$$

then the totality of operators $L_1, \ldots, L_p$ generates the p-parametric Lie group in such a way that any operator

$$L(\xi) = \lambda_1 L_1(\xi_1) + \ldots + \lambda_p L_p(\xi_p)$$

with constants coefficient $\lambda_1, \ldots, \lambda_p$ generates a one-parametric group (cf [31]).

3. Extended group. Thus far all variables x^{α} were equivalent. Let us define some space $X_n = \{x^{\alpha}\}$ corresponding to the separation in the points space $X_{m+2}(x^1, x^2, u^1, \ldots, u^m)$ of the two-dimensional manifolds

$$u^l = u^l(x^1, x^2) \quad (l = 1, \ldots, m). \tag{1}$$

Point $x = \{x^1, \ldots, x^n\} \in X_n$ will be defined by the coordinates

$$x^s = x^s, \quad x^{2+l} = u^l, \tag{2a}$$

$$x^{2+m+l} = \frac{\partial u^l}{\partial x^1} = p^l_1, \quad x^{2+2m+l} = \frac{\partial u^l}{\partial x^2} = p^l_2 \tag{2б}$$

$$(s = 1, 2; \; l = 1, \ldots, m; \; n = 3m + 2).$$

Among all transformations in the space X_n, we separate out the transformations that leave the following forms invariant:

$$\Omega^l(x, dx) = dx^{2+l} - x^{2+\beta m+l} dx^{\beta} \quad (l = 1, \ldots, m; \; \beta = 1, 2). \tag{3}$$

We will call these transformations tangential. Relations (2b) remained unchanged for them.

Among the tangential transformations we single out the subclass of extended transformations, which are characterized by the fact that the subspace $X_{m+2} = \{x^1, x^2, x^3, \ldots, x^{m+2}\} \subset X_n$ remains invariant. We will call these transformations in X_{m+2} point transformations.

Let us show that the point transformations completely define the extended transformations.

Suppose $\xi = \{\xi^1, \ldots, \xi^{m+2}\}$ is the direction vector of the point transformation $G_1(\xi)$ at X_{m+2}, and $\bar{\xi} = \{\bar{\xi}_1, \ldots, \bar{\xi}^{m+2}, \bar{\xi}^{m+3}, \ldots \bar{\xi}^{3m+2}\}$ is the direction vector of the corresponding extended transformation $\bar{G}_1(\xi)$.

From the invariancy of X_{m+2} it follows that

$$\bar{\xi}^l = \xi^l \quad (l = 1, \ldots, m+2). \tag{4}$$

From the condition of the invariancy Ω relative to $\bar{G}_1(\xi)$, we find

$$d\xi^{2+l} - x^{2+\beta m+l} d\xi^\beta - \bar{\xi}^{2+\beta m+l} dx^\beta = 0 \quad (l = 1, \ldots, m;\ \beta = 1, 2). \tag{5}$$

From this it follows that

$$\bar{\xi}^{2+sm+l} = \frac{\partial \xi^{2+l}}{\partial x^s} + \frac{\partial \xi^{2+l}}{\partial u^\alpha} p_s^\alpha - p_\beta^l \left[\frac{\partial \xi^\beta}{\partial x^s} + \frac{\partial \xi^\beta}{\partial u^\alpha} p_s^\alpha\right] \tag{6}$$
$$(s, \beta = 1, 2;\ l, \alpha = 1, \ldots, m).$$

The statement is proven.

Finally, we can consider the further extensions of the group in the space of the components of derivatives of second and higher orders.

4. Proofs of transformations admissible by the system of differential equations. For simplicity of presentation, we will limit ourselves to quasilinear equations. Using the notations (13.3.2), let us write the system of quasilinear equations

$$\frac{\partial u^l}{\partial x^2} + a_\alpha^l (x^1, x^2; u^1, \ldots, u^m) \frac{\partial u^\alpha}{\partial x^1} = f^l (x^1, x^2; u^1, \ldots, u^m) \tag{1}$$
$$(l, \alpha = 1, \ldots, m)$$

in the form of equations in differentials:

$$\Omega^i(x, dx) = dx^{2+i} - x^{2+\beta m+i}\, dx^\beta = 0 \quad (i = 1, \ldots, m;\ \beta = 1, 2) \tag{2}$$

and finite relations:

$$F^i(x^1, \ldots, x^n) = x^{2m+2+i} + a_\alpha^i x^{m+2+\alpha} - f^i = 0 \quad (i, \alpha = 1, \ldots, m). \tag{3}$$

Then the integral manifold of the system (2), (3) can be determined as the surface in the space $X_n = X_{3m+2}$ of variables $x^1, \ldots, x^{3m+2}$:

$$x^i = x^i(x^1, x^2) \quad (i = 1, \ldots, n = 3m+2), \tag{4}$$

on which the forms Ω^i tend to zero and which itself lies on the manifold Φ given by equation (3).

Definition. System (1) admits of the group $G_1(\xi)$ in the space X_{m+2} of variable $x^i (i = 1, \ldots, m+2)$ if the extended group $\bar{G}_1(\xi)$ corresponding to it leaves invariant the manifold Φ, i.e., the group $G_1(\xi)$ translates the integral surface of the equation (1) into an integral surface. The totality of groups $G_1(\xi)$ forms a set of admissible transformations of system (1).

The condition for the invariancy of manifold Φ is of the form:

$$\xi^{2m+2+i} + a_\alpha^i \xi^{m+2+\alpha} + \frac{\partial a_\alpha^i}{\partial x^\beta} \xi^\beta x^{m+2+\alpha} - \frac{\partial f^i}{\partial x^\beta} \xi^\beta = 0$$
$$(\beta = 1, \ldots, m+2;\ i, \alpha = 1, \ldots, m). \tag{5}$$

Substituting in (5) the expressions for $\bar{\xi}^{2+\beta m+i}$ from (13.3.6) and expressing by means of (3), $x^{2m+2+i} = p_2^i$ in terms of $x^{m+2+i} = p_1^i$, we arrive at the system of equations

$$A^i + A_\alpha^i p_1^\alpha + A_{\alpha_1\alpha_2}^i p_1^{\alpha_1} p_1^{\alpha_2} = 0 \quad (i, \alpha, \alpha_1, \alpha_2 = 1, \ldots, m), \tag{6}$$

where

$$\begin{aligned} & [illegible] (x^1, \ldots, x^{m+2}) \frac{\partial \xi^{\alpha}}{\partial x^{\alpha}} + b_{\beta}^{i}(x^1, \ldots, x^{m+2}) \xi^{\beta}, \\ & [illegible] (x^1, \ldots, x^{m+2}) \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} + b_{\beta}^{i_1}(x^1, \ldots, x^{m+2}) \xi^{\beta}, \\ & [illegible] (x^1, \ldots, x^{m+2}) \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} + b_{\beta}^{i_1 i_2}(x^1, \ldots, x^{m+2}) \xi^{\beta} \\ & (i, i_1, i_2 = 1, \ldots, m; \ \alpha, \beta = 1, \ldots, m+2); \end{aligned} \qquad (7)$$

the coefficients of b(x) depend on

$$a_{\alpha}^{i}(x), \ f^{i}(x), \ \frac{\partial a_{\alpha}^{i}(x)}{\partial x^{\beta}}, \ \frac{\partial f^{i}(x)}{\partial x^{\beta}} \quad (i, \alpha = 1, \ldots, m; \ \beta = 1, \ldots, m+2).$$

On the (2m + 2)-dimensional manifold Φ the independent parameters are the quantities $x^1, \ldots, x^{m+2}, p_1^1, \ldots, p_1^m$. Since the ratio (6) must be satisfied identically relative to these parameters, from (6) it follows that

$$A^{i} = 0, \ A_{\alpha}^{i} = 0, \ A_{\alpha_1 \alpha_2}^{i} + A_{\alpha_2 \alpha_1}^{i} = 0 \quad (i, \alpha, \alpha_1, \alpha_2 = 1, \ldots, m). \qquad (8)$$

Equations (8) constitute a linear homogeneous system of equations relative to $\xi^1, \ldots \xi^{m+2}$; this system is called determining and, generally speaking, is overdetermined.

The satisfaction of (8) is necessary and sufficient for the group $G_1(\xi)$ to be admissible for the given system (1).

Let us consider a number of examples.

L. V. Ovsyannikov [31], based on this algorithm, investigated a group of transformations for a system of one-dimensional plane equations of gas dynamics and the polytropic equation of state:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\\ &\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0,\\ &\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\gamma p\frac{\partial u}{\partial x}=0. \end{aligned}\right\} \tag{9}$$

System (9) for any γ value admits of the operators

$$\left.\begin{aligned} &L_1=\frac{\partial}{\partial t},\ L_2=\frac{\partial}{\partial x},\ L_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\ L_4=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\\ &L_5=-t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}-2\rho\frac{\partial}{\partial \rho},\ L_6=\rho\frac{\partial}{\partial \rho}+p\frac{\partial}{\partial p}. \end{aligned}\right\} \tag{10}$$

When $\gamma = 3$, system (9) admits of also a single independent operator:

$$L_7=t^2\frac{\partial}{\partial t}+tx\frac{\partial}{\partial x}+(x-tu)\frac{\partial}{\partial u}-t\rho\frac{\partial}{\partial \rho}-3tp\frac{\partial}{\partial p}. \tag{11}$$

The operators L_1, L_2 are operators of displacement with respect to the axes x, t; the operator L_3 corresponds to Galileo's transformation; operator L_4 is an operator of the transformation of similitude (homothety) in the x, t plane. These operators obtain for any equation of state. The operators L_5, L_6 correspond to the polytropic equation of state; operator L_7 corresponds to the specific value $\gamma = 3$.

The Euler-Poisson equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}+\frac{a}{(x_1-x_2)^2}u=0 \qquad (a=\text{const}) \tag{12}$$

admits of the operators

$$\left.\begin{aligned} &L_1=\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2},\quad L_2=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2},\\ &L_3=x_1^2\frac{\partial}{\partial x_1}+x_2^2\frac{\partial}{\partial x_2},\quad L_4=u\frac{\partial}{\partial u},\quad L_5=\varphi(x_1,x_2)\frac{\partial}{\partial u}. \end{aligned}\right\} \tag{13}$$

where $\varphi(x_1, x_2)$ is the arbitrary solution (12). Operator L_1 is an operator of consistent displacement with respect to axes x_1, x_2; L_2 is the operator of similitude in the x_1, x_2 plane; L_3 is the operator of inversion. Operator L_5 contains in its coefficients two arbitrary functions of the same argument and generates in a totality with the operators L_1, L_2, L_3, L_4 the infinite Lie group.

Let us show, following L. V. Ovsyannikov, how the value of the group of transformation that admits by the Euler-Poisson equation (12) allows us to define for it Riemann's function.

As we know (subsuction 3 of section XII), Riemann's function is the solution $u(\xi_1, \xi_2; x_1, x_2)$ (12) satisfying the condition

$$u(\xi_1, \xi_2; \xi_1, x_2) = u(\xi_1, \xi_2; x_1, \xi_2) = 1 \qquad (14)$$

The operators L_1, L_2, L_3 generate a three-parametric group of consistent fractionally linear transformations

$$\bar{x}_1 = \frac{ax_1 + b}{cx_1 + d}, \quad \bar{x}_2 = \frac{ax_2 + b}{cx_2 + d} \qquad (15)$$

where a, b, c, d are arbitrary constants and $ad - bc \neq 0$. Since equation (12) admits of the operators L_1, L_2, L_3 then transformation (15) leaves unchanged. Employing this, let us choose the transformation (15) in such a way that to the values $x_1 = \xi_1$, $x_2 = \xi_2$ there correspond the values $x_1 = 0$, $x_2 = \infty$. This is achieved by the transformation

$$\bar{x}_1 = \frac{x_1 - \xi_1}{x_1 - \xi_2}, \quad \bar{x}_2 = \frac{x_2 - \xi_1}{x_2 - \xi_2}. \qquad (16)$$

As a result, the solution $u(\xi_1, \xi_2; x_1, x_2)$ to the problem (12), (14) changes into the solution $v(\bar{x}_1, \bar{x}_2)$ to the problem

$$\left.\begin{aligned} &\frac{\partial^2 v}{\partial \bar{x}_1 \partial \bar{x}_2} + \frac{a}{(\bar{x}_1 - \bar{x}_2)^2} v = 0, \\ &v(0, \bar{x}_2) = 1, \quad u(\bar{x}_1, \infty) = 1. \end{aligned}\right\} \qquad (17)$$

Problem (17) admits of the elongation operator L_2 and thus is self-modeling. Its solution can be sought for in the form

$$v(\bar{x}_1, \bar{x}_2) = v(\xi), \quad \xi = \frac{\bar{x}_1}{\bar{x}_2}. \qquad (18)$$

Substituting (18) into the first of the relation (17), we arrive at the equation

$$\xi v'' + v' - \frac{a}{(\xi - 1)^2} v = 0, \qquad (19)$$

where the stroke denotes differentiation relative to ξ. The desired solution $v(\bar{x}_1, \bar{x}_2) = v(\xi)$ satisfies the second and third of the condition (17), if

$$v(0) = 1. \qquad (20)$$

Let us represent a in the form $a = m(m-1)$. (21)

Equation (19) is reduced to the hypergeometric equation by the substitution

$$v = (1 - \xi)^m w \qquad (22)$$

and takes on the form

$$\xi(\xi - 1) w'' + [-1 + (2m+1)\xi] w' + m^2 w = 0. \qquad (23)$$

Parameters α, β, γ of this equation are found from the relations

$$\gamma = 1, \quad \alpha + \beta + 1 = 2m + 1, \quad \alpha\beta = m^2 \qquad (24)$$

and take on the values $\alpha = \beta = m$, $\gamma = 1$. (25)

Thus, taking (20) into account, we find $w = F(m, m, 1, \;)$, (26)

where $F(\alpha, \beta, \gamma, \xi)$ is a hypergeometric function. Next, returning to the functions $v(\xi)$, $v(\bar{x}_1, \bar{x}_2)$, $u(\xi_1, \xi_2; x_1, x_2)$, we have

$$v(\xi) = (1 - \xi)^m F(m, m; 1, \xi), \qquad (27)$$

$$v(\bar{x}_1, \bar{x}_2) = \left(1 - \frac{\bar{x}_1}{\bar{x}_2}\right)^m F\left(m, m; 1, \frac{\bar{x}_1}{\bar{x}_2}\right), \qquad (28)$$

$$u(\xi_1, \xi_2; x_1, x_2) = \left[\frac{(\xi_1 - \xi_2)(x_1 - x_2)}{(\xi_1 - x_2)(x_1 - \xi_2)}\right]^m F\left(m, m; 1, \frac{(x_1 - \xi_1)(x_2 - \xi_2)}{(x_1 - \xi_2)(x_2 - \xi_1)}\right). \qquad (29)$$

For the case of conservative systems, it is possible to construct transformations by only in the space of the extended system, but also in the space of potentials.

The conservative homogeneous system

$$\frac{\partial u^l}{\partial x^2}+\frac{\partial v^l(u^1,\ldots,u^n)}{\partial x^1}=\frac{\partial u^l}{\partial x^2}+\frac{\partial v^l}{\partial u^\alpha}\cdot\frac{\partial u^\alpha}{\partial x^1}=0 \quad (l,\ \alpha=1,\ 2,\ \ldots,\ n) \qquad (30)$$

can, by introducing the potential φ^i(cf subsection 3 of section 5), be converted to the system of equations in differentials:

$$\omega^l=d\varphi^l-u^l\,dx^1+v^l(u^1,\ \ldots,\ u^n)\,dx^2=0. \qquad (31)$$

Let us consider the operator

$$L=\xi^\beta\frac{\partial}{\partial x^\beta}+\xi^{2+\alpha}\frac{\partial}{\partial u^\alpha}+\xi^{n+2+\alpha}\frac{\partial}{\partial\varphi^\alpha} \quad (\alpha=1,\ \ldots,\ n;\ \beta=1,\ 2). \qquad (32)$$

in the space x^1, x^2, u^i, φ^i ($i = 1, \ldots, n$). To the operator L there correspond the admissible transformations, if the Lie derivative of the form ω^i equals zero:

$$\frac{\delta\omega^l}{\delta\tau}=L\omega^l=0. \qquad (33)$$

From condition (33) it follows that

$$d\xi^{n+2+l}-u^l\,d\xi^1+v^l\,d\xi^2-\xi^{2+l}\,dx^1+a^l_\alpha\xi^{2+\alpha}\,dx^2=0. \qquad (34)$$

Next let us proceed to the equations for ξ^i($i = 1, 2, \ldots, 2n+2$):

$$\left.\begin{aligned}\frac{d\xi^{n+2+l}}{dx^1}-u^l\frac{d\xi^1}{dx^1}+v^l\frac{d\xi^2}{dx^1}&=\xi^{2+l},\\ \frac{d\xi^{n+2+l}}{dx^2}-u^l\frac{d\xi^1}{dx^2}+v^l\frac{d\xi^2}{dx^2}&=-a^l_\alpha\xi^{2+\alpha},\end{aligned}\right\} \qquad (35)$$

where

$$\left.\begin{aligned}
&\frac{d}{dx^1}=\frac{\partial}{\partial x^1}+p_1^\alpha\frac{\partial}{\partial u^\alpha}+u^\alpha\frac{\partial}{\partial\varphi^\alpha},\\
&\frac{d}{dx^2}=\frac{\partial}{\partial x^2}+p_2^\alpha\frac{\partial}{\partial u^\alpha}-v^\alpha\frac{\partial}{\partial\varphi^\alpha},\\
&p_s^\alpha=\frac{\partial u^\alpha}{\partial x^s}\quad(\alpha=1,\ \ldots,\ n;\ s=1,\ 2).
\end{aligned}\right\}\tag{36}$$

The quantities p_s^i are associated, by virtue of (30), by the relations

$$p_2^i=-a_\alpha^i p_1^\alpha,\quad a_\alpha^i=\frac{\partial v^i}{\partial u^\alpha}\quad(i=1,\ \ldots,\ n).\tag{37}$$

Substituting (36) into (35), we find

$$\left.\begin{aligned}
&\frac{\partial\xi^{n+2+i}}{\partial x^1}+u^\alpha\frac{\partial\xi^{n+2+i}}{\partial\varphi^\alpha}-u^i\left(\frac{\partial\xi^1}{\partial x^1}+u^\alpha\frac{\partial\xi^1}{\partial\varphi^\alpha}\right)+\\
&+v^i\left(\frac{\partial\xi^2}{\partial x^1}+u^\alpha\frac{\partial\xi^2}{\partial\varphi^\alpha}\right)+p_1^\alpha\left(\frac{\partial\xi^{n+2+i}}{\partial u^\alpha}-u^i\frac{\partial\xi^1}{\partial u^\alpha}+v^i\frac{\partial\xi^2}{\partial u^\alpha}\right)=\xi^{2+i},\\
&\frac{\partial\xi^{n+2+i}}{\partial x^2}-v^\alpha\frac{\partial\xi^{n+2+i}}{\partial\varphi^\alpha}-u^i\left(\frac{\partial\xi^1}{\partial x^2}-v^\alpha\frac{\partial\xi^1}{\partial\varphi^\alpha}\right)+\\
&+v^i\left(\frac{\partial\xi^2}{\partial x^2}-v^\alpha\frac{\partial\xi^2}{\partial\varphi^\alpha}\right)+p_2^\alpha\left(\frac{\partial\xi^{n+2+i}}{\partial u^\alpha}-u^i\frac{\partial\xi^1}{\partial u^\alpha}+v^i\frac{\partial\xi^2}{\partial u^\alpha}\right)=\\
&\qquad=-a_\alpha^i\xi^{2+\alpha}.
\end{aligned}\right|\tag{38}$$

Requiring that (38) be satisfied identically relative to p_1^α, we arrive at a system of determining equations

$$\left.\begin{aligned}
&\frac{\partial\xi^{n+2+i}}{\partial x^1}+u^\alpha\frac{\partial\xi^{n+2+i}}{\partial\varphi^\alpha}-u^i\left(\frac{\partial\xi^1}{\partial x^1}+u^\alpha\frac{\partial\xi^1}{\partial\varphi^\alpha}\right)+\\
&\qquad+v^i\left(\frac{\partial\xi^2}{\partial x^1}+u^\alpha\frac{\partial\xi^2}{\partial\varphi^\alpha}\right)=\xi^{2+i},\\
&\frac{\partial\xi^{n+2+i}}{\partial x^2}-v^\alpha\frac{\partial\xi^{n+2+i}}{\partial\varphi^\alpha}-u^i\left(\frac{\partial\xi^1}{\partial x^2}-v^\alpha\frac{\partial\xi^1}{\partial\varphi^\alpha}\right)+\\
&\qquad+v^i\left(\frac{\partial\xi^2}{\partial x^2}-v^\alpha\frac{\partial\xi^2}{\partial\varphi^\alpha}\right)=-a_\alpha^i\xi^{2+\alpha},\\
&\frac{\partial\xi^{n+2+i}}{\partial u^\alpha}-u^i\frac{\partial\xi^1}{\partial u^\alpha}+v^i\frac{\partial\xi^2}{\partial u^\alpha}=0.
\end{aligned}\right|\tag{39}$$

5. **Partially invariant and invariant solutions.** We will seek on the manifold Φ given by equation (13.4.3) submanifolds $\varphi \in \Phi$ that are invariant relative to some totality of admissible transformations. Without delving into details, let us note the path it follows in finding invariant submanifolds and their corresponding admissible transformations.

Suppose

$$x^i = x^i(\varphi^1, \ldots, \varphi^{2m+2}) \quad (i = 1, \ldots, 3m+2) \tag{1}$$

is the parametric representation of Φ,

$$\varphi^j = \varphi^j(v^1, \ldots, v^p) \quad (j = 1, \ldots, 2m+2) \tag{2}$$

is the parametric representation of the desired manifold φ and $\bar{G}_1(\xi)$ induced on φ the group $G_1(\eta)$, $\eta = \{\eta^1, \ldots, \eta^p\}$. The correspondences (1), (2) translate the forms Ω_i from (13.4.2) to the forms

$$\omega^i = c^i_\alpha(v)\, dv^\alpha \quad (i = 1, \ldots, m;\ \alpha = 1, \ldots, p). \tag{3}$$

Here the coefficients c^i_α are known functions of $\varphi^j (j = 1, \ldots, 2m+2)$ and linear functions of $\frac{\partial \varphi^j}{\partial v^k} (k = 1, \ldots, p)$. The conditions for the invariancy of the forms ω^i relative to $G_1(\eta)$ are of the form

$$\frac{\partial c^i_\alpha}{\partial v^\beta}\eta^\beta dv^\alpha + c^i_\alpha d\eta^\alpha = 0 \quad (i = 1, \ldots, m;\ \alpha, \beta = 1, \ldots, p). \tag{4}$$

By virtue of the correspondences (1), (2) ξ and η are associated by the function

$$\xi^i = \frac{\partial x^i}{\partial z^\alpha} \cdot \frac{\partial \varphi^\alpha}{\partial v^\beta}\eta^\beta$$
$$(i = 1, \ldots, 3m+2;\ \alpha = 1, \ldots, 2m+2;\ \beta = 1, \ldots, p), \tag{5}$$

where

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \quad (\alpha, \beta = 1, \ldots, 2m+2). \tag{6}$$

In particular, when i = 1, ..., m+2, we have

$$\bar{\xi}^i = \xi^i = \frac{\partial\varphi^i}{\partial v^\beta}\pi^\beta. \qquad (7)$$

Substituting (7) into the determining system (13.4.8), we get some linear homogeneous system of equations of the first order for :

$$B^{(s)}_{\beta\gamma}\frac{\partial\pi^\beta}{\partial v^\gamma} + B^{(s)}_\beta \pi^\beta = 0 \quad (\beta = 1, \ldots, p;\ \gamma = 1, \ldots, p). \qquad (8)$$

whose coefficients B depend on φ^i and are linear relative to $\frac{\partial\varphi^i}{\partial v^\nu}$ (i = 1, ..., 2m+2; ν = 1, ..., p).

From conditions (4) we find

$$\frac{\partial c^i_j}{\partial v^\beta}\pi^\beta + c^i_\beta\frac{\partial\pi^\beta}{\partial v^j} = 0 \quad (i = 1, \ldots, m;\ j, \beta = 1, \ldots, p). \qquad (9)$$

The consistency conditions for the system (8), (9) lead to several equations for the functions $\varphi^j(v^1, \ldots, v^p)$. The solutions (integral manifolds) lying in φ are called partially invariant (cf [31]).

In the particular case when the manifold φ is an integral manifold, we have an invariant solution. Invariant solutions are a generalization of the familiar self-modeling solutions. We see that the theory of partially invariant and invariant solutions is intimately bounded up with the method of differential relations, since the equations giving manifolds φ are none other than differential relations of the first order.

L. V. Osvyannikov (cf [31]) pointed to the relation of simple and double waves with partially invariant solutions familiar in gas dynamics.

CHAPTER TWO CLASSICAL AND GENERALIZED SOLUTIONS OF ONE-DIMENSIONAL GAS DYNAMICS

Section I. General Remarks on the Mathematical Description of the Motion of Compressible Gases

1. Gas as a continuous medium. Gas is an aggregate of a large number of particles (molecules, atoms, ions) in continuous chaotic motion.

To characterize the state of gas as a given instant of time, we must specify the position and the velocity of each gas particle.

The problem of taking account of the interactions in motion of each gas particle is incredibly difficult; therefore in describing the state of a gas we use the statistical approach.

In the statistical description of the state of gas it is convenient to assume that its constituent particles continuously fill the space they occupy. Naturally, here we consider only the volumes whose dimensions are sufficiently large compared with the distances between the gas particles.

Therefore the expressions "small volume" and "infinitely small volume" of gas used in the following must be understood as being sufficiently large in the sense indicated above.

The motion of gas particles can be characterized by the number of particles of each species present in a given location in space and with a given velocity. This quantity is proportional to the distribution function, which satisfies the integro-differential equations of transport (so-called kinetic equations). The simplest example of a kinetic equation for gases is Boltzmann's equation (cf [1], [2]).

The description of the state of gas, employing distribution functions, and the solution of kinetic equation is also a very arduous task.

At the same time we know that there exist such flows that can be described with good accuracy by means of several specific distribution functions. This description is attained by employing the concept of the state of thermodynamic equilibrium as a state in which the distribution functions are wholly determinate.

Let us recall in passing some thermodynamic fundamentals (cf [3]).

Among the parameters characterizing the state of gas, some are defined only by bodies that are external relative to the gas mass under consideration and do not depend at all on the gas itself. These parameters are called external. External parameters include, for example, the volume occupied by the gas, the intensities of the external electromagnetic or gravity fields, and so on.

In contrast to the external, internal parameters are defined by the state of the gas itself (for example, gas energy, temperature, and pressure).

The state of the gas is called equilibrium if it does not change with time and also with the exchange of energy with external bodies does not occur. We stress that simple invariance of this state (steady-state) of itself does not signify that the gas is at equilibrium.

The equilibrium state is the state from which the gas cannot depart spontaneously.

If a gas present in an arbitrary state is left to itself (i.e., the exchange of energy was external by these is precluded and external parameters are fixed), then in some time interval (so-called relaxation time) it arrives at the state of equilibrium.

The exchange of energy between the gas and the external bodies occurs, first of all, by a heat transfer, and secondly, when work is done on the gas (or by the gas on external bodies). Work is done by the gas only when there is change in the external parameters a_i and for infinitely small changes in the latter is equal to the quantity $\delta W = \sum_i A_i da_i$, (1)
where A_i are the so-called generalized forces.

If δQ is the amount of the heat communicated to the gas, then the change in internal energy of the gas E (the kinetic energy of motion of the molecules plus the potential energy of molecular interaction), by the law of conservation of energy, is written in the form

$$dE = \delta Q - \delta W = \delta Q - \sum_i A_i da_i. \qquad (2)$$

As for the quantities A_i, let us note that for an arbitrary state of gas they, in addition to a_i, depend also on the position and velocities of individual gas molecules, i.e., on the microscopic state of the gas.

The issue becomes simplified if we consider the equilibrium states of the gas and infinitely small departures from them. Then based on the familiar fundamental theorem of thermodynamics, all internal parameters, including A_i, are single-valued functions of the external parameters a_i and energy (or temperature T) of the gas.

Usually, the generalized forces A_i themselves are taken as the equilibrium internal parameters. Thus, in the equilibrium state

$$A_i = A_i(T, a_1, \ldots, a_n), \qquad (3)$$
$$E = E(T, a_1, \ldots, a_n), \qquad (4)$$

and equality (2) becomes $dE = \delta Q - \sum_i A_i(T, a)\, da_i$. (5)

Relations (3) and (4) are determined by the microscopic structure of the gas under consideration and called the equations of state.

Relations (3) are called thermal equations of the state of the gas, and equation (4) is the caloric equation of state.

We will consider the case when the gas physically and chemically is homogeneous in its microscopic composition and does not interact with any fields (i.e., forces of gravity, electromagnetic fields, and so on are absent). Then the only external parameter of the gas is the volume V occupies, and the force A is the pressure p, so that the manifold of thermodynamic states is two-dimensional. Therefore $d\mathcal{E} = \delta Q - p\, dV$, (6)
If we consider a unit mass of gas, then the quantity $V = 1/\rho$ is called the specific volume, ρ is the density of gas, and the variable $\mathcal{E}$ is the specific internal energy of the gas.

According to the second law of thermodynamics, the quantity

$$dS = \frac{\delta Q}{T} = \frac{1}{T}(d\varepsilon + p\,dV) \qquad (7)$$

is the total differential of the function S = S(V, T), called the entropy per unit mass of gas.

Thus, the second law of thermodynamics is written as the equality

$$T\,dS = d\varepsilon + p\,dV, \qquad (8)$$

where p, ε , S are given by the equations

$$p = p(V, T), \qquad (9)$$
$$\varepsilon = \varepsilon(V, T), \qquad (10)$$
$$S = S(V, T). \qquad (11)$$

From the second law of thermodynamics (8) it follows as equations of state (9) and (10) are not independent, since the integrability condition of relation (8) imposes the following restriction on them:

$$\frac{\partial}{\partial V}\left(\frac{1}{T}\frac{\partial\varepsilon}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{1}{T}\frac{\partial\varepsilon}{\partial V} + \frac{p}{T}\right), \quad \text{or} \quad \frac{\partial\varepsilon}{\partial V} = T^2\frac{\partial}{\partial T}\left(\frac{p}{T}\right). \qquad (12)$$

If, for example, equation of state (9) is given, then condition (12) defines the caloric equation (10) with an accuracy up to an additive function of temperature.

Thus, in the state of thermodynamic equilibrium a gas is described by the folloing variables: density ρ — the mass contained per unit volume; specific volume V = $1/\rho$; pressure p -- force acting on unit area; ε — the internal energy per unit mass of gas; T — gas temperature; and S -- entropy per unit mass of gas.

By equalities (9) - (11), there are only two independent variables among all these thermodynamic quantities.

From equality (8) it follows that

$$\frac{\partial S}{\partial T}=\frac{1}{T}\frac{\partial \varepsilon}{\partial T}=\frac{c_V}{T};\quad \frac{\partial S}{\partial V}=\frac{1}{T}\left(\frac{\partial \varepsilon}{\partial V}+p\right). \tag{13}$$

Thus, for assigned equations of state (9) and (10), entropy S is determined with an accuracy up to the additive constant, which is canceled out if the entropy is normalized by the Nernst relation: $S \to 0$ when $T \to 0$. Equations of state of a gas (9) - (11) can be assigned also for a different selection of independent parameters, for example:

$$p=p(V,S),\quad \varepsilon=\varepsilon(V,S),\quad T=T(V,S) \tag{14}$$

or

$$\varepsilon=\varepsilon(p,V),\quad S=S(p,V),\quad T=T(p,V). \tag{15}$$

Under this assignment of equations of state, the second law of thermodynamics (8) requires that the equalities

$$\left.\begin{array}{c}\frac{\partial \varepsilon(V,S)}{\partial V}=-p(V,S),\quad \frac{\partial \varepsilon(V,S)}{\partial S}=T(V,S),\\ \frac{\partial T(V,S)}{\partial V}+\frac{\partial p(V,S)}{\partial S}=0\end{array}\right\} \tag{16}$$

or, correspondingly,

$$\left.\begin{array}{c}T(p,V)\frac{\partial S(p,V)}{\partial V}=p+\frac{\partial \varepsilon(p,V)}{\partial V};\quad T\frac{\partial S(p,V)}{\partial p}=\frac{\partial \varepsilon(p,V)}{\partial p},\\ \frac{\partial T(p,V)}{\partial p}\frac{\partial S(p,V)}{\partial V}-\frac{\partial T(p,V)}{\partial V}\frac{\partial S(p,V)}{\partial p}=\frac{\partial[T,S]}{\partial[p,V]}=1.\end{array}\right\} \tag{17}$$

be satisfied.

2. Nonequilibrium states and processes in gases. In the nonequilibrium state of a gas the fundamental concepts of thermodynamics -- temperature, pressure, and entropy -- lose their significance.

Generally, a nonequilibrium state of a gas is not described completely in terms of thermodynamic (i.e., macroscopic) concepts and requires microscopic analysis.

However, for the purposes of classical gas dynamics, the approach of nonequilibrium thermodynamics is sufficient and generally accepted. Let us imagine that a mass of gas under study is subdivided into large number of elementary cells of extremely small dimensions, each of which we will assume to be in a state of thermodynamic equilibrium.

This assumption is warranted by the fact that the relaxation time of the system decreases with decrease in its dimensions, so that for a small portion of gas it is close to zero.

Understanding the term "point" in a gas to denote a "infinitely small" volume in the sense indicated in subsection 1, we can thus introduce for each gas point at each time instant the concepts of pressure, temperature, and energy. Now they take on the meaning of functions of coordinates of point and time;

$$p = p(x_1, x_2, x_3, t); \quad T = T(x_1, x_2, x_3, t); \quad S = S(x_1, x_2, x_3, t).$$

As for density $\rho(x_1, x_2, x_3, t)$ and energy $\varepsilon = \varepsilon(x_1, x_2, x_3, t)$, these quantities obviously retain their significance regardless of our assumption.

And thus, the nonequilibrium status of a gas is understood as the absence of equilibrium between individual gas particles, each of which itself is at equilibrium.

From the assumption of equilibrium of small gas portion it follows that the functions $p(V, T)$, $\varepsilon(V, T)$, and $S(V, T)$ satisfy the equation of state (1.1.9) - (1.1.11).

Thus, when modified these thermodynamic parameters satisfy the equations of state of a gas which are defined for an equilibrium gas. This process is called an equilibrium or reversible process.

The above discussion on relaxation time, however, does not afford a grasp of the limits of applicability of thermodynamic concepts. This appreciation can be realized on the basis of a more general gas model, the statistical model.

Macroscopic consideration leads to the conclusion that thermodynamic concepts of temperature and entropy are meaningful if the changes in parameters characterizing a gas state for lengths of the order of the length of the free path of a gas molecule in space and for times of the order of time between molecular collisions are small compared with these latter quantities.

Let us discuss, as an example, one-dimensional nonequilibrium gas flow. Suppose u(x, t) is the gas velocity. One of the approximations reducing Boltzmann's equation to gas dynamics equations assumes that temperature T and pressure p satisfying the equations of state (1.1.9) and (1.1.10) can be introduced for the treatment of a gas, but the momentum and energy flows are defined also by nonequilibrium components associated with molecular diffusion. For a materially fixed gas particle in a momentum flow, instead of the pressure p we introduce the quantity

$$\bar{p} = p - \mu \operatorname{div} u = p - \mu \frac{\partial u}{\partial x}, \qquad (1)$$

and the energy flow is defined by the quantity

$$\bar{p}u - \varkappa \operatorname{grad} T = pu - \mu u \frac{\partial u}{\partial x} - \varkappa \frac{\partial T}{\partial x} \qquad (2)$$

In this approximation the function p(V, T) satisfies the equation of state (11.9); μ and $\varkappa$ are the coefficients of viscosity and thermoconductivity, respectively, proportional to the free path length of the gas molecules.

Hence it follows that if for path length and for times of the order of the time between collisions changes in thermodynamic quantities are small, then the approximation of equilibrium thermodynamics is valid.

Actually, for a large number of problems in the dynamics of gases and liquids, this requirement can be regarded as met. In this case process is quasiequilibrium in nature and we can introduce temperature and entropy, which will satisfy all thermodynamic relations with a high degree of accuracy.

On the other hand, as we will see in this chapter, zones of abrupt and rapid change in the quantities characterizing the flow arise in flows of gases and liquids. In these regions we can no longer neglect nonequilibrium components in momentum and energy flows. However, these zones have dimensions of the order of the length of the free path of gas molecules. Therefore if this length is small compared with the characteristic dimensions of a problem, we can represent the zone of nonequilibrium status as a discontinuity surface partitioning the zones of smooth change in flow parameters.

In this approach we assume that these parameters everywhere satisfy thermodynamic relations, and that the conditions for continuity of flows of mass, momentum, and energy must be satisfied at the discontinuity surfaces. Thus, we will treat of discontinuous flows in gases and liquids whose viscosity and thermal conductivity are sufficiently small.

In this chapter we will consider principally exactly this case.

Finally, for the case of sufficiently large coefficients of viscosity and thermal conductivity we must consider nonequilibrium components in momentum and energy flows. It may be the case that this consideration will prove insufficient and we must bring in Boltzmann's integro-differential equation.

Thus, there exist flows in which temperature and entropy retain their thermodynamic definitions, and all thermodynamic relations are satisfied; here the regions of abrupt variation are treated as discontinuity surfaces of flow parameters.

This chapter then is devoted to this type of flow.

In an equilibrium process the following relation holds:

$$\frac{dS}{dt}=\frac{1}{T}\left(\frac{d\varepsilon}{dt}+p\frac{dV}{dt}\right) \quad (3)$$

or

$$\frac{dS}{dt}=\frac{1}{T}\frac{dQ}{dt}, \quad (4)$$

where dQ/dt is the heat inflow velocity to the gas portion under study.

If the gas portion under study is thermally insulated ($dQ = 0$), the equilibrium process is called adiabatic. For an adiabatic process

$$dS/dt = 0. \quad (5)$$

Relation (4) does not obtain for a nonequilibrium process, and by the second law, for a thermally insulated system $dS/dt > 0$. (6)

Suppose the mass of gas participates in a nonequilibrium process, by exchanging heat with external bodies. In this case the second law of thermodynamics demands that the condition

$$\frac{dS}{dt}+\frac{dS_e}{dt}>0, \tag{7}$$

be satisfied, where S_e is the entropy of the external bodies. The quantity dS_e/dt can be considered as the flow of entropy from external bodies through the gas mass.

We demonstrate the calculation of entropy flow dS_e/dt with the example of the exchange of heat by a gas with a thermostat at constant temperature T_0. In this case

$$\frac{dS_e}{dt}=-\frac{1}{T_0}\frac{dQ}{dt}, \tag{8}$$

where dQ/dt is the amount of heat flowing from external bodies to the gas portion under consideration. Therefore for the case of a thermostat as the external body, the second law of thermodynamics requires that

$$\frac{dS}{dt}>\frac{1}{T_0}\frac{dQ}{dt}. \tag{9}$$

These considerations will be employed in our analysis of an isothermal gas in section IV.

3. Methods of describing flows. Eulerian and Lagrangian variables. The flow of a continuous medium can be described by two different methods.

In the first method, at each time instant we determine the parameters of the gas state as a function of the coordinates x_1, x_2, x_3 of a point in some fixed coordinate system. Thus, $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3, t)$ signifies under this method of description the velocity of a particle present at time instant t at the point (x_1, x_2, x_3). Analogously, all the remaining variables characterize the state of the gas particle present at time t at the point (x_1, x_2, x_3).

This method of describing the motion of a continuous medium is called Eulerian, and the coordinates x_1, x_2, x_3 are called Eulerian coordinates.

Another method of description, called Lagrangian, presupposes the assignment of thermodynamic quantities and velocity u of the gas for each particle as functions of time t.

Suppose we distinguish the gas particle from other particles by means of searching parameters y_1, y_2, y_3. Then we seek all variables characterizing the flow as functions of variables y_1, y_2, y_3, t.

Under this method of description, for example, the vector $\mathbf{u}(y_1, y_2, y_3)$ for fixed y_1, y_2, y_3 denotes the rate of translation in space of a wholly specific gas particle. The coordinates y_1, y_2, y_3 are called Lagrangian.

In most cases, we select as the coordinates y_1, y_2, y_3 the Eulerian coordinates of the point at which the gas particle exists at any specific time instant, for example, at the instant $t = 0$.

If we adopt this choice of Lagrangian coordinates y_1, y_2, y_3, then we can easily compute position of the particle also at the time instant $t \neq 0$. Since for fixed y_1, y_2, y_3 the velocity $\mathbf{u}(y_1, y_2, y_3, t)$ is the particle velocity, then

$$x_i = y_i + \int_0^t u_i(y_1, y_2, y_3, \tau)\, d\tau \quad (i = 1, 2, 3). \tag{1}$$

Here $x_i = x_i(y_1, y_2, y_3, t)$ are the coordinates at time instant t (Eulerian coordinates) of a particle which at time instants $t = 0$ is at a point with coordinates $x_i = y_i$; $u_i(y_1, y_2, y_3, t)$ are the components of the velocity vector $\mathbf{u}(y_1, y_2, y_3, t)$.

Formulas (1) thus establish a relation between Lagrangian coordinates y_i of the particle and its Eulerian coordinates.

Suppose $f(x_1, x_2, x_3, t)$ is any function of the Eulerian coordinates, and $\bar{f}(y_1, y_2, y_3, t)$ is the representation of the same function in the Lagrangian coordinates. Then, according to (1),

$$f\left(y+\int_0^t u(y,\tau)\,d\tau,\ t\right)=\bar{f}(y,\ t), \tag{2}$$

where for simplicity we denote

$$y=\{y_1,\ y_2,\ y_3\};\quad x=\{x_1,\ x_2,\ x_3\};\quad u=\{u_1,\ u_2,\ u_3\}.$$

Differentiating (2) relative to variable t (naturally, here we assume the differentiability of f), we obtain

$$\frac{\partial \bar{f}(y,t)}{\partial t}=\frac{\partial f(x,t)}{\partial t}+u(y,\ t)\operatorname{grad} f(x,\ t)_x \tag{3}$$

since by (1), $x = y + \int_0^t u\,d\tau$. By the very concept of the velocity of a gas particle **u**(y, t) = **u**(x, t) (here we symbolize different functions with the same letter **u**: **u**(y, t) -- in Lagrangian coordinates, and **u**(x, t) -- in Eulerian), therefore we rewrite (3) as

$$\frac{\partial \bar{f}(y,t)}{\partial t}=\frac{\partial f(x,t)}{\partial t}+u\nabla f(x,\ t), \tag{4}$$

where $u\nabla$ is the differentiation operator over the space in the direction of the vector **u**:

$$u\nabla=u_1\frac{\partial}{\partial x_1}+u_2\frac{\partial}{\partial x_2}+u_3\frac{\partial}{\partial x_3}. \tag{5}$$

Ordinarily, in gas dynamics all quantities are denoted by the same letters both in Eulerian and Lagrangian representation. Therefore to avoid any confusion, the quantity $\frac{\partial \bar{f}(y,\ t)}{\partial t}$ is denoted by $\frac{df(y,\ t)}{dt}$. Under this notation, formula (3) becomes

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+u\nabla f=\frac{\partial f}{\partial t}+u_\alpha\frac{\partial f}{\partial x_\alpha}. \tag{6}$$

The quantity df/dt is equal to the total time derivative of the function f along the trajectory of particle $x = y + \int_0^t u\,d\tau$ and is called the substantive derivative.

By formulas (1), specific Eulerian coordinates correspond to the Lagrangian coordinates y, t, and this transformation y, t $\rightarrow$ x, t is unique. The inverse transformation x, t $\rightarrow$ y, t is not, generally speaking, always defined.

Actually, we can easily imagine a case when gas fills the entire space y_1, y_2, y_3 at the time instant t = 0, but as the result of motion at t > 0, some portion of the space x_1, x_2, x_3 proves to be free of the gas. This means that in this portion of the space for a given t > 0 no y_1, y_2, y_3 values correspond to the coordinates x_1, x_2, x_3.

As we will see below, the condition for the mutually unique mapping x, t --- y, t is best in the form $\rho(x, t) \neq 0$.

4. Equations of state of gases. Ideal gas. Van der Waals' gas. Normal gas. Limiting ourselves to these brief remarks on methods of describing the motion of gases and liquids, let us dwell briefly on several of the simplest equations of state of gases.

The term ideal gas refers to a gas for which Clapeyron's law is valid:

$$pV = RT, \qquad (1)$$

where R is the gas constant per gram. Then from the relation (1.1.12) it follows that

$$\frac{\partial \varepsilon(V, T)}{\partial V} = 0 \quad \text{or} \quad \varepsilon = \varepsilon(T), \qquad (2)$$

i.e., the internal gas energy is a function solely of temperature; here the specific heat capacity of the gas $c_V = \frac{\partial \varepsilon}{\partial T} = c_V(T)$ is also a function only of temperature.

A gas is called polytropic if c_V does not depend on temperature. Then

$$\varepsilon = c_V T, \qquad (3)$$

i.e., the internal energy is proportional to gas temperature.

The kinetic theory of gases leads to equations of state of a polytropic gas on the following assumption:

1) The potential energy of molecular interaction is negligibly small compared with the kinetic energy of the molecules. As a consequence of this assumption, the energy of a given mass of gas is the sum of the kinetic energies of the gas molecules comprising it.

2) Only pairwise collisions between molecules are possible, and they occur under the laws of elastic collision. The internal molecular structure remains unchanged and overall momentum and kinetic energy of the molecules are conserved.

The second approximation denotes, in particular, that the volume of the molecules themselves is small compared with the volume occupied by the gas.

The kinetic theory yields the following expressions for the coefficients c_V, c_p, R of the equations of state (1), (2) of an ideal gas:

$$c_V = \frac{f}{2} k \frac{N}{M}, \tag{4}$$

$$R = c_p - c_V, \tag{5}$$

where f is the number of degrees of freedom of the molecule (f = 3 for a monoatomic gas, f = 5 for a diatomic gas, and so on), k is Boltzmann's constant, N is Avogadro's number, and M is molecular weight.

Taking account of equation of state (1), from the fundamental relation (1.1.13) we have

$$S = c_V \ln T + R \ln V + \text{const} = c_V \ln T + c_p \ln V - c_V \ln V + \text{const} = c_V \ln p + c_p \ln V + \text{const}. \tag{6}$$

The simplest correction to the equation of state for an ideal gas, associated with the allowance for molecular volume and forces of molecular attraction, is given by Van der Waals' equation:

$$p = \frac{RT}{V-b} - \frac{a}{V^2}. \tag{7}$$

Here a is a quantity proportional to the force of attraction of the gas molecules and b is a quantity proportional to the volume of the molecules themselves.

We can analogously derive expressions for ε and S:

$$\varepsilon = \int c_V(T)\,dT - \frac{a}{V}, \tag{8}$$

$$S = \int \frac{c_V(T)}{T}\,dT + R \ln (V-b) + \text{const}. \tag{9}$$

If some gas element is subject to slow expansion or compression such that heat exchange does not occur with the ambient environment, the element executes adiabatic transition from one thermodynamic state to another. Here the slow process is reversible and the entropy of the element remains unchanged. Therefore this transition is called isoentropic.

All thermodynamic states through which a given gas element passes in this case will lie on the curve $S = \text{const},$ (10)

which is called Poisson's adiabat.

For an ideal polytropic gas, as readily follows from (6), the equation of Poisson's adiabat is of the form

$$p = \frac{A^2}{\gamma} V^{-\gamma}, \quad \gamma = \frac{c_p}{c_V} = 1 + \frac{R}{c_V} > 1, \quad A^2 = A^2(S) = a^2 e^{\frac{S}{c_V}} = \text{const}. \tag{11}$$

We can readily see that the following relations hold along Poisson's adiabat for a polytropic gas (Figure 2.1):

$$\frac{dp}{dV} = \frac{\partial p(V, S)}{\partial V} < 0 \quad \text{(property I)}, \tag{12}$$

$$\frac{d^2p}{dV^2} = \frac{\partial^2 p(V, S)}{\partial V^2} > 0 \quad \text{(property II)}. \tag{13}$$

Thus, Poisson's adiabat $p = p(V, S_0)$ is a curve monotonically decreasing relative to V, with its convexity facing downward. It is easy to see that the axes $V = 0$ and $p = 0$ are asymptotes of Poisson's adiabat, i.e., $p \to 0$ as $V \to \infty$ and $p \to \infty$ as $V \to 0$ (property III).

If some gas element is subject to compression or expansion such that the temperature of the element remains unchanged in the process, the transition made by the gas is called an isothermal process.

In an ideal gas undergoing an isothermal process, V and p are associated by the relation

$$p = c^2 \frac{1}{V} = c^2 \rho, \quad c^2 = (c_p - c_V) T = RT = \text{const}. \tag{14}$$

Therefore in several cases an ideal isothermal gas can be formally regarded as a polytropic gas with exponent $\gamma = 1$.

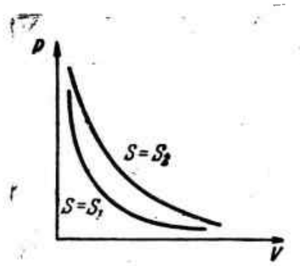

Figure 2.1

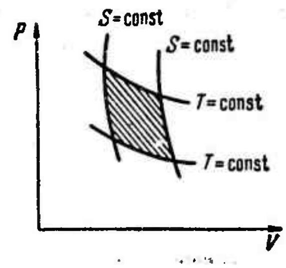

Figure 2.2

Curve (14) is called an iscthermal. It satisfies all three properties of Poisson's adiabat. The isothermal T and Poisson's adiabat A intersect in the V, p plane just as shown in Figure 2.2.

Thus, isotherms and Poisson's adiabats of an ideal gas form a regular net in the V, p plane.

Clearly, equations of state of an ideal gas provide for the satisfaction of the properties:

$$\left.\begin{array}{ll} \frac{\partial p(V,S)}{\partial S} > 0 & \text{(property IV)}, \\ c_V = \frac{\partial \varepsilon(V,T)}{\partial T} \quad 0 & \text{(property V)}. \end{array}\right\} \qquad (15)$$

Van der Waals' gas isotherms satisfy properties I-III now no longer in the entire phase space V, p (Figure 2.3).

Property II can be violated in the square A'CB', and property I in the BCA square; in general property III is not satisfied, since the straight line V = b is an isotherm asymptote. Let us note that the violation of property I expressed by inequality (12) points to the impossibility of thermodynamic equilibrium. Actually, let some volume of gas from which property I is violated undergo compression under the action of external pressure, which we will assume to be constant and exceeding the initial gas pressure. If the compression occurs slowly and if $\frac{\partial p(V,S)}{\partial V} > 0$, the pressure in the gas decreases, i.e., its drag relative to external compression is reduced. As a result, total collapse of the gas volume occurs. Conversely, the volume of the gas exhibiting an excess of pressure over the ambient environment must grow unboundedly.

If we now imagine a gas for which $\frac{\partial p(V, S)}{\partial V} > 0$, at thermodynamic equilibrium and constant pressure then this equilibrium will prove to be absolutely unstable, since the slightest departure from equilibrium leads to a situation in which one part of the gas will be compressed, while another will be expanded without turning to its initial state.

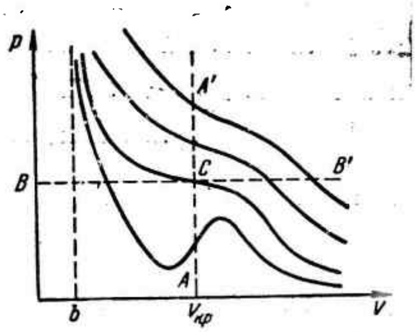

Figure 23.

The situation is otherwise in the case of a gas satisfying condition I. In adiabatic compression it will increase its pressure and oppose compression. On departure from equilibrium, fluctuations will arise in the gas, leading it to its initial state.

Thus, property I is the stability condition of thermodynamic equilibrium and is satisfied for all real substances.

In the following we will limit ourselves to a consideration of gases whose equations of state satisfy, besides thermodynamic relations in subsection 1 that are valid for any substances, several assumptions not stemming from the demands of thermodynamics.

Specifically, we will require that properties I-V be satisfied:

I.	$\frac{\partial p(V,S)}{\partial V} < 0$		(16)
II.	$\frac{\partial^2 p(V,S)}{\partial V^2} > 0$		(17)
III.	$p(V,S) \to \infty$	when $V \to 0$	(18)
IV.	$\frac{\partial p(V,S)}{\partial S} > 0$		(19)
V.	$c_V = \frac{\partial \varepsilon(V,T)}{\partial T} > 0$		(20)

Additionally, we will demand that the region of variables V, T in which properties I-V are satisfied will be convex (property VI).

A gas whose equation of state satisfies properties I-VI will be called a normal gas*).

Property I obviously denotes that along Poisson's adiabat A pressure p decreases monotonically with increase in V; property II requires that this curve phase convexity-downward; property III requires that V = 0 be the asymptote to any Poisson's abiabat; finally, property IV denotes that Poisson's adiabat corresponding to greater entropy lie higher in the plane of variables V, p (cf Figure 2.1), i.e.,

$$\frac{\partial S(V,p)}{\partial p} > 0, \quad \frac{\partial S(V,p)}{\partial V} > 0. \tag{21}$$

Let us consider the behavior of the surface S = S(V, p) in the three-dimensional space of variables V, p, S, assuming that the equations of state satisfy properties I-V.

*) Relations I-V were formulated by Bethe and Weyl (cf [4]).

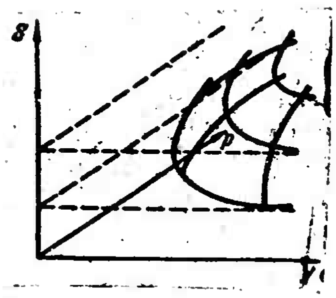

Figure 2.4

In accordance with properties I-IV, each horizontal section of the "relief" of entropy is a monotonic convex curve with asymptote V = 0. Each section of the plane V = constant or p = constant is a curve monotonically increasing relative to p, and so relative to V.

Thus, any higher-tying horizontal section is projected within the underlying section (Figure 2.4).

Based on relation (1.1.7): $TdS = d\mathcal{E} + pdV$, it follows that by virtue of V,

$$\frac{\partial S(V, T)}{\partial T} = \frac{1}{T}\frac{\partial \varepsilon(V, T)}{\partial T} = \frac{c_V}{T} > 0. \qquad (22)$$

From whence

$$\frac{dp(V, T)}{dT} = \frac{\partial p(V, S)}{\partial S}\frac{\partial S(V, T)}{\partial T} > 0. \qquad (23)$$

Taking formulas (21) and (22) into account, we find

$$\frac{\partial S(V, T)}{\partial V} > 0. \qquad (24)$$

The last inequality denotes that the isotherm and adiabats A form a regular net of the same type as for the case of the ideal gas (cf Figure 2.2).

Differentiating the identity S = S(V, p) = S(V, p(V, S)) relative to the variables V, S, we get

$$\frac{\partial S(V,p)}{\partial p}\frac{\partial p(V,S)}{\partial S}=1, \quad \frac{\partial S(V,p)}{\partial V}+\frac{\partial S(V,p)}{\partial p}\frac{\partial p(V,S)}{\partial V}=0 \qquad (25)$$

and differentiating the second equation in (25) relative to the variable V, we find

$$S_{VV}+2S_{Vp}p_V+S_{pp}p_V^2=-S_p p_{VV}<0. \qquad (26)$$

Substituting (25) into (26), we get

$$S_{VV}S_p^2-2S_{Vp}S_VS_p+S_{pp}S_V^2=-S_p^3 p_{VV}\leqslant 0. \qquad (27)$$

Let us look at the pattern of variation in entropy S along the straight lines

$$p = p_0 + at, \quad V = V_0 + bt \qquad (28)$$

in the plane of the variables V, p.

The following statements are valid:

a) if $a \geqslant 0$, $b \geqslant 0$ ($a \leqslant 0$; $b \leqslant 0$), then

$$\frac{dS}{dt}\geqslant 0 \quad \left(\frac{dS}{dt}\leqslant 0\right). \qquad (29)$$

This follows from inequalities (21).

b) if $\left(\frac{dS}{dt}\right)_{t=t_0}=0$, then

$$\left(\frac{d^2S}{dt^2}\right)_{t=t_0}=\frac{\partial^2 S}{\partial p^2}a^2+2\frac{\partial^2 S}{\partial p\,\partial V}ab+\frac{\partial^2 S}{\partial V^2}b^2<0. \qquad (30)$$

Actually, from $\left(\frac{dS}{dt}\right)_{t=t_0}=\frac{\partial S}{\partial p}a+\frac{\partial S}{\partial V}b = 0$ we have $\frac{\partial S}{\partial p}$ = kb, $\frac{\partial S}{\partial V}$ = -ka, where k is some proportionality constant. Substituting $b = \frac{1}{K}\frac{\partial S}{\partial p}$, $a = -\frac{1}{K}\frac{\partial S}{\partial V}$ into (30), we get (27).

c) if $\frac{dS}{dt} \leqslant 0$ when $t = 0$, then $dS/dt < 0$ when $t > 0$.

Let us assume the converse. Then there exist $t_1 \geqslant 0$ and $\varepsilon > 0$ such that $dS/dt < 0$ in the interval $0 \leqslant t < t_1$, $dS/dt = 0$ when $t = t_1$, $dS/dt > 0$ when $t_1 < t < t_1 + \varepsilon$. Therefore, $d^2S/dt^2 \geqslant 0$ when $t = t_1$, which contradicts property b). The assertion is proven.

We note that properties a), b), and c) follow readily from the nature of the "relief" of the function S(V, p) (Figure 2.4).

By virtue of properties I and II of Poisson's adiabats, the ray passing through the point V_0, p_0 of the adiabat $S = S_0$ does not intercept it at any point if it lies in the first and third quadrants (Figure 2.5), and intercepts it at only one point if it lies in the second and fourth quadrants (here we include the case when the ray is tangent to adiabat A, and then we will regard the tangency point as a paired points.

In the first case S is a monotonic function of the parameter t, in the second case S has a single maximum. The point of the maximum is the point of tangency by the ray at some adiabat.

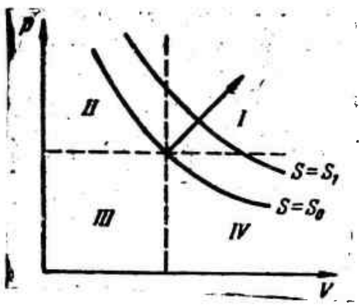

Figure 2.5

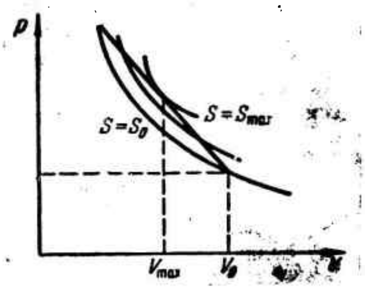

Figure 2.6

Here let us note the following: the point $S = S_{max}$ at the ray divides into two halves, so that an arbitrary adiabat intersects its upper branch in the upper half, and lower branch in the lower (Figure 2.6).

In the upper half of the ray we have

$$\frac{\partial p(V, S)}{\partial V} < \frac{p - p_0}{V - V_0} \quad (V < V_{max}); \tag{31}$$

and in the lower half of the ray the opposite equality holds

$$\frac{\partial p(V, S)}{\partial V} > \frac{p - p_0}{V - V_0} \quad (V > V_{max}). \tag{32}$$

Section II. Integral laws of conservation. Equations of the hydrodynamics of one-dimensional flows

1. General assumptions on the flow of compressible gases. We wil represent the motion of a gas as the motion of a continuous medium in the three-dimensional space x_1, x_2, x_3. In accordance with section I, the motion is wholly determined if we know the quantities $u = u(x, t)$, $\rho = \rho(x, t)$, $p = p(x, t)$, $\varepsilon = \varepsilon(x, t)$, $S = S(x,t)$. Almost everywhere in this chapter we will assume that internal friction and thermal conductivity are absent in the gas, i.e., that the particles of the fluid are thermally insulated from each other.

Under these conditions each element of a fluid does not participate in heat exchange and does not lose its energy in friction, therefore for smooth changes in state the entropy of each element was remained constant.

Still, we must take particular note of the fact that this consequence is valid only for slow, smooth changes in the parameters of the gas particle, therefore, if a gas particle abruptly changes its thermodynamic parameters, its entropy no longer remains constant in time.

In accordance with section I, we assume that everywhere, except for the discontinuity surface the flow is sufficiently smooth (quasiequilibrium). Therefore the particle changes its entropy only by intersecting the discontinuity surface; away from the discontinuity surface the concepts of temperature, pressure, and entropy are defined, which together with the given equations of state satisfy all thermodynamic relations.

And thus, we will consider a fluid devoid of internal friction and thermal conductivity; however, the flow particles can change their entropy by intersecting the discontinuity surfaces. In actuality, the discontinuity surface of the flow parameters is a narrow zone (zone of nonequilibrium status) in which the effect of viscosity and thermal conductivity are substantial, however small they may be. Actually, as we have already pointed out in section 1, entropy changes defined by the quantities μ div **u**, $\varkappa$ grad T, which are finite in this narrow zone.

When in fact we considered these zones of large gradients as discontinuity surfaces of thermodynamic parameters, we avoid any detailed consideration of nonequilibrium flow in these zones; however, let us consider this nonequalibrium status overall, which in fact leads to an increase in the entropy of a particle when it intersects a discontinuity surface (nonequilibrium-state zone).

As a consequence (of section V), we see that allowing for viscosity and thermoconductivity in a fluid and the further passing to the limit as $\mu, \varkappa \to 0$ lead us exactly to this flow pattern.

And thus, flows of a fluid to which we will in general consider here are limiting flows of a viscous and thermally conducting fluid as the coefficients of viscosity and thermal conductivity approach zero.

On the other hand, a detailed description of flows requires the employment only of general laws that are valid simultaneously both for equilibrium and for nonequilibrium processes.

Such laws are the fundamental laws of conservation in physics: the law of conservation of the mass of a fluid, the law of conservation of momentum, and the law of conservation of energy, whose derivation we commence in subsection 2.

2. Laws of the conservation of mass, momentum, and energy in a three-dimensional space. We will study one-dimensional flows of compressible gases. However, in order to achieve a unique derivation of equations for different forms of flow symmetry (plane, spherical, and cylindrical), initially we obtain equations for an arbitrary three-dimensional unsteady flow, and then from these we derive the equations of interest to us for one-dimensional flows.

Thus, suppose the gas moves in a three-dimensional space with Cartesian coordinates (x_1, x_2, x_3) measured off in some fixed system. We will call such coordinates Eulerian.

We will assume that external forces acting on the gas are absent and that in the space occupied by the gas there are no sources of mass, momentum, and energy.

Suppose G is some part of the space (x_1, x_2, x_3) bounded by a closed smooth surface Σ_G. The amount of gas present at time instant t in the volume G is equal to the quantity

$$\int\int_G\int \rho\, dx_1\, dx_2\, dx_3 = \int_G \rho(P, t)\, dG. \qquad (1)$$

In formula (1) let dG denote volume of element and P -- a point with coordinates (x_1, x_2, x_3).

The quantity

$$\int_G [\rho(P, t_2) - \rho(P, t_1)] \, dG \qquad (2)$$

is equal to the increment in the mass of the gas in the volume G during the time interval from $t = t_1$ to $t = t_2$.

Since sources are nonexistent in the volume G, the increment in the mass of the gas (2) must equal the gas mass escaping in the time interval from $t = t_1$ to $t = t_2$ through the surface Σ_G of the volume G.

We let $d\Sigma$ stand for the vector that has the direction of the external normal to Σ and that is equal in magnitude to the area of the isolated small piece of surface Σ. Through the area $d\Sigma$ there flows per unit time into the volume G a mass of gas equal to the quantity $-\rho u d\Sigma$. Therefore equating the quantity (2) to the amount of gas flowing into G, we get the equation

$$\int_G [\rho(P, t_2) - \rho(P, t_1)] \, dG + \int_{t_1}^{t_2} \left[\int_{\Sigma_G} \rho u \, d\Sigma \right] dt = 0. \qquad (3)$$

Equation (3) expresses the law of conservation of a mass of gas.

Change in momentum in the volume G during the time interval from $t = t_1$ to $t = t_2$ is equal to the quantity

$$\int_G \left[\rho u \Big|_{t=t_2} - \rho u \Big|_{t=t_1} \right] dG = \int_G \rho u \Big|_{t_1}^{t_2} dG \qquad (4)$$

and is due to the gas escaping from the surface Σ_G. This gas transports the momentum

$$-\int_{t_1}^{t_2} \left[\int_{\Sigma_G} \rho u (u \, d\Sigma) \right] dt, \qquad (5)$$

and the increment in momentum is also due to the pressure forces p exerted by the remaining mass of gas onto the gas in the volume G along the normal to the surface Σ_G. The total momentum of the pressure forces acting on the gas in the volume G is equal to the quantity

$$-\int_{t_1}^{t_2}\left[\int_{\Sigma_G} p\,d\Sigma\right]dt. \qquad (6)$$

Equating the quantity (4) to the sum of the quantity (5) and (6), we get an integral relation expressing the law of conservation of momentum:

$$\int_G \rho u \Big|_{t_1}^{t_2} dG + \int_{t_1}^{t_2}\left[\int_{\Sigma_G} p\,d\Sigma + \rho u(u\,d\Sigma)\right]dt = 0. \qquad (7)$$

In our derivation of relation (7), we used the fact that momentum sources are not present in the volume G.

The value of momentum in the space G is a vector; therefore equation (7) contains three scalar equations for each of the momentum components.

The energy contained per unit volume is equal to $\rho(\frac{u^2}{2}+\varepsilon)$ the quantity

$$\int_G \rho\left(\frac{u^2}{2}+\varepsilon\right)\Big|_{t_1}^{t_2} dG \qquad (8)$$

is equal to the increment in the total energy in the volume G during the time interval from $t = t_1$ to $t = t_2$. This increment is due to the transport of energy by the moving gas in the quantity

$$\mp\int_{t_1}^{t_2}\left[\int_{\Sigma_G} \rho\left(\varepsilon+\frac{u^2}{2}\right)u\,d\Sigma\right]dt \qquad (9)$$

and due to the work done by the pressure forces p.

The force $-p d\Sigma$ exerted by the surrounding gas acts on the gas in the volume G across the surface element $d\Sigma$; gas particles at the surface Σ_G move at a velocity **u**. Therefore per unit time the pressure forces perform work on the gas, $\int_{\Sigma_G} pu\,d\Sigma$, and the total work done by the pressure forces in the time interval from $t = t_1$ to $t = t_2$ is given by the formula

$$-\int_{t_1}^{t_2}\left[\int_{\Sigma_G} pu\,d\Sigma\right]dt. \qquad (10)$$

Equating the quantity (8) to the sum of the quantities (9) and (10), we obtain an integral relation expressing the law of conservation of energy:

$$\int_{\Omega} \rho\left(\varepsilon + \frac{u^2}{2}\right)\Big|_{t_1}^{t_2} d\Omega + \int_{t_1}^{t_2}\left[\oint_{\Sigma} \rho\left(\varepsilon + \frac{u^2}{2} + \frac{p}{\rho}\right) \mathbf{u}\, d\boldsymbol{\sigma}\right] dt = 0 \qquad (11)$$

In the derivation of (11) it was assumed that the gas has no thermal conductivity. Formulas (3), (7), and (11) mathematically express the laws of conservation of mass, momentum, and energy for gases devoid of friction and thermal conductivity and are the fundamental equations defining the motion of a gas.

Equations of state of the gas are added to the equations (3), (7), and (11). For example, if we give the relation $p = p(V, \varepsilon)$, $V = 1/\rho$, (12) then the problem of determining flow reduces to finding five variables: three velocity **u** components and two thermodynamic variables: ρ and ε, since by (12) p is function of ρ and ε. To this end, we make use of five scalar relations (3), (7), and (11).

Let us incidentally note that the equations of state can be assigned in any form of those considered in section I, and not necessarily in the form of (12).

The vector $\rho\mathbf{u}$ is called the mass flow vector, and $\rho u(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2})$ is the energy flow vector; the tensor

$$\Pi_{ik} = \delta_{ik} p + \rho u_i u_k$$

is the tensor of momentum flow.

Now we will proceed to a closer study of one-dimensional flows, i.e., flows in which the quantities **u**, ρ, p, ε depend only on a single spatial coordinate x and on time t.

We will discuss three cases of one-dimensional flows:

1) Plane one-dimensional flow, when the quantity **u**, ρ, p, ε are constant in the planes $x = x_1$ = const; $\mathbf{u} = \{u, 0, 0\}$. We will call this situation the case of plane symmetry.

2) A cylindrical one-dimensional flow (case of cylindrical symmetry). In this case u, ρ, p, ε are constant at the cylindrical surfaces $x = \sqrt{x_1^2 + x_2^2}$ = const (for fixed t). Let us assume

$$u = u(x, t)\left\{\frac{x_1}{x}, \frac{x_2}{x}, 0\right\}; \quad \rho = \rho(x, t); \quad p = p(x, t), \ldots, \tag{13}$$

3) Spherically symmetric flow (case of spherical symmetry) is obtained if the following formulas hold:

$$u = u(x, t)\left\{\frac{x_1}{x}, \frac{x_2}{x}, \frac{x_3}{x}\right\}; \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2};$$
$$\rho = \rho(x, t); \quad p = p(x, t), \ldots$$

3. Integral laws of conservation for one-dimensional flows in Eulerian coordinates. For the case of plane one-dimensional flow all the quantities depend only on x and t. Integral laws of conservation (2.2.3), (2.2.7), and (2.2.11) are rewritten in form*)

$$\int_{x_1}^{x_2} \rho \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \rho u \Big|_{x_1}^{x_2} dt = 0 \tag{1}$$

is the law of conservation of mass,

$$\int_{x_1}^{x_2} \rho u \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} [p + \rho u^2] \Big|_{x_1}^{x_2} dt = 0 \tag{2}$$

is the law of conservation of momentum, and

$$\int_{x_1}^{x_2} \left[\rho\left(\varepsilon + \frac{u^2}{2}\right)\right] \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \rho u \left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) \Big|_{x_1}^{x_2} dt = 0 \tag{3}$$

is the law of conservation of energy.

In equations (1)-(3) x_1 and x_2 fix the isolated volume G, and t_1 and t_2 are arbitrary instants of time.

---

*) In the following, we will understand x_1 and x_2 to refer to two values of the single coordinate x, and not too different Cartesian coordinates considered in the preceding subsection.

In the case of cylindrical symmetry we obtain integral relations, by writing the laws of conservation (2.2.3), (2.2.7), and (2.2.11) as applied to volume G shown in Figure 2.7.

In view of the constancy of all variables at the cylindrical surfaces x = const, the law of conservation of mass is written for volume G in the form

$$2\alpha l \int_{x_1}^{x_2} \rho \Big|_{t_1}^{t_2} x\, dx + 2\alpha l \int_{t_1}^{t_2} (\rho u x) \Big|_{x_1}^{x_2} dt = 0$$

or, after canceling out $2\alpha l$, in the final form:

$$\int_{x_1}^{x_2} \rho \Big|_{t_1}^{t_2} x\, dx + \int_{t_1}^{t_2} (\rho u x) \Big|_{x_1}^{x_2} dt = 0. \qquad (4)$$

The integral law of conservation of momentum is derived in somewhat more complex fashion. Let us write equality (2.2.7) as applied to volume G (Figure 2.7) only for the component of momentum in the direction of the vector e. At once we note that for these two other directions orthogonal to e equality (2.2.7) reduces to an identity.

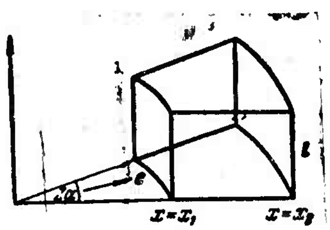

Figure 2.7

By virtue of (2.2.13), $\int \rho u_e \, dG$ is written in the form

$$\int_G \rho u_e \, dG = 2l \int_0^{\alpha} \cos\varphi \, d\varphi \int_{x_1}^{x_2} \rho u x \, dx =$$

$$= 2l \sin\alpha \int_{x_1}^{x_2} \rho u x \, dx.$$

The integrand in $\int\limits_{\Sigma_G} \rho u_e (u\, d\Sigma)$ is distinct from zero only for the parts $x = x_1$, $x = x_2$ of surface Σ_G; therefore

$$\int\limits_{\Sigma_G} \rho u_e (u\, d\Sigma) = 2l \sin\alpha\, [\rho u^2 x]\Big|_{x_1}^{x_2}.$$

Finally, the components of $\int\limits_{\Sigma_G} p\, d\Sigma$ in the direction of the vector e consists of two terms: integrals over the parts $x = x_1$, $x = x_2$ of surface Σ_G:

$$2l \int\limits_0^{\alpha} \cos\varphi\, d\varphi \left[px\Big|_{x_1}^{x_2}\right] = 2l \sin\alpha \left[px\Big|_{x_1}^{x_2}\right],$$

and integrals over the parts $\varphi = \pm\alpha$ surface Σ_G (Figure 2.7):

$$-2l \sin\alpha \int\limits_{x_1}^{x_2} p\, dx.$$

Substituting all these expressions in (2.2.7), we obtain after canceling out $2l\sin\alpha$ the final formula:

$$\int\limits_{x_1}^{x_2} [\rho u]\Big|_{t_1}^{t_2} x\, dx + \int\limits_{t_1}^{t_2} [(p + \rho u^2)x]\Big|_{x_1}^{x_2} dt = \int\limits_{t_1}^{t_2}\int\limits_{x_1}^{x_2} p\, dx\, dt. \qquad (5)$$

The law of conservation of energy (2.2.11) as applied to volume G for a flow with cylindrical symmetry is written in the form

$$\int\limits_{x_1}^{x_2} \left[\rho\left(\varepsilon + \frac{u^2}{2}\right)\right]\Big|_{t_1}^{t_2} x\, dx + \int\limits_{t_1}^{t_2} \left[\rho u \left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) x\right]\Big|_{x_1}^{x_2} dt = 0. \qquad (6)$$

And thus, relations (4), (5), and (6) afford a representation of the laws of conservation of mass, momentum, and energy for a flow with cylindrical symmetry.

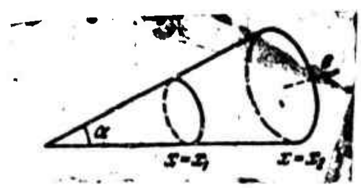

Figure 2.8

For a spherically symmetrical flow, the laws of conservation of mass, momentum, and energy are written for volume G excised from a right cone with an apex angle α by the spheres $x = x_1$ and $x = x_2$ (Figure 2.8).

Operations analogous to those just concluded, lead to the following equations:

$$\int_{x_1}^{x_2} \left(\rho\Big|_{t_1}^{t_2}\right) x^2\,dx + \int_{t_1}^{t_2} (\rho u x^2)\Big|_{x_1}^{x_2} dt = 0, \tag{7}$$

$$\int_{x_1}^{x_2} \left(\rho u\Big|_{t_1}^{t_2}\right) x^2\,dx + \int_{t_1}^{t_2} [(p+\rho u^2)x^2]\Big|_{x_1}^{x_2} dt = \int_{t_1}^{t_2}\int_{x_1}^{x_2} 2px\,dx\,dt, \tag{8}$$

$$\int_{x_1}^{x_2} \left[\rho\left(e+\frac{u^2}{2}\right)\right]\Big|_{t_1}^{t_2} x^2\,dx + \int_{t_1}^{t_2} \left[\rho u\left(e+\frac{p}{\rho}+\frac{u^2}{2}\right)x^2\right]\Big|_{x_1}^{x_2} dt = 0. \tag{9}$$

Now it is easy to note that for all three cases of homogeneous flows the laws of conservation of mass, momentum, and energy are written by the general formulas:

$$\int_{x_1}^{x_2} \rho\Big|_{t_1}^{t_2} x^\nu\,dx + \int_{t_1}^{t_2} (\rho u x^\nu)\Big|_{x_1}^{x_2} dt = 0, \tag{10}$$

$$\int_{x_1}^{x_2} \rho u\Big|_{t_1}^{t_2} x^\nu\,dx + \int_{t_1}^{t_2} [(p+\rho u^2)x^\nu]\Big|_{x_1}^{x_2} dt = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \nu p x^{\nu-1}\,dx\,dt, \tag{11}$$

$$\int_{x_1}^{x_2} \left[\rho\left(e+\frac{u^2}{2}\right)\right]\Big|_{t_1}^{t_2} x^\nu\,dx + \int_{t_1}^{t_2} \left[\rho u\left(e+\frac{p}{\rho}+\frac{u^2}{2}\right)x^\nu\right]\Big|_{x_1}^{x_2} dt = 0. \tag{12}$$

In the formulas (10)-(12) we must set $\nu = 0$ for the case of plane, $\nu = 1$ for the case of cylindrical, and $\nu = 2$ for the case of spherical symmetry of flow.

Let us consider in the plane of variables x, t the rectangular loop C and the domain G_C it bounds (Figure 2.9). Obviously, equalities (10)-(12) can be rewritten as

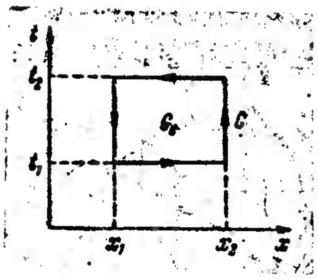

Figure 2.9

$$\oint_C \rho x^\nu dx - \rho u x^\nu dt = 0, \quad (13)$$

$$\oint_C \rho u x^\nu dx - (p + \rho u^2) x^\nu dt = - \iint_{G_C} \nu p x^{\nu-1} dx\, dt, \quad (14)$$

$$\oint_C \rho\left(\varepsilon + \frac{u^2}{2}\right) x^\nu dx - \rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) x^\nu dt = 0, \quad (15)$$

if the loop C is of the form shown in Figure 2.9.

In its physical meaning the quantities $\int_{x_1}^{x_2} \rho x^\nu dx$, $\int_{x_1}^{x_2} \rho u x^\nu dx$ and $\int_{x_1}^{x_2} \rho (\varepsilon + \frac{u^2}{2}) x^\nu dx$ are continuous functions of the variables t, x_1, and x_2, and $\int_{t_1}^{t_2} \rho u x^\nu dt$, $\int_{t_1}^{t_2} (p + \rho u^2) x^\nu dt$, and $\int_{t_1}^{t_2} \rho u (\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}) dt$ are continuous functions of the variables x, t_1, and t_2. Therefore, assuming at the quantities u, ρ, ε, p are bounded and piecewise-continuous functions*) of the variables x, t, we conclude that the relations (13)-(15) will be satisfied for any close piecewise-smooth loop C and the domain G_C bounded by it.

4. Integral laws of conservation in Lagrangian coordinates. Suppose r denotes the initial position of a gas particle, for example, at the time instant t = 0, and suppose x = x(r, t) is the position of the same particle at time instant t. The Lagrangian coordinate r in the Eulerian coordinate x are related, as we have seen in subsection 3 of section I, by the relation

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*) For the case of spherical and cylindrical symmetry, the boundedness can be violated at the straight line x = 0. This of course is not central to our following treatment.

$$x = r + \int u(r, \tau)\,d\tau = x(r, t) \qquad (1)$$

In (1) the velocity $u = u(r, t)$ is given as functions of Lagrangian variables. For a known velocity $u(r, t)$ equation (1) defines the particle trajectory. Obviously, the mass of gas enclosed in the volume bounded by the sections $x = x(r_0, t)$ and $x = x(r, t)$ remains constant in time. Therefore we can write *):

$$\int_{x(r_0, t)}^{x(r, t)} \rho(x, t)\,x^{\nu}\,dx = \int_{r_0}^{r} \rho_0(r)\,r^{\nu}\,dr = M = \text{const.} \qquad (2)$$

where $\rho_0(r)$ denotes the density at the time instant $t = 0$.

Differentiating (2) relative to r, we get

$$\frac{\partial x(r, t)}{\partial r} = \frac{r^{\nu}}{x^{\nu}}\,\frac{\rho_0(r)}{\rho(r, t)}, \qquad (3)$$

since, obviously, $\rho(x(r, t), t) = \rho(r, t)$.

Formula (3) shows that mapping of Lagrangian coordinates are onto Eulerian coordinates x is mutually unique given the condition $\rho(r, t) \neq 0$.

In the regions where $\rho = 0$ (vacuum regions) no Lagrangian coordinates r and t correspond to the points x and t, i.e., no flow trajectories pass through these points.

By formula (1)

$$\frac{\partial x(r, t)}{\partial t} = u(r, t). \qquad (4)$$

Therefore from formulas (3) and (4) we conclude that the transition from Eulerian coordinates x, t to Lagrangian coordinates r, t is given by the relation

*) The quantity M has the dimension of mass only of the case of spherical symmetry.

$$dx = \frac{r^\nu}{x^\nu}\frac{\rho_0(r)}{\rho(r, t)}dr + u(r, t)\,dt. \qquad (5)$$

The substitution of formula (5) into the law of conservation of mass (2.3.13) transform it into an identity. However, from (5) follows the integral relation

$$\oint_C x^\nu dx = \oint_C \frac{\rho_0(r)}{\rho(r, t)} r^\nu dr + u(r, t)\, x^\nu dt = 0, \qquad (6)$$

which is equivalent to the law of conservation of mass, since it is a consequence of relation (2). In inequality (6) C is an arbitrary piecewise-smooth closed loop in the plane of variables r, t; x = x(r, t) is defined by using formula (1).

Let us note that equality (6) is also called the integral law of conservation of the volume occupied by a gas.

Converting from variables x, t to Lagrangian variables r, t in equation (2.3.14), we get

$$\oint_C u\rho_0(r) r^\nu dr - px^\nu(r, t)\,dt = -\iint_{G_C} \nu p x^{\nu-1}\, dt \frac{\partial x(r, t)}{\partial r} dr =$$
$$= -\iint_{G_C} \nu p x^{\nu-1} \frac{r^\nu}{x^\nu}\frac{\rho_0(r)}{\rho(r, t)}\,dt\,dr =$$
$$= -\iint_{G_C} \nu p \frac{r^\nu}{x(t, r)}\frac{\rho_0(r)}{\rho(r, t)}\,dt\,dr \qquad (7)$$

which is the equation of conservation of momentum. In formula (7) C is the loop of the plane r, t, and G_C is the region of these variables bounded by it. Finally, the law of conservation of energy (2.3.15) is written in Lagrangian variables in the form

$$\oint_C \left(\varepsilon + \frac{u^2}{2}\right)\rho_0(r) r^\nu dr - upx^\nu(r, t)\,dt = 0. \qquad (8)$$

Equations (6)-(8) constitute the laws of conservation of volume, mome tum, and energy in Lagrangian variables.

The laws of conservation (6)-(8) become somewhat simplified if we introduce the notation

$$V(r, t)=V=\frac{1}{\rho(r, t)}; \quad q=q(r)=\int_0^r \rho_0(r) r^\nu dr. \qquad (9)$$

The variable q coincides with M and is called the mass Lagrangian coordinate.

Converting to these variables, we get:

$$\oint_C V\,dq + x^\nu(q, t) u\,dt = 0, \qquad (10)$$

$$\oint_C u\,dq - px^\nu\,dt = -\int\int_{G_C} \frac{\nu p V}{x}\,dq\,dt, \qquad (11)$$

$$\oint_C \left(\varepsilon + \frac{u^2}{2}\right) dq - upx^\nu\,dt = 0. \qquad (12)$$

In equations (10)-(12) C is an arbitrary piecewise-smooth closed loop in the plane of variables q, t.

5. Differential equations one-dimensional flows. Let us assume that in so e region of variables x, t the functions u, ρ, p, ε describing gas flow are continuously differentiable. Then by Green's formula the contour integrals in the equalities (2.3.13)-(2.3.15) are transformed into integrals over the region G_C; here the integrands will contain the first derivatives u, ρ , p, and ε. In view of the arbitrary mass of the region G_C, these integrand expressions must tend to zero. Therefore for smooth flows (u, ρ , p, $\varepsilon \in C_1$), from the condition that the integral of laws of conservation (2.3.13)-(2.3.14) are satisfied follows the fulfillment of the differential equations

$$\frac{\partial}{\partial t}(x^\nu \rho) + \frac{\partial}{\partial x}(x^\nu \rho u) = 0, \qquad (1)$$

$$\frac{\partial}{\partial t}(x^\nu \rho u) + \frac{\partial}{\partial x}[x^\nu (p + \rho u^2)] = \nu x^{\nu-1} p, \qquad (2)$$

$$\frac{\partial}{\partial t}\left[x^\nu \rho\left(\varepsilon + \frac{u^2}{2}\right)\right] + \frac{\partial}{\partial x}\left[x^\nu \rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right)\right] = 0. \qquad (3)$$

Differential equations (1)-(3) are written in Eulerian coordinates x, t and are valid for smooth flows. Analogously, from the laws of conservation (2.4.10)-(2.4.12) follow differential equations in Lagrangian coordinates q, t:

$$\frac{\partial V}{\partial t} - \frac{\partial}{\partial q} x^\nu u = 0, \qquad (4)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial q} x^\nu p = \frac{\nu p V}{x}, \qquad (5)$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{u^2}{2}\right) + \frac{\partial}{\partial q}(u p x^\nu) = 0. \qquad (6)$$

In equations (4)-(6) the Eulerian particle coordinate x must be considered as a function of Lagrangian coordinate q and time t, i.e., $x = x(q, t)$; from formula (2.4.4) we obtain a differential equation for x(q, t):

$$\frac{\partial x(q, t)}{\partial t} = u(q, t). \qquad (7)$$

Equations (4)-(7) describe a smooth one-dimensional gas flow in Lagrangian coordinates.

Equations (1)-(3) are transformed to the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\nu \rho u}{x} = 0, \quad (8)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad (9)$$

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0. \quad (10)$$

where S is the entropy defined by the equation $TdS = d\varepsilon + pdV$. Analogously, by combining equations (4)-(7) in Lagrangian coordinates, we obtain

$$\frac{\partial S(q, t)}{\partial t} = 0.$$

This equation shows that entropy S of each gas particle remains constant in time through the entire region of flow smoothness. Hence it follows that if the flow is smooth, the entropy of each gas particle remains constant.

We note that if the gas exhibits finite viscosity and thermal conductivity, equations (1)-(3) for a viscous thermally conductive gas are replaced by the following:

$$\frac{\partial}{\partial t}(x^{\nu}\rho) + \frac{\partial}{\partial x}(x^{\nu}\rho u) = 0, \quad (11)$$

$$\frac{\partial}{\partial t}(x^{\nu}\rho u) + \frac{\partial}{\partial x}\left[x^{\nu}\left(p + \rho u^2 - \mu\frac{\partial u}{\partial x}\right)\right] = \nu x^{\nu-1}\left(p - \mu\frac{\partial u}{\partial x}\right), \quad (12)$$

$$\frac{\partial}{\partial t}\left[x^{\nu}\rho\left(\varepsilon + \frac{u^2}{2}\right)\right] + \frac{\partial}{\partial x}\left[x^{\nu}\rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) - \mu x^{\nu} u \frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x} x^{\nu}\varkappa\frac{\partial T}{\partial x}, \quad (13)$$

where $\mu > 0$, $\varkappa > 0$ are the coefficients of viscosity and thermal conductivity, respectively, and T is gas temperature.

The corresponding equation in the Lagrangian coordinates are of the form

$$\frac{\partial V}{\partial t}-\frac{\partial}{\partial q}x^{\nu}u=0. \tag{14}$$

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial q}\left[x^{\nu}\left(p-\mu\rho x^{\nu}\frac{\partial u}{\partial q}\right)\right]=\frac{\nu V}{x}\left[p-\mu\rho x^{\nu}\frac{\partial u}{\partial q}\right]. \tag{15}$$

$$\frac{\partial}{\partial t}\left(\varepsilon+\frac{u^{2}}{2}\right)+\frac{\partial}{\partial q}\left[ux^{\nu}\left(p-\mu\rho x^{\nu}\frac{\partial u}{\partial q}\right)\right]=\rho x^{\nu}\frac{\partial}{\partial q}\left[x^{2\nu}\rho\varkappa\frac{\partial T}{\partial q}\right]. \tag{16}$$

Now, by combining -- on analogy with the preceding -- equations (11)-(13), we find:

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}=\frac{\mu}{T\rho}\left(\frac{\partial u}{\partial x}\right)^{2}+\frac{1}{T}\frac{1}{x^{\nu}\rho}\frac{\partial}{\partial x}\varkappa x^{\nu}\frac{\partial T}{\partial x} \tag{17}$$

or, in Lagrangian coordinates,

$$\frac{\partial S(q,t)}{\partial t}=\frac{\mu\rho x^{\nu}}{T}\left(\frac{\partial u}{\partial q}\right)^{2}+\frac{1}{T}\frac{\partial}{\partial q}\varkappa x^{2\nu}\rho\frac{\partial T}{\partial q}. \tag{18}$$

From equations (17) and (18) it follows that for a thermally insulated gas mass exhibiting viscosity and thermal conductivity, its total entropy does not decrease with time t.

Actually, by writing equation (18) as

$$\frac{\partial S(q,t)}{\partial t}=\frac{\mu\rho x^{\nu}}{T}\left(\frac{\partial u}{\partial q}\right)^{2}+\frac{\partial}{\partial q}\varkappa x^{2\nu}\rho\frac{1}{T}\frac{\partial T}{\partial q}+\frac{\varkappa x^{2\nu}\rho}{T^{2}}\left(\frac{\partial T}{\partial q}\right)^{2} \tag{19}$$

and integrating it within the limits from $q = q_1$ and $q = q_2$, we get

$$\frac{\partial}{\partial t}\int_{q_1}^{q_2}S(q,t)\,dq=$$
$$=\frac{\varkappa x^{2\nu}\rho}{T}\frac{\partial T}{\partial q}\Big|_{q_1}^{q_2}+\int_{q_1}^{q_2}\left[\frac{\mu\rho x^{\nu}}{T}\left(\frac{\partial u}{\partial q}\right)^{2}+\frac{\varkappa x^{2\nu}\rho}{T^{2}}\left(\frac{\partial T}{\partial q}\right)^{2}\right]dq. \tag{20}$$

Denoting $S_n = \int_{q_1}^{q_2} S(q, t)\,dq$ as the total entropy of the given mass of gas, we obtain from(20)

$$\frac{dS_n}{dt} > -\left[\frac{W}{T}\right] \qquad (21)$$

where $W = -\mathcal{H} x^{2\nu} \rho \frac{\partial T}{\partial q}$ is the heat flow and the quantity W/T is the entropy flow. Thus, inequality (21) indicates that for a given mass of viscous thermally conductive gas the output of entropy exceeds its inflow through the boundaries of this gas mass. If the mass of gas enclosed between the sections $q = q_1$ and $q = q_2$ is thermally insulated, then $\mathcal{H} \frac{\partial T}{\partial q} x^{2\nu} \rho = 0$ for $q = q_1$, $q = q_2$. Therefore from (20) follows that

$$dS_n/dt \geqslant 0 \qquad (22)$$

Thus, if we consider the motion of a gas devoid of viscosity and thermal conductivity as the limiting motion of a viscous and thermally conductive gas as $\mu, \mathcal{H} \to 0$, then from equations(17), (18), (20) follows that $S(q, t) = S(q, 0)$ only for the case when $\sqrt{\mu}\, \frac{\partial u}{\partial q} \to 0, \sqrt{\mathcal{H}}\, \frac{\partial T}{\partial q} \to 0$, as $\mu, \mathcal{H} \to 0$. As we will see later, the motion of a gas devoid of viscosity and thermal conductivity is not smooth, since within the gas there form regions where the gradients $\partial u/\partial q$, $\partial T/\partial q$ are not bounded. For this reason, if we are considering flow of an inviscid and nonthermally conductive gas as the limiting flow when $\mu, \mathcal{H} \to 0$, the conservation of entropy S for each gas particle obtains only as long as the particle trajectory is in the region of flow smoothness.

But if the particle trajectory passes through the zone of unbounded gradients or else through the discontinuity surface of the hydrodynamic variables, the entropy of this particle increases.

This principle of entropy increment when a particle passes through a discontinuity surface (zone of nonequilibrium status) will be employed by us in the following (section IV) in selecting stable discontinuities when we investigate the flow of a gas devoid of viscosity and thermal conductivity.

Now let us note that for a unique determination of flow we must add to equations (1)-(3) or to (4)-(6) for an inviscid and nonthermally conductive gas, and also to equations (11)-(13) or (14)-(16) for a gas with viscosity and thermal conductivity, equations of state of the gas. In the following we will assign equations of state in one of the following forms:

$$p = p(\rho, T), \quad \varepsilon = \varepsilon(\rho, T); \tag{23}$$

$$\varepsilon = \varepsilon(\rho, p), \quad \text{or} \quad p = p(\rho, \varepsilon); \tag{24}$$

$$\varepsilon = \varepsilon(V, S), \quad p = p(V, S), \quad \text{or} \quad p = p(\rho, S). \tag{25}$$

Equations of state can be given in any of these forms, however for the case of the gas with viscosity and to thermoconductivity it is especially convenient to employ them in the form (23), since this law is to select temperature T as the main thermodynamic variable.

For the case when various gases participate in the motion, we must assume that these functions are distinct in regions occupied by the different gases. Since these regions are not known in advance, we cannot, in general, assign these relations as functions of the Eulerian coordinates x, t.

Let us note here the advantages of Lagrangian coordinates in which the equations of state (23)-(25) can be assumed to be assigned in the form of functions of q, t, for example: $\varepsilon = \varepsilon(V, p, q)$. (26)

In this case function (26) must be assumed to be continuous relative to variable q at the points $q = q_i$ = const, which are the interfaces of the different gases.

6. Study of equations in Eulerian coordinates. Characteristic form. Characteristics. We will assume that the equation of state is given in the form (2.5.25): $p = p(\rho, S)$. Then

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial S}\frac{\partial S}{\partial x}.$$

Substituting this expression into the system of equations (2.5.8)-(2.5.10), we obtain equations of one-dimensional flow in Eulerian variables written in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = -\frac{\nu \rho u}{x}, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial S}\frac{\partial S}{\partial x} = 0, \tag{2}$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0. \tag{3}$$

Let us reduce the system of three quasilinear equations (1)-(3) relative to three unknowns ρ, u, S to the characteristic form. To do this, according to section II of chapter one we must compute the roots ξ_1, ξ_2, ξ_3 of the equation

$$\begin{vmatrix} u-\xi & \rho & 0 \\ \frac{1}{\rho} p'_\rho & u-\xi & \frac{1}{\rho} p'_S \\ 0 & 0 & u-\xi \end{vmatrix} = (u-\xi)^3 - p'_\rho (u-\xi) = 0. \tag{4}$$

As we already stated in section I, we will assume that

$$\frac{\partial p(\rho, S)}{\partial \rho} > 0. \tag{5}$$

Then denoting

$$c^2 = c^2(\rho, S) = \frac{\partial p(\rho, S)}{\partial \rho}, \tag{6}$$

let us write equation (4) in the form

$$(u-\xi)[(u-\xi)^2 - c^2] = 0, \tag{7}$$

from whence we get:

$$\xi_1 = u - c; \quad \xi_2 = u; \quad \xi_3 = u + c; \tag{8}$$
$$\xi_1 < \xi_2 < \xi_3 \quad (c > 0).$$

And thus, provided condition (5) is satisfied the system of equations (1)-(3) is given by a hyperbolic system.

In section I we saw that the condition $\frac{\partial p}{\partial \rho} = -V^2 \frac{\partial p}{\partial V} > 0$ is the stability condition for the thermodynamic state of a gas. This same condition insures the hyperbolicity of equations in gas dynamics and, therefore, the correctness of Cauchy's problem for gas dynamics equations. If however $\frac{\partial p}{\partial \rho} < 0$, then Cauchy's problem for the system (1)-(3) would, generally speaking, be incorrect.

Let us reduce this system of equations to the characteristic form. To this end, multiply equation (1) by the quantity $-c/\rho$, add it to equation (2), and to the result add equation (3) multiplied by the quantity $-\frac{1}{\rho c}\frac{\partial p}{\partial S}$. Then we arrive at an equation in the characteristic form:

$$\frac{\partial u}{\partial t}+(u-c)\frac{\partial u}{\partial x}-\frac{c}{\rho}\left[\frac{\partial p}{\partial t}+(u-c)\frac{\partial p}{\partial x}\right]-$$
$$-\frac{\partial p}{\partial S}\frac{1}{\rho c}\left[\frac{\partial S}{\partial t}+(u-c)\frac{\partial S}{\partial x}\right]=\frac{\nu c u}{x}. \qquad (9)$$

The second characteristic equation corresponding to the eigenvalue $\xi_2 = u$ is equation (3): $\partial S/\partial t + u\partial S/\partial x = 0$, (10)
and the third is obtained by multiplying (1) by c/ρ, adding the results to (2), after which equation (3) multiplied by $\frac{1}{\rho c}\frac{\partial p}{\partial S}$ is added onto the result. So we obtain a third equation in the characteristic form:

$$\frac{\partial u}{\partial t}+(u+c)\frac{\partial u}{\partial x}+\frac{c}{\rho}\left[\frac{\partial p}{\partial t}+(u+c)\frac{\partial p}{\partial x}\right]+$$
$$+\frac{1}{\rho c}\frac{\partial p}{\partial S}\left[\frac{\partial S}{\partial t}+(u+c)\frac{\partial S}{\partial x}\right]=-\frac{\nu c u}{x}. \qquad (11)$$

Thus, equations (9), (10), and (11) constitute the characteristic form of gas dynamics equations (1)-(3) in Eulerian coordinates. The quantity

$$c=c(\rho, S)=\sqrt{\frac{\partial p(\rho, S)}{\partial \rho}}$$

is called the speed of sound, since small perturbations in the solution are propagated along the characteristics; but the inclinations of the characteristics are the quantities $u - c$, u, $u + c$. Therefore small perturbations are propagated relative to the substance at the velocity 0, $\pm c(\rho, S)$. Integral curves of the equations

$$\frac{dx}{dt}=u-c, \quad \frac{dx}{dt}=u, \quad \frac{dx}{dt}=u+c \qquad (12)$$

are called characteristics of the system of equations (1)-(3) or (9)-(11), where the line $dx/dt = u$ is also called the trajectory.

By equation (10) the entropy S is constant along the trajectory. We obtain a conclusion that we already noted above: the entropy of a particle remains constant as long as the flow is smooth.

Since $c > 0$, the characteristics $dx/dt = u - c$ of the first family relative to the substance move leftward with time t, while the characteristics of the third family $dx/dt = u + c$ move to the right relative to the substance.

Finally, let us note further one form of the notation of the characteristic system (9)-(11) which is often encountered in the literature:

$$\begin{aligned} &dx=(u-c)\,dt, \quad du-\frac{c}{\rho}\,d\rho-\frac{\partial p}{\partial S}\frac{1}{\rho c}\,dS=\frac{\nu c u}{x}\,dt; \\ &dx=u\,dt, \quad dS=0; \\ &dx=(u+c)\,dt, \quad du+\frac{c}{\rho}\,d\rho+\frac{\partial p}{\partial S}\frac{1}{\rho c}\,dS=-\frac{\nu c u}{x}\,dt. \end{aligned} \tag{13}$$

In this notation it is indicated along which lines differential relations between the desired functions are satisfied.

7. Isentropic and isothermal flows. Riemann's invariants. Suppose that the initial values of the gas dynamic variables are assigned at the straight line

$$t = 0: \; u = u_0(x), \quad \rho = \rho_0(x), \quad S = S_0(x). \tag{1}$$

If we assume that the initial functions u_0, ρ_0, S_0 have a bounded first derivative (or are Lipschitz-continuous), then from the results of chapter one follows the existence in some strip $0 \leq t \leq t_0$ of the differentiable (or Lipschitz-continuous) solution to the system of equations (2.6.9)-(2.6.11).

Let us assume that $S_0(x) = S_0$ = const. Then from equation (2.6.10) it follows that throughout the region of variables x, t where a differentiable solution exists to system (2.6.9)-(2.6.11), entropy remains constants:

$$S(x, t) = S_0(x) = S_0 = \text{const}. \tag{2}$$

This rule is called isentropic.

The problem of defining an isentropic flow reduces, obviously, to integrating the system of two quasilinear equations

$$\frac{\partial u}{\partial t}+(u-c)\frac{\partial u}{\partial x}-\frac{c}{\rho}\left[\frac{\partial \rho}{\partial t}+(u-c)\frac{\partial \rho}{\partial x}\right]=\frac{\nu c u}{x}, \tag{3}$$

$$\frac{\partial u}{\partial t}+(u+c)\frac{\partial u}{\partial x}+\frac{c}{\rho}\left[\frac{\partial \rho}{\partial t}+(u+c)\frac{\partial \rho}{\partial x}\right]=-\frac{\nu c u}{x}, \tag{4}$$

where

$$c=c(\rho, S_0)=\sqrt{\frac{\partial p(\rho, S_0)}{\partial \rho}}=c(\rho). \tag{5}$$

Just as any system of two quasilinear equations, system (3), (4) reduces to Riemann's invariants (in chapter one, section III). Introducing the function

$$\varphi(\rho)=\int_{\rho_0}^{\rho}\frac{c(\rho)\,d\rho}{\rho}=\int_{\rho_0}^{\rho}\frac{c^2(\rho)\,d\rho}{\rho c(\rho)}=\int_{\rho_0}^{\rho}\frac{dp(\rho, S_0)}{\rho c(\rho)} \qquad (6)$$

and the new variables s, r: $s = u - \varphi(\rho)$, $r = u + \varphi(\rho)$, (7)

let us write system (3), (4) as

$$\frac{\partial s}{\partial t}+(u-c)\frac{\partial s}{\partial x}=\frac{\nu c u}{x}, \quad \frac{\partial r}{\partial t}+(u+c)\frac{\partial r}{\partial x}=-\frac{\nu c u}{x} \qquad (8)$$

The variables r, s are called Riemann's invariants.

From the known r, s u, ρ are uniquely defined, i.e., transformation (7) has an inverse. Actually, $u = (r + s)/2$ (9)

and

$$\varphi(\rho)=\frac{r-s}{2}. \qquad (10)$$

Since

$$\varphi'(\rho)=\frac{c(\rho)}{\rho}>0, \qquad (11)$$

then from formula (10) ρ is uniquely defined as a function of r-s, i.e., we can assume that

$$\rho=\varphi^{-1}(r-s), \quad c(\rho)=\psi(r-s)=c(\varphi^{-1}(r-s)), \qquad (12)$$

where

$$(\varphi^{-1})'=\frac{\rho}{2c(\rho)}>0, \quad \frac{\partial\psi}{\partial r}=-\frac{\partial\psi}{\partial s}=\frac{\rho c'(\rho)}{c(\rho)}. \qquad (13)$$

Thus, the system of equations for isentropic flow can be written via Riemann's invariants in the form

$$\frac{\partial s}{\partial t}+\left[\frac{r+s}{2}-\psi(r-s)\right]\frac{\partial s}{\partial x}=\frac{\nu\psi(r-s)(r+s)}{2x}, \qquad (14)$$

$$\frac{\partial r}{\partial t}+\left[\frac{r+s}{2}+\psi(r-s)\right]\frac{\partial r}{\partial x}=-\frac{\nu\psi(r-s)(r+s)}{2x}. \qquad (15)$$

Equations are specially simplified for isentropic plane-symmetric flow. In this case $\nu = 0$ and the right side in the equations (14) and (15) cancel out:

$$\frac{\partial s}{\partial t} + \left[\frac{r+s}{2} - c\right]\frac{\partial s}{\partial x} = 0,$$
$$\frac{\partial r}{\partial t} + \left[\frac{r+s}{2} + c\right]\frac{\partial r}{\partial x} = 0, \quad c = \psi(r - s). \tag{16}$$

From equations (16) it follows that Riemann's invariants r, s retain contant values along the corresponding characteristics: the invariant s is constant along the characteristics

$$\frac{dx_1}{dt} = u - c = \frac{s+r}{2} - \psi(r - s),$$

invariant r is constant along the characteristics

$$\frac{dx_2}{dt} = u + c = \frac{r+s}{2} + \psi(r - s).$$

Now let us consider further a special case --the case of isothermal gas. We assume that gas exhibits extermely high thermal conductivity and is enclosed in a thermstat which is kept at a constant temperature T_0. Owing to the high thermal conductivity, the temperature in the gas will be very rapidly equalized, and we can consider it approximately as constant and equal to T_0. This means that we consider the limiting case of infinite thermal conductivity.

In contrast to the approximation of a locally adiabatic process in which the conservation of particle entropy is violated in the region of steep gradients, this model is physically meaningful and remains noncontradictory also for discontinuous flows.

Therefore, let us establish integral laws of conservati n which are valid in this case.

It is quite understandable that the laws of conservation of mass and momentum are valid also for the case of this flow model. As for the law of conservation of energy, it must be modified in this case, since gas maintains a constant temperature T_0 by receiving from or supplying energy to the thermostat.

The law of conservation of energy now is meaningful only for a closed gas-thermostat system. As for the gas itself, the application of the law of conservation of energy to any gas mass specifies only the amount of heat communicated to or received from the thermostat. In an example of this approach is in subseotion 6 of section IV.

Writing the equation of state in the form $p = p(\rho, T_0)$, we see that in this case pressure can be taken as a function of only density ρ and that $\frac{\partial p}{\partial \rho} > 0$.

From the laws of conservation (2.3.13) and (2.3.14) follow differential equations (2.5.1) and (2.5.2), which are reduced to the characteristic form (3) and (4), where c must now signify the quantity

$$c = c_T(\rho) = \sqrt{\frac{\partial p(\rho, T_0)}{\partial \rho}}; \qquad (17)$$

and $c = c_T(\rho)$ is the so-called "isothermal speed of sound."

Equations (3) and (4) naturally can be written in the form (14) and (15), and for the case of an isothermal gas, here formulas (7) are valid for r and s, formula (12) for $c_T(\rho)$, and formula (6) for $\varphi(\rho)$ if $c(\rho)$ is understood to stand for quantity (17).

8. Equations in Lagrangian coordinates. Case of variable entropy. We will take as the initial equations in Lagrangian variables equations (2.5.4), (2.5.5), and (2.5.6) in which q is the mass coordinate. Let us write the following form:

$$\frac{\partial V}{\partial t} - x^\nu \frac{\partial u}{\partial q} = \nu x^{\nu-1} u \frac{\partial x}{\partial q} = \frac{\nu u V}{x}, \qquad (1)$$

$$\frac{\partial u}{\partial t} + x^\nu \frac{\partial p}{\partial q} = 0, \qquad (2)$$

$$\frac{\partial S}{\partial t} = 0, \quad V = \frac{1}{\rho}; \qquad (3)$$

where the Eulerian coordinate $x = x(q, t)$ must be considered as the solution to differential equation (2.5.7):

$$\frac{\partial x(q, t)}{\partial t} = u(q, t). \qquad (4)$$

Satisfying the initial condition following from (2.4.9):

$$\int_0^{x(q, 0)} \rho_0(r) r^\nu \, dr = q. \qquad (5)$$

i.e., we can assume that $x(q, 0) = x_0(q)$. (6)

where $x_0(q)$ is a monotonically increasing and continuous function of the variable q. To equations (1)-(4), as always, we adjoin the equation of state, which we will assume to be given in the form $p = p(V, S)$, $p'_V(V, S) < 0$. (7)

The variable $x = x(q, t)$ is defined from equation (4); we have the following expression for the derivative $\partial x/\partial q$:

$$\frac{\partial x(q, t)}{\partial q} = \frac{1}{x^\nu} V. \tag{8}$$

Eigenvalues ξ_1, ξ_2, ξ_3, as usually, are defined from the equation

$$\begin{vmatrix} -\xi & -x^\nu & 0 \\ x^\nu p'_V & -\xi & x^\nu p'_S \\ 0 & 0 & -\xi \end{vmatrix} = 0, \tag{9}$$

i.e.

$$\xi\left[\xi^2 + x^{2\nu} p'_V\right] = 0. \tag{10}$$

Since

$$p'_V(V, S) = -\rho^2 p'_\rho(\rho, S) = -\rho^2 c^2(\rho, S), \tag{11}$$

then

$$\xi_1 = -\rho c x^\nu = -\frac{c x^\nu}{V}, \quad \xi_2 = 0, \quad \xi_3 = \rho c x^\nu = \frac{c x^\nu}{V}.$$

The same formulas for ξ_1, ξ_2, ξ_3 can be obtained from the results in subsection 6 by applying the transformation of coordinates x, t into q, t by formula (2.4.5).

The reduction of system (1)-(3) to the characteristic form is carried out as in subsection 6. We present here the final form:

$$\frac{\partial u}{\partial t} - x^\nu \rho c \frac{\partial u}{\partial q} + \rho c\left(\frac{\partial V}{\partial t} - x^\nu \rho c \frac{\partial V}{\partial q}\right) - \frac{p'_S(V, S)}{\rho c}\left(\frac{\partial S}{\partial t} - x^\nu \rho c \frac{\partial S}{\partial q}\right) = \frac{\nu c u}{x}, \tag{12}$$

$$\frac{dS}{dt} = 0, \tag{13}$$

$$\frac{\partial u}{\partial t} + x^\nu \rho c \frac{\partial u}{\partial q} - \rho c\left(\frac{\partial V}{\partial t} + x^\nu \rho c \frac{\partial V}{\partial q}\right) + \frac{p'_S(V, S)}{\rho c}\left(\frac{\partial S}{\partial t} + x^\nu \rho c \frac{\partial S}{\partial q}\right) = -\frac{\nu c u}{x}. \tag{14}$$

Equation (4) with initial condition (6) is adjoined to equations (12)-(14).

For the isentropic flow $S(q, t) = S_0$. Riemann invariants

$$s = u - \int_{\rho_0}^{\rho} \frac{c\,d\rho}{\rho}, \qquad r = u + \int_{\rho_0}^{\rho} \frac{c\,d\rho}{\rho}$$

coincide with those introduced in subsection 7 (cf (2.7.6), (2.7.7)). Incidentally, this is a general property of Riemann invariants; they are invariant upon interchange of dependent and independent variables.

Thus, isentropic flow satisfies a system of two quasilinear equations in invariants

$$\frac{\partial s}{\partial t} - x^{\nu}\rho c \frac{\partial s}{\partial q} = \frac{\nu c u}{x}, \qquad (15)$$

$$\frac{\partial r}{\partial t} + x^{\nu}\rho c \frac{\partial r}{\partial q} = -\frac{\nu c u}{x}, \qquad (16)$$

and the function $x = x(q, t)$ is defined by formulas (4) and (6). The quantity c depends only on $\rho = 1/V$ and is a unique function of $r - s$:

$$\rho c = \psi(r - s)\varphi^{-1}(r - s) = \xi(r - s).$$

Equations (15) and (16) are significantly simplified for the case of plane symmetry when $\nu = 0$.

In contrast to the case of Eulerian coordinates, in Lagrangian coordinates the problem of defining a smooth flow with variable entropy is also reduced to a system of two quasilinear equations and therefore admits of the introduction of invariants.

Actually, suppose we know the entropy distribution at initial time instant $t = 0$:

$$S(q, 0) = S_0(q), \qquad (17)$$

and $S_0(q)$ is a differentiable function.

In the region of flow smoothness, from equation (13) follows

$$S(q, t) = S_0(q), \qquad (18)$$

and the problem reduces to solving the two equations (12) and (14). Introducing, as above, Riemann invariants

$$s = u + \int_{V_0}^{V} \sqrt{-\frac{\partial p(V, S_0(q))}{\partial V}}\, dV, \qquad (19)$$

$$r = u - \int_{V_0}^{V} \sqrt{-\frac{\partial p(V, S_0(q))}{\partial V}}\, dV, \qquad (20)$$

let us write equations (12) and (14) as

$$\frac{\partial s}{\partial t} - x^{\nu}\rho c \frac{\partial s}{\partial q} = f_1, \quad \frac{\partial r}{\partial t} + x^{\nu}\rho c \frac{\partial r}{\partial q} = f_2, \qquad (21)$$

where

$$f_1 = \frac{\nu c u}{x} - \frac{dS_0(q)}{dq}\left[x^{\nu} p'_S(V, S_0) - x^{\nu} c\rho \int_{\rho_0}^{\rho} \frac{\partial c(\rho, S_0(q))}{\partial S_0} \frac{d\rho}{\rho}\right], \qquad (22)$$

$$f_2 = -\frac{\nu c u}{x} - \frac{dS_0(q)}{dq}\left[x^{\nu} p'_S(V, S_0) - x^{\nu} c\rho \int_{\rho_0}^{\rho} \frac{\partial c(\rho, S_0(q))}{\partial S_0} \frac{d\rho}{\rho}\right]. \qquad (23)$$

Functions f_1 and f_2 are represented in the form

$$f_i = f_i\left(x, u, r - s, \frac{\partial S_0(q)}{\partial q}, S_0(q)\right), \qquad (24)$$

and functions (24) are linear relative to variables u, $\frac{\partial S_0}{\partial q}$. For greater clarity, let us write these equations for the case of plane symmetry, i.e., when $\nu = 0$:

$$\frac{\partial s}{\partial t} - \rho c \frac{\partial s}{\partial q} = f_1, \quad \frac{\partial r}{\partial t} + \rho c \frac{\partial r}{\partial q} = f_2, \qquad (25)$$

$$f_1 = f_2 = -\frac{\partial S_0}{\partial q}\left[p'_S - c\rho \int_{\rho_0}^{\rho} \frac{\partial c(\rho, S_0(q))}{\partial S} \frac{d\rho}{\rho}\right]. \qquad (26)$$

The quantities ρ and c are expressed uniquely in terms of r, s.

For the case of isothermal flow equations in Lagrangian variables as before are of the form (15) and (16), only the quantity c now appearing in the equations and in the definition of Riemann invariants are r, s is the isothermal speed of sound c_T (formula (2.7.17)).

9. Equations in invariants for polytropic and isothermal gases. For a polytropic gas pressure p is given by the formula

$$p=\frac{A^2(S)}{\gamma}\rho^\gamma, \quad \gamma=\frac{c_p}{c_V}>1, \tag{1}$$

therefore

$$c^2=\frac{\partial p}{\partial \rho}=A^2(S)\rho^{\gamma-1}, \quad c(\rho, S)=A(S)\rho^{\frac{\gamma-1}{2}}. \tag{2}$$

For an isentropic flow Riemann invariants are defined by formulas (2.7.7). Calculating $\varphi(\rho)$, we get

$$\varphi(\rho)=\int_{\rho_0}^{\rho}\frac{A(S)\rho^{\frac{\gamma-1}{2}}d\rho}{\rho}=\frac{2A(S)}{\gamma-1}\left[\rho^{\frac{\gamma-1}{2}}-\rho_0^{\frac{\gamma-1}{2}}\right].$$

since $\gamma>1$. For simplicity let us set $\rho_0=0$, then

$$\varphi(\rho)=\frac{2A(S)}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \quad c(\rho)=A(S)\rho^{\frac{\gamma-1}{2}}=\frac{\gamma-1}{2}\varphi(\rho), \tag{3}$$

i.e.

$$c(\rho)=\frac{\gamma-1}{4}(r-s). \tag{4}$$

Thus, isentropic flow equations (2.7.14) and (2.7.15) take on the following form for a polytropic gas:

$$\frac{\partial s}{\partial t}+(\alpha s+\beta r)\frac{\partial s}{\partial x}=\frac{\nu(\gamma-1)(r^2-s^2)}{4x}, \tag{5}$$

$$\frac{\partial r}{\partial t}+(\alpha r+\beta s)\frac{\partial r}{\partial x}=-\frac{\nu(\gamma-1)(r^2-s^2)}{4x}, \tag{6}$$

where

$$\alpha=\frac{1}{2}+\frac{\gamma-1}{4}>\frac{1}{2}>0, \quad \beta=\frac{1}{2}-\frac{\gamma-1}{4}. \tag{7}$$

Let us write these equations for the case of plane symmetry ($\nu = 0$):

$$\frac{\partial s}{\partial t}+(\alpha s+\beta r)\frac{\partial s}{\partial x}=0, \quad \frac{\partial r}{\partial t}+(\alpha r+\beta s)\frac{\partial r}{\partial x}=0. \tag{8}$$

Let us also write formulas (2.7.7) for invariants r and s:

$$s=u-\varphi(\rho)=u-\frac{2}{\gamma-1}c, \quad r=u+\varphi(\rho)=u+\frac{2}{\gamma-1}c, \tag{9}$$

and the reciprocal formulas:

$$u=\frac{r+s}{2}, \quad c=c(\rho, S_0)=\frac{\gamma-1}{4}(r-s). \tag{10}$$

For the case of an isothermal ideal gas $p = R\rho T_0$ (11)

therefore

$$c_T=\sqrt{RT_0}=\text{const}=c_0, \tag{12}$$

$$\varphi(\rho)=\int_{\rho_0}^{\rho}\frac{c_T\,d\rho}{\rho}=c_0\ln\frac{\rho}{\rho_0}. \tag{13}$$

Assuming for definiteness $\rho_0 = 1$, we will have

$$2\varphi(\rho)=(r-s)=2c_0\ln\rho, \quad \rho=\exp\left\{\frac{r-s}{2c_0}\right\}. \tag{14}$$

Equations in invariants for an isothermal gas are of the form

$$\frac{\partial s}{\partial t}+\left(\frac{r+s}{2}-c_0\right)\frac{\partial s}{\partial x}=\frac{\nu c_0}{2x}(r+s), \tag{15}$$

$$\frac{\partial r}{\partial t}+\left(\frac{r+s}{2}+c_0\right)\frac{\partial r}{\partial x}=-\frac{\nu c_0}{2x}(r+s). \tag{16}$$

Special cases of system (8) are of interest ($\nu = 0$). One of the cases, $\gamma = -1$ (corresponding to so-called Chaplygin's gas*)), reduces to a weakly nonlinear system:

$$\frac{\partial s}{\partial t}+r\frac{\partial s}{\partial x}=0, \quad \frac{\partial r}{\partial t}+s\frac{\partial r}{\partial x}=0. \tag{17}$$

Another interesting case, $\gamma = 3$, reduces to a degenerating system of equations

$$\frac{\partial s}{\partial t}+s\frac{\partial s}{\partial x}=0, \quad \frac{\partial r}{\partial t}+r\frac{\partial r}{\partial x}=0, \tag{18}$$

whose characteristics, as we have seen in chapter one, are straight lines.

In concluding this subsection, let us derive equations in Lagrangian variables for the motion of a polytropic gas in the case of plane symmetry ($\nu = 0$) with variable entropy.

From formulas (2.8.25) and (2.8.26), in this case, we obviously obtain

$$\frac{\partial s}{\partial t}-B(S_0(q))(r-s)^{\frac{\gamma+1}{\gamma-1}}\frac{\partial s}{\partial q}=f_1, \tag{19}$$

$$\frac{\partial r}{\partial t}+B(S_0(q))(r-s)^{\frac{\gamma+1}{\gamma-1}}\frac{\partial s}{\partial q}=f_2, \tag{20}$$

where

$$B(S_0(q))=\left[\frac{\gamma-1}{4A(S_0(q))}\right]^{\frac{\gamma+1}{\gamma-1}}A(S_0(q)), \tag{21}$$

$$f_1=f_2=\frac{dS_0(q)}{dq}\left[\frac{1}{\gamma(\gamma-1)}\rho^{\gamma}\right]\cdot 2A(S_0(q))A'(S_0(q)), \tag{22}$$

and ρ, is always, is defined as some function of the difference r - s and the quantity $S_0(q)$.

Let us also observe that for the case of the isentropic flow of Chaplygin's gas ($\gamma = -1$), equations (19) and (20) become linear:

$$\frac{\partial s}{\partial t}-B\frac{\partial s}{\partial q}=0, \quad \frac{\partial r}{\partial t}+B\frac{\partial r}{\partial q}=0. \tag{23}$$

Hence we have the general integral $s = s_0(q + Bt)$, $r = r_0(q - Bt)$, where s_0, r_0 are arbitrary functions of the same argument. From this it follows that if the solution r, s to system (23) is smooth at the instant $t = t_0$, it remains smooth than for any t.

*) To lend physical meaning to the equation of state, here we must assume that $p = p_0 - \frac{A^2(S)}{\nu\rho}$, p_0 = const > 0.

System (23) is the weakly nonlinear system (17) transform into Lagrangian variables. Consequently, solutions to system (17) also retain their smoothness. This fact can be viewed as an illustration of the theorem on weakly nonlinear systems demonstrated in section X of chapter one.

III. Study of the Simplest Plane One-Dimensional Flows

In this section we will study the quality of properties of the simplest, mainly isentropic or isothermal, flows for the case of plane symmetry ($\nu = 0$).

1. General properties. Integration for the case $\gamma = 3$. In Eulerian coordinates isentropic flow is described by a system of two equations, which for the case of a polytropic gas is of the form

$$\frac{\partial s}{\partial t}+(\alpha s+\beta r)\frac{\partial s}{\partial x}=0, \quad \frac{\partial r}{\partial t}+(\alpha r+\beta s)\frac{\partial r}{\partial x}=0, \tag{1}$$

$$\alpha=\frac{1}{2}+\frac{\gamma-1}{4}, \quad \beta=\frac{1}{2}-\frac{\gamma-1}{4}, \quad \gamma\neq 1. \tag{2}$$

We will assume that the initial conditions $s(x, 0) = s_0(x)$, $r(x, 0) = r_0(x)$, (3) are imposed on system (1) and that the functions s_0 and r_0 are bounded and have a continuous first derivative.

Then from the results in Chapter One there follows the existence of solutions to Cauchy's problem (1), (2) in some strip $0 \leq t < t_0$; the variable t_0 is the time instant at which derivatives of the solution become unbounded.

We assume that the solution $s(x, t)$, $r(x, t)$ is known. We will call the integral curve of the equation $\frac{dx}{dt}=\xi_1(s, r)=\alpha s(x, t)+\beta r(x, t)=u-c$

The s-characteristic, analogous to the integral curve of the equation

$$\frac{dx}{dt}=\xi_2(s, r)=\alpha r(x, t)+\beta s(x, t)=u+c$$

of the r-characteristic.

The invariant $s = s(x, t)$ is constant along the s-characteristic $x = x_s(t, x_0)$, as follows from equations (1), i.e., $s(x_s(t, x_0), t) \equiv s_0(x_0)$, if $x_s(0, x_0) = x_0$.

Analogously $r(x_r(t, x_0), t) = r_0(x_0)$, if $x = x_r(t, x_0)$ is the equation of the r-characteristic passing through the point $x = x_0$ of the initial axis $t = 0$.

The estimates

$$\left.\begin{aligned}\min_x s_0(x) \leqslant s(x, t) \leqslant \max_x s_0(x),\\ \min_x r_0(x) \leqslant r(x, t) \leqslant \max_x r_0(x),\end{aligned}\right\} \quad (4)$$

obtain for the solution s, r, and are valid only in the domain $0 \leq t \leq t_0$ in which the solution remains continuous.

Since

$$s = u - \frac{2}{\gamma - 1} c, \quad r = u + \frac{2}{\gamma - 1} c, \quad u = \frac{r + s}{2}, \quad c = \frac{\gamma - 1}{4}(r - s), \quad (5)$$

from (4) there follow the estimates for velocity u and the speed of sound c (when $\gamma > 1$):

$$\begin{aligned}\min_x \left[u_0(x) - \frac{2}{\gamma - 1} c_0(x)\right] + \min_x \left[u_0(x) + \frac{2}{\gamma - 1} c_0(x)\right] \leqslant 2u(x, t) \leqslant\\ \leqslant \max_x \left[u_0(x) - \frac{2}{\gamma - 1} c_0(x)\right] + \max_x \left[u_0(x) + \frac{2}{\gamma - 1} c_0(x)\right], \quad (6)\end{aligned}$$

$$\begin{aligned}\min_x \left[u_0(x) + \frac{2}{\gamma - 1} c_0(x)\right] - \max_x \left[u_0(x) - \frac{2}{\gamma - 1} c_0(x)\right] \leqslant \frac{4}{\gamma - 1} c(x, t) \leqslant\\ \leqslant \max_x \left[u_0(x) + \frac{2}{\gamma - 1} c_0(x)\right] - \min_x \left[u_0(x) - \frac{2}{\gamma - 1} c_0(x)\right], \quad (7)\end{aligned}$$

where $c_0(x)$, $u_0(x)$ are the initial values of the speed of sound and flow velocity.

If we approximate these estimates, then we readily get

$$\begin{aligned}\min_x u_0(x) - \frac{1}{\gamma - 1}\left[\max_x c_0(x) - \min_x c_0(x)\right] \leqslant u(x, t) \leqslant\\ \leqslant \max_x u_0(x) + \frac{1}{\gamma - 1}\left[\max_x c_0(x) - \min_x c_0(x)\right], \quad (8)\end{aligned}$$

$$\begin{aligned}\min_x c_0(x) - \frac{\gamma - 1}{4}\left[\max_x u_0(x) - \min_x u_0(x)\right] \leqslant c(x, t) \leqslant\\ \leqslant \max_x c_0(x) + \frac{\gamma - 1}{4}\left[\max_x u_0(x) - \min_x u_0(x)\right]. \quad (9)\end{aligned}$$

Denoting

$$\min_x u_0(x) = u_0, \quad \max_x u_0(x) = U_0, \quad \min_x c_0(x) = c_0,$$
$$\max_x c_0(x) = C_0, \quad \Delta u_0 = U_0 - u_0, \quad \Delta c_0 = C_0 - c_0,$$

Let us rewrite inequalities (8) and (9) more concisely:

$$u_0 - \frac{1}{\gamma-1}\Delta c_0 < u(x, t) < U_0 + \frac{1}{\gamma-1}\Delta c_0, \tag{10}$$

$$c_0 - \frac{\gamma-1}{4}\Delta u_0 < c(x, t) < C_0 + \frac{\gamma-1}{4}\Delta u_0. \tag{11}$$

We must bear in mind that in its physical meaning the speed of sound $c \geqslant 0$. Therefore if $c_0 - \frac{\gamma-1}{4}\Delta u_0 < 0$ or

$$\min_x\left[u_0(x) + \frac{2}{\gamma-1}c_0(x)\right] - \max_x\left[u_0(x) - \frac{2}{\gamma-1}c_0(x)\right] < 0,$$

then in the left side of inequalities (11) and (7) we must insert zero. From formulas (5) it therefore also follows

$$\min_x\left[u_0(x) - \frac{2}{\gamma-1}c_0(x)\right] < u(x, t) < \max_x\left[u_0(x) + \frac{2}{\gamma-1}c_0(x)\right].$$

However, when $c_0 - \frac{\gamma-1}{2}\Delta u_0 < 0$, estimate $c(x, t) \geqslant 0$ also proves to be rough. Therefore we will now show that if the initial functions $u_0(x)$ and $c_0(x)$ possess bounded derivatives, then $c(x, t) \neq 0$ for no finite value $t > 0$ whatever. At the same time we will establish useful estimates for derivatives of the solution to Cauchy's problem (1), (3).

Since the assertion to be proved is general in scope, we will consider the case of any $\gamma \geqslant -1$. When $\gamma \geqslant -1$, $\alpha \geqslant 0$.

Theorem. If in an isentropic flow of a polytropic gas ($\gamma \geqslant -1$) no shock waves are induced (no characteristics of the same family are intersected) and if the initial values $r_0(x)$ and $s_0(x)$ of the Riemann invariants are differentiable, there exists the functions $P(t) < \infty$ and $\rho_0(t) > 0$ such that

$$\frac{\partial u(x, t)}{\partial x} < P(t), \quad \rho(x, t) > \rho_0(t).$$

Proof. Let $\frac{\partial s}{\partial x} = p_1$, $\frac{\partial r}{\partial x} = p_2$, and assume

$$\frac{ds_0(x)}{dx} = p_1(x, 0) < P_0, \quad \frac{dr_0(x)}{dx} = p_2(x, 0) < P_0 \tag{12}$$

and also

$$0 < \rho_0 < \rho(x, 0) = \rho_0(x) = \left[\frac{\gamma-1}{4A}(r_0(x) - s_0(x))\right]^{\frac{2}{\gamma-1}} < R_0 \quad (13)$$

We denote

$$\frac{\partial}{\partial t} + (\alpha s + \beta r)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x} = \left(\frac{d}{dt}\right)_s,$$

$$\frac{\partial}{\partial t} + (\alpha r + \beta s)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x} = \left(\frac{d}{dt}\right)_r.$$

Then, by differentiating equations (1) relative to variable x, we get

$$\left(\frac{dp_1}{dt}\right)_s = -\alpha p_1^2 - \beta p_1 p_2, \quad \left(\frac{dp_2}{dt}\right)_r = -\alpha p_2^2 - \beta p_1 p_2. \quad (14)$$

Since $\beta = 1 - \alpha$, these equations can be rewritten in the form

$$\left(\frac{d}{dt}\ln p_1\right)_s = -\alpha(p_1 - p_2) - p_2, \quad \left(\frac{d}{dt}\ln p_2\right)_r = -\alpha(p_2 - p_1) - p_1. \quad (15)$$

Writing equations (1) as

$$\left(\frac{ds}{dt}\right)_s = \frac{\partial s}{\partial t} + (u - c)\,p_1 = 0, \quad \left(\frac{dr}{dt}\right)_r = \frac{\partial r}{\partial t} + (u + c)\,p_2 = 0,$$

we can easily establish the formulas

$$p_1 = \frac{1}{2c}\left(\frac{ds}{dt}\right)_r = -\frac{1}{2c}\left[\frac{d}{dt}(r - s)\right]_r = -\frac{2}{\gamma-1}\frac{1}{c}\left(\frac{dc}{dt}\right)_r = -\left[\frac{d}{dt}\ln c^{\frac{2}{\gamma-1}}\right] = -\left(\frac{d\ln\rho}{dt}\right)_r,$$

$$p_2 = -\frac{1}{2c}\left(\frac{dr}{dt}\right)_s = -\frac{1}{2c}\left[\frac{d}{dt}(r - s)\right]_s = -\left[\frac{d}{dt}\ln\rho\right]_s.$$

Let us substitute these formulas into equations (15); then they will become

$$\left[\frac{d}{dt}\ln\frac{p_1}{\rho}\right]_s = -\alpha(p_1 - p_2), \quad \left[\frac{d}{dt}\ln\frac{p_2}{\rho}\right]_r = -\alpha(p_2 - p_1). \quad (16)$$

when $\nu \geqslant -1, \alpha \geqslant 0$. Therefore, if at any point x, t $p_1(x, t) \geqslant p_2(x, t)$, at this point

$$\left(\frac{d}{dt}\ln\frac{p_1(x,t)}{\rho(x,t)}\right)_s \leqslant 0,$$

i.e., the quantity p_1/ρ does not increase along the s-characteristic with increase in variable t.

In general, from equations (16) it follows that

$$\frac{\max\{p_1(x,t),\ p_2(x,t)\}}{\rho(x,t)} \leqslant \frac{P_0}{\rho_0} \tag{17}$$

Let us prove this inequality in somewhat greater detail. Suppose A(x, t) is an arbitrary point on the half-plane $t \geqslant 0$. Let us pass through this point s-characteristic I and r-characteristic II (Figure 2.10).

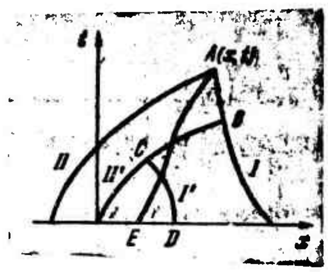


Figure 2.10

Suppose, for example, that at the point A(x, t) $p_1(x, t) \geqslant p_2(x, t)$, and suppose that this relation is also satisfied over the segment AB of characteristic I, i.e., $p_1(x', t') \geqslant p_2(x', t')$, and below point B (Figure 2.10) on characteristic I this relation changes into its inverse ($p_1 \leqslant p_2$) and $p_1 = p_2$ at point B. Then by the first equation in (16)

$$\frac{p_1(x,t)}{\rho(x,t)} \leqslant \frac{p_1(B)}{\rho(B)} = \frac{p_2(B)}{\rho(B)}.$$

Suppose that over the segment BC of characteristic II' running through point B, $p_2 \geqslant p_1$. Then by the second equation in (16)

$$\frac{p_2(B)}{\rho(B)} \leqslant \frac{p_2(C)}{\rho(C)}.$$

If at the point C, $p_1(C) = p_2(C)$ and that CD, $p_1 \geqslant p_2$, then by the first equation in (16)

$$\frac{p_2(C)}{\rho(C)} = \frac{p_1(C)}{\rho(C)} \leqslant \frac{p_1(D)}{\rho(D)}.$$

Thus we sooner or later arrive at the initial axis t = 0. For the case shown in Figure 2.10 we will have

$$\frac{p_2(x,t)}{\rho(x,t)} < \frac{p_1(x,t)}{\rho(x,t)} \leqslant \frac{p_2(B)}{\rho(B)} < \frac{p_1(C)}{\rho(C)} \leqslant \frac{p_1(D)}{\rho(D)} \leqslant \frac{P_0}{\rho_0}. \quad (18)$$

This chain of inequalities convinces us of the validity of assertion (17). And so, from (17) there follow the estimates of the derivatives

$$p_1(x,t) = \frac{\partial s(x,t)}{\partial x} \leqslant \frac{P_0\rho(x,t)}{\rho_0}; \quad p_2(x,t) = \frac{\partial r(x,t)}{\partial x} \leqslant \frac{P_0\rho(x,t)}{\rho_0}, \quad (19)$$

where the quantities p_0 and ρ_0 are given by equations (12) and (13). Since $u = \frac{1}{2}(r+s)$, then from (19) there also follows the one-sided estimate of the derivative $\partial u/\partial x$:

$$\frac{\partial u}{\partial x} \leqslant \frac{P_0\rho(x,t)}{\rho_0}. \quad (20)$$

Writing the continuity equation

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} = -\rho\frac{\partial u}{\partial x}$$

in the form

$$\left(\frac{d\rho}{dt}\right)_0 = -\rho\frac{\partial u}{\partial x}, \quad \left(\frac{d}{dt}\right)_0 = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},$$

we conclude that according to (20)

$$\left(\frac{d\rho}{dt}\right)_0 \geqslant -\frac{P_0\rho^2(x,t)}{\rho_0}, \quad \text{i.e.,} \left(\frac{d}{dt}\frac{1}{\rho}\right) \leq \frac{P_0}{\rho_0} \quad (21)$$

Integrating inequality (21) from arbitrary point A(x, t) to point E on the initial axis along the trajectory AE (Figure 2.10), we obviously obtain

$$\frac{1}{\rho(x,t)} - \frac{1}{\rho_0(E)} \leqslant \frac{P_0 t}{\rho_0},$$

i.e.,

$$\rho(x,t) \geqslant \frac{\rho_0(E)\rho_0}{\rho_0 + \rho_0(E)P_0 t} \geqslant \frac{\rho_0^2}{\rho_0 + R_0 P_0 t}. \quad (22)$$

Inequality (22) states that density $\rho(x, t)$ cannot tend to zero if the initial values possess a bounded derivative (obviously, we must assume that $P_0 \geqslant 0$). Inequalities (20) and (22) prove the theorem.

Inequality (22) tells us that under these conditions when $t \geqslant 0$ a mutually unique correspondence between Eulerian and Lagrangian coordinates holds.

This fact is physically interpreted thusly. A gas continuously filling at time instants $t = 0$ the entire space with density different from zero cannot in its motion "collapse", i.e., form a vacuum region in which $\rho = 0$ if and only if when $t = 0$ velocity discontinuities $u_0(x)$ such that $u_0(x - 0) < u_0(x + 0)$ are absent.

This fact obtains also in more general cases. It is valid, in particular, both for nonisentropic continuous flows and for ideal gases, as well as for discontinuous solution (flows with shock waves).

Since $c = A\rho^{\frac{\gamma-1}{2}}$, then when $\gamma > 1$, from (22) follows the estimate from below also for the quantity $c(x, t)$. Since when $\gamma > 1$, $c(x, t)$ and therefore $\rho(x, t)$ are bounded from above by means of inequalities (11), then from (20) there follows the estimate of the derivative

$$\frac{\partial u(x, t)}{\partial x} \leqslant P_0\left[\frac{C_0}{c_0} + \frac{\gamma-1}{4}\frac{\Delta u_0}{c_0}\right]^{\frac{2}{\gamma-1}}.$$

Let us note that when $\gamma < 1$ inequalities (11) vary in such a manner that together with (22) they bound the value $c(x, t)$ from above and from below.

In the foregoing we established estimates for p_1, p_2 from above. In estimating these quantities from below, we note that from system (14) it follows that if $P(t)$ stands for the quantity

$$P(t) = \max_{-\infty < x < \infty} \max\{|p_1(x, t)|, |p_2(x, t)|\},$$

then

$$\frac{dP(t)}{dt} \leqslant (|\alpha| + |\beta|) P^2(t).$$

Hence

$$\left.\begin{aligned} P(t) &\leqslant \frac{P(0)}{1 - (|\alpha| + |\beta|) t \cdot P(0)}, \\ p_i(x, t) &\geqslant \frac{-P(0)}{1 - (|\alpha| + |\beta|) t \cdot P(0)}. \end{aligned}\right\} \quad (23)$$

Estimates (19) and (23) bound derivatives of the solution from above and from below. From (23) it follows that when $t < 1/(|\alpha|+|\beta|)P(0)$ the derivatives remain bounded both from above and from below. Thus, the strip $0 \leq t \leq 1/(|\alpha|+|\beta|)P(0)$ is a strip in which a classical solution to Cauchy's problem (1), (2) clearly exist.

Naturally, estimate (23) is to growth. Thus, for example, if $ds_0/dx \geqslant 0$, $dr_0/dx \geqslant 0$, then $p_1 \geqslant 0$, $p_2 \geqslant 0$, as follows from equations (14) and as a consequence of (18) a classical solution exists for all $t > 0$.

Let us clarify our general remarks with the example of the flow of a polytropic gas with exponent $\gamma = 3$. In this case $\alpha = 1$, $\beta = 0$ and the system of equations (1) decompose into two individual equations:

$$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = 0, \quad \frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0. \tag{24}$$

Since invariant s is constant along the s-characteristic, and the slope of the s-characteristic is s, these characteristics are straight lines.

Along the line $x = x_0 + s_0(x_0)t = x_s(t, x_0)$, we have $s(x, t) = s_0(x_0)$; similarly, $r(x, t) = r_0(x_0)$ along the line $x = x_0 + r_0(x_0)t$. Thus, the solution to Cauchy's problem for system (24) is given implicitly by the formulas

$$s(\xi + s_0(\xi)t;\ t) = s_0(\xi),$$

$$r(\eta + r_0(\eta)t;\ t) = r_0(\eta).$$

For the explicit expression of solution $s(x, t)$, $r(x, t)$ we must solve the functions

$$x = \xi + s_0(\xi)t, \quad x = \eta + r_0(\eta)t \tag{25}$$

relative to the quantities ξ, η. Suppose we define them from (25):

$$\xi = \xi(x, t), \quad \eta = \eta(x, t);$$

then

$$s(x, t) = s_0(\xi(x, t)), \quad r(x, t) = r_0(\eta(x, t)). \tag{26}$$

The geometrical significance of the quantities $\xi(x, t)$, $\eta(x, t)$ is clear from Figure 2.11.

Let us verify that several of the estimates obtained above are satisfied. Differentiating relations (25) relative to the variable x, we get

$$\frac{\partial \xi(x, t)}{\partial x} = \frac{1}{1 + t\,\frac{ds_0}{dx}(\xi(x, t))}.$$

Therefore, from (26) it follows that

$$\frac{\partial s(x, t)}{\partial x} = \left(\frac{\frac{ds_0(x_0)}{dx}}{1 + t\,\frac{ds_0}{\partial x}(x_0)} \right)_{x_0 = \xi(x, t)}$$

and we see that estimates (23) hold in this case. By Figure 2.11

$$r(x, t) - s(x, t) = \frac{\xi(x, t) - \eta(x, t)}{t} = r_0(\eta(x, t)) - s_0(\xi(x, t)). \qquad (27)$$

Let us make the following estimates;

$$r_0(\eta) - s_0(\xi) = r_0(\eta) - s_0(\eta) + s_0(\eta) - s_0(\xi) = 2c_0(\eta) + s_0(\eta) - s_0(\xi). \qquad (28)$$

From (27) and (28) therefore must follow

$$2c(x, t) \geqslant 2c_0(\eta) - P_0[\xi(x, t) - \eta(x, t)]. \qquad (29)$$

But from (27) we have $\xi(x, t) - \eta(x, t) = 2c(x, t)t$. (30)
Combining equality (30) with inequality (29), we get

$$c(x, t) > \frac{c_0(\eta)}{1 + P_0 t} > \frac{\rho_0}{1 + P_0 t} > \frac{\rho_2^0}{\rho_0 + R_0 P_0 t}. \qquad (31)$$

Since when $\nu = 3$, $\rho = c$, equality (31) is a special case of estimate (22). Similarly, for the case $\nu = 3$ we can easily verify all the other estimates obtained above.

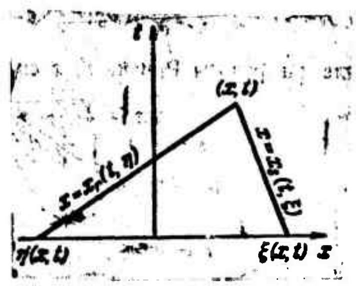


Figure 2.11

2. Travelling waves (Riemann waves). Waves of compression and rarefaction. Here we look at several of the simplest isentropic flows for the case of plane symmetry ($\nu = 0$). In studying travelling waves it is of practical indifference as to which variables -- Eulerian or Lagrangian -- we conducted our examination. Here let us use Lagrangian. Then isentropic flow equations are written as

$$\frac{\partial s}{\partial t} - \xi(r-s)\frac{\partial s}{\partial q} = \frac{\partial s}{\partial t} - c\rho\frac{\partial s}{\partial q} = 0, \qquad (1)$$

$$\frac{\partial r}{\partial t} + \xi(r-s)\frac{\partial r}{\partial q} = \frac{\partial r}{\partial t} + c\rho\frac{\partial r}{\partial q} = 0, \qquad (2)$$

where Riemann invariants r, s are associated with u, V by the formulas

$$s = u + \int_{\bar{V}}^{V} \frac{c}{V}\,dV = u + \int_{\bar{V}}^{V} \sqrt{-\frac{\partial p}{\partial V}}\,dV,$$

$$r = u - \int_{\bar{V}}^{V} \frac{c}{V}\,dV = u - \int_{\bar{V}}^{V} \sqrt{-\frac{\partial p}{\partial V}}\,dV.$$

For an isentropic flow p is a function of the single variable V.

We will assume that $p = p(V)$ is an arbitrary differentiable function satisfying, however, conditions I and II:

$$\frac{dp}{dV} < 0, \quad \frac{d^2p}{dV^2} > 0. \qquad (3)$$

Then, denoting

$$\varphi(V) = -\int_{V}^{V} \frac{c(V)\,dV}{V} = \frac{r-s}{2},$$

we get

$$\varphi'(V) = -\frac{c(V)}{V} < 0$$

and

$$\xi'(r-s) = -\frac{1}{4}\frac{\frac{d^2p}{dV^2}}{\frac{dp}{dV}} > 0.$$

Thus, from conditions (3) it follows that $\xi(r - s)$ is a monotonically increasing function of the difference $r - s$.

Isentropic flow in which one of the Riemann invariants is constant is called a Riemann wave for a travelling wave. Suppose for definiteness that $r = r_0 = \text{const}$; then equation (2) is satisfied identically, and equation (1):

$$\frac{\partial s}{\partial t} - \xi(r_0 - s)\frac{\partial s}{\partial q} = 0, \qquad (4)$$

serves for the determination of the function $s(q, t)$.

Characteristics are integral curves of the equation

$$\frac{dq}{dt} = -\xi(r_0 - s(q, t)),$$

and are obviously straight lines in the q, t plane since the invariant $s(q, t)$ is constant along these. Hence it follows that along the straight lines

$$\frac{dq}{dt} = \frac{q - q_0}{t - t_0} = -\xi(r_0 - s(q, t)) \qquad (5)$$

both $s(q, t)$ and $r = r_0$ are constants, and therefore, all hydrodynamic variables V, p, ρ, c, u are constants. Let us note at once that the s-characteristics will be straight lines also in the plane of Eulerian coordinates x, t.

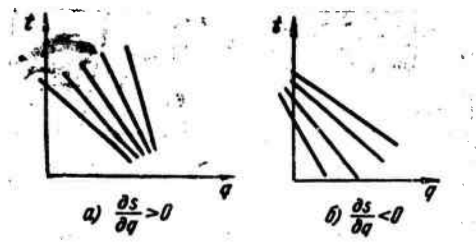


Figure 2.12

If in some region $\partial s/\partial q > 0$, the characteristics (5) form a divergent bundle of straight lines (Figure 2.12, a); if however $\partial s/\partial q < 0$, then they form a convergent bundle of lines (Figure 2.12, b). Since $u = \frac{1}{2}(r + s)$ $= \frac{1}{2}(r_0 + s)$, then from $\partial s/\partial q > 0$ follows $\partial u/\partial q > 0$. From the continuity equation, $\partial V/\partial t = \partial u/\partial q$; therefore it follows that when $\partial s/\partial q > 0, \partial V/\partial t > 0$, and density ρ decreases. Therefore the Riemann wave in which $\partial s/\partial q > 0$ is called the rarefaction wave, and the Riemann wave in which $\partial s/\partial q < 0$ is the compression wave.

The case when $s = s_0 =$ constant is treated quite similarly. In this case also when $\partial r/\partial q < 0$ we have a compression wave, and when $\partial r/\partial q > 0$ -- a rarefaction wave. Thus, the general characteristic of the compression wave $\partial u/\partial q < 0$ or, which amounts to the same thing, $\partial u/\partial x < 0$ leads to the conditions $\partial s/\partial q < 0$ for the Riemann s-wave $(r = r_0)$ and to $\partial r/\partial q < 0$ for the Riemann r-wave $(s = s_0)$.

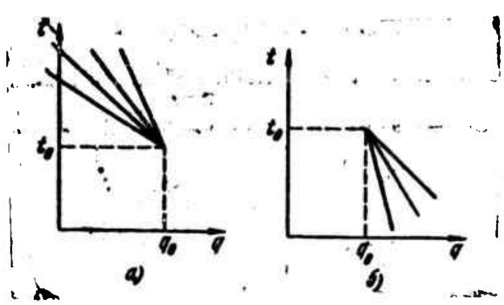


Figure 2.13

For Chaplygin's gas, $p = AV + p_0$, $d^2p/dV^2 = 0$, and $\xi = \sqrt{-A}$ is the constant. The slope of the characteristics in this case is fixed and therefore all s-characteristics are parallel to each other, just as are the r-characteristics.

Nevertheless condition $\partial s/\partial q > 0$ again distinguishes the rarefaction region in the Riemann wave $r = r_0$.

The Riemann wave ($r = r_0$) is called centered if the s-characteristics form a bundle of straight lines departing from a single point (q_0, t_0) let (Figure 2.13). Since the invariant s(q, t) is constant along the s-characteristic, it follows that in the central the Riemann wave

$$s = s\left(\frac{q-q_0}{t-t_0}\right), \quad r = r_0$$

or

$$r = r\left(\frac{q-q_0}{t-t_0}\right), \quad s = s_0.$$

Solutions dependent only on the variable $y = \frac{q - q_0}{t - t_0}$ are called self-modeling.

Let us show that centered Riemann waves yield all self-modeling solutions to gas dynamics equations.

Writing equations (2.5.4)-(2.5.6) in the form

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial q} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial q} = 0, \quad \frac{\partial S}{\partial t} = 0 \quad (v = 0) \tag{6}$$

and assuming that the variables V, u, p, S depend only on $y = \frac{q - q_0}{t - t_0}$, we make the substitution by the formulas

$$\frac{\partial}{\partial t} = -\frac{y}{t-t_0}\frac{d}{dy}, \quad \frac{\partial}{\partial q} = \frac{1}{t-t_0}\frac{d}{dy},$$

after which we arrive at the equations:

$$y\frac{dV}{dy} + \frac{du}{dy} = 0, \quad y\frac{du}{dy} - \frac{dp}{dy} = 0, \quad \frac{dS}{dy} = 0.$$

From the last equation it follows that $S = S_0$ = constant, and the self-modeling flow is, therefore, an isentropic flow.

Converting in the remaining two equations to Riemann invariants, let us rewrite them in the form

$$[y+\xi(r-s)]\frac{ds}{dy}=0, \quad [y-\xi(r-s)]\frac{dr}{dy}=0.$$

If $y \neq -\xi(r-s)$, then $ds/dy = 0$ and $s = s_0$. Since $\xi(r-s) > 0$, then when $y = -\xi(r-s)$ we have $r = r_0$ = constant. And so, there are only two possibilities for the self-modeling solution: either

$$s=s_0=\text{const}, \quad y=\xi(r-s_0), \tag{7}$$

or

$$r=r_0=\text{const}, \quad y=-\xi(r_0-s), \tag{8}$$

and, therefore, any self-modeling solution to equations (6) is a Riemann wave.

Since $\xi'(r-s) > 0$, then by differentiating equalities (7) and (8) relative to q, we conclude that any self-modeling solution when $t < t_0$ is a compression wave, and when $t > t_0$ -- a rarefaction wave.

Thus, if we consider the half-plane $t \geq 0$, then in it any self-modeling solution dependent on the variable $y = q/t$ ($q_0 = t_0 = 0$), is a Riemann rarefaction wave.

In Eulerian variables a centered Riemann wave is given by the conditions

$$s=s_0=\text{const}, \quad y=\frac{x-x_0}{t-t_0}=u+c$$

or

$$r=r_0=\text{const}, \quad y=\frac{x-x_0}{t-t_0}=u-c.$$

Let us consider several relations holding for an arbitrary Riemann wave for the case of a polytropic gas.

For a polytropic gas

$$s=u-\frac{2}{\gamma-1}c, \quad r=u+\frac{2}{\gamma-1}c.$$

Suppose that in the Riemann wave $r = r_0$ = constant, and suppose u_0, c_0 are the values of the velocity and the speed of sound at some point in the Riemann wave. Then

$$u_0+\frac{2}{\gamma-1}c_0=u+\frac{2}{\gamma-1}c,$$

or

$$c=c_0\left(1-\frac{\gamma-1}{2}\frac{u-u_0}{c_0}\right). \tag{9}$$

Since $p = \frac{1}{\gamma} A^{-\frac{2}{\gamma-1}} c^{\frac{2\gamma}{\gamma-1}}$, then from (9) we have a relation between pressure p and velocity u in the Riemann s-wave:

$$p = p_0\left[1 - \frac{\gamma-1}{2}\frac{u-u_0}{c_0}\right]^{\frac{2\gamma}{\gamma-1}} \quad (r = r_0 = \text{const}). \tag{10}$$

Similarly, in the Riemann r-wave:

$$p = p_0\left[1 + \frac{\gamma-1}{2}\frac{u-u_0}{c_0}\right]^{\frac{2\gamma}{\gamma-1}} \quad (s = s_0 = \text{const}). \tag{11}$$

Finally, let us note yet another important property of the Riemann wave.

Any continuous flow adjoining the zone of constant flow is a Riemann wave.

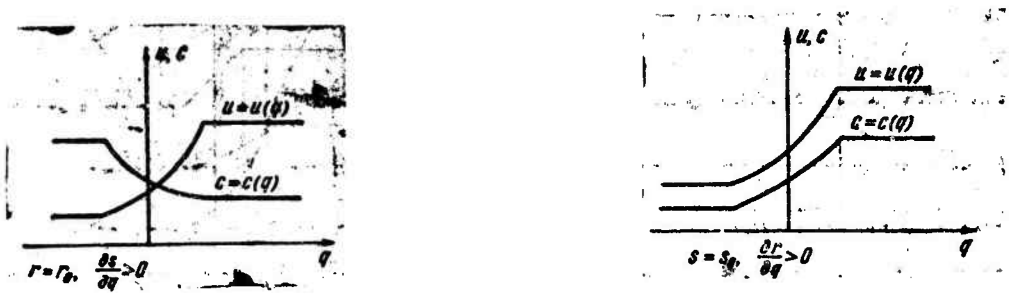
Actually, suppose the flow is continuous and constant leftward from the line AB (Figure 2.14). This means that the line AB is a line through which the solution to a system of equations in gas dynamics is not uniquely extended and, therefore, the line AB is a characteristic. Since the flow leftward from AB is constant, AB is a straight line.

Suppose, for example, that AB is an s-characteristic, then to the right of AB $r = r_0$ and the flow to the right of AB is a Riemann s-wave.

Figure 2.14

3. Profiles in a Riemann wave. Radiant catastrophe. Let us consider the behavior of hydrodynamic variables in traveling waves of compression and rarefaction. Suppose, for example, $r = r_0$ = constant and $\frac{\partial s}{\partial q} > 0$, i.e., we are considering the case of a rarefaction s-wave.

As we saw above, in this case $\frac{\partial u}{\partial q} > 0$, $\frac{\partial V}{\partial t} > 0$. Since $\xi^2(r - s) = -dp/dV$, $d^2p/dV^2 > 0$, V decreases with increase in ξ and, therefore, c increases. Since, moreover, $\xi'(r - s) > 0$, then from the condition $\frac{\partial s}{\partial q} > 0$, $r = r_0 =$ constant it follows that $\frac{\partial V}{\partial q} > 0$, $\frac{\partial p}{\partial q} < 0$, $\frac{\partial c}{\partial q} < 0$. Thus, profiles of velocity u and the speed of sound c in the rarefaction-ness wave (r = constant, 0) are of the form shown in Figure 2.15, a.



(a) (b)

Figure 2.15

We similarly obtain profiles for the case of a rarefaction r-wave (Figure 2.15, b).

Figure 2.16, a and b shows the profiles of hydrodynamic variables in the case of a compression wave.

And thus, in the rarefaction wave $\partial u/\partial q > 0$, and in the compression wave $\frac{\partial u}{\partial q} < 0$. The wave r = constant differ from the wave s = constant by the sign of the variable $\frac{\partial u}{\partial q} \cdot \frac{\partial c}{\partial q}$; when $r = r_0$ $\frac{\partial u}{\partial q} \cdot \frac{\partial c}{\partial q} < 0$, when $s = s_0$ $\frac{\partial u}{\partial q} \cdot \frac{\partial c}{\partial q} > 0$.

Let us note that the sign du dc coincides with the sign du dp. Therefore in the plane of variables p, u (p, u-diagram), the family of states in the Riemann s-wave is described by curve 3.2.10, of the form shown in Figure 2.16, c. Similarly, for the r-wave du dp > 0 (Figure 2.16, d).

Let us consider the time variation of the profiles of hydrodynamic quantities.

Since the solution is constant in the traveling wave along the straight lines and since these lines diverge with increase in t in the rarefaction wave, in the rarefaction wave the gradients of all hydrodynamic variables decrease in absolute value with increase in time t.

Conversely, in the compression wave the characteristics of the corresponding family diverge with increase in time t and the gradients of all hydrodynamic variables increase in absolute value. Figure 2.17, a and b shows the variation in the profile $s = s(q, t)$ with increase in t in the waves of compression and rarefaction for the case $r = r_0 = $ constant.

The characteristics in the compression wave intersect each other at some finite value $t = t_0$. At the point of intersection of the characteristics the derivatives of the hydrodynamic variables become unbounded.

It is not difficult to calculate the instant of formation of the unbounded derivatives. Suppose $r = r_0$, and $s(q, 0) = s_0(q)$. Then by (3.2.5) two s-characteristics departing from the points $q = q_1^0$ and $q = q_2^0$ of the initial axis $t = 0$ intersect at instant t_n:

$$t_n = \frac{q_2^0 - q_1^0}{\xi(r_0 - s_0(q_1^0)) - \xi(r_0 - s_0(q_2^0))}. \quad (1)$$

For the case when $s_0(q)$ is differentiable, we obtain the smallest time instant t, for which the s-characteristics intersect each other:

$$t_{min} = \frac{1}{\max_q \left[-\frac{\partial \xi(r_0 - s_0(q))}{\partial q}\right]}. \quad (2)$$

Formula (2) is meaningful only for the case when $\max_q [-\partial \xi / \partial q] > 0$. If $t_{min} \leq 0$, the s-characteristics do not intersect each other when $t > 0$ and the derivatives in the Riemann wave remain bounded.

Thus, in any traveling compression wave gradients increase and become unbounded in a finite time interval. This phenomenon is sometimes called gradient catastrophe.

When $\frac{\partial^2 p}{\partial V^2} > 0$, gradient catastrophe commences in a traveling compression wave, and when $\frac{\partial^2 p}{\partial V^2} < 0$ -- in a traveling rarefaction wave. Thus, when $\frac{\partial^2 p}{\partial V^2} > 0$ a compression shock wave is formed, and when $\frac{\partial^2 p}{\partial V^2} < 0$ — a rarefaction shock wave (cf section IV).

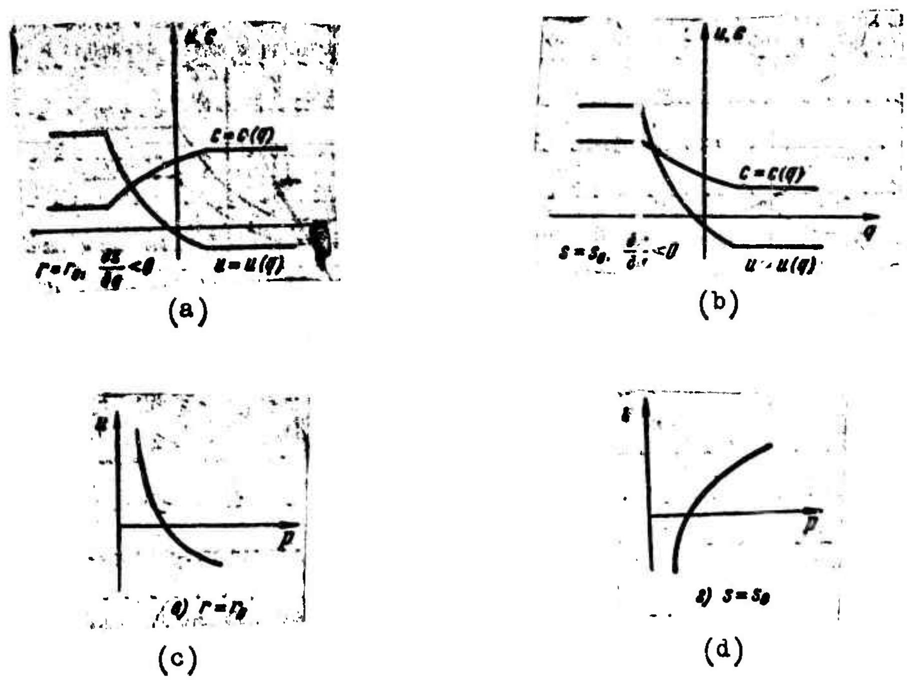


Figure 2.16

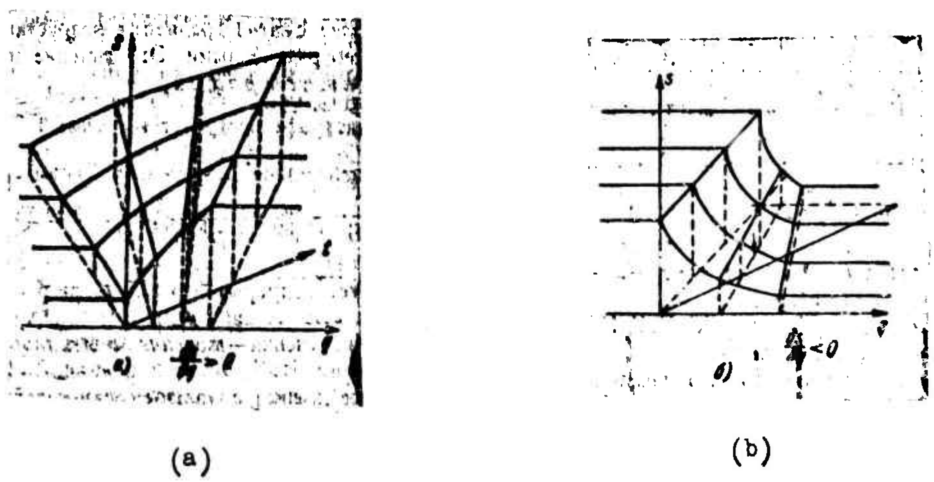
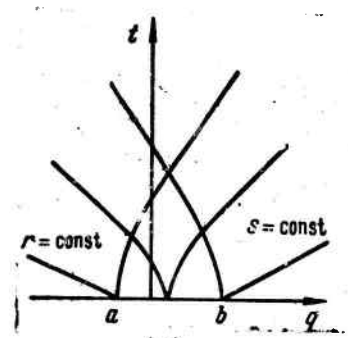
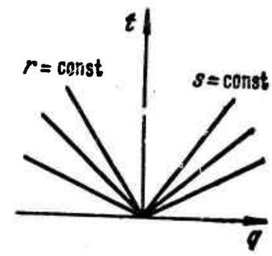


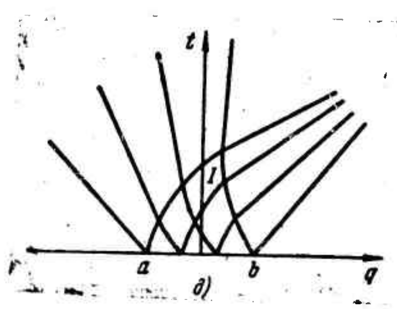
Figure 2.17



(c)



(d)



(e)

Figure 2.17

No continuous solution exists when $t > t_{min} > 0$: the solution becomes discontinuous.

Let us consider a more complex form of flow -- flow with local initial data.

We will state that initial data are local if initial functions $r_0(q)$, $s_0(q)$ are variable only over a finite interval $a \leq q \leq b$ of the axis $t = 0$, i.e.,

$$r_0(q) = \begin{cases} r^- & \text{when } q < a, \\ r^+ & \text{when } q > b, \end{cases} \qquad s_0(q) = \begin{cases} s^- & \text{when } q < a, \\ s^+ & \text{when } q > b. \end{cases} \tag{3}$$

Flow arising under the initial conditions (3) is not described by traveling Riemann waves. However, in several cases of flow, for sufficiently large $t > 0$, it can consist only of traveling waves.

If, for example, $r_0'(q) \geqslant 0$, $s_0'(q) \geqslant 0$, the solution has bounded derivatives for any $t \geqslant 0$, as follows from the estimates (cf section I). Values of the invariant s are transported along the s-characteristics, with velocity $-\rho c < 0$; the values of invariant r are transported along the r-characteristics with velocity $\rho c > 0$. Therefore for a certain $t_1 > 0$ the zones of variability of the variants r and s diverge and the solution will consist of two traveling rarefaction waves separated by a constant-flow zone (Figure 2.17, c).

Now with $a \to 0$, $b \to 0$, i.e., if we proceed to the problem with piecewise-constant initial data, solution $r(q, t)$, $s(q, t)$ will obviously tend to the self-modeling solution consisting of two centered rarefaction waves (Figure 2.17, d).

Cauchy's problem with piecewise-constant initial data

$$r_0(q) = \begin{cases} r^-, & q < 0, \\ r^+, & q > 0, \end{cases} \quad s_0(q) = \begin{cases} s^-, & q < 0, \\ s^+, & q > 0 \end{cases} \tag{4}$$

is called the problem of the decay of an arbitrary discontinuity and is studied in detail in section IV of this chapter.

The above analysis allows us to assert that if $s^- \leq s^+$, $r^- \leq r^+$, t[illegible] problem of decay has a solution that is consistent when $t > 0$, consisting of two centered rarefaction waves (Figure 2.17, d).

An analogous approach to the consideration of the problem of decay as a limiting problem with local initial data enables us to predict the quantitative behavior of the solution even when we reject the inequalities $r^- \leq r^+$, $s^- \leq s^+$. If, for example, $r^- > r^+$, then after the interaction of traveling waves (zone I) the compression r-wave will propagate toward the right (Figure 2.17, e). As we saw above, the gradients in the compression waves increase unboundedly. This leads to a discontinuity in the solution. Shock waves appear in the solution.

Based on this, we can assert that if $s^- > s^+$ or $r^- > r^+$, shock waves necessarily emerge in the solution to the decay problem.

4. Piston problem. Escape of gas into vacuum. Traveling waves find numerous applications in a number of these simplest problems, and also in the qualitative investigation of thus far extremely complicated flows. We will consider here several of the simple problems whose solutions are described with the aid of traveling waves.

Suppose a gas exists on one side (to the right) of a rigid wall(piston), which beginning at the initial time instant $t = 0$ moves according to a certain law.

We will assume that at the initial instant the gas is at rest and exhibits constant density, pressure, and entropy, i.e., we will presuppose that

$$u(q, 0) = 0, \quad p(q, 0) = p_0, \quad \rho(q, 0) = \rho_0, \quad S(q, 0) = S_0. \tag{1}$$

We will assume for the gas that its equation of state satisfying conditions I and II is given:

$$\frac{\partial p}{\partial V} < 0, \quad \frac{\partial^2 p}{\partial V^2} \geq 0, \quad p(\infty, S) = 0. \tag{2}$$

The piston is at the gas boundary, whose coordinate q we will take as equal to zero. The law of piston motion is expressed by the function $u(0, t) = U(t)$, (3) where $U(t)$ is the piston velocity, given as a function of the time t, and $u(q, t)$ is the gas velocity*). We will assume that $U(t)$ is a continuously differentiable function satisfying the condition $U(0) = 0$. (4)

Initially let us consider the case when $U'(t) \leq 0$. The construction of this problem reduces to finding the solution to equations of an isentropic flow satisfying the initial conditions (1) and boundary condition (3). In Riemann invariants the problem reduces to finding the solution to the equations

$$\frac{\partial s}{\partial t} - \xi(r - s)\frac{\partial s}{\partial q} = 0, \quad \frac{\partial r}{\partial t} + \xi(r - s)\frac{\partial r}{\partial q} = 0, \tag{5}$$

Satisfying the initial conditions

*) Equality (3) holds only when the gas does not separate from the piston.

$$s(q, 0) = -r(q, 0) = -\varphi(V_0), \quad V_0 = \frac{1}{\rho_0}, \tag{6}$$

and the boundary condition assigned at the straight line q = 0:

$$s(0, t) + r(0, t) = 2U(t). \tag{7}$$

Zone I (Figure 2.18), bounded on the left by the characteristic OA ($q = c_0 \rho_0 t$) is obviously the zone of flow constancy, i.e., in zone I

$$s = s_0, \quad r = r_0 = -s_0, \quad u = 0, \quad \rho = \rho_0, \quad c = c_0. \tag{8}$$

Zone II of an inconstant flow is bounded along the r-characteristic of the OA from the constant-flow zone. Therefore, flowing zone II is a Riemann wave. As we can see from Figure 2.18, the Riemann invariant s(q, t) is constant in zone II, and the values of this invariant are transported to zone II along the s-characteristics from the initial axis t = 0.

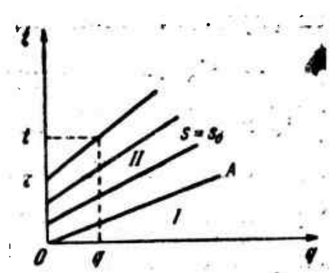


Figure 2.18

Thus, in zone II, $s(q, t) = s_0$ = constant, and it remains for us to integrate only the second equation in system (5):

$$\frac{\partial r}{\partial t} + \xi(r - s_0)\frac{\partial r}{\partial q} = 0. \tag{9}$$

If the r-characteristic intersects *) the axis q = 0 at the point $\tau > 0$, then at this point $s(0, \tau) = s_0$ and from (7) we define $r(0, \tau)$:
[*) at following page]

$$r(0, \tau) = 2U(\tau) - s_0 = 2U(\tau) + \varphi(V_0) \quad (10)$$

In zone II of the Riemann wave, invariant r(q, t) is constant along the r-characteristics, which are given by the straight lines:

$$\frac{q}{t-\tau} = \xi(r(0, \tau) - s_0) = \xi(r(q, t) - s_0) \quad (11)$$

Along the straight line (11) r(q, t) = r(0, τ) where r(0, τ) is given by formula (10). Formula (11) yields the solution r = r(q, t) parametrically by means of the parameter -- the ordinates of the point of intersection of characteristic (11) with axis q = 0. From (11) it follows that if U' (t) < 0, in zone II $\frac{\partial r(q, t)}{\partial q} > 0$, i.e., motion in zone II is a rarefaction wave and the pattern of characteristics is of the form shown in Figure 2.18. When solving boundary condition (7) it was assumed that the r-characteristics departing from the ray q $\geqslant$ 0 of the initial axis intersect the line q = 0 when t > 0.

According to its physical meaning, the quantity $c \geqslant 0$ (12) therefore formula (10) is meaningful only when the inequality

$$c = \psi(2U(t) - 2s_0) = \psi(2U(t) + 2\varphi(V_0)) > 0. \quad (13)$$

is satisfied.

According to (2), ψ (r - s) is a monotonically increasing function. Therefore inequality (13), generally speaking, bounds from below piston velocity U(t) for which the boundary condition (3) can be satisfied, i.e., bounds from below the gas velocity at the line q = 0.

For U(t) < 0 that are sufficiently small modulewise, inequality (13) is obviously clearly satisfied, since when U (t) = 0

$$\psi(2\varphi(V_0)) = \psi(r_0 - s_0) = c_0 > 0.$$

Suppose that for any t = t_1 inequality (13) is converted into an equality

*) As we will see below, for U(t) < 0 that are sufficiently large modulewise, this is not the case, i.e., the r-characteristics do not intersect the axis q = 0.

even when $t > t_1$, $U(t) < U(t_1)$. Then when $t > t_1$ boundary condition (3) loses its meaning.

In this case the piston separates some of the gas, and a vacuum region is induced between the piston and the gas. Formula (11) parametrically defines the solution $r(q, t)$ throughout the zone II assuming that the parameter τ lies within the segment $0 \leq \tau \leq t_1$.

Note that from the assumptions (2) it follows that if $\sqrt{-\frac{\partial p}{\partial V}} = \xi(r-s) = c\rho = 0$, then $p = 0$, therefore the boundary $q = 0$ providing $t > t_1$ can be viewed as the free boundary of the gas, i.e., the boundary of the gas with the vacuum in which $p = 0$, $\rho = 0$. The inconvenience of Lagrangian coordinates is manifested in this case in that in the q, t plane the boundary of the gas $q = 0$ coincides with the piston position, since in the region between them $\rho = 1/V = 0$.

The pattern becomes more graphic if we represent gas and piston motions in Eulerian coordinates. Figure 2.19 shows the pattern of the r-characteristics in the case of gas separation from the piston, in Eulerian coordinates.

In zone I we have as before a gas at rest, zone II bounded by the r-characteristics OC and AB is a zone of the rarefaction wave, and zone III (between the pisont trajectory $z = x(t)$ and the characteristic AB) is the vacuum zone. At the point $A(t = t_1)$ separation of a gas from the piston occurs. Note that obviously the straight line AB, which is the boundary between gas and vacuum is simultaneously an r- and a s-characteristic. For a polytropic gas, $\varphi(V) = \frac{2}{\gamma-1} c$, $\psi(r-s) = c = \frac{\gamma-1}{4}(r-s)$; therefore condition (13) can be written when $\gamma>1$ in the form

$$\left[U(t) + \frac{2}{\gamma-1} c_0\right] \geqslant 0. \tag{14}$$

i.e.

$$U(t) \geqslant -\frac{2}{\gamma-1} c_0. \tag{15}$$

And so, when $U(t) < -\frac{2}{\gamma-1} c_0$ the separation of gas from the piston commences. Note that if we consider the isothermal gas formally as a gas for which $\gamma = 1$, then from (15) it follows that the separation of an isothermal gas from a piston generally does not occur, since the quantity $\frac{2}{\gamma-1} c_0 \longrightarrow \infty$ when $\gamma \to 1$. The same conclusion can be obtained also from the formula for ρ, since

$$\rho = \exp\left\{\frac{r-s}{2c_T}\right\} > 0, \qquad \text{where } c_T = \sqrt{RT} = \text{constant}.$$

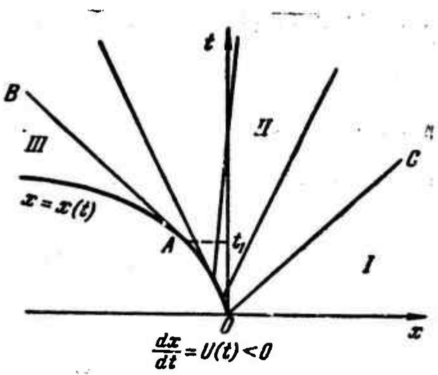


Figure 2.19

Another physical problem about pistons is solved in almost similar fashion, when it is not the piston velocity that is given, but the pressure on the piston. This leads to a boundary condition in Lagrangian variables: $p(0, t) = p_0(t)$. (16)

If we assume that under the conditions of the preceding problem $U(t)$ decreases monotonically over the segment $0 \leq t \leq t_1$, but when $t \geq t_1$, $U(t) = U_0$ = constant.

The pattern of characteristics in the q, t plane is presented for this case in Figure 2.20, a. We see that if $U_0 \geq -\frac{2}{\gamma-1} c_0$, then the flow is constant in the zone III.

If now the quantity t_1 tends to zero, at the limit as $t_1 \to 0$ we obtain a self-modeling solution, whose pattern of characteristics is shown in Figure 2.20, b. This self-modeling solution corresponds to the problem of the piston impelled from a gas at constant velocity $U_0 \geq -\frac{2}{\gamma-1} c_0$. For the case $U_0 < -\frac{2}{\gamma-1} c_0$ zone III vanishes, giving way to the rarefaction wave, i.e., zone II in this case displaces zone III. Providing $U_0 < -\frac{2}{\gamma-1} c_0$ at the boundary $q = 0$, condition $\rho = 0$ is satisfied, and the gas is not in contact with the piston, i.e., they are separated by a vacuum region.

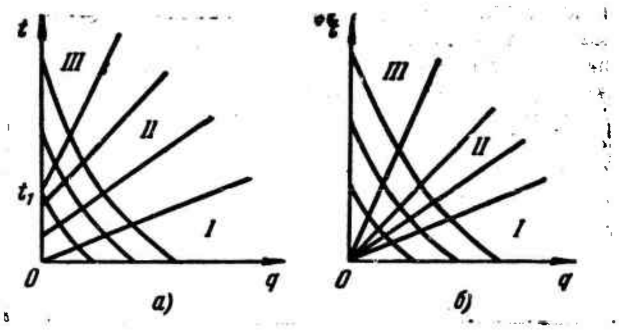


Figure 2.20

This self-modeling solution yields the solution to another problem, called the problem of the escape of gas into a vacuum. In this problem we can assume thh that at the initial time instant $t = 0$ the wall bounding the gas on the left (at the point $q = 0$) is taken away.

Now let us consider the second case, when $U'(t) \geqslant 0$, $U(0) = 0$. (17)

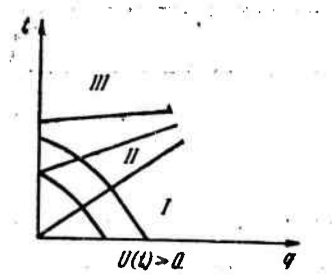


Figure 2.21

Boundary condition (7) can now be always satisfied, since providing $U(t)>0$ inequality (13) is always satisfied. We can readily see that when $U(t)>0$, $\frac{\partial r(q,t)}{\partial q} < 0$, and the flow in zone II is therefore a compression wave. Figure 2.21 gives the pattern of characteristics in the q, t plane. Since r-characteristics intersect each other when $t > 0$, the classical solution to the piston problem for the case $U(t)>0$ exists only for a limited time interval.

Beyond the intersection of two characteristics, the solution to the piston problem become discontinuous.

In general note that if we are considering an arbitrary law of piston motion $u = U(t)$ and if $U'(t) > 0$ for certain t, then discontinuities necessarily emerge in the solution to this problem. Consequently, the complete solution to the piston problem in these cases is described by discontinuous solutions of gas dynamics equations containing shock waves. We will study the properties of shock waves in section IV.

Note still further that if we assume that the piston begins to move toward the gas side with finite velocity $U(0) > 0$, the solution is discontinuous for all $t > 0$. So in this case consideration of the isentropic problem is meaningless.

5. Problem with two pistons. Reflection and refraction of a traveling wave at the contact boundary. We continue our qualitative study of the simplest flows with the example of these two problems in which we must consider interaction of two Riemann waves.

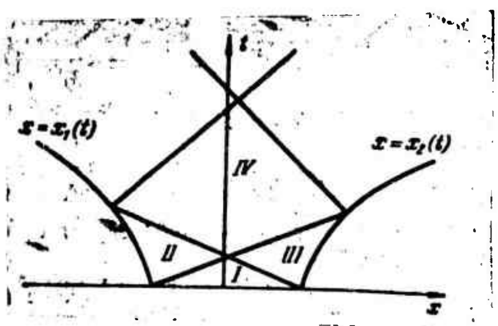


Figure 2.22

As we already took note of in subsection 1, for the case of a polytropic gas with adiabatic index $\gamma = 3$, straight lines in the plane of the Eulerian coordinates x, t along which the corresponding Riemann invariant is constant are characteristics. This fact enables us to derive a solution to Cauchy's problem, and also a solution to any correctly formulated boundary problem for the case when shock waves do not form in the solution. Suppose a polytropic gas with index $\gamma = 3$ at the initial time instants $t = 0$ is in a quiescent state ($u = 0$, $\rho = \rho_0$, $p = p_0$, $S = S_0$), bounded on two sides by pistons whose trajectory of motion are given by: $x = x_1(t)$, $x = x_2(t)$ (Figure 2.22). We now employ Eulerian coordinates x, t.

Obviously, in zone I we have a constant flow coinciding with the initial flow, in zones II and III -- Riemann waves, and zone IV is the zone of Riemann wave interference.

Consider qualities of another extremely simple problem: the interaction of Riemann waves with a contact boundary, i.e., with boundary of two distinctive gases (for identical gases with different densities). For simplicity, we will regard both gases as polytropic: the gas to the left of the boundary $x = x_0$ (Figure 2.23) has the adiabatic index $\gamma = \gamma_1$, and the gas to the right -- the index $\gamma = \gamma_2$, i.e., we assume that to the left of $x = x_0$, $p = A_1\rho^{\gamma_1}$, and to the right -- $p = A_2\rho^{\gamma_2}$. We can readily understand that if two gases adjoin each other, the boundary between them is a trajectory, and therefore the following conditions must be satisfied at the interface of the two gases:

$$u_- = u_+, \quad p_- = p_+, \qquad (1)$$

where u_-, p_-, u_+, p_+ are the velocity and pressure in the gases to the left and to the right of the contact boundary, respectively. As will be shown in section IV, these equations derive from the laws of conservation of mass, momentum, and energy.

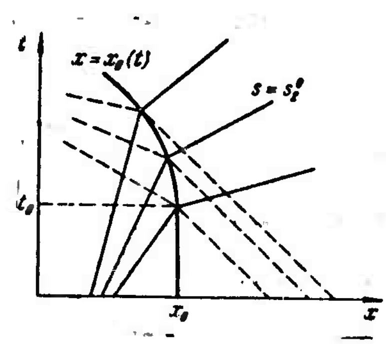


Figure 2.23

Denoting the Riemann invariants s, r to the left and to the right of the contact boundary by, respectively, s_1, r_1, s_2, and r_2, we conclude that at the continuity of velocity at the contact boundary require by the first of the conditions (1) is described by the equality

$$s_1 + r_1 = s_2 + r_2 = 2U(t), \qquad (2)$$

where $U(t)$ is the velocity of the contact boundary. Take some point $(x_0(t),t)$ at the contact boundary (Figure 2.23) after a Riemann r-wave with the velocity of the left gas arrives at it. Since s-characteristics transport to the contact boundary $x = x_0(t)$ in gas II constant values in the invariant $s = s_2 = s_2^0$, motion to the right of the contact boundaries naturally a Riemann r-wave ($s = s_2^0 =$ constant). Therefore, by formula (3.2.11) the pressure against the contact boundary is defined by the function

$$p_2 = p_0\left[1 + \frac{\gamma_2 - 1}{2}\frac{U(t)}{c_2^0}\right]^{\frac{2\gamma_2}{\gamma_2 - 1}}, \qquad (3)$$

where p_0 is initial pressure in gases I and II, and c_2^0 is the initial speed of sound in gas II (the initial velocity in gases I and II is zero).

Calculate pressure p_1 in gas I. Since

$$\frac{p_1}{p_0} = \left(\frac{c_1}{c_1^0}\right)^{\frac{2\gamma_1}{\gamma_1 - 1}},$$

where c_1^0 is the speed of sound in gas I at the initial time instant, and $c_1 = \frac{\gamma_1 - 1}{4}(r_1 - s_1)$, therefore

$$p_1 = p_0\left[\frac{\gamma_1 - 1}{4}\frac{r_1 - s_1}{c_1^0}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}}. \qquad (4)$$

Canceling out the quantity s_1 at this point by means of (2), we get the formula

$$p_1 = p_0\left[\frac{r_1}{r_1^0} - \frac{\gamma_1 - 1}{2}\frac{U(t)}{c_1^0}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}}, \qquad (5)$$

where $r_1^0 = \frac{2}{\gamma_1 - 1} c_1^0$ is the value of invariant r in the constant-flow zone of gas I ahead of the Riemann r-wave (we assume here that $U(0) = 0$). Equating pressures p_1 and p_2, we derive the equation

$$\left[\frac{r_1}{r_1^0} - \frac{\gamma_1 - 1}{2}\frac{U(t)}{c_1^0}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = \left[1 + \frac{\gamma_2 - 1}{2}\frac{U(t)}{c_2^0}\right]^{\frac{2\gamma_2}{\gamma_2 - 1}}. \qquad (6)$$

If it is assumed that $r_1 = r_1(t)$ is a known function, then from this stage we can define $U(t)$. Actually, the quantity $r_1(t)$ is an unknown, since the interaction of two Riemann waves to the left of the contact boundary (Figure 2.23) leads to curving of r-characteristics. However, we can make a qualitative investigation of the dependence of $U(t)$ on $r_1(t)$ and study the flow pattern qualitatively.

Suppose, for example, that a rarefaction r-wave arrives at the contact boundary. Then, as is readily seen from Figure 2.23, $r_1(t)$ is monotonically decreasing function of the variable t $\left(\frac{dr_1(t)}{dt} < 0\right)$. When $t = t_0$ (Figure 2.23), $r_1(t_0) = r_1^0$; therefore from (6) it follows that $U(t_0) = 0$. Since we assume that $\gamma_1, \gamma_2 > 1$, the left side of equality (6) decreases with increase in $U(t)$, while the right side -- in contrast -- increases monotonically. Hence it follows that if $r_1(t)$ is a monotonically decreasing function of the variable t, $U(t) < 0$ and $U(t)$ is also monotonically decreasing.

This means that flow in gas II can be regarded as motion induced by the movement of a piston with velocity $U(t) < 0$. Therefore the Riemann r-wave in gas II is a rarefaction wave. Quite analogously, we can ascertain that if the incident wave is a compression wave, $U(t) > 0$ and a compression r-wave is propagated in gas II.

It is somewhat more difficult to establish the pattern of an s-wave reflected from the contact boundary, that is to say, the sign of $\frac{ds_1(t)}{dt}$. Differentiating equality (2) relative to variable t, we get

$$\frac{ds_1(t)}{dt} = 2U'(t) - \frac{dr_1(t)}{dt}. \tag{7}$$

Hence we conclude that if $2U'(t) > \frac{dr_1(t)}{dt}$, the reflected wave is a rarefaction wave, otherwise -- a compression wave. For simplicity, we consider only the case of identical gases ($\gamma_1 = \gamma_2 = \gamma$) whose entropies are distinct, i.e., $c_1^0 \neq c_2^0$. Then equation (6) can be rewritten as

$$\frac{\gamma-1}{2}\frac{r_1(t)-U(t)}{c_1^0} = 1 + \frac{\gamma-1}{2}\frac{U(t)}{c_2^0} \tag{8}$$

and

$$U(t) = \frac{r_1(t) - \frac{2}{\gamma-1}c_1^0}{1 + c_1^0/c_2^0} \quad \text{and} \quad U'(t) = \frac{1}{1 + c_1^0/c_2^0} r_1'(t).$$

Hence we conclude that if $c_1^0/c_2^0 < 1$, the quantity $\frac{ds_1(t)}{dt}$ has the sign of $\frac{dr_1(t)}{dt}$; but if $c_1^0/c_2^0 > 1$, then the signs of $\frac{ds_1(t)}{dt}$ and $\frac{dr_1(t)}{dt}$ are opposed.

For the same gas ($\gamma_1 = \gamma_2 = \gamma$), inequality $c_1^0 < c_2^0$ means that the density ρ_1^0 of gas I is greater than the density ρ_2^0 of gas II, since $p = \frac{c^2\rho}{\gamma}$. Therefore we can formulate our result thusly:

From a denser medium ($\rho_2^0 > \rho_1^0$, $c_2^0 < c_1^0$) a rarefaction (compression) wave is reflected also in the form of a rarefaction (compression) wave, and conversely, from a less dense medium ($\rho_2^0 < \rho_1^0$, $c_2^0 > c_1^0$) a rarefaction (compression) wave is reflected in the form of a compression (rarefaction) wave.

Note one special case, $c_2^0 = 0$ ($\rho_2^0 = \infty$) for which $U(t) = 0$. This case can be treated as the reflection of a Riemann wave from an infinitely dense gas or as a reflection from a rigid wall ($U(t) = 0$). We conclude that compression (rarefaction) waves are reflected from a rigid wall always in the form of compression (rarefaction) waves.

Formula (6) enables us to make a qualitative investigation of the problem of the reflection of Riemann waves at a contact boundary. But the total solution of this problem is complicated by as a fact that actually the function $r_1(t)$ is unknown, since the r-characteristics are curved in the zone of interaction of the incident Riemann wave with the wave reflected from the contact boundary.

Accordingly, let us note a special case when the solution to this problem reduces to the solution of ordinary differential equation.

Consider the case when $\gamma_1 = 3$, but $\gamma_2 > 1$ arbitrarily. Then the characteristics in gas I are straight lines, and we can assume that $r_1(x, t) = f_1(x, t)$, where $f_1(x, t)$ is defined uniquely in the Riemann wave. However, in equation (6) $r_1(t)$ is the value of invariant $r_1(x, t)$ at the contact boundary, i.e., at the point $(x_0(t), t)$: $r_1(t) = r_1(x_0(t), t) = f_1(x_0(t), t)$. (9)

Since by the physical meaning of quantity $U(t)$

$$\frac{dx_0(t)}{dt} = U(t), \qquad (10)$$

and from equations (6) and (9) $U(t)$ is expressed as a certain function of the variables t, $x_0(t)$, therefore the solution of the entire problem is reduced to integral equation (10). As the initial condition for equation (10), here the condition (Figure 2.23) $x_0(t_0) = x_0$ is imposed. Thus, for example, if $\nu_1 = \nu_2 = 3$ and if a centered rarefaction wave arrives, equation (10) takes on, according to (8), the form

$$\frac{dx_0(t)}{dt} = \frac{\frac{x_0(t)}{t} - c_1^0}{1 + c_1^0/c_2^0}, \quad x_0(t_0) = c_1^0 t_0, \ t_0 > 0, \tag{11}$$

since in the centered rarefaction wave when $\nu = 3$, $r(x, t) = x/t$.

6. Remarks on boundary conditions for equations in gas dynamics and an illustration of their solvability with the example of the piston problem. In analyzing the solvability of problems with boundary and wall conditions, the conclusions which we drew in considering this problem area for hyperbolic systems of quasilinear equations in section XI of Chapter One are wholly applicable for gas dynamics equations.

Nevertheless, when studying the motion of gases and liquids, certain classes of boundary and wall conditions are especially to be anticipated in therefore are particularly important. Therefore we will here consider typical boundary conditions for equations in gas dynamics.

The wall conditions are most naturally formulated for the boundary of an isolated gas volume, i.e., for a trajectory. In Lagrangian coordinates, a fixed coordinate q corresponds to a trajectory; therefore this kind of boundary condition in Lagrangian coordinates is imposed on the straight lines q = constant.

Two types of boundary conditions can be distinguished:

1) External boundary conditions, or conditions at external boundaries. This will naturally be boundary conditions effectively describing the influence of an external environment on a given gas volume. For the one-dimensional flow under consideration, such conditions are the conditions at the left and right boundaries of the volume in which the gas is situated.

2) Internal boundary conditions, or conditions at internal boundaries. These include conditions at contact boundaries between gases exhibiting different properties (different entropies, different equations of state, and so on). Among the internal boundary conditions we can include also conditions at the discontinuity lines of a solution. These conditions will be discussed in detail in the

next section.

As for the conditions at the contact boundaries, in gas dynamics they are always identical and require continuity of flow velocity u and pressure p. An example of the use of these conditions is given by subsection 5, where the problem of reflecting Riemann waves from a contact boundary was studied qualitatively.

Let us dwell on external boundary conditions. For the case of the piston problem taken up in subsection 4, the boundary condition is imposed at the point $q = 0$. We can easily observe that if the pressure $p(0, t) = p(t) \geq 0$ (1) is assigned at the piston, this boundary problem is solvable, and uniquely so, since from condition (1) it follows that $c \geq 0$. But for the case when the piston velocity

$$u\,(0, t) = U(t), \tag{2}$$

is given, we see that provided $U(t) < 0$ separation of gas from the piston is possible, that is to say, condition (2) is not satisfied, but is replaced by condition $p = 0$.

The very same situation obtains for systems of linear equations as well. If boundary conditions are imposed outside the domain of dependence of initial data, these conditions generally speaking do not uniquely define the solution.

For the piston problem, the physical formulation of the problem yielded the correct solution, by replacing, where required, condition (2) by the "free boundary" condition $p = 0$.

When $U(t) > 0$, the piston problem with boundary condition (2), conversely, is always solvable, even though the solution will be discontinuous. This conclusion must be compared with the case of a system of linear equations for which the boundary problem generally speaking is not solvable either in the class of smooth or in the class of discontinuous solutions, if the boundary condition is posed in the domain of determinacy of the solution to Cauchy's problem.

Section IV. Discontinuities in a One-Dimensional Flow of Compressible Gases. Shock Waves

1. Hugoniot's Conditions. In the examples of these simplest flows in section III we saw that as a rule solutions to equations in gas dynamics remain continuous for a limited time, then discontinuities emerge in the solutions.

Naturally, differential equations lose their meaning for discontinuous flows; however, as we already stated above, integral laws of conservation of mass, momentum, and energy remain in force for discontinuous flows as well.

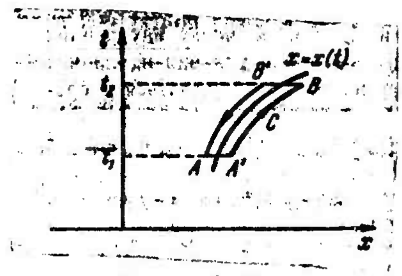


Figure 2.24

Let us derive conditions that must be met at the discontinuity lines of solutions to equations in gas dynamics as consequences of integral laws of conservation.

Suppose $x = x(t)$ is the equation of one of discontinuity lines of the hydrodynamic variables, which we will assume over the segment $t_1 \leq t \leq t_2$ under consideration as possessing a continuous tangent (Figure 2.24). Suppose $f(x, t)$ suffers a discontinuity at the line $x = x(t)$. Let us denote

$$f_1(t) = f(x(t) - 0,\ t); \quad f_2(t) = f(x(t) + 0,\ t); \quad [f] = f_2(t) - f_1(t). \tag{1}$$

Integral laws of conservation in Eulerian coordinates (2.3.13) - (2.3.15) are of the form

$$\left.\begin{aligned} &\oint_C \rho x^\nu\,dx - \rho u x^\nu\,dt = 0, \\ &\oint_C \rho u x^\nu\,dx - (p + \rho u^2)\,x^\nu\,dt = -\iint_{G_C} \nu p x^{\nu-1}\,dx\,dt, \\ &\oint_C \rho\left(\varepsilon + \frac{u^2}{2}\right) x^\nu\,dx - \rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) x^\nu\,dt = 0. \end{aligned}\right\} \tag{2}$$

Let us write the laws of conservation (2) for the path $AA'BB'$, assuming that the lines $A'B$ and $B'A$ adjoin infinitely close to the discontinuity line $x(t)$, respectively, to the right and to the left of it.

In view of the boundedness of all hydrodynamic variables, integrals vanishh over the parts AA' and BB' of the path C, as does the $\iint_{G_C} \nu \ p x^{\nu-1} \ dx \, dt$. Along the line $x = x(t)$ we have $dx = D \, dt$, where $D = D(t) = x'(t)$. Therefore, for example, from the first equation in (2) we get

$$\int_{t_1}^{t_2} x^{\nu}(t) \{(\rho_2(t) - \rho_1(t)) D(t) - (\rho_2(t) u_2(t) - \rho_1(t) u_1(t))\} dt = 0. \quad (3)$$

In view of the arbitrariness of the limits of integration in (3), we must equate the integrand to zero, i.e., $x^{\nu}(t) \{ D(t) [\rho] - [\rho u]\} = 0$.

Cancelling from this equality, we see that the conditions at the discontinuity line are identical for three symmetry cases $\nu = 0, 1, 2$. Proceeding in similar fashion with all laws of conservation (2), we get conditions at discontinuity line $x = x(t)$:

$$D[\rho] = [\rho u], \quad (4)$$

$$D[\rho u] = [p + \rho u^2], \quad (5)$$

$$D\left[\rho\left(\varepsilon + \frac{u^2}{2}\right)\right] = \left[\rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right)\right], \quad (6)$$

which associate discontinuities in hydrodynamic variables at the discontinuity line $x = x(t)$ and the velocity $D = x'(t)$ of the discontinuity line.

Relations (4)-(6) are called conditions for hydrodynamic consistency of a discontinuity, or Hugoniot's conditions, after the French scientist who first derived them*).

According to the notation for the quantity $D = D(t)$, $D[f] = [Df]$. This means that equation (4) can be rewritten as $[\rho(u - D)] = 0$. (7)
Multiplying equation (7) by D and subtracting from (5), we get

---

*) In American literature this condition is often called Rankine's or Rankine-Hugoniot conditions.

$$[p+\rho(u-D)^2]=0. \qquad (8)$$

Finally, multiplying (8) by D, subtracting the results from (6), and considering that $[D] = [D^2] = 0$, we get

$$\left[\rho(u-D)\left(\varepsilon+\frac{p}{\rho}+\frac{(u-D)^2}{2}\right)\right]=0. \qquad (9)$$

Taking the notation (1) into account, we can rewrite Hugoniot's conditions (7)-(9) in the form of the equalities:

$$\rho_2(u_2-D)=\rho_1(u_1-D)=m, \qquad (10)$$

$$p_2+\rho_2(u_2-D)^2=p_1+\rho_1(u_1-D)^2=j, \qquad (11)$$

$$\rho_2(u_2-D)\left(\varepsilon_2+\frac{p_2}{\rho_2}+\frac{(u_2-D)^2}{2}\right)= \\ =\rho_1(u_1-D)\left(\varepsilon_1+\frac{p_1}{\rho_1}+\frac{(u_1-D)^2}{2}\right)=f. \qquad (12)$$

According to subsection 2 of section II, the quantities m, j, f denote, respectively, flows of mass, momentum, and energy calculated in a coordinate system that travels at the velocity D relative to the system in which the flow velocity u is measured. So Hugoniot's conditions (10)-(12) require continuities of the flows of mass, momentum, and energy at the discontinuity line of the hydrodynamic variables.

Note, finally, that integral laws of conservation written in Lagrangian coordinates lead to the same Hugoniot's conditions (4)-(6), if we take into account the relationship between Lagrangian and Eulerian coordinates.

2. Different kinds of discontinuities; shock waves, contact discontinuities. Different forms of Hugoniot's conditions. Hugoniot's adiabat. We will distinguish solution to continuities as a function of the satisfaction of the conditions $m = 0$, $m \neq 0$.

If $m(t) = m = 0$, then this kind of discontinuity will be called a contact discontinuity; if $m(t) \neq 0$, we will call the discontinuity a shock wave.

For the case of a contact discontinuity, from (4.1.10) it follows that

$$D = u_1 = u_2 = x'(t), \qquad (1)$$

i.e., the discontinuity line coincides with the particle trajectory (the contact discontinuity is represented in Lagrangian coordinates, therefore the straight line q = constant).

Putting $u_1 = D$, $u_2 = D$, we get from (4.1.11)

$$p_1 = p_2 = j. \qquad (2)$$

and condition (4.1.12) when $u_1 = u_2 = D$ is satisfied by identity. And so, the following two conditions are satisfied that the contact discontinuity:

$$u_1 = u_2, \qquad p_1 = p_2, \qquad (3)$$

i.e., flow pressure and velocity are continuous. We can readily show the converse: if conditions (3) are satisfied at the discontinuity, then the discontinuity is of the contact type.

The quantities ρ, ε, S can experience a longitudinal shock at a contact discontinuity, however satisfying in the process the condition of pressure continuity (2). A contact discontinuity, in particular, can be the interface between two different gases satisfying different equations of state. The conditions for velocity and pressure continuity (3) in this instance can be regarded as internal boundary conditions at the interface of the distinct gases.

For the case of a shock wave $m \neq 0$, the substance flows across the discontinuity line $x = x(t)$. For the case $m > 0$, the substance flows across the discontinuity line from left to right; therefore we will state that when $m > 0$ the shock wave will travel from right to left relative to the substance; conversely, when $m < 0$ we will state that the shock wave will travel to the right.

Let us consider different representations of Hugoniot's conditions (4.1.10)-(4.1.12) for the case of a shock wave.

The condition of momentum flow continuity (4.1.11) can be written in the form of the following equivalent equalities:

$$
\left.\begin{aligned}
p_2+m(u_2-D)&=p_1+m(u_1-D).\\
p_2+V_2m^2&=p_1+V_1m^2.\\
m^2&=\frac{p_2-p_1}{V_1-V_2}.
\end{aligned}\right\}\tag{4}
$$

where $V = 1/\rho$. Condition (1.12) after being divided by m becomes

$$
\varepsilon_2+p_2V_2+\frac{(u_2-D)^2}{2}=\varepsilon_1+p_1V_1+\frac{(u_1-D)^2}{2}. \tag{5}
$$

If we let the letter $\mathcal{J}$ stand for the quantity $\varepsilon + pV$, called enthalpy, then (5) can be written as

$$
\mathcal{J}_2+\frac{(u_2-D)^2}{2}=\mathcal{J}_1+\frac{(u_1-D)^2}{2}. \tag{6}
$$

From (4.1.10) we have

$$
(u_2-D)(u_1-D)=\frac{m^2}{\rho_1\rho_2}=m^2V_1V_2. \tag{7}
$$

Substituting here the third formula (4), we get

$$
(u_2-D)(u_1-D)=\frac{p_2-p_1}{V_1-V_2}V_1V_2=\frac{p_2-p_1}{\rho_2-\rho_1}. \tag{8}
$$

According to (4.1.10)

$$
(u_1-D)=mV_1;\ (u_2-D)=mV_2;\ (u_1-D)^2=m^2V_1^2;\ (u_2-D)^2=m^2V_2^2. \tag{9}
$$

Substituting here the third formula in (4), we obtain

$$
(u_1-D)^2=\frac{p_2-p_1}{V_1-V_2}V_1^2,\quad (u_2-D)^2=\frac{p_2-p_1}{V_1-V_2}V_2^2. \tag{10}
$$

Let us note several other useful formulas. From (9) we have

$$
u_2-u_1=[(u_2-D)-(u_1-D)]=m(V_2-V_1).
$$

Therefore

$$\left.\begin{aligned}(u_2-u_1)(u_1-D)=m^2V_1(V_2-V_1)=(p_1-p_2)V_1,\\(u_2-u_1)(u_2-D)=m^2V_2(V_2-V_1)=(p_1-p_2)V_2.\end{aligned}\right\}\qquad(11)$$

and, finally,

$$\left.\begin{aligned}(u_2-u_1)^2=(p_2-p_1)(V_1-V_2),\\(u_2-D)^2-(u_1-D)^2=(p_1-p_2)(V_1+V_2).\end{aligned}\right\}\qquad(12)$$

From the resulting formulas we can draw several conclusions: 1) from formulas (4) it follows that during a transition across a shock front pressure increases or decreases simultaneously with density.

2) From formula (7) it follows that the differences $(u_2 - D)$ and $(u_1 - D)$ have the same sign.

3) For a finite m, the differences $(V_2 - V_1)$ and $(p_2 - p_1)$ are of the same order, so that as $V_2 - V_1 \to 0$, $p_2 - p_1 \to 0$.

Equalities (10) express the relative velocities $u_2 - D$ and $u_1 - D$ in terms of thermodynamic quantities. Therefore substituting these formulas into equalities (5) and (6), we obtain relations containing only thermodynamic variables:

$$\varepsilon_2-\varepsilon_1=\frac{1}{2}(p_2+p_1)(V_1-V_2).\qquad(13)$$

$$\mathcal{J}_2-\mathcal{J}_1=\frac{1}{2}(p_2-p_1)(V_2+V_1).\qquad(14)$$

Equality (13) or (14) is called the condition of a Hugoniot's adiabat.

Let us introduce into consideration the function

$$H(p,V;\,p_0,V_0)=\varepsilon(p,V)-\varepsilon(p_0,V_0)+(V-V_0)\frac{p+p_0}{2},\qquad(15)$$

which we will consider as a function of two variables p, V that depend parametrically on p_0, V_0.

Suppose $M_1 = \{p_1, V_1\}$, $M_2 = \{p_2, V_2\}$ are points on the p, V plane characterizing the thermodynamic state of substance on different size of discontinuity line. Then by virtue of (13) the following relation is valid:

$$H(M_2, M_1) = H(p_2, V_2;\ p_1, V_1) = -H(M_1, M_2) = -H(p_1, V_1;\ p_2, V_2) = 0. \qquad (16)$$

The points M_1 and M_2 associated by relation (16) will be referred to as conjugate. The property of conjugateness is not transitive, since from the relations $H(M_0, M_1) = 0$, $H(M_1, M_2) = 0$, $H(M_0, M_2) = 0$ does not follow.

Let us fix the point $M_0(p_0, V_0)$ of the p, V plane and considering a set of points $M(p, V)$ conjugate to M_0. They must lie on the curve

$$H(M, M_0) = H(p, V, p_0, V_0) = 0. \qquad (17)$$

The curve given by equality (17) will be called Hugoniot's adiabat with a set of points $M_0(p_0, V_0)$. According to equality (13), Hugoniot's adiabat is a geometrical locus of points (p, V) characterizing the thermodynamic state of the substance on one side of the discontinuity (shock wave) front if the state p_0, V_0 on the other side of the front is assigned.

Suppose Hugoniot's adiabat $H(M, M_0)$ passes through the point M_1. Then Hugoniot's adiabat $H(M, M_1)$ passes through the point M_0, but does not coincide with the adiabat $H(M, M_0)$ (Figure 2.25). This situation reflects that Hugoniot's adiabat is not a line of constancy of the function of two variables, but is the line of constancy of a function of two variables dependent also on two parameters. Therefore, if we select the points (p, V) that are conjugate to the point (p_0, V_0) as the new centers of Hugoniot's adiabats, then we obtain a one-parametric family of Hugoniot's adiabats passing through the point (p_0, V_0) (bundle of adiabats).

3. Hugoniot's adiabat for a normal gas. Our preceding remarks on Hugoniot's adiabats apply to a substance with an arbitrary equation of state.

For a more detailed study of Hugoniot's adiabat, we will assume that the equations of state of the substance $p = p(V, S)$, $\varepsilon = \varepsilon(V, T)$ satisfy the requirements that were formulated in subsection 4 of section I for a normal gas:

$$\left.\begin{array}{llll} \dfrac{\partial p(V,S)}{\partial V} < 0 & \text{(I)}, & \dfrac{\partial^2 p(V,S)}{\partial V^2} > 0 & \text{(II)}, \\ p(V,S) \to \infty & \text{when} & V \to 0 & \text{(III)}, \\ \dfrac{\partial p(V,S)}{\partial S} > 0 & \text{(IV)}, & c_V = \dfrac{\partial \varepsilon(V,T)}{\partial T} > 0 & \text{(V)}. \end{array}\right\} \qquad (1)$$

Condition VI is the requirement of convexity of the domain of variables p, V in which requirements I-V are satisfied.

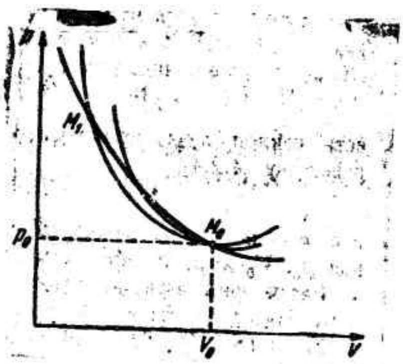


Figure 2.25

Our entire subsequent consideration, with the exception of cases that will be given special reservations, will apply to a normal gas*).

Let us examine, following [4], the total differential dH of the function $H(p, V, p_0, V_0)$ of two variables p, V, assuming p_0, V_0 to be fixed:

$$dH = d\varepsilon + \frac{p + p_0}{2} dV + \frac{V - V_0}{2} dp. \quad (2)$$

using the fundamental relation $d\varepsilon = TdS - pdV$, we obtain the following expression for dH:

$$dH = T\,dS + \frac{V - V_0}{2} dp - \frac{p - p_0}{2} dV. \quad (3)$$

Equation (3) can also be written as

$$dH = T\,dS + \frac{(V - V_0)^2}{2} dK, \quad (4)$$

*) Properties of the adiabat for gas with arbitrary equation of state was studied in the work of G. Ya. Galin [43].

where

$$K = \frac{p - p_0}{V - V_0} \tag{5}$$

is the slope of the ray passing through the center $M_0(p_0, V_0)$. According to equality (4.2.4),

$$K = -m^2, \tag{6}$$

where m is a mass flow across the shock front separating the states p_0, V_0 and p, V (here $H(p, V, p_0, V_0) = 0$).

Let us consider the mutual arrangement of the following curves:

a) Poisson's adiabat given by the equation dS = 0 and passing through the point $M_0(p_0, V_0)$; we will denote this curve with the letter A; b) the Hugonict's adiabat dH = 0 passing through the point $M_0(p_0, V_0)$; let H stand for this curve.

By formulas (3) and (4), when $V = V_0$, $p = p_0$, and dH = TdS. Thus, the curves A and H at the point of their intersection $M_0(p_0, V_0)$ have a common tangent.

Let us consider the behavior of the differentials dH, dS, dK at each of these indicated curves.

a) at the adiabat A, dS = 0. According to (4),

$$dH = \frac{(V - V_0)^2}{2} dK. \tag{7}$$

According to properties I and II, along A (i.e., when S = constant) $dK/dV > 0$, (8)

therefore at the adiabat A $dH/dV > 0$. (9)

Consequently, $H < 0$ in the upper part of the adiabat A ($V < V_0$), $H > 0$ in the lower section of adiabat A ($V > V_0$). (10)

b) At the ray dK = 0, dH = TdS. (11)

This equality signifies that along the ray K the signs of dH and dS coincide and, in particular, the stationary points of the functions H and S coincide.

As will be shown in subsection 4 of section I, entropy S has no stationary points at the ray K = constant > 0 and increases monotonically with p; at the ray K < 0 entropy S has a single stationary point at which S reaches the maximum. The same is valid, therefore, for the function H, so that H along the ray K= const < 0 has no stationary points; H has a single maximum at the ray K = constant > 0, and this at the same point as does S. Hence it follows that at each ray, K = const<0 there exists a single point M_H of Hugoniot's adiabat H lying between the point M_A of Poisson's adiabat and the center M_0.

Thus, there exists a curve H -- Hugoniot's adiabat passing through point M_0. The smoothness of Hugoniot's curve follows from the fact that it is an integral curve of the ordinary differential equation (2) passing through the center M_0.

c) At the adiabat H, from relation (4) follows

$$T\,dS = -\frac{(V-V_0)^2}{2}\,dK. \qquad (12)$$

Hence it follows that if the point M(p, V) travels along the adiabat H such that the ray $K(M, M_0)$ travels clockwise ($dK<0$), entropy S increases monotonically.

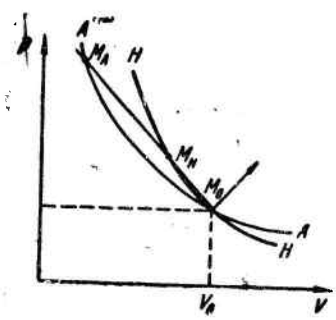


Figure 2.26

Summing up our conclusions on the behavior of differentials dH, dS, dK at the curves A, K, H, we conclude that in the neighborhood of the point M_o where the adiabats intersect tangentially, the curves A and H lie as shown in Figures 2.26.

The mutual disposition of adiabats A and H in the neighborhood of the center M_o indicates that the tangency of the adiabats A and H must be of an order not less than two. Let us confirm this by computation.

From (3) we have along H

$$T\,dS = -\frac{V-V_0}{2}\,dp + \frac{p-p_0}{2}\,dV. \tag{13}$$

Differentiating equality (13) relative to variable V, we find

$$dT\,dS + T\,d^2S = -\frac{1}{2}\,dV\,dp - \frac{V-V_0}{2}\,d^2p + \frac{1}{2}\,dp\,dV = -\frac{V-V_0}{2}\,d^2p, \tag{14}$$

i.e., $d^2S = 0$ at the point (p_o, V_o) and the tangency of the curves A and H is of the second order. Finally, differentiating (14) once more relative to V, we find

$$d^2T\,dS + 2\,dT\,d^2S + T\,d^3S = -\frac{1}{2}\,dV\,d^2p - \frac{V-V_0}{2}\,d^3p.$$

Since at the point M_o, $dS = d^2S = 0$, from this we find

$$T\,d^3S = -\frac{1}{2}\,dV\,d^2p \qquad \text{and} \qquad \frac{d^3S}{dV^3} = -\frac{1}{2T}\,\frac{d^2p}{dV^2} < 0$$

at the point M_o according to property II of the equation of state.

Thus, only the third derivative d^3S/dV^3 is distinct from 0 at the point M_o of the curve H.

It is not difficult to see that in the neighborhood of the center $M_o(p_o, V_o)$ the following properties of adiabat H hold:

1) For any point M on the curve H

$$K = \frac{p - p_o}{V - V_o} < 0 \qquad (15)$$

2) No ray M_oM_H touches the adiabat H when $M_H \neq M_o$.

3) Each ray M_oM_H intersects the adiabat H at no more than a single point $M_H \neq M_o$.

4) Each ray intersecting the upper branch of adiabat A intersects also the upper branch of adiabat H. Here, the functions H and S at the ray M_oM_H have profiles of the form shown in Figure 2.27.

Observe that from properties 2) and 3) does not follow the convexity of the curve H, since these properties are valid for rays departing only from a certain point on an adiabat, namely from its center.

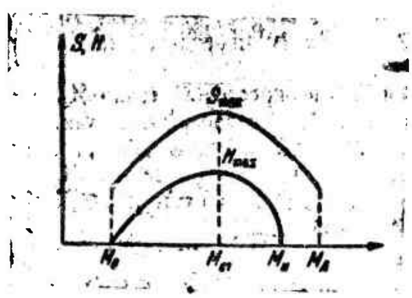


Figure 2.27

Let us prove that these properties of adiabat I obtain not only locally, but also globally. Let M_H be an arbitrary point on adiabat H. Since

$$H(M_H, M_o) = H(M_o, M_o) = 0,$$

therefore at the ray M_oM_H there exists the stationary point of the function H (and also by virtue of (11), the stationary point of the function S). Therefore, $K < 0$

and property 1) of adiabat H has been proven.

Let us prove property 2). The stationary point M_{ct} [ct = stationary] lies strictly within the segment M_oM_H and is the sole stationary point. The tangency of adiabat H with ray K would signify the existence at the ray K of two stationary points of function H and, consequently, two stationary points of S, which is impossible. Property 2) stands proven.

From the existence of a ray intersecting curve H at two points distinct from M_o there necessarily follows the existence of a ray tangent to H at the point distinct from M_o. This is impossible by virtue of property 2) and therefore property 3) is also proven.

Finally, since $S(M_H) > S(M_o)$ when $p_H > p_o$, then any ray intersecting the upper part of adiabat H intersects at first adiabat H at point M_H, and then adiabat A at point M_A. Property 4) stands proven.

In particular, from this it follows that the slope of the ray K intersecting adiabat H at the point $M_H \neq M_o$ lies within the limits

$$-\infty < K = \frac{p - p_0}{V - V_0} < \frac{\partial p(V_0, S_0)}{\partial V} < 0, \tag{16}$$

Now let us consider the following problem:

The state is given along one side of the discontinuity line (shock wave). Suppose it is characterized by the parameter $u = u_o$, $p = p_o$, and $\varepsilon = \varepsilon_o = \varepsilon(p_o, V_o)$. The flow of mass m across the discontinuity front is also given. It is required to define the state (u, V, p, ε) along the other side of this discontinuity line based on Hugoniot's conditions.

We assume that $K = -m^2$ satisfies condition (16). Let us show that for a normal gas this problem always has a solution, and one that is unique at that.

Thermodyanmic parameters of the gas p, V are defined by the point of intersection of Hugoniot's adiabat $H(p, V, p_0, V_0) = 0$ with the ray $K = \frac{p - p_0}{V - V_0} = -m^2$. Since K satisfies condition (16), then by properties 1) - 4) of adiabat H, the point (p, V) exists and is uniquely defined.

So, thermodynamic parameters p, V, ε are defined uniquely. Next, all the remaining shock wave parameters are uniquely defined. The differences $u_0 - D = mV_0$ and $u - D = mV$ are defined by formulas (4.2.9). Since u_0 is known, from this the velocity D of the shock wave is determined.

This means that the formulated problem has a unique solution provided condition (16) is satisfied.

Let us consider the limiting case when $K = -m^2 = \frac{\partial p(V_0, S_0)}{\partial V}$. At this K value the ray is tangent to Hugoniot's adiabat (and simultaneously, to Poisson's adiabat) at the point $M_0(p_0, V_0)$. The conjugate point M(p, V) characterizing the state of the substance along the other side of the front coincide to $M_0(p_0, V_0)$, i.e., the shock wave is infinitely weak.

In this case $u = u_0$ and by formula (4.2.8) we have

$$(u - D)^2 = \frac{\partial p(\rho_0, S_0)}{\partial \rho} = c_0^2, \qquad \text{i.e.,} \qquad u - D = \pm c_0 = \pm c.$$

Thus, an infinitely shock wave will travel at the speed of sound relative to the substance, i.e., it is a weak discontinuity.

When $K = -\infty$, the shock wave is called infinitely strong.

4. Stable and unstable discontinuities. Stability conditions and Champlin's theorem. Suppose the variables $u_1, p_1, V_1, \varepsilon_1; u_2, p_2, V_2, \varepsilon_2$, and the velocity of the shock wave D satisfy Hugoniot's conditions (4.1.10) - (4.1.12). We can easily note that Hugoniot's conditions do not change, whether or not we assume the state $u_1, p_1, V_1, \varepsilon_1$ as the state of the gas left of the shock wave front, and the state $u_2, p_2, V_2, \varepsilon_2$ as the right side state, or, vice versa, $u_1, p_1, V_1,$

$\mathcal{E}_1$ as the right, and u_2, p_2, V_2, $\mathcal{E}_2$ as the left. However, as we now see, as to their physical meaning these cases differ widely so that one of them must even be regarded as impossible.

Let us begin with our assumption of some definite sign for the mass flow $m = \rho_1(u_1 - D) = \rho_2(u_2 - D)$. Suppose, for example, $m < 0$. In this case the wave relative to the substance travels to the right and, thus, in its motion the substance cut across the shock wave front, moving from right to left relative to the front.

As we state in section I of this chapter, we represent the discontinuity surface of the flow at the narrow zone of large gradients in which the action of dissipative forces -- viscosity and thermal conductivity -- is substantial. The action of these processes leads, as we know from thermodynamics, to entropy gain, which characterizes the irreversibility of processes with viscosity and thermal conductivity. Since when $m < 0$ gas particles in the course of motion move from a right position relative to the shock wave front to a left position, then obviously, due to the irreversibility of the processes occurring in the narrow zone which we associate with the shock wave we must require that $S_{прав} < S_{лев}$ [прав = right; лев = left], where $S_{прав}$ and $S_{лев}$ are the entropy of the gas to the right and to the left, respectively, of the wave front.

The inequality $S_{прав} < S_{лев}$ $(m < 0)$ (1)

no longer allows us to interchange the states u_1, p_1, V_1, $\mathcal{E}_1$; u_2, p_2, V_2, $\mathcal{E}_2$ but rather indicates the specific position of these states relative to the front.

We will call the shock wave when $m < 0$ for which condition (1) is satisfied a stable discontinuity; if condition (1) is violated, then we will call this discontinuity unstable.

Wholly analogous arguments lead us to the conclusion that when m C

$$S_{лев} < S_{прав} \quad (m > 0). \qquad (2)$$

We will call conditions (1) - (2) shock wave stability conditions.

If we agree to label the state ahead of the shock wave front the state to the right of it when $m < 0$ and to the left of it when $m > 0$, and label the second state as the state behind the shock wave front, then inequalities (1) and (2) require that the entropy of the gas present behind the front will be greater than the entropy of the gas ahead of the wave front.

We will denote the state ahead of the front by the letters u_0, p_0, V_0, ε_0, and the state behind the front by u, p, V, ε; then the stability conditions (1) and (2) can be written in a single inequality: $S > S_0$. (3)

In the following, we will understand shock wave to refer only to a stable shock wave, i.e., a discontinuity satisfying Hugoniot's conditions and condition (3).

Let us turn to the problem of determining the state along one side of the front, if the state on the other side is given along with the mass flow m considered in the preceding subsection. If u_0, p_0, V_0, ε_0 is the state ahead of the shock wave front, the problem of defining the state behind the front satisfying stability condition (3) is solved, and done so uniquely, providing the condition

$$-m^2 = K = \frac{p - p_0}{V - V_0} < \frac{\partial p}{\partial V}(V_0, S_0). \qquad (4)$$

is satisfied. For a normal gas stability condition (3) leads to the following consequences:

1) To each D value and to each state ahead of the front u_0, p_0, V_0, ε_0 there corresponds one and only one state behind the front u, p, V, ε, if $|D - u_0| > c_0$.

Actually, $m = \rho_0(u_0 - D)$ and $m^2 > \rho_0 c_0^2 = -\frac{\partial p(V_0, S_0)}{\partial V}$. Hence follows (4.3.16) and the validity of the assertion.

2) With increase in $|D - u_0|$ from c_0 to ∞, entropy behind the front increases monotonically.

3) Shock waves lead only to compression of the substance and to a pressure rise.

Actually, at the upper branch of adiabat H, which corresponds to states behind the front, we have $S > S_0$, $p > p_0$, $V < V_0$, i.e., $\rho > \rho_0$.

4) The shock wave travels at supersonic velocity with respect to the gas ahead of the front and with subsonic velocity in the medium behind the front.

This assertion (Champlin's theorem) is written as the inequalities

$$|u_0 - D| > c_0, \quad |u - D| < c. \tag{5}$$

The first inequality, as we saw, is equivalent to the condition $S > S_0$. In subsection 3 it was shown that

$$-\frac{p - p_0}{V - V_0} < -\frac{\partial p}{\partial V} = \rho^2 c^2 \tag{6}$$

with p and V lying to the right of Poisson's adiabat A. Since in accordance with subsection 3, M_H lies to the right of adiabat A, inequality (6) has been satisfied. Therefore

$$\frac{p - p_0}{V - V_0} = -m^2 = -\rho^2 (u - D)^2 > -\rho^2 c^2.$$

Hence $|u - D| < c$, which was required to be proved.

Now let us note that we can consider also the problem of defining the state u_0, p_0, V_0, ε_0 ahead of the front with respect to the given state u, V, p, ε behind the front and the mass flow m (or velocity D) across the discontinuity front. Stability condition (3) in this case can be replaced by the equivalent inequality:

$$K = -\frac{p - p_0}{V - V_0} > \frac{\partial p(V, S)}{\partial V}.$$

and the point (p, V) (if it exists) is unique and lies on the lower branch of the Hugoniot's adiabat with its center at the point M(p, V). Recalling that the velocities of the characteristics ξ_1, ξ_2, ξ_3 are equal to

$$\xi_1 = u - c, \quad \xi_2 = u, \quad \xi_3 = u + c, \tag{7}$$

we can write inequality (5) also in the form

$$\xi_{1\,лев} > D > \xi_{1\,прав} \quad \text{when} \quad D < u_{прав} \tag{8}$$

$$\xi_{3\,лев} > D > \xi_{3\,прав} \quad \text{when} \quad D > u_{лев} \tag{9}$$

where $\xi_{лев}$ and $\xi_{прав}$ are the velocities of the characteristics, respectively, to the left and to the right of the discontinuity.

a) $m < 0$ b) $m > 0$

Figure 2.28

Finally, let us note further that the stability conditions of discontinuity (1), (2), or (3) are equivalent to the inequality $u_{лев} > u_{прав}$ (10)
Actually, suppose, for example, $m < 0$. Then by (4.2.10)

$$u_{лев} - u_{прав} = u - u_0 = m(V - V_0).$$

Since $V > V_0$, from this follows (10). The case $m > 0$ is similarly proven.

Inequalities (8) - (10) enables us to schematically represent the behavior of characteristics and streamlines in the vicinity of the discontinuity line.

Figures 2.28, a and 2.28, b show the mutual disposition of discontinuity line $x = x(t)$, streamlines $dx/dt = u$ and characteristics $dx/dt - \xi_1 = u - c$, $dx/dt = \xi_3 = u + c$ in the regions to the left and to the right of the shock front in Eulerian coordinates for the case of a shock wave that travels to the right ($m<0$) and to the left ($m > 0$) relative to the gas.

Figure 2.29 gives the disposition of the corresponding curves in the plane of Lagrangian coordinates q, t for the case $m<0$. In this figure $q = q(t)$ is the discontinuity line, q = constant is the streamline, and $dq/dt = \pm \rho c$ are the characteristics of the first and third families.

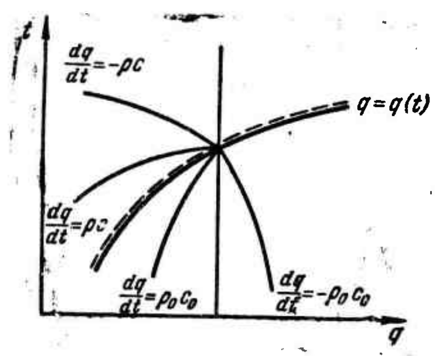


Figure 2.29

Let us note one typical feature that follows from the pattern of the disposition of the discontinuity line and the characteristics shown in Figures 2.28 and 2.29. Four characteristics arrive at each of the points on the discontinuity line $x = x(t)$ from below (from the region of lower time values t); only two characteristics depart (defined at large t values), and one of these is the streamline. For the case $m < 0$, two arriving characteristics of the third family intersect at the discontinuity line, and when $m > 0$ -- two incoming characteristics of the first family. We state that when $m < 0$ the characteristics of the third family form a "herringbone pattern", and when $m > 0$ characteristics of the first family form the "herringbone."

This situation is strongly related to shock wave stability. In particular, it indicates that the solution in this case is necessarily discontinuous, since arriving characteristics of the same family intersect at the discontinuity line. These considerations enable us to formally define these stabilities of a discontinuous solution to a hyperbolic system of equations as the satisfaction of the "herringbone condition" at discontinuity lines. This approach to discontinuous solutions is employed and extensively discussed in Chapter Four.

5. Hugoniot's conditions for a polytropic gas. For the case of polytropic gas the equation of state is of the form

$$\varepsilon = \frac{1}{\gamma - 1}\frac{p}{\rho} = \frac{pV}{\gamma - 1} = \frac{1}{\gamma(\gamma - 1)}c^2, \quad c^2 = \frac{\partial p(\rho, S)}{\partial \rho}, \tag{1}$$

For enthalpy we have the expression

$$\mathcal{J} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{1 - h}{2h}c^2, \quad 0 < h = \frac{\gamma - 1}{\gamma + 1} < 1, \tag{2}$$

and the equation of Hugoniot's adiabat with its center at the point (p_0, V_0) is of the form

$$pV + h(p_0V - pV_0) - p_0V_0 = 0 \tag{3}$$

or

$$(p + hp_0)(V - hV_0) = (1 - h)^2 p_0V_0. \tag{4}$$

In Figure 2.30 we give the plots of Hugoniot's adiabat and Poisson's adiabat for the case of a polytropic gas. Hugoniot's adiabat H, according to (4), is a hyperbola with the asymptotes $V = hV_o$, $p = -hp_o$, (5)

and Poisson's adiabat is given by the equation $pV = a^2(S_o) =$ constant (6)

and has its asymptotes, axis $p = 0$, $V = 0$.

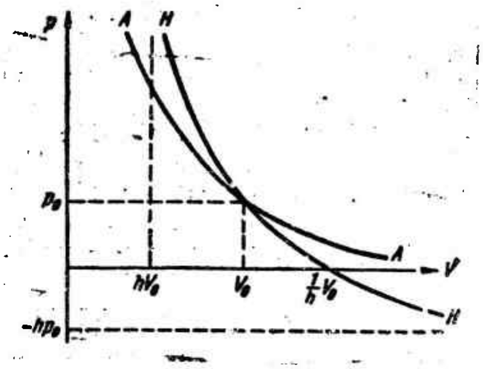


Figure 2.30

The following properties are valid for a polytropic gas.

1) Pressure p varies along Hugoniot's adiabat from 0 to ∞ when V is varied from V_o/h to hV_o; $p = \infty$, $V = hV_o$ corresponds to an infinitely strong shock wave at p_o, V_o values ahead of the front: $p = 0$, $V = V_o/h$ corresponds to an infinitely strong shock wave for the given state p_o, V_o behind the front. From Figure 2.30 we conclude that the limiting compression of a polytropic gas by a shock wave is equal to

$$\frac{\rho}{\rho_0} = \frac{V_0}{V} = \frac{1}{h} = \frac{\gamma+1}{\gamma-1} > 1. \quad (7)$$

2) Along Hugoniot's adiabat $dp/dV < 0$ (8)

This means that with an increase in the quantity $|D - u_o|$ pressure p and density ρ behind the wave front increases monotonically.

In subsections 3 and 4 we showed that for a normal gas, the state behind the shock wave front is uniquely determined from Hugoniot's conditions if the state in front of the front (u_o, p_o, V_o, ε_o) and the quantity characterizing the strength of the shock wave (mass flow m or velocity of shock wave D) are given.

Let us here present the corresponding working formulas for the case of a polytropic gas. We will characterize the strength of the shock wave by the dimensionless quantity

$$M_0 = \frac{|u_0 - D|}{c_0}, \tag{9}$$

which, by virtue of Champlin's theorem, is greater than or equal to 1.

If we know M_o, then we find the modulus of the mass flow M across the wave front:

$$|m| = \rho_o c_o M_o. \tag{10}$$

Now let us express explicitly the quantities p, V, c, u in terms of p_o, V_o, c_o, u_o, M_o. Rewriting the equation for the adiabat (4) in the form

$$\frac{p + hp_0}{V_0} = \frac{p_0 + hp}{V} = \frac{(1-h)(p-p_0)}{V_0 - V} = (1-h)m^2, \tag{11}$$

where we use (4.2.4), we find

$$p + hp_0 = (1-h)\rho_0(u_0 - D)^2 = (1+h)p_0 M_0^2, \tag{12}$$

since $p_o = c_o^2 \rho_o \frac{1-h}{1+h}$. Hence

$$p = p_0\left[(1+h)M_0^2 - h\right]. \quad (13)$$

$$\frac{\rho}{\rho_0} = \frac{V_0}{V} = \frac{p+hp_0}{p_0+hp} = \frac{M_0^2}{(1-h)+hM_0^2}, \quad (14)$$

$$\frac{c^2}{c_0^2} = \frac{p\rho_0}{p_0\rho} = \frac{\left[(1+h)M_0^2-h\right]\left[(1-h)+hM_0^2\right]}{M_0^2}. \quad (15)$$

(13) - (15) express thermodynamic quantities behind the wave front in terms of those known ahead of the front, and in terms of M_0. To determine velocity u behind the front, let us employ formula (4.2.10), according to which

$$u = u_0 + m(V - V_0) = u_0 \pm \rho_0 c_0 M_0 V_0\left(\frac{V}{V_0} - 1\right) = \\ = u_0 \pm c_0 M_0\left(\frac{V}{V_0} - 1\right) = u_0 \pm (1-h)c_0\left(M_0 - \frac{1}{M_0}\right) \quad (16)$$

(the symbol + sign in formula (16) as taken for the case of a wave traveling toward the right, i.e., when $m < 0$; if the wave moves to the left relative to the gas ($m > 0$), then the - sign is taken).

Equalities (13) - (16) explicitly express the state behind the front in the form of rational functions of M_0. The quantities

$$\frac{p}{p_0}, \quad \frac{\rho}{\rho_0}, \quad \frac{c}{c_0}, \quad \left|\frac{u-u_0}{c_0}\right|$$

are, as we can readily see, monotonically increasing functions of the parameter M_0.

By virtue of the symmetry of Hugoniot's conditions, the state ahead of a shock wave front u_0, p_0, V_0, ε_0 with respect to the given state u, p, V, ε behind the front is defined by the very same formulas, if instead of M_0 we introduce the number

$$M = \frac{|u-D|}{c},$$

i.e.,

$$\left.\begin{aligned} &\frac{p_0}{p} = (1+h)M^2 - h, \quad \frac{\rho_0}{\rho} = \frac{M^2}{(1-h)+hM^2}, \\ &\frac{c_0^2}{c^2} = \left[(1+h) - \frac{h}{M^2}\right][(1-h)+hM^2], \\ &u_0 = u \pm (1+h)c\left[M - \frac{1}{M}\right]; \end{aligned}\right\} \qquad (17)$$

where from Champlin's theorem it follows that $M \leq 1$.

6. Hugoniot's conditions for an isothermal gas. An isothermal gas is the limiting case of a thermally conducted gas when the thermal conductivity coefficient tends to infinity, and the temperature of the gas is kept constant due to external heat sources. From integral of laws of conservation in this case follow only two laws of conservation at the discontinuity front -- those of mass and momentum:

$$\rho_1(u_1 - D) = \rho_2(u_2 - D) = m, \qquad (1)$$

$$p_1 + \rho_1(u_1 - D)^2 = p_2 + \rho_2(u_2 - D)^2 = j, \qquad (2)$$

and pressure p is given by the formula $p = p(V, T_0) = F(V)$. (3)

Since in an isothermal gas the temperatures identical ahead of and behind the shock wave front, the role of Hugoniot's adiabat H in this case is played by isotherm (3), whose equation can be written as

$$p - F(V) = p_0 - F(V_0) = 0. \qquad (4)$$

In this case, i.e. two points (p_0, V_0) and (p, V) lying on isotherm (3) will be conjugate.

On the assumption that the gas exhibits properties I - V of subsection 3, the isotherm satisfies the conditions (cf section I, subsection 4)

$$\frac{\partial p}{\partial V} = \frac{dF}{dV} < 0; \quad \frac{\partial^2 p}{\partial V^2} = \frac{d^2 F}{dV^2} > 0; \quad p \to \infty \quad \text{when } V \to 0 \qquad (5)$$

Consequently, isotherm (3) is a convex curve, and any ray drawn from the point (p_o, V_o) intersects it at not more than one point.

Therefore, the ray $K = \frac{p - p_o}{V - V_o}$ intersects the upper part of isotherm (3) only at a single point (Figure 2.32) if

$$-\infty < K = -m^2 < \left(\frac{\partial p}{\partial V}\right)_{V_0} = -\left[\frac{c_T(V_0)}{V_0}\right]^2, \qquad (6)$$

where c_T is the isothermal speed of sound. Providing $K = -\infty$ (an infinitely strong shock wave), compression will also be infinite. To the same conclusion formally derived from the compression formulas for a polytropic gas as $\gamma \to 1$, since $h = \frac{\gamma - 1}{\gamma + 1} \to 0$ at $\gamma \to 1$.

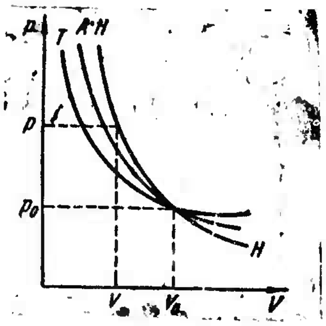


Figure 2.31

Let us present working formulas for a shock wave in an isothermal gas, again taking as the parameter defining the strength of the wave the quantity

From (2) follows

$$M_0 = \frac{|u_0 - D|}{c_T(V_0)},$$

$$\frac{p(V) - p(V_0)}{V_0 - V} = m^2 = \rho_0^2 c_T^2(V_0) M_0^2 \qquad (7)$$

Hence we can determine V, after which u is found from the formula

$$u - u_0 = m(V - V_0) = -\rho_0 c_T(V_0) M_0 (V - V_0), \qquad (8)$$

where for definiteness we put $m < 0$ (shock wave travels from left to right relative to the gas). In the case of an ideal gas

$$p = P(V) = \frac{RT_0}{V} = \frac{c_T^2}{V} = c_T^2 \rho, \qquad (9)$$

the isothermal speed of sound $c_T = \sqrt{RT_0}$ is constant and formulas (7) and (8) become

$$\frac{V_0}{V} = \frac{\rho}{\rho_0} = M_0^2, \qquad (10)$$

$$u - u_0 = c_T M_0 \left(1 - \frac{V}{V_0}\right) = c_T M_0 \left(1 - \frac{1}{M_0^2}\right) = c_T \left(M_0 - \frac{1}{M_0}\right). \qquad (11)$$

Converting to Riemann invariants (section 2, subsection 9)

$$s = u - c_T \ln \rho, \quad r = u + c_T \ln \rho, \qquad (12)$$

let us give Hugonict's conditions (10) and (11) a symmetric form:

$$s - s_0 = c_T\left[-\ln M_0^2 + \left(M_0 - \frac{1}{M_0}\right)\right] = \psi(M_0). \qquad (13)$$

$$r - r_0 = c_T\left[\ln M_0^2 + \left(M_0 - \frac{1}{M_0}\right)\right] = \varphi(M_0). \qquad (14)$$

where s_0, r_0 are values of Riemann invariants calculated at the points u_0, p_0, V_0.

To solve the problem of these stabilities of shock wave in an isothermal gas, it is sufficient to involve the requirement of the second law of thermodynamic for a nonequilibrium isothermal process (formula(1.2.29)):

$$\frac{dS}{dt} > \frac{1}{T}\frac{dQ}{dt}. \qquad (15)$$

Let us use this inequality for a gas particle intersecting a shock wave in its motion. Since per unit time the gas mass $|\rho_0(u_0 - D)| = |\rho(u - D)| = |m|$ intersects the shock wave, then, taking this quantity as the mass of this particle, we set equal to 1 the time of its transition from the state ahead of the wave (u_0, p_0, V_0, ε_0) to its state behind the wave (u, p, V, ε). By the law of conservation of energy (formula (1.1.2)), $\Delta Q = \Delta E + \Delta A$, where ΔE is the increment in total energy of the particle, and ΔA is the work performed by it per unit of time on the surrounding gas. Since

$$\Delta E = |m|\left(\varepsilon - \varepsilon_0 + \frac{u^2 - u_0^2}{2}\right), \qquad \Delta A = (pu - p_0 u_0)\operatorname{sign} m,$$

inequality (15) gives

$$T_0(S - S_0) + \frac{p_0 u_0 - pu}{m} + \varepsilon_0 - \varepsilon + \frac{u_0^2 - u^2}{2} > 0$$

(where $m > 0$ if the wave moves from right to left relative to the gas, $m < 0$ otherwise). Taking relations (1) and (2) into account, we get

$$T_0(S-S_0)+\frac{p_0}{\rho_0}-\frac{p}{\rho}+\varepsilon_0-\varepsilon+\frac{(u_0-D)^2}{2}-\frac{(u-D)^2}{2}>0. \quad (16)$$

Substituting here formulas (4.2.10):

$$(u-D)^2=\frac{p_0-p}{V-V_0}V^2, \quad (u_0-D)^2=\frac{p_0-p}{V-V_0}V_0^2,$$

we give inequality (16) the form

$$T_0(S-S_0)-\left[(\varepsilon-\varepsilon_0)+(V-V_0)\frac{p+p_0}{2}\right]= \\ =T_0(S-S_0)-H(p, V, p_0, V_0)>0. \quad (17)$$

Let us compute the quantity $T_0(S - S_0)$. To do this, we integrate the thermodynamic equality $TdS = d\mathcal{E} + pdV$ along isotherm T from the point (p_0, V_0) to the point (p, V) (Figure 2.32) (we assume the gas to be normal, so that $F = F(V)$ is a curve with convexity facing downward). Along isotherm $p = F(V)$, $T = T_0$ = constant, therefore

$$T_0(S-S_0)=\varepsilon-\varepsilon_0+\int_{V_0}^{V}F(V)\,dV. \quad (18)$$

Substituting this into inequality (17), we get it the final form:

$$\frac{p+p_0}{2}(V_0-V)-\int_{V}^{V_0}F(V)\,dV>0. \quad (19)$$

Obviously, for a normal gas this requirement is tantamount to the condition $V < V_0$ i.e., only the upper half of isotherm T corresponds to the state behind the front.

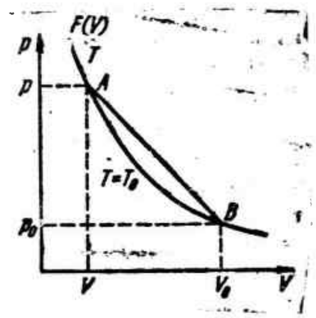


Figure 2.32

Hence, as earlier, it follows that the shock wave in the case of an isothermal gas leads to an increment in pressure and in density; motion ahead of the wave front is supersonic, and subsonic behind the front (the isothermal speed of sound $c_T = V\sqrt{-\partial F/\partial V}$ is taken as the speed of sound here).

Stability condition (19) can be given a different form. Let us consider in the plane of Lagrangian invariables (q, t) the gas element situated between the straight lines $q = q_1$ and $q = q_2$ $(q_1 < q_2)$. For any successive time instants t_1, t_2 $(t_1 < t_2)$, inequality (15) applied to this gas element, gives:

$$\int_{q_1}^{q_2}\left(\varepsilon+\frac{u^2}{2}-T_0S\right)_{t=t_1}dq-\int_{q_1}^{q_2}\left(\varepsilon+\frac{u^2}{2}-T_0S\right)_{t=t_2}dq \geqslant \Delta A, \tag{20}$$

where ΔA is the work done by this element on the ambient gas in the time $t_2 - t_1$:

$$\Delta A=\int_{t_1}^{t_2}(pux^{\nu})_{q=q_2}\,dt-\int_{t_1}^{t_2}(pux^{\nu})_{q=q_1}\,dt \tag{21}$$

(the equality sign in (20) corresponds to the case of a smooth, i.e., quasi-equilibrium, flow). From (20) and (21) we readily obtain the following integral condition equivalent to condition (19):

$$\oint_C \left(\varepsilon + \frac{u^2}{2} - T_0 S\right) dq - p u x^{\nu}\, dt \geq 0 \tag{22}$$

(C is an arbitrary piecewise-smooth contour drawn in the positive direction).

The quantity $\varepsilon + u^2/2 + T_0 S = E - T_0 S$ is called free energy (per unit mass) in thermodynamics, and inequality (22) expresses the familiar thermodynamic law: in an isothermal process the work done by a system is smaller than the loss internal energy and is equal to it only for the case of a quasiequilibrium process.

In this latter case the flow is smooth, and from (22) there derives the differential relation

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{u^2}{2} - T_0 S\right) + \frac{\partial}{\partial q}(p u x^{\nu}) = 0. \tag{23}$$

It must be satisfied by identity in the smooth portion of the flow, i.e., it must be a consequence of differential equations of motion (2.5.4) and (2.5.5). This actually does hold: from equations (2.5.4) and (2.5.5) we readily obtain the relation

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2} - \int^{V} P(V)\, dV\right) + \frac{\partial}{\partial q}(p u x^{\nu}) = 0$$

which, by virtue of (18), coincides with (23).

Note that conditions (19) and (22), as explained in section 1, can be understood as the requirement for an increment in the energy of the total system consisting of a gas particle and the external heat sources (thermostat). Equality

(23) then denotes the constancy of entropy of this system for the case of smooth flow.

7. Strong and weak shock waves. A comparison of shock wave and Riemann compression wave. For simplicity we will assume that the gas ahead of the shock wave front is at rest, i.e., $u_0 = 0$. For an infinitely strong shock wave $p/p_0 = \infty$. We would assume p to be finite, and $p_0 = 0$.

Passing in equations (4.5.13) - (4.5.16) to the limit as $M_0 \longrightarrow \infty$, $p_0 \longrightarrow 0$, and $c_0 \longrightarrow 0$, we obtain Hugoniot's conditions for an infinitely long shock wave in a polytropic gas:

$$p = \lim (1+h)\, p_0 M_0^2 = (1-h)\,\rho_0 (u_0 - D)^2 = \frac{2}{\gamma+1}\rho_0 D^2, \quad (1)$$

$$\frac{\rho}{\rho_0} = \frac{V_0}{V} = \frac{1}{h} = \frac{\gamma+1}{\gamma-1}, \quad (2)$$

$$c = \sqrt{h(1+h)}\,|D|, \quad (3)$$

$$u = (1-h)\,D. \quad (4)$$

We see that the kinetic energy of a gas behind the front of the shock wave is equal to the internal energy, since

$$\varepsilon = \frac{(1-h)^2}{2h(1+h)}c^2 = \frac{(1-h)^2}{2}D^2 = \frac{u^2}{2}.$$

For weak shock waves we will take the quantity $M_0 = \frac{|u_0 - D|}{c_0}$ as been close to unity. Putting $M_0 = 1 + \varepsilon$, $0 < \varepsilon \ll 1$, we perform an expansion in formulas (4.5.13) - (4.5.16) with respect to parameter ε with an accuracy up to members of the order ε^3:

$$\frac{p}{p_0} = 1 + (1 + h)(2\varepsilon + \varepsilon^2),$$

$$\frac{\rho}{\rho_0} = \frac{M_0^2}{(1-h) + hM_0^2} = \frac{1 + 2\varepsilon + \varepsilon^2}{1 + h\varepsilon(2 + \varepsilon)} \simeq [1 - 2h\varepsilon - h\varepsilon^2(1 - 4h)] \times$$
$$\times (1 + 2\varepsilon + \varepsilon^2) = 1 + 2\varepsilon(1 - h) + \varepsilon^2[1 - 5h + 4h^2] + O(\varepsilon^3),$$

$$\frac{c}{c_0} = \sqrt{\left(1 + h - \frac{h}{M_0^2}\right)(1 - h + hM_0^2)} = 1 + 2h\varepsilon + h\varepsilon^2 + O(\varepsilon^3),$$

$$u - u_0 = (1 - h)c_0\left(M_0 - \frac{1}{M_0}\right) = (1 - h)c_0(2\varepsilon - \varepsilon^2) + O(\varepsilon^3)$$
$$(m < 0).$$

Let us compute the discontinuities in Riemann invariants r, s at the front of a weak shock wave:

$$r - r_0 = u - u_0 + \frac{(1-h)}{h}(c - c_0) =$$
$$= (1 - h)c_0[(2\varepsilon - \varepsilon^2) + (2\varepsilon - \varepsilon^2)] + O(\varepsilon^3) =$$
$$= 4(1 - h)c_0\varepsilon - 2(1 - h)c_0\varepsilon^2 + O(\varepsilon^3) \quad (m < 0). \quad (9)$$

$$s - s_0 = u - u_0 - \frac{(1-h)}{h}(c - c_0) = (1 - h)c_0(2\varepsilon - \varepsilon^2) -$$
$$- c_0\frac{1-h}{h}[2h\varepsilon - h\varepsilon^2] + O(\varepsilon^3) = O(\varepsilon^3) \quad (m < 0). \quad (10)$$

From the general properties of Hugoniot's adiabat H, as we saw in subsection 3, there follows

$$S - S_0 = O(h^3). \quad (11)$$

Thus, in a weak shock wave propagating to the right relative to a gas, invariant r has a first-order discontinuity relative to ε, and invariant s and entropy S have third-order discontinuities. Just as for a wave traveling to the left, in variant s and entropy S have third-order discontinuities.

Formulas (10) and (11) show that a weak shock wave behaves as a "short" travelling compression wave. Actually, as we saw in section III, constancy of entropy S and one Riemann invariant characterizes a Riemann compression wave. This is violated for weak shock waves only in the third order, and therefore in approximate terms functions in a weak shock wave can be regarded just as functions in a travelling compression wave. This enables us to approximately replace a weak shock wave with a travelling wave. In several cases, this proves useful, especially when analyzing the interactions of shock waves with Riemann waves.

Since in our approximation

$$D = c_0(1 + \varepsilon) \qquad (m < 0),$$

we see that the velocity of a weak shock wave is the arithmetic mean of the velocities of r-characteristics, since the equality

$$D = \frac{u_0 + c_0 + u + c}{2}$$

is satisfied with an accuracy up to third-order terms.

8. Examples. Let us consider now two of the simplest problems on flow containing shock waves:

1) Motion of a piston in a quiescent gas.

In a quiescent polytropic gas located to the right of the piston and characterized by the parameters $u_0 = 0$, p_0, V_0, c_0, a piston advances with a velocity $U > 0$ when $t > 0$ and $U = 0$ when $t = 0$.

A shock wave will be propagated relative to the gas with a constant velocity D, leaving behind the front a state with parameters u, p, V, c. Clearly, $u = U$. From relation (4.5.16) therefore we determine M_0 as a positive root of the equation

$$M_0^2 - \frac{U}{(1-h)\,c_0} M_0 - 1 = 0,$$

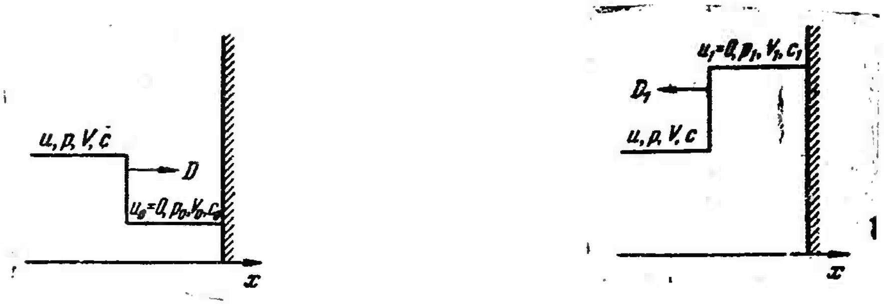
i.e.,

$$M_0 = \frac{U}{2(1-h)\,c_0} + \sqrt{\frac{U}{4(1-h)^2 c_0^2} + 1}. \qquad (1)$$

Next, the quantities p, V are defined from formulas (4.5.13) and (4.5.14), and $D = c_0 M_0$. Note that provided $U > 0$ $V < V_0$, and the stability condition for the shock wave is satisfied.

2) Reflection of a shock wave from a rigid wall.

A shock wave moves toward the right with a velocity $D > 0$ in a gas at rest ($u_0 = 0$, p_0, V_0, c_0), leaving behind the front the state u, p, V, c calculated above (Figure 2.33, a). At the instant $t = 0$ the shock wave approaches the rigid wall $x = x_0$ bounding the gas on the right. The condition $u(x_0, t) = 0$ is assigned at the rigid wall. Therefore the shock wave is reflected from the wall in the form of a shock wave propagating to the left with a velocity $D_1 < 0$. Let us denote the state behind the front of the reflected shock wave as $u_1 = 0$, p_1, V_1, c_1 (Figure 2.33, b).



a) t< 0 b) t >0

Figure 2.33

Thus, the problem reduces to determining precisely these parameters. Note that the problem of determining u_1, p_1, V_1, c_1 from known u, p, V, c reduces to the preceding task, since we know that $u_1 = 0$.

Let us introduce for consideration the quantities

$$M_0 = \frac{D - u_0}{c_0} = \frac{D}{c_0}, \quad M = \frac{D - u}{c}, \quad M_1 = \frac{u - D_1}{c}. \tag{2}$$

The following relations are valid:

$$M_0 > 1, \quad 0 < M < 1, \quad M_1 > 1. \tag{3}$$

$$u - u_0 = u = (1 - h)\, c_0 \left(M_0 - \frac{1}{M_0}\right). \tag{4}$$

$$u_0 - u = -u = (1 - h)\, c \left(M - \frac{1}{M}\right). \tag{5}$$

$$u - u_1 = u = (1 - h)\, c \left(M_1 - \frac{1}{M_1}\right). \tag{6}$$

From equations (5) and (6) it follows that the quantities M_1 and $1/M$ satisfy the same quadratic equation

$$\alpha^2 - \frac{u}{(1-h)c}\alpha - 1 = 0. \tag{7}$$

Since $M_1 > 1$, $1/M > 1$, then $M_1 = 1/M$, i.e., $MM_1 = 1$. (8)

Let us calculate the pressure rise in the reflection of a shock wave. Using (8), we get:

$$\left.\begin{aligned} \frac{p}{p_0} &= (1+h)M^2 - h, \quad M^2 = \frac{\frac{p}{p_0}+h}{1+h}, \\ \frac{p_1}{p} &= (1+h)M_1^2 - h = (1+h)\frac{1}{M^2} - h = \frac{(1+2h)\frac{p}{p_0} - h}{1+h\frac{p}{p_0}}, \end{aligned}\right\} \tag{9}$$

$$\frac{p_1 - p_0}{p - p_0} = 1 + \frac{(1+h)}{h + \frac{p_0}{p}}. \tag{10}$$

For the case of a weak wave $p/p_0 \to 1$, $\frac{p_1 - p_0}{p - p_0} \to 2$, which corresponds to the acoustic law of reflection.

For the case of strong wave, when $\frac{p_0}{p} \to 0$, $\frac{p_1 - p_0}{p - p_0} \to 2 + \frac{1}{h} = 2 + \frac{\nu + 1}{\nu - 2}$. For gases with index close to 1, we obtain a strong pressure rise. However, it must not be assumed that for an isothermal gas the pressure rise will be infinite, since the analogy of an isothermal gas and a polytropic gas with $\nu = 1$ is inapplicable here. Actually, if for an isothermal gas $p_0 \to 0$, then $\rho_c \to 0$, since $c_T^2 = RT =$ constant. Therefore no isothermal shock wave traveling with finite velocity relative to the background $p_0 = 0$ (vacuum) exists.

Section V. Study of Shock Transition. Width of Shock Wave

1. Formulation of the problem for a normal gas. We will assume that the equations of state of a gas satisfy conditions I - VI formulated in subsection 4 of section I, i.e., that the gas is normal.

As we already stated, we considered discontinuous flows as limiting flows of a viscid and thermally conductive fluid as the coefficients of viscosity and thermal conductivity tend to zero. Therefore we will now study several of the simplest solutions to equations in gas dynamics for a gas exhibiting viscosity and thermal conductivity, and then we will obtain discontinuous flows by means of passage to the limit. Here we can estimate the width of the shock wave zone for gases exhibiting finite viscosity and thermal conductivity*).

We will consider only the case of plane symmetry ($\nu = 0$) since as sufficiently small sections and sufficiently small time intervals, any shock wave, when $\nu \neq 0$, can be considered approximately as planar.

Let us write differential equations for a viscid thermally conductive liquid (2.5.11) - (2.5.13):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0. \qquad (1)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left[p - \mu \frac{\partial u}{\partial x} + \rho u^2\right] = 0. \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\rho e + \rho \frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left[\rho u \left(e + \frac{p}{\rho} + \frac{u^2}{2}\right) - \mu u \frac{\partial u}{\partial x} - \varkappa \frac{\partial T}{\partial x}\right] = 0. \qquad (3)$$

To equations (1) - (3) are added equations of state

*) Our presentation in subsections 1-4 follows [5].

$$p = p(\rho, T), \qquad \varepsilon = \varepsilon(\rho, T), \tag{4}$$

which we assume to satisfy requirements I-VI. Multiply (1) by $u^2/2 - \varepsilon$, (2) by $-u$, and summing the results with equation (3), we get

$$\rho\left(\frac{\partial\varepsilon}{\partial t} + u\frac{\partial\varepsilon}{\partial x}\right) + p\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\mu u\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right) - u\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) = \mu\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right). \tag{5}$$

Expressing $\partial u/\partial x$ from (1) and substituting into (5), we give it the form

$$\frac{\partial\varepsilon}{\partial t} + u\frac{\partial\varepsilon}{\partial x} + p\left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x}\right) = \frac{\mu}{\rho}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right). \tag{6}$$

Since TdS = d + pdV, from (6) follows

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = \frac{\mu}{\rho T}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{\rho T}\frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right) = \frac{\mu}{\rho T}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\varkappa}{\rho T^2}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\varkappa\frac{\partial \ln T}{\partial x}\right). \tag{7}$$

Equation (7) shows that entropy S of a thermally insulated mass of gas increases. Actually, by integrating equation (7), we obtain

$$\frac{\partial}{\partial t}\int_{q_1}^{q_2} S(q, t)\,dq = \varkappa\frac{\partial \ln T}{\partial x}\Big|_{q_1}^{q_2} + \int_{x_1}^{x_2}\left[\frac{\mu}{T}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\varkappa}{T^2}\left(\frac{\partial T}{\partial x}\right)^2\right]dx, \tag{8}$$

where q is the Lagrangian coordinate ($dq = \rho\,dx$), q_1 and q_2 are the bounds of the isolated gas volume.

If the gas volume under consideration is thermally insulated, then $\varkappa\frac{\partial T}{\partial x}\Big|_{q_1}^{q_2} = 0$, and from (8) follows the increment in total gas entropy. In particular, from equation (7) it follows that in a viscid, but nonthermally conductive gas, the entropy of each particle does not decrease with time.

Let us consider the simplest solutions to the system of equations (1)-(3), specifically, stationary solution. Since system (1)-(3) is invariant relative to Galileo's transformation $x' = x - Ut$, $u' = u - \bar{U}$, we will consider steady flows in this system of coordinates in which flow has fixed profiles of all quantities u, p, ρ, ε. Omitting the strokes at the variable x' and u' and putting $\frac{\partial \rho}{\partial t} = \frac{\partial u}{\partial t} = \frac{\partial \varepsilon}{\partial t} = 0$, we get equations for the determination of steady flows:

$$\frac{d}{dx}\rho u = 0, \qquad (9)$$

$$\frac{d}{dx}\left[p + \rho u^2 - \mu \frac{du}{dx}\right] = 0, \qquad (10)$$

$$\frac{d}{dx}\left[\rho u\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) - \mu u \frac{\partial u}{\partial x} - \varkappa \frac{\partial T}{\partial x}\right] = 0. \qquad (11)$$

These equations have first integrals:

$$\rho u = C_1, \quad p + \rho u^2 - \mu \frac{du}{dx} = C_2, \quad C_1\left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right) - \mu u \frac{du}{dx} - \varkappa \frac{dT}{dx} = C_3. \qquad (12)$$

The first integral obviously expresses the constancy of the mass flow, the second -- momentum flow, and the third -- the constancy of the energy flow across and arbitrary cross section x = constant. Transform equations (12) into a form suitable for the following treatment. Simple transformations lead to two ordinary differential equations:

$$\mu \frac{dV}{dx} = \frac{1}{C_1}\left[C_1^2\left(V - \frac{C_2}{C_1^2}\right) + p(V, T)\right] = \mathcal{M}(V, T), \tag{13}$$

$$\varkappa \frac{dT}{dx} = C_1\left[\varepsilon(V, T) - \frac{1}{2}C_1^2\left(V - \frac{C_2}{C_1^2}\right)^2 - \frac{C_3}{C_1} + \frac{1}{2}\frac{C_2^2}{C_1^2}\right] = \mathcal{L}(V, T), \tag{14}$$

which relate to functions V and T. As a consequence of equations (13) and (14), we get an equation in the V, T plane:

$$\frac{\varkappa}{\mu}\frac{dT}{dV} = \frac{\mathcal{L}(V, T)}{\mathcal{M}(V, T)}, \tag{15}$$

which plays an important role in the analysis of stationary solutions and conditions for their existence.

Let us introduce the dimensionless variables:

$$\bar{V} = V\frac{C_1^2}{C_2}, \quad \bar{p} = \frac{p}{C_2}, \quad \bar{\varepsilon} = \varepsilon\frac{C_1^2}{C_2^2}, \quad \bar{T} = R\frac{C_1^2}{C_2^2}T. \tag{16}$$

Converting to dimensionless variables $\bar{V}$, $\bar{p}$, $\bar{\varepsilon}$, $\bar{T}$, we rewrite equations (13) and (14) as

$$\bar{\mu}\frac{d\bar{V}}{dx} = \bar{V} + \bar{p}(\bar{V}, \bar{T}) - 1 = \bar{\mathcal{M}}(\bar{V}, \bar{T}), \tag{17}$$

$$\bar{\varkappa}\frac{d\bar{T}}{dx} = \bar{\varepsilon}(\bar{V}, \bar{T}) - \frac{1}{2}(\bar{V} - 1)^2 - \bar{\beta} = \bar{\mathcal{L}}(\bar{V}, \bar{T}), \tag{18}$$

where we denote:

$$\bar{\mu} = \frac{\mu}{C_1}, \quad \bar{\varkappa} = \frac{\varkappa}{C_1 R}, \quad \bar{\beta} = \frac{C_3 C_1}{C_2^2} - \frac{1}{2}, \qquad \bar{p}(\bar{V}, \bar{T}) = p(V, T), \quad \bar{\varepsilon}(\bar{V}, \bar{T}) = \varepsilon(V, T). \tag{19}$$

To simplify notation, in the following we will omit the bar over all quantities and rewrite system (17) and (18) in its final form:

$$\mu \frac{dV}{dx} = p(V, T) + V - 1 = \mathcal{M}(V, T), \tag{20}$$

$$\varkappa \frac{dT}{dx} = \varepsilon(V, T) - \frac{1}{2}(V - 1)^2 - \beta = \mathcal{L}(V, T). \tag{21}$$

However, we must remember that the functions $p(V, T)$ and (V, T) in (20) and (21) are obtained from the equations of state with allowance for (16).

Dimensionless variables $p(V, T)$ and $\varepsilon(V, T)$ appear in equations (20) and (21), where V and T are also dimensionless. We can easily verify that these functions are such that conditions I-VI are satisfied, if we satisfy the initial equations of state. (Here we must bear in mind that $C_2 > 0$, $C_3 C_1 > 0$.)

We set up the following boundary problem for the system (20) and (21):

Find the solution $V(x)$, $T(x)$ of system (20) and (21), which at infinity tends to constant values, i.e.,

when $x \to +\infty$, $\quad V(x) \to V_1$, $\; T(x) \to T_1$, (22)

but when $x \to -\infty$, $\; V(x) \to V_2$, $T(x) \to T_2$. (23)

A necessary condition for the existence of the solution is obviously the requirement that the point (V_1, T_1), (V_2, T_2) be stationary points of the system (20) and (21), i.e.,

$$\mathcal{M}(V_1, T_1) = \mathcal{L}(V_1, T_1) = \mathcal{M}(V_2, T_2) = \mathcal{L}(V_2, T_2) = 0. \quad (24)$$

In other words, the points (V_1, T_1) and (V_2, T_2) must be points of intersection of the curves $\mathcal{M}(V, T) = 0$, $\mathcal{L}(V, T) = 0$ (25)

Suppose such points (V_1, T_1) and (V_2, T_2) exist. Then at these points, according to (20) and (21), $\mu\, dV/dx = \mathcal{H}\, dT/dx = 0$; therefore from integrals (12) it follows that: $\rho_1 u_1 = \rho_2 u_2 = C_1$. (26)

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 = C_2, \quad (27)$$

$$\rho_1 u_1 \left(\varepsilon_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2}\right) = \rho_2 u_2 \left(\varepsilon_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}\right) = C_3 \quad (28)$$

(here all functions u, ρ, p, ε are initial functions, without transformation to dimensionless variables).

Hence it follows that the states $u_1, \rho_1, p_1, \varepsilon_1$; $u_2, \rho_2, p_2, \varepsilon_2$ must satisfy Hugoniot's conditions. Actually, since we are considering the stationary solution of system (1)-(3) in a moving system of coordinates, then by denoting the rate of displacement U of the coordinate system by D, we obtain the results that in conditions (26)-(28) the quantities u_1, u_2, in the transformation to the fixed coordinate system, are replaced by $u_1 - D$, $u_2 - D$, after which conditions (26)-(28) take on the usual form of Hugoniot's conditions (4.1.10)-(4.1.12). Hence, in particular, it follows that the points (p_1, V_1) and (p_2, V_2) in the plane of variables p, V must lie on the Hugoniot's adiabat.

Thus, all points of intersection of the curves (25) lie on Hugoniot's adiabat passing through one of them.

2. Properties of the curve $\mathcal{M} = 0, \mathcal{L} = 0$ for a normal gas. Let us show that for a normal gas functions $\mathcal{L}$ and $\mathcal{M}$ exhibit the following properties:

1) $\frac{\partial \mathcal{L}(V, T)}{\partial T} > 0, \frac{\partial \mathcal{M}(V, T)}{\partial T} > 0.$

2) The curves $\mathcal{M} = 0$ and $\mathcal{L} = 0$ either have no intersection points, or have two and only two common points (V_1, T_1) and (V_2, T_2). We will denote them by (V_0, T_0) and (V_1, T_1), assuming that $V_0 > V_1$.

3) $\frac{\partial \mathcal{L}(V, T)}{\partial V} > 0$ at the curve $\mathcal{L} = 0$ while $V_1 < V < V_0$.

4) $\frac{\mathcal{L}_V}{\mathcal{L}_T} < \frac{\mathcal{M}_V}{\mathcal{M}_T}$ at the point (V_0, T_0) and $\frac{\mathcal{L}_V}{\mathcal{L}_T} > \frac{\mathcal{M}_V}{\mathcal{M}_T}$ at the point (V_1, T_1).

From property 1 it follows that temperature T is a unique function of variable V along the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$. Let us denote:

$$\left.\begin{aligned} T &= l(V) \text{ along the curve } \mathcal{L} = 0. \\ T &= m(V) \text{ along the curve } \mathcal{M} = 0. \end{aligned}\right\} \quad (1)$$

From properties 1 and 3 it follows that the curve $\mathcal{L} = 0$ at the section $[V_1, V_0]$ diminishes monotonically, i.e., the function $l(V)$ decreases monotonically. Actually, along the curve $\mathcal{L} = 0$ we have

$$\frac{dl(V)}{dV} = \frac{dT}{dV} = -\frac{\frac{\partial \mathcal{L}}{\partial V}}{\frac{\partial \mathcal{L}}{\partial T}} < 0. \quad (2)$$

Finally, property 4 means that the slope of the curve $\mathcal{M} = 0$, at the point (V_0, T_0) is less than the slope of the curve $\mathcal{L} = 0$, but is greater at the point (V_1, T_1).

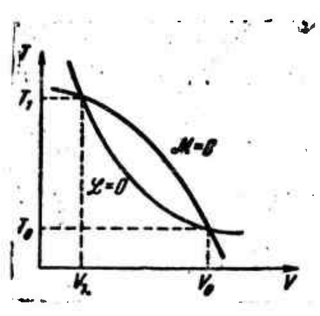


Figure 2.34.

Thus, properties 1-4 are functions $\mathcal{L}$ and $\mathcal{M}$ denote that the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$ at the V, T plane lie approximetely thusly (Figure 2.34).

Let us prove that the properties 1-4 of functions $\mathcal{L}$ and $\mathcal{M}$ follow from the properties I-V of the equations of state.

Property 1 for $\mathcal{L}$ and $\mathcal{M}$ is written thusly:

$$\frac{\partial \mathcal{L}(V, T)}{\partial T} > 0, \quad \frac{\partial \mathcal{M}(V, T)}{\partial T} > 0, \tag{3}$$

and since dimensionless variables (5.1.16) differ from the dimensional variable only by their positive cofactors, inequalities (3) follow from properties IV and V (cf also formula (1.4.23)).

Let us now prove property 2. All points of intersection of the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$ must lie in the plane V, T at Hugoniot's adiabat H passing through one of them. The line $\mathcal{M} = 0$ obviously is the straight line

$$p = 1 - V \tag{4}$$

with negative slope in the plane of the dimensionless variables p, V. Thus we have seen in subsection 3 of section IV, any straight line with negative slope intersects Hugoniot's adiabat H exactly at two points. Thus, if the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$ have even one point in common, then they have also the second point in common, but no more. Property 2 stands proven.

Now let us demonstrate properties 3 and 4. Using the relation d = TdS - pdV in expressions (5.1.13) and (5.1.14) for $\mathcal{L}$ (V, T), $\mathcal{M}$(V, T) (in dimensionless variables), we get

$$d\mathcal{L} = C_1' [T\, dS - C_{10} \mathcal{M}\, dV]. \tag{5}$$

Whence

$$\frac{\partial \mathcal{L}(V, T)}{\partial V} = C_1 T \frac{\partial S(V, T)}{\partial V} - C_{10}^2 \mathcal{M}. \tag{6}$$

Let us consider the intersection of the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$ in the plane of dimensional variables V, p (Figure 2.35). From (5) it follows that at the curve

$$\mathcal{L} = 0 \qquad dS/dV = C_1/T M. \qquad (7)$$

Hence it follows that dS/dV = 0 at the points of intersection of the curve = 0 with the straight line $\mathcal{L} = 0$, i.e., the curve $\mathcal{M} = 0$ is tangent to Poisson's adiabat A passing through the point of intersection, and consequently also the adiabat H. Since the slope of adiabat A for a normal gas is greater at the point (p_1, V_1) than the slope of the ray $\mathcal{M} = 0$, but at the point (p_o, V_o) is less than the slope of $\mathcal{M} = 0$, hence it follows that the inclination of the curve $\mathcal{L} = 0$ at the point (p_1, V_1) is greater than the inclination of the straight line $\mathcal{M} = 0$, but is less than the point (p_o, V_o).

In mapping the plane V, p on the plane V, T, corresponding to the lower half-plane $\mathcal{M}(V, p) < 0$ is the domain $\mathcal{M}(V, T) < 0$, which lies beneath the curve $\mathcal{M} = 0$ by virtue of the relation $\frac{\partial p(V, T)}{\partial T} > 0$, which is valid for the normal gas. Hence it follows that the pattern of the intersection of the curves $\mathcal{M} = 0$ and $\mathcal{L} = 0$ in the V, p plane is qualitatively the same as in the V, T plane (Figure 2.34).

Thus, properties 3 and 4 have been proven*).

3. Qualitative investigation of integral curves of the shock transition. The solution of systems (5.1.20) and (5.1.21) satisfying the boundary conditions (5.1.22) and (5.1.23) describes a stationary shock wave in a viscid thermally conductive gas. We will let V_o, T_o stand for the state ahead of the wave front,

---

*) Let us note that study of the curves $\mathcal{L} = 0$ and $\mathcal{M} = 0$ in the V, p plane is much simpler than in the V, T plane. However, in investigating isothermal discontinuity it is more convenient to employ the variables V, T.

and V_1, T_1 represent the state ahead of the front, and for definiteness we will assume that $m = C_1 > 0$, i.e., the shock wave propagates to the left with respect to the gas. Then dimensionless variables $\mu > 0, \varkappa > 0$.

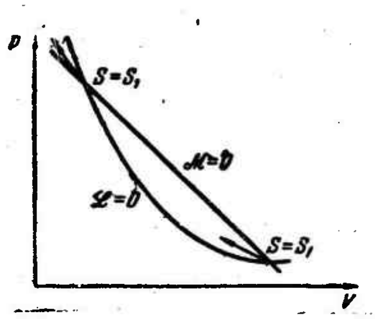


Figure 2.35

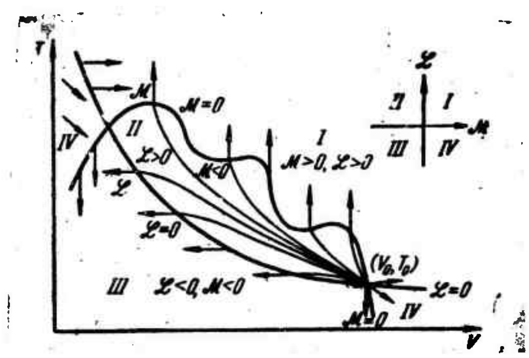


Figure 2.36

The solution $V = V(x)$, $T = T(x)$ of system (5.1.20), (5.1.21) can be considered as a parametric assignment of the integral curve of the equation

$$\frac{\varkappa}{\mu}\frac{dT}{dV}=\frac{\mathscr{L}(V,T)}{\mathscr{M}(V,T)}. \qquad (1)$$

Conversely, to each solution $T = T(V)$ of equation (1) there corresponds the solution $T = T(x)$, $V = V(x)$ of system (5.1.20), (5.1.21) defined with an accuracy up to the displacement.

To the shock transition will correspond the solution $T = T(V)$ of equation (1) passing through the points (V_0, T_0), (V_1, T_1) of the intersection of the curves $\mathscr{L} = 0, \mathscr{M} = 0$. Clearly, these points are singular points of equation (1), and this makes the present formulation of the problem possible.

Let us consider in the plane of variables V, T the field of vector directions $\{\mathscr{M}, \mathscr{L}\}$. The curves $\mathscr{M} = 0, \mathscr{L} = 0$ divide up the quadrant $V > 0$ into four regions I-IV. The number of the regions corresponds to the number of the quadrant in which the vector $\{\mathcal{M}, \mathcal{L}\}$ lies (Figure 2.36).

From the graph of the direction field given in Figure 2.36 for equation (1) it follows that in each of the regions the function T = T(V) is monotonic at the integral curve. Here the integral curves can pass from one region into another in the following order: from region IV to regions I and III, from region II to regions I and III. The following transitions are impossible: from regions I and III to regions II and IV, from region IV to II and conversely from region II to region IV.

Hence it follows that the integral curve connecting the points (V_0, T_0), (V_1, T_1) must lie entirely in the region II. Thus, the following relations must obtain for the sharp transition:

$$\frac{dV}{dx} < 0, \quad \frac{dT}{dx} > 0 \quad (m = C_1 > 0). \tag{2}$$

Let us prove the existence and uniqueness of the integral curve of the shock transition. To do this, we establish the type of singular points (V_0, T_0), (V_1, T_1) of equation (1). Thus we know, the type of the singular point is determined by the characteristic equation, which is of the form

$$\begin{vmatrix} \frac{\partial \mathcal{M}}{\partial V} - \lambda & \frac{\partial \mathcal{M}}{\partial T} \\ \frac{\partial \mathcal{L}}{\partial V} & \frac{\partial \mathcal{L}}{\partial T} - \lambda \end{vmatrix} = 0. \tag{3}$$

We have for the roots λ_1, λ_2 of the characteristic equation the expression

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{\partial \mathcal{M}}{\partial V} + \frac{\partial \mathcal{L}}{\partial T}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\partial \mathcal{M}}{\partial V} + \frac{\partial \mathcal{L}}{\partial T}\right)^2 + \frac{\partial \mathcal{L}}{\partial V}\frac{\partial \mathcal{M}}{\partial T} - \frac{\partial \mathcal{M}}{\partial V}\frac{\partial \mathcal{L}}{\partial T}}.$$

We can readily see that by virtue of properties 1 and 3 of functions $\mathcal{L}$ and $\mathcal{M}$, the characteristic roots λ_1 and λ_2 are real and distinct. They are of the same sign, if

$$\Delta = \begin{vmatrix} \frac{\partial \mathcal{M}}{\partial V} & \frac{\partial \mathcal{M}}{\partial T} \\ \frac{\partial \mathcal{L}}{\partial V} & \frac{\partial \mathcal{L}}{\partial T} \end{vmatrix} > 0,$$

and are of different signs if $\Delta < 0$. Therefore in view of property 4 of functions $\mathcal{L}$ and $\mathcal{M}$, we conclude that at the point (V_o, T_o) λ_1, λ_2 are of the same sign (positive), but of dissimilar signs at the point (V_1, T_1). Thus, the point (V_o, T_o) is a node, and the point (V_1, T_1) is a saddle point for the equation (1). Let us show that there exists an integral curve of equation (1) connecting the points (V_o, T_o) and (V_1, T_1). According to Figure 2.36, for the field of directions of equation (1), through any point M on curve $\mathcal{M} = 0$ passes an integral curve of equation (1), passing simultaneously through the point (V_o, T_o). Shifting the point M along the curve $\mathcal{M} = 0$ to the point (V_1, T_1), we obtain by a continuity consideration the result that the points (V_1, T_1) and (V_o, T_o) are connected by the integral curve of equation (1).

Similarly, through any point L on curve $\mathcal{L} = 0$ there passes the integral curve (1) passing through the point (V_o, T_o). Shifting the point L along the curve $\mathcal{L} = 0$ to the point (V_1, T_1), we obtain at the limit that there may exist an integral curve of equation (1) connecting the points (V_o, T_o), (V_1, T_1) and which curve is distinct from the preceding.

Since the point (V_1, T_1) is a saddle point, then by the qualitative theory of differential equations, through it pass only two integral curves of equation (1).

Therefore the points (V_0, T_0) and (V_1, T_1) connected by one and only one integral curve of equation (1). Actually, if these points were connected by two distinct integral curves of equation (1), then through these points would pass also any integral curve drawn through the point lying with the region bounded by these two integral curves. This contradicts our conclusion that the point (V_1, T_1) is a saddle point.

By virtue of properties 1 and 3 of functions $\mathcal{M}$ and $\mathcal{L}$, curves $\mathcal{M} = 0$, $\mathcal{L} = 0$ are zeros of the first order of these equations. Hence it follows that in the neighborhood of singular points (for example, (V_0, T_0)), the functions $\mathcal{L}$ and $\mathcal{M}$ can be represented in the form

$$\mathcal{L} = a_{11}(V - V_0) + a_{12}(T - T_0),$$
$$\mathcal{M} = a_{21}(V - V_0) + a_{22}(T - T_0),$$

and along the integral curve connecting the points (V_0, T_0) and (V_1, T_1)

$$\mathcal{M} = b(V - V_0), \quad b \neq 0.$$

From equation (5.1.21) it follows that the "width" of the shock transition zone is infinite, since the integral

$$\Delta x = \int_{V_0}^{V_1} \frac{\mu \, dV}{\mathcal{M}(V, T)}$$

diverges. Therefore plots of the shock transition are of the form shown approximately in Figure 2.37, where it is clear that values V_0, T_0 ahead of the front and V_1, T_1 behind the front are attained asymptotically at infinity.

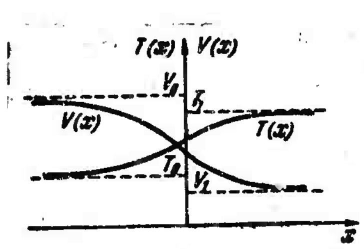


Figure 2.37

4. Limiting case. Isothermal discontinuity. Let us consider the behavior of the shock transition at two limiting cases: as $\mathcal{H} \to 0$ and as $\mu \to 0$.

If we fix $\mu > 0$ and vary $\mathcal{H}$, then the dependence of the shock transitions on $\mathcal{H}$ will be monotonic in the sense that to the lesser value of $\mathcal{H}$ they will correspond the integral curve of equation (5.3.1) lying closer to the curve $\mathcal{L} = 0$.

Let us show that for sufficiently small $\mathcal{H}$ integral curve (5.3.1) lies in the ε-strip above the curve $\mathcal{L} = 0$. Actually, in the left point of region II we will have over the ε-strip of the curve $\mathcal{L} = 0$, $\mathcal{L} > \delta(\varepsilon)$. Then, by selecting as large $N > 0$ as we like, we can select a small $\mathcal{H} = \mathcal{H}(\varepsilon, N)$ such that at any point over the ε-strip

$$\frac{dT}{dV} = -\frac{\mu}{\varkappa}\left|\frac{\mathcal{L}}{\mathcal{M}}\right| < -N,$$

i.e., the slope of integral curve of equation (5.3.1) over the ε-strip can be made larger than the maximum slope of the curve bounding the ε-strip from above*). Then, if the integral curve exists at any point from the ε-strip, it will no longer enter at the point (V_1, T_1). This means that as $\mathcal{H} \to 0$ the shock transition tends to the curve $\mathcal{L} = 0$.

Let us consider the second limiting case when $\mu \to 0$, $\mathcal{H} \neq 0$. If the segment of the line $\mathcal{M} = 0$ includes between the singular points is a curve monotonically diminishing toward the side of increasing V, then by analogous arguments we can show that as $\mu \to 0$ the shock transition tends toward the curve $\mathcal{M} = 0$.

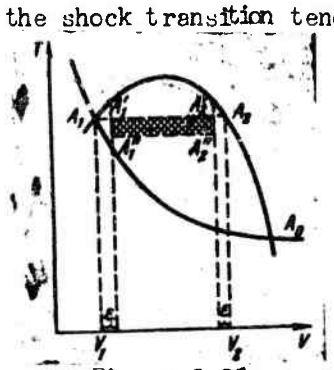


Figure 2.38

*) /on following page/

For the case when this section of the curve $\mathcal{M} = 0$ is not a monotonic curve, the situation turns out differently. Let us consider for simplicity the case when the curve $\mathcal{M} = 0$ at the segment $V_1 \leq V \leq V_0$ has one maximum (Figure 2.38). Clearly, in this case the upper bound of the integral curves is the curve $A_1A_2A_0$ consisting of the chord A_1A_2 parallel to the V axis, and the arc A_0A_2 of the curve $\mathcal{M} = 0$. Reasoning analogously to the above treatment, we can show that this boundary is a point, i.e., as $\mu \to 0$ the integral curve tends to the curve $A_1A_2A_0$, and does so uniformly. In particular, at the interval $[V_1 + \varepsilon, V_2 - \varepsilon]$ the shock transition lies in the ε-strip $A_1'A_1''A_2'A_2''$ (cross-hatched in Figure 2.38). Since in this strip $|\mathcal{M}(V, T)| > M_0(\varepsilon)$, $|\mathcal{L}(V, T)| < L_0(\varepsilon)$, then the inequality

$$\frac{x}{\mu}\left|\frac{dT}{dV}\right| < \left|\frac{L_0}{M_0}\right|.$$

is valid. Hence it follows that we can make μ so small that at the inteval $[V_1 + \varepsilon, V_2 - \varepsilon]$ we will have

$$\left|\frac{dT}{dV}\right| < \delta_1 = \delta_1(\varepsilon, \mu), \quad \delta_1 \to 0 \quad \text{where } \mu \to 0.$$

Consequently, T in this interval lies within the limit $T_1 - \alpha(\varepsilon, \mu) \leq T \leq T_1 + \alpha(\varepsilon, \mu), \alpha(\varepsilon, \mu) \to 0$ as $\mu \to 0$. The segment Δx corresponding to the integral $[V_1 + \varepsilon, V_2 - \varepsilon]$ has the value

$$\Delta x = \Delta(\varepsilon, \mu) = \int_{V_1+\varepsilon}^{V_2-\varepsilon} \frac{\mu\, dV}{\mathcal{M}(V, T(V))}.$$

Since at the interval $[V_1 + \varepsilon, V_2 - \varepsilon]$ $\mathcal{M}(V, T) > M_0(\varepsilon) > 0$, hence it follows that $\Delta(\varepsilon, \mu) \to 0$ as $\mu \to 0$. The relations derived are valid for arbitrary $\varepsilon \to 0$.

Thus, when there is a maximum at the segment $V_1 \leq V \leq V_0$ on the curve

*) For definiteness we can assume that the ε-strip is obtained by displacing the curve $\mathcal{L} = 0$ with respect to the variable T by the quantity ε.

$\mathcal{K} = 0$, the shock transition in the case $\mathcal{H} \neq 0$, $\mu = 0$ consists of a smooth variation over the interval $(-\infty, x_0)$ from the values V_0, T_0 ahead of the front to the values V_2, $T_2 = T_1$ (point A_2 in Figure 2.38) and at the discontinuity of specific volume at the point x_0 from values V_2 on the left to V_1 on the right at constant temperature $T = T_1$ (Figure 2.39). Thus, when only thermal conductivity is present, a discontinuity of density at constant temperature is possible. This discontinuity is called isothermal.

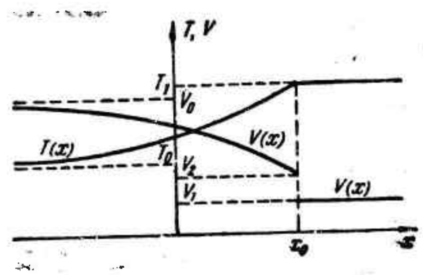


Figure 2.39

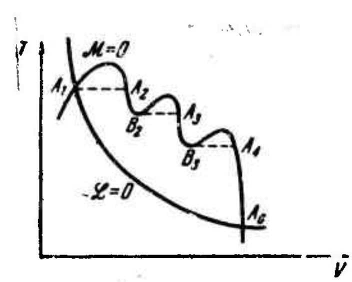


Figure 2.40

As a consequence of the possitiveness of the coefficient of thermal conductivity it follows that the specific volume V_2 ahead of the front of the isothermal discontinuity is greater than the specific volume V_1 behind the front of the isothermal discontinuity. Thus we obtain the result that the limiting discontinuity as $\mu \to 0$ satisfy the condition for the stability of discontinuity in an isothermal gas which we derived in subsection 6 of section IV.

For the case when the curve $\mathcal{M} = 0$ has several maxima, the shock transition will have several isothermal discontinuities (Figure 2.40) simply did by smooth flow zones. In this case the shock transition tends, as $\mu \to 0$, to the curve $A_1A_2B_2A_3B_3A_4A_o$, which is clear from the fact that the integral curve of equation (5.3.1) always lies within the zone II.

Let us also note that the effect of isothermal discontinuity depends not only on the equations of state, but also on the constants C_1, C_2, C_3 defining the flow. Generally speaking, an isothermal discontinuity is induced only for strong enough shock waves. We will show this below for the example of an ideal gas.

In concluding this section, we note that we have obtained steady flows of a viscid and heat-conducting liquid in the form of a "blurred" shock wave, of the type shown in Figure 2.37. Actually, the values V_o, T_o, u_o ahead of the front and V_1, T_1, u_1 behind the front satisfy the Hugoniot's conditions and the stability condition $S_1 > S_o$ $(V_1 < V_o)$. Therefore when $\mu \to 0$, $\mathcal{H} \to 0$ we obtain at the limit from solutions $T = T(x)$, $V = V(x)$ of systems (5.1.20) and (5.1.21) a stable this continuity (shock wave) satisfying the conditions for dynamic compactibility and the stability condition.

In general the assumption exists that any solutions of equations in gas dynamics containing stable discontinuities can be viewed as limiting solutions of equations in gas dynamics with viscosity and thermal conductivity as the coefficients of viscosity μ and of thermal conductivity $\mathcal{H}$ tend to zero.

Thus far there is not a single example refuting this hypothesis, although neither is there a proof of it. The latter circumstance stems from the difficulties arising in an exact consideration of Cauchy's problem for nonlinear equations describing flows of a viscid heat-conducting gas.

In this section we have shown only that this hypothesis valid for stationary flows, i.e., for a constant shock wave which exists for an indefinitely long time.

5. Shock transition for the case of an ideal gas (Becker's study). The first investigation of shock transition in a viscid heat-conducting gas was made by Becker [6] in 1922. He considered the case of an ideal gas

$$p = \frac{RT}{V} = \frac{R\varepsilon}{c_V V}, \quad \varepsilon = c_V T. \tag{1}$$

In dimensionless variables equations (1) become

$$T = pV, \quad \varepsilon = \frac{1}{\gamma - 1} T = \frac{1}{\gamma - 1} pV. \tag{2}$$

and the functions $\mathcal{L}$ and $\mathcal{M}$ are specialized thusly:

$$\mathcal{M}(V, T) = \frac{T}{V} + V - 1, \tag{3}$$

$$\mathcal{L}(V, T) = \frac{1}{\gamma - 1} T - \frac{1}{2}(V - 1)^2 - \beta. \tag{4}$$

Thus, the curve $\mathcal{M}(V, T) = 0$ is a parabola

$$T = V - V^2 = -\left(V - \frac{1}{2}\right)^2 + \frac{1}{4}, \tag{5}$$

with its convexity facing upward, with its axis as the straight line $V = \frac{1}{2}$ and the apex as the point $V = \frac{1}{2}$, $T = \frac{1}{4}$ (Figure 2.41). The curve $\mathcal{L}(V, T) = 0$ is also a parabola

$$T = \frac{\gamma - 1}{2}(V - 1)^2 + \beta(\gamma - 1). \qquad (6)$$

with its convexity facing downward, with axis V = 1 and the apex at the point V = 1, $T = \beta(\gamma - 1)$. Thus, the shape of parabola (6) is unchanged, and its position depends on β, i.e., on the flow constants.

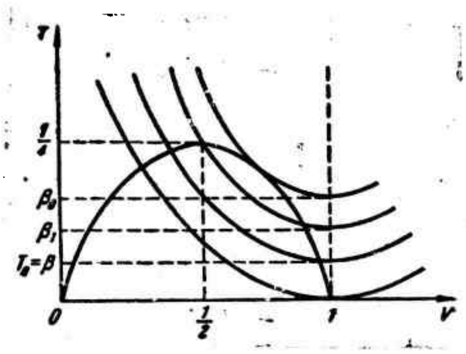


Figure 2.41

Parabolas $\mathcal{L} = 0$ and $\mathcal{M} = 0$ intersect two points when $0 \leq \beta \leq \beta_0$; the cases $\beta > \beta_0$ and $\beta < 0$ are physically unattainable, since in the first case there are no points of intersection and, therefore, there are no asymptotic values of V and T, and in the second $T < 0$. The case $\beta = 0$ corresponds to an infinitely strong shock wave, since $T_1/T_0 = p_1/p_0 = \infty$, and the case $\beta = \beta_0 = \frac{1}{2(\gamma^2 - 1)}$ corresponds to an infinitely weak shock transition, since in this case the points (V_0, T_0) and (V_1, T_1) merge, i.e., $T_1/T_0 = p_1/p_0 = 1$.

For the case $\mathcal{H} = 0$, $\mathcal{M} \neq 0$, equations (5.1.20) and (5.1.21) are integrated. Actually,

$$T(x) = \frac{\gamma - 1}{2}[V(x) - 1]^2 + \beta(\gamma - 1). \qquad (7)$$

and V(x) satisfies the equation

$$\mu \frac{dV}{dx} = \frac{\gamma+1}{2} \frac{(V-V_0)(V-V_1)}{V}$$ where $V_1 \leq V \leq V_0$. (8)

Whence

$$\frac{2}{\gamma+1} \int \frac{V\,dV}{(V-V_0)(V-V_1)} = \frac{2}{\gamma+1} \ln\left[(V_0 - V)^{\frac{V_0}{V_0-V_1}} (V - V_1)^{-\frac{V_1}{V_0-V_1}}\right] = \frac{x}{\mu} + \text{const.} \qquad (9)$$

Figure 2.42, a gives the approximate form of the profile $V = V(x)$ yielded by formula (9). The proximity of the graph $V(x)$ to the constant values $V = V_1$ and $V = V_0$ occurs as $x \longrightarrow \pm\infty$, so that the "width" of the shock wave, strictly speaking, is infinite. However, this proximity occurs exponentially, i.e., quite rapidly. To determine the order of the "width" of the shock wave zone, it is defined as the quantity

$$l = \frac{V_0 - V}{\max\left|\frac{dV(x)}{dx}\right|} \qquad (10)$$

(cf also Figure 2.42, a).

Calculations made by Becker led to an amazing result. It turned out that for most gases at not very high temperatures and densities the quantities μ and $\mathcal{H}$ are such that the width of a shock transition zone proves to be of the order of 10^{-4} - 10^{-6} cm, i.e., of the order of a gas molecule path length.

If we exclude from consideration of the fact of nonequilibrium status of hydrodynamic flows at distances of the order of a molecular path length, and this indicates that with a high degree of accuracy the shock transition can be effectively replaced by a mobile discontinuity (shock wave), whose left and right limiting values satisfy Hugoniot's condition and the condition of entropy increase (the stability condition).

Thus, justification is gained for our point of view according to which the flow is decomposed into regions of reversible processes where equations of hydrodynamics are in effect without allowance for dissipative terms, and into regions of irreversible processes, which constitute narrow zones and can be effectively described by mobile discontinuity surfaces.

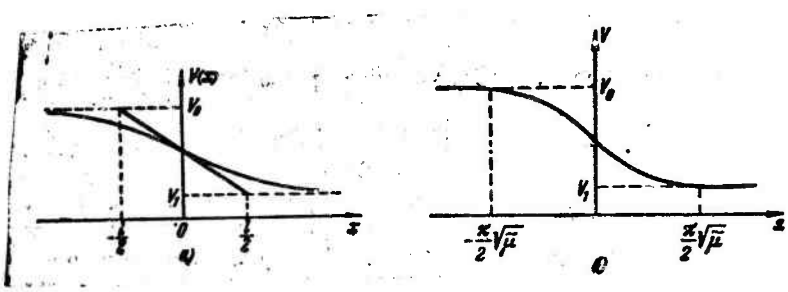


Figure 2.42

Let us additionally note that a more exact representation of a shock transition zone in a real gas can be obtained only by using the Boltzman equation, however the estimate of the width of a shock wave transition zone remains the same.

Let us note in concluding this subsection that in the case $\mu = 0$, $\mathcal{H} \neq 0$, the isothermal discontinuity, as can be readily seen in Figure 2.41, exists only for sufficiently strong shock waves when $0 \leq \beta \leq \beta_1$.

Let us present formulas (5.1.13) and (5.1.14) in Lagrangian variables q, t which we will need in the following treatment:

$$\bar{\mu}\frac{dV}{dq} = C_1\left(V - \frac{C_2}{C_1^2}\right) + \frac{p(V,T)}{C_1}, \qquad \bar{\mu} = \mu\rho = \frac{\mu}{V}; \quad (11)$$

$$\bar{\varkappa}\frac{dT}{dq} = C_1\varepsilon - \frac{1}{2}C_1^3\left(V - \frac{C_2}{C_1^2}\right)^2 - C_3 + \frac{1}{2}\frac{C_2^2}{C_1}, \qquad \bar{\varkappa} = \rho\varkappa. \quad (12)$$

For an ideal gas $\varepsilon = \frac{1}{\gamma - 1}$ pV. Let us consider the case $\mathcal{H} = 0$, $\mu \neq 0$. From (11) and (12) it follows the equation for V:

$$\bar{\mu} \frac{dV}{dq} = \frac{\gamma+1}{2} m \frac{(V - V_1)(V - V_0)}{V}, \qquad (13)$$

where m = C_1 is the mass flow rate of the shock wave; V_0, V_1 are the values of the specific volume V ahead of and behind the shock wave front.

For the case μ = constant, from (13) we have the integral

$$\ln \left(\frac{V_0 - V}{V - V_1} \right)^{\frac{1}{V_0 - V_1}} = \frac{\gamma+1}{2\mu} mq + \text{const}; \qquad (14)$$

if, for example, $\bar{\mu} = \mu\rho$ = constant, then from (13) follows the formula close to (9):

$$\ln \frac{(V_0 - V)^{\frac{V_0}{V_0 - V_1}}}{(V - V_1)^{\frac{V_1}{V_0 - V_1}}} = \frac{\gamma+1}{2\bar{\mu}} mq + \text{const}. \qquad (15)$$

From formula (13), on the assumption μ = constant, and using the definition of the width of the shock wave zone by formula (10), we obtain an expression for the zone width Δq:

$$\Delta q = \frac{8\mu}{\gamma+1} \frac{1}{|\Delta u|},$$

where $\Delta u = u_1 - u_0$ is the discontinuity in velocity at the shock wave.

6. Stationary solutions of equations in hydrodynamics with Neymar-Richtmeyer viscosity. Now let us consider the stationary solution of equations in gas dynamics for a polytropic gas whose thermal conductivity is equal to zero, and whose "viscosity" appearing in equations (5.1.1) - (5.1.3) is of a special form*):

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*) /on following page/

$$\mu = \frac{\lambda}{V}\frac{\partial u}{\partial x}\left|\frac{\partial u}{\partial x}\right|, \tag{1}$$

i.e., we can formally assume in the preceding consideration that the coefficient μ is equal to the quantity

$$\mu = \frac{\lambda}{V}\left|\frac{\partial u}{\partial x}\right|. \tag{2}$$

Since according to (5.1.12) and (5.1.16), $u = mV$, (3)

then we can easily see that for a polytropic gas the determination of a stationary solution reduces to integrating the equation

$$\tilde{\mu}\frac{dV}{dx}\left|\frac{dV}{dx}\right| = (V - V_0)(V - V_1) \tag{4}$$

(cf equation (5.5.8)), where $\tilde{\mu} = \frac{2}{\gamma + 1} m\lambda$. For the case $m > 0$, $\tilde{\mu} > 0$ $dV/dx < 0$ (cf Figure 2.42, a), therefore equation (4) is represented in the form

$$\tilde{\mu}\left(\frac{dV}{dx}\right)^2 = (V - V_1)(V_0 - V) > 0 \tag{5}$$

and is readily integrated:

$$\arcsin\frac{V - \frac{V_0 + V_1}{2}}{\frac{V_0 - V_1}{2}} = \frac{x}{\sqrt{\tilde{\mu}}} + C. \tag{6}$$

In Figure 2.42, b is given the profile $V = V(x)$ for the solution (6). The substantial difference from the preceding case here is the finite width $\pi\sqrt{\tilde{\mu}}$ of the shock transition. Accordingly, we observe that the increasing proximity

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*) The viscosity of this type was first discussed by Neyman and Richtmayer [7].

of the solution $V = V(x)$ to the constant values V_0 and V_1 is not analytic, since the second derivative $V''(x)$ at the point $x = \pm \frac{\pi\sqrt{\lambda}}{2}$ suffers a discontinuity, which can be readily verified both from the equation (4) as well as from formula (6) for the solution.

If in equation (5.5.13) we quite analogously set

$$\mu = \frac{\lambda}{V}\left|\frac{\partial v}{\partial q}\right|, \qquad (7)$$

then we arrive at the equation

$$\tilde{\mu}\left|\frac{dV}{dq}\right|\frac{dV}{dq} = (V - V_0)(V - V_1), \quad \tilde{\mu} = \frac{2\lambda}{\gamma+1}, \qquad (8)$$

whose integral is the equality

$$\arcsin \frac{V - \frac{V_0 + V_1}{2}}{\frac{V_0 - V_1}{2}} = \frac{q}{\sqrt{\tilde{\mu}}} + C, \qquad (9)$$

and the width of the shock transition zone is $\pi\sqrt{\tilde{\mu}} = \pi\sqrt{\frac{2\lambda}{\gamma+1}}$.

note that for the case of Lagrangian coordinates, the Neyman-Richtmayer viscosity results in a finite width of the shock front independent of the strength of a shock wave.

The finiteness of the width of the shock transition in several cases appears to be substantial. Therefore the Neyman-Richtmayer "viscosity" (1) is widely used for numerical calculations of discontinuous solutions of equations in gas dynamics. The artificial "viscosity" of the type (1) with a small coefficient λ is introduced into equations of gas dynamics for gases devoid of internal friction and thermal conductivity. This permits blurring of the shock transitions over a finite region, which proves to be convenient in the numerical calculations.

This is discussed in greater detail in Chapter **Four.**

Section VI Problem of the Decay of an Arbitrary Discontinuity.

1. General properties of the solution of discontinuity decay problem. In this section we consider in detail the problem of the decay of an initial discontinuity. For the case of an isothermal gas this problem was posed and solved by Riemann in his work [8]. A qualitative examination of the problem of discontinuity decay for polytropic gases was made by N. Ye. Kochin [9], and for normal gases by L. D. Landau and Ye. M. Lifschitz [10].

The term arbitrary discontinuity is given to the initial state of two infinite gas masses characterized by constant parameters u_1, p_1, V_1, ε_1, T_1; u_0, p_0, V_0, ε_0, T_0, and adjoining at initial instant t = 0 along the plane x = 0. Here the quantities to the left and to the right of the discontinuity are arbitrary and are governed only by the equations of state of the gases, which can differ for the adjoining gases.

The determination of flow arising when t > 0 under these initial conditions is called the problem of the decay of an arbitrary discontinuity.

Thus, the problem of discontinuity decay is one of determining one-dimensional flow with plane symmetry (ν = 0) satisfying the integral laws of conservation:

$$\left.\begin{array}{c} \oint\limits_C \rho\, dx - \rho u\, dt = 0, \quad \oint\limits_C \rho u\, dt - [p + \rho u^2]\, dx = 0, \\ \oint\limits_C \rho\left(\varepsilon + \frac{u^2}{2}\right) dt - \left[\rho u \left(\varepsilon + \frac{p}{\rho} + \frac{u^2}{2}\right)\right] dx = 0 \end{array}\right\} \qquad (1)$$

and the piecewise-constant initial conditions:

where t = 0

$$\left.\begin{array}{ll} u = u_1,\ V = V_1,\ p = p_1,\ T = T_1,\ \varepsilon = \varepsilon_1 & \text{where } x < 0; \\ u = u_2,\ V = V_2,\ p = p_2,\ T = T_2,\ \varepsilon = \varepsilon_2 & \text{where } x > 0. \end{array}\right\} \quad (2)$$

In section IV we saw that conditions of dynamic compatibility (Hugoniot's conditions) must be complied with for stable discontinuity. For the case of a shock wave at the discontinuity the stability condition must also be complied with (entropy gain), and at a contact discontinuity (interface of two gases), pressure and velocity are continuous.

Therefore, if an arbitrary discontinuity is not of the contact or shock wave type, then it decays, forming some kind of configuration of stable discontinuities and continuous gas dynamic flows.

We can readily see that if we perform the similarity transformation of the independent variables: $t' = kt$, $x' = kx$ $(k>0)$, then in the new variables x', t' as well, seeking the solution of the problem of decay reduces to finding the solution to the laws of conservation (1) satisfying initial conditions (2) if x and t are now understood as x' and t'.

If we presuppose the uniqueness of the solution of problems (1) and (2), from this it follows that $z(x, t) = \bar{z}(x', t') = \bar{z}(kx, kt)$. (3)

Here we will denote as the vector $\bar{z}$ the totality of hydrodynamic variables $z = \{\rho, u, p, \varepsilon, \dots\}$, and by the letter $\bar{z}$ -- on the same quantities in the variables x', t'.

Putting in the identity (3) $k = 1/t > 0$, we get

$$z(x, t) = \bar{z}\left(\frac{x}{t}, 1\right) = z_0\left(\frac{x}{t}\right). \quad (4)$$

Thus, from the presupposition of the uniqueness of this solution stems the self-modeling status of the solution of the problem of arbitrary discontinuity decaly, i.e., the dependence of all hydrodynamic variables only on the single variable y = x/t. In particular, from this it follows that the discontinuity lines -- shock waves and contact discontinuities -- are straight lines in the plane of variables x, t, i.e., the velocities of the shock waves and the contact discontinuity are constant.

In subsection 2 of section III, we saw that the self-modeling solution continues when $t > 0$ is a centered Riemann rarefaction wave characterized by the constancy of entropy S and one of the Riemann invariants (r or s).

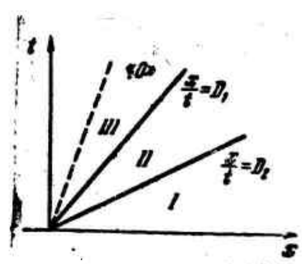

Figure 2.43

Thus, the self-modeling solution of the problem of discontinuity decay contains as its elements shock waves, rarefaction waves, and contact discontinuity.

Let us establish on general properties of the self-modeling solution of the decay problems that are valid for normal gases.

1) In each of the gases "1" (left) and "0" (right) not more than one shock wave is propagated.

Actually, we signify by the term shock wave only a stable shock wave. As can be seen from subsection 4 of section IV, from this follows Champlin's theorem.

Let us presuppose, for example, that in the gas "0" two shock waves are propagated: $x/t = D_1$, $x/t = D_2$; $D_2 > D_1$ (Figure 2.43), and that the flows in the zones I, II and III are constant. Denoting the speed of sound c in the zones I, II, and III, respectively, by c_I, c_{II}, and c_{III}, and velocity u by u_I, u_{II}, and u_{III}, we will have

$$c_I < c_{II} < c_{III}, \quad u_I < u_{II} < u_{III}. \tag{5}$$

However, Champlin's theorem requires that

$$c_I < D_2 - u_I, \quad D_2 - u_{II} < c_{II}, \tag{6}$$

$$c_{II} < D_1 - u_{II}. \tag{7}$$

Inequality (7) obviously is incompatible with (6) given the condition $D_1 < D_2$, which in fact proves our assertion.

2) In each of the gases not more than one centered rarefaction wave is propagated; in the gas "0" (at the right) s = constant in the rarefaction wave; r = constant in gas "1" (at the left).

The assertion readily follows from the equalities

$$u + c = y = \frac{x}{t} \quad (s = \text{const}), \qquad u - c = y = \frac{x}{t} \quad (r = \text{const}), \tag{8}$$

that are valid in centered rarefaction waves.

3) The presence in one of the gases of a shock wave precludes the possibility of the propagation in the same gas of a rarefaction wave, and, conversely, the propagation in one of the gases of a centered rarefaction wave precludes the possibility of a shock wave propagating in it.

As a consequence of these properties, we obtain the result that the self-modeling solution of the problem of discontinuity decay contains a contact discontinuity partitioning the gases ("0" and "1"); in each of the gases not more than one wave (traveling or shock) adjoining the constant-flow zones **is propagated.**

Thus, construction of a self-modeling solution of the discontinuity decay problem consists of "splicing" elementary solutions (constant flows, centered waves) and determining the parameters characterizing the solutions and discontinuities. Since discontinuities and elementary solutions are determined by a finite number of parameters, this problem becomes purely algebraic. We will show below that for normal gases a self-modeling solution of the problem of the decay of an arbitrary discontinuty exists and is unique, that is, is uniquely determined by initial data (2).

We will begin our examination with the case of polytropic gas. The index of Poisson's adiabat γ for the gas "0" lying to the right of the point x = 0 will be denoted by γ_0 and the gas "1" -- by γ_1.

We present the following method of considering the decay problem: starting from the specific case of conditions (2) in which the position of the shock waves and the rarefaction waves (configuration) is obvious, by continuously varying the parameters in problem (2), we will continuously vary the solution, thus passing through critical values of the parameters that distinguish one configuration from another.

2. Configuration A. Since gas velocities u_1 and u_0 are determined with an accuracy up to the additive constant, then we will put $u_0 = 0$. It is sufficient to consider only the case when

$$p_1 \geqslant p_0. \qquad (1)$$

We will begin our examination of the problem of discontinuity decay, formed by two quiescent masses of polytropic gases. The problem is posed thusly:

At the point x = 0 we have a partition separating two masses of gas characterized by parameters γ_1, S_1, ρ_1, p_1, T_1, $u_1 = 0$ to the left and, respectively, by γ_0, S_0, ρ_0, p_0, T_0, $u_0 = 0$ to the right, where the condition

$$p_1 > p_0. \qquad (2)$$

is satisfied.

At the instant t = 0 the partition is pierced, and the gases are set into motion. Presupposing the self-modeling status of the motion (or, which amounts to same thing, the uniqueness of the solution of the decay problem), we calculate it.

Since through the contact boundary (the boundary between gases "0" and "1") no substance passes, therefore for each gas mass the contact boundary can be considered as a piston. By virtue of condition (2) the piston will travel toward the side of gas "0" and advance relative to gas "1". If the piston velocity U is assigned as constant (owing to the self-modeling status), the problem is uniquely solved for each of the gases taken separately. To obtain a solution to the discontinuity decay problem, we must "splice" the solutions of the these two piston problems, requiring that at the contact boundary the pressure p_- to the left is equal to the pressure p_+ to the right. From this condition we determine the velocity U of the contact boundary and all parameters defining motion.

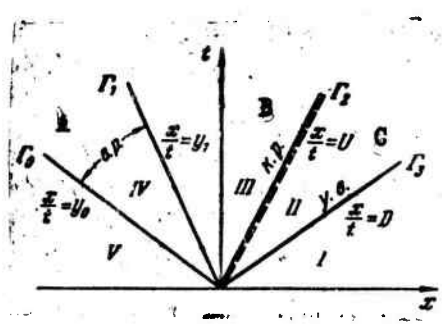

Figure 2 44

KEY:

A) Rarefaction wave

B) Contact discontinuity

C) Shock wave

Solutions to the piston problems are known to us (cf section III, subsection 4 and section IV, subsection 8); therefore the disposition of the discontinuity will be of the form shown in Figure 2.44 (configuration A).

Four rays: Γ_0, Γ_1, Γ_2, and Γ_3 -- partition the upper half-plane into five regions. We have constant flows in regions I, II, III, V, and a centered rarefaction wave in region IV. Ray Γ_3 is a shock wave, Γ_2 is a contact discontinuity, and Γ_1, Γ_0 are the lines of weak discontinuity at which the solution is continuous. Hugoniot's conditions must be complied with at the line Γ_3 for the shock wave and the stability condition, at Γ_2 the continuity of pressure and velocity must be observed, and at Γ_1, Γ_0 -- the continuity of all hydrodynamic variables.

In region IV, p and u are associated by a relation stemming from the constancy of Riemann invariant r (cf formulas (3.2.10) and (3.2.11):

$$p = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{u - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{u}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} (u_1 = 0). \qquad (3)$$

Since zone III is a constant-flow zone, velocity u at characteristic Γ_1 is equal to the velocity U of contact boundary Γ_2. Therefore if we denote the pressure p in the zone III as p_-, then by (3) we have

$$p_- = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{U - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{U}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}}. \qquad (4)$$

At shock wave Γ_3 we have (cf formulas (4.5.13) - (4.5.16)):

$$p = p_0\left[(1 + h_0) M_0^2 - h_0\right], \quad M_0 = \frac{|D - u_0|}{c_0} = \frac{D}{c_0}, \quad h_0 = \frac{\gamma_0 - 1}{\gamma_0 + 1}. \qquad (5)$$

$$u = u_0 + c_0(1 - h_0)\left(M_0 - \frac{1}{M_0}\right) = c_0\left(M_0 - \frac{1}{M_0}\right)(1 - h_0). \qquad (6)$$

Since the zone II is a zone of flow constancy and since at the contact boundary Γ_2 pressure and velocity are continuous, we arrive at the equation

$$p_-(M_0)=p_1\left[1-\frac{\gamma_1-1}{2}\,\frac{(1-h_0)c_0\left(M_0-\frac{1}{M_0}\right)}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1-1}} = p_+(M_0)=p_0\left[(1+h_0)M_0^2-h_0\right] \quad (7)$$

for the determination of M_0.

The function $p_-(M_0)$ appearing in the left side of equation (7) is a monotonically decreasing function of M_0, $p_+(M_0)$ is a function of M_0 monotonically increasing to $+\infty$. Since when $M_0 = 1$

$$p_-(1)=p_1>p_+(1)=p_0, \quad (8)$$

by virtue of assumption (2), then it follows that equation (7) has one and only one root $M_0 > 1$.

Determine the quantities y_0, y_1, U, D as functions of M_0 and show that the configuration A conditions: $y_0 < y_1 < U < D$, (9)

are satisfied.

Inequalities (9) are always satisfied when $U>0$, $U - u_1 = U>0$. Actually, inequality $U = c_0(1 - h_0)(M_0 - 1/M_0) < D = c_0M_0$ is obvious. For y_0, y_1 we have

$$y_0=u_1-c_1=-c_1, \quad y_1=U-c_-, \quad (10)$$

where c_- is the speed of sound in zone III, $c_- > 0$. Hence it follows that y_1 U. Finally, $r_1 = (r_1)_-$; therefore

$$y_1-y_0=U-c_-+c_1=U+\frac{\gamma_1-1}{2}(U-u_1)= =U+\frac{\gamma_1-1}{2}U=\frac{\gamma_1+1}{2}U>0. \quad (11)$$

And thus, all conditions (9) are satisfied and configuration A is compatible.

Now we will vary the problem parameters. Fix p_1, p_0 ($p_1 > p_0$) and vary velocity u_1 of gas "1". Then, putting $u_1 \neq 0$ in (3) and (4), we arrive at the equation

$$p_-(M_0) = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{U - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = p_1\left[1 - \frac{\gamma_1 - 1}{2}\frac{c_0(1-h_0)\left(M_0 - \frac{1}{M_0}\right) - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = p_+(M_0) = p_0\left[(1+h_0)M_0^2 - h_0\right] \quad (12)$$

instead of equation (7). As before, $p_-(M_0)$, $p_+(M_0)$ are monotonic functions of M_0.

Let us trace the variation of root M_0 as a function of parameter u_1. The following assertion is valid: the root M_0 of equation (12) is a monotonically **rising** function of u_1. Actually, $P_-(M_0)$ is a monotonic function of u_1. In particular, the value of $p_-(1) = p_1\left[1 + \frac{\gamma_1 - 1}{2}\frac{u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}}$,

to which corresponds point B in Figure 2.45, also monotonically increases with u_1. Point B increases with u_1, and together with it the entire curve $p = p_-(M_0, u_1)$ is monotonically elevated (Figure 2.45). Since the curve $p = p_+(M_0)$ is fixed, the point C of intersection of the curves $p = p_-(M_0, u_1)$ and $p = p_+(M_0)$ are shifted toward the right with increase in u_1, i.e., the curve M_0 of equation (12) increases with u_1, which was what we set out to prove.

Let $u_B < 0$ stand for the value of u_1 at which $p_-(1, u_1) = p_+$, i.e.

$$1 + \frac{\gamma_1 - 1}{2}\frac{u_B}{c_1} = \left(\frac{p_0}{p_1}\right)^{\frac{\gamma_1 - 1}{2\gamma_1}},$$

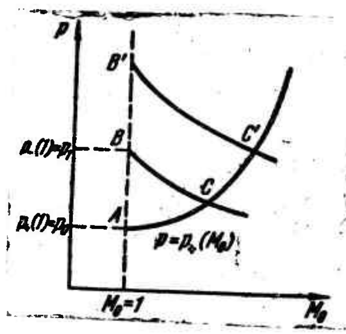

Figure 2.45

or

$$u_B = \frac{2}{\gamma_1 - 1} c_1 \left[\left(\frac{p_0}{p_1} \right)^{\frac{\gamma_1 - 1}{2\gamma_1}} - 1 \right] < 0. \tag{13}$$

Then, in accordance with Figure 2.45, it is obvious that where

$$u_B < u_1 \leqslant 1 \tag{14}$$

equation(12) always has the single root $M_0 > 1$. When $M_c > 1$ shock wave Γ_3 (Figure 2.44) satisfies the stability condition; when $M_c < 1$, shock wave Γ_3 is unstable and therefore configuration A is impossible. However, the entire chain of inequalities (9) must be satisfied to make configuration A possible (Figure 2.44).

Therefore let us consider the satisfaction of these inequalities when $u_1 > 0$. The quantities U, D increase with u_1 up to infinity so that the inequality $U < D$ is preserved. The inequality $y_1 < U$ is also preserved. It remains only to investigate the difference

$$y_1 - y_0 = (U - c_-) - (u_1 - c_1) = (U - u_1) - (c_- - c_1). \tag{15}$$

Due to the constancy of invariant r in zone IV, we have

$$r = u_1 + \frac{2}{\gamma_1 - 1} c_1 = U + \frac{2}{\gamma_1 - 1} c_-, \quad U - u_1 = \frac{2}{\gamma_1 - 1}(c_1 - c_-). \tag{16}$$

Substituting (16) into (15), we get

$$y_1 - y_0 = \frac{\gamma_1 + 1}{2}(U - u_1). \tag{17}$$

We have shown that M_o increases with u_1. But from equation (12) it follows that the difference $U - u_1$ decreases with increase in M_o. Consequently, from (17) it follows that $y_1 - y_o$ decreases with increase in u_1. As long as $U - u_1 > 0$, $y_1 - y_o > 0$. When $U - u_1 = 0$, $y_1 - y_o = 0$. The latter **is satisfied given** the condition that the equalities

$$p_1 = p_0 [(1 + h_0) M^2_{0\,kp} - h_0], \tag{18}$$

$$u_1 = U = (1 - h_0) c_0 \left(M_{0\,kp} - \frac{1}{M_{0\,kp}} \right) = u_Б, \tag{19}$$

in which $M_{o\ kp}$ [kp = critical] appears as a parameter are simultaneously satisfied.

It is not difficult to express $u_Б$ explicitly:

$$u_Б = (1 - h_0) c_0 \left[\sqrt{\frac{\frac{p_1}{p_0} + h_0}{1 + h_0}} - \sqrt{\frac{1 + h_0}{\frac{p_1}{p_0} + h_0}} \right] > 0. \tag{20}$$

Thus, if the conditions $u_B < u_1 < u_Б$, (21)

are met, then conditions (9) for the compatibility of configuration A are met.

So assuming that inequalities (21) are satisfied, the pattern of discontinuity is of the form shown in Figure 2.44, and the formulas obtained above enable us to wholly calculate the flow under configuration A conditions.

3. Configuration B. When $u_1 = u_B$, $y_1 - y_o = 0$, i.e., the zone of the rarefaction wave vanishes, and the solution is constructed from the single shock wave and the contact discontinuity (Figure 2.46). With a further increase in u_1, the difference $u_1 - U$ becomes negative. Therefore when $u_1 > u_B$, the contact boundary $dx/dt = U$ must be considered as a piston simultaneously advancing both in gas "0" and in gas "1". Thus, in accordance with the solution of the piston problem (cf section IV, subsection 8), the solution to the discontinuity decay problem when $u_1 > u_B$ must be sought for in the form of two shock waves, one propagating in gas "0", and the other in gas "1" (cf Figure 2.47). We will call this case configuration B.

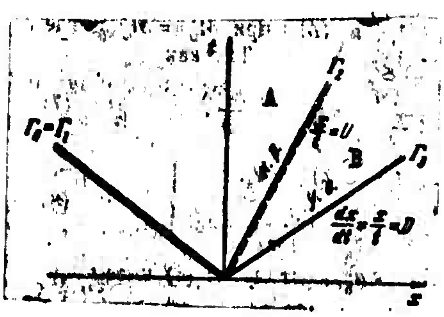

Figure 2.46

KEY:

A) Contact discontinuity B) Shock wave

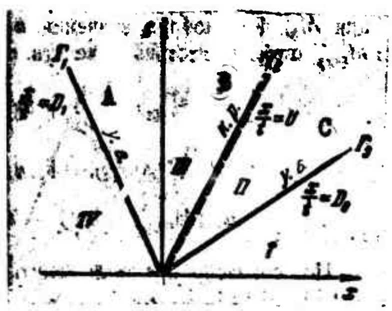

Figure 2.47

KEY:

A) Shock wave

B) Contact discontinuity

C) Shock wave

In the case of configuration B, we have four constant-flow zones I-IV, partitioned by shock waves Γ_1 and Γ_3, and by contact discontinuity Γ_2. Let us prove the compatibility of configuration A provided we have the condition $u_1 > u_B$. Let us write the conditions at the shock waves Γ_1 and Γ_3:

$$p_- = p_1 [(1 + h_1) M_1^2 - h_1]; \quad (1)$$

$$u_- = u_1 - c_1 (1 - h_1)\left(M_1 - \frac{1}{M_1}\right), \quad M_1 = \frac{|D_1 - u_1|}{c_1}; \quad (2)$$

$$p_+ = p_0 [(1 + h_0) M_0^2 - h_0]; \quad M_0 = \frac{D_0}{c_0}; \quad (3)$$

$$u_+ = c_0 (1 - h_0)\left(M_0 - \frac{1}{M_0}\right). \quad (4)$$

Equating each other the pressures p_- and p_+ at the boundary Γ_2, we get:

$$M_1^2 - a M_0^2 \ldots \text{[illegible]} \tag{5}$$

Hence it follows that M_1 is a monotonically increasing function of M_0. To determine M_c, let us write the condition for continuity of velocity at the contact boundary Γ_2:

$$u_-(M_1) = u_1 - c_1(1-h_1)\left(M_1 - \frac{1}{M_1}\right) = u_+(M_0) = c_0(1-h_0)\left(M_0 - \frac{1}{M_0}\right). \tag{6}$$

The left side of this equality is a monotonically decreasing function of M_1 and by virtue of (5), a monotonically decreasing function of M_0; the right side is a function of M_0 monotonically increasing up to ∞. Let us consider the values $u_-(M_c)$ and $u_+(M_0)$ when $M_c = M_{0\,kp}$, where $M_{0\,kp}$ is determined from equalities (6.2.18) and (6.2.19). Since

$$M_{0\,kp}^2 = \left(\frac{p_1}{p_0} + h_0\right)\frac{1}{1+h_0} = \frac{p_1 + p_0 h_0}{p_0(1+h_0)}, \tag{7}$$

then

$$\left.\begin{aligned} &M_1(M_{0\,kp}) = 1, \quad u_-(M_1(M_{0\,kp})) = u_1, \\ &u_+(M_{0\,kp}) = c_0(1-h_0)\left(M_{0\,kp} - \frac{1}{M_{0\,kp}}\right) = u_B. \end{aligned}\right\} \tag{8}$$

But, according to our assumption, $u_1 > u_B$, therefore from (8) we have

$$u_-(M_1(M_{0\,kp})) > u_+(M_{0\,kp}). \tag{9}$$

So equation (6) providing $u_1 > u_B$ will always have, and only one at that, the root $M_o > M_{o\ kp} > 1$.

After determining M_o, by formula (5) we determine M_1 in all flow parameters in the zones II and III. The conditions for **compatibility** of configuration B

$$D_1 < u_1 - c_1, \quad D_1 < U < D_0, \quad c_0 < D_0 \qquad (10)$$

are easily verified and are always satisfied providing $u_1 > u_B$, if we consider that if $M_1 > 1$, and also $M_o > M_{o\ kp} > 1$. And so, for any u_1 satisfying the condition $u_1 > u_B$, we have configuration B.

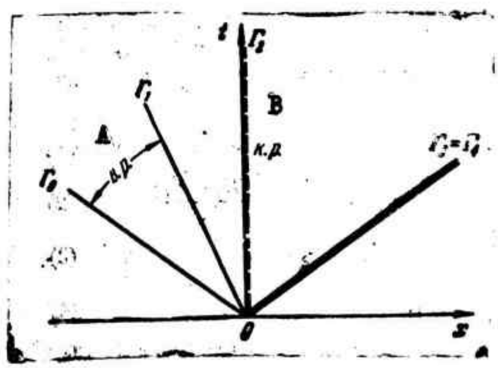

Figure 2.48

KEY:

A) Rarefaction wave B) Contact discontinuity

4. Configuration C. When $u_1 = u_C < 0$ (cf formula (6.2.13)), $M_o = 1$, $U = 0$, i.e., the contact boundary is a fixed piston for gas "0". When $u_1 = u_C$, we therefore have a solution when in gas "0" the shock wave vanishes and it remains fixed, retaining its initial parameters, and the rarefaction wave $\Gamma_0 0 \Gamma_1$ (Figure 2.48) propagates in gas "1".

With a further reduction in u_1 ($u_1 < u_C < 0$), the contact boundary begins to advance to the left relative to gas "C" ($U < 0$), so that it can be considered as a piston simultaneously traveling from gases "C" and "1". In accordance with the solution of the piston problem (section III, subsection 4), in this case the solution to the decay problem consists of two centered rarefaction waves propagating in gases "1" and "C" (Figure 2.49). We will call this condition configuration C. The upper half-plane is partitioned into the six regions I-VI, divided by four lines of weak discontinuities Γ_0, Γ_1, Γ_3, Γ_4, and by contact boundary Γ_2. Zones I, III, IV, and VI are constant-flow zones, zones II and V are rarefaction wave regions; invariant is constant in zone II, and variant r is constant in zone V.

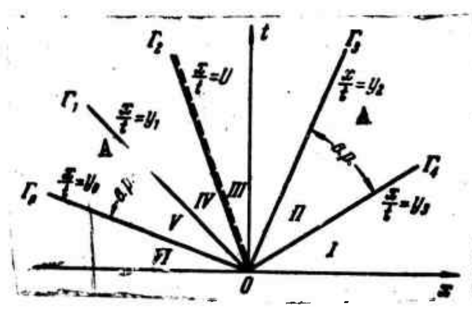

Figure 2.49

KEY:

A) Rarefaction wave

Let us show that given the condition $u_1 < u_C$ (1)

configuration C is compatible.

In zones II and V we have

$$p_+ = p_0\left[1 + \frac{\gamma_0 - 1}{2}\,\frac{u - u_0}{c_0}\right]^{\frac{2\gamma_0}{\gamma_0 - 1}} = p_0\left[1 + \frac{\gamma_0 - 1}{2}\,\frac{u}{c_0}\right]^{\frac{2\gamma_0}{\gamma_0 - 1}} \quad (s = \text{const}), \tag{2}$$

$$p_- = p_1\left[1 - \frac{\gamma_1 - 1}{2}\,\frac{u - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} \quad (r = \text{const}). \tag{3}$$

In particular, at the contact boundary Γ_2 when $u = u_- = u_+ = U$ we must obtain $p_- = p_+$, i.e., we must arrive at the equation

$$p_- = p_-(U) = p_1\left[1 - \frac{\gamma_1 - 1}{2}\,\frac{U - u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}} = p_+ = p_+(U) = p_0\left[1 + \frac{\gamma_0 - 1}{2}\,\frac{U}{c_0}\right]^{\frac{2\gamma_0}{\gamma_0 - 1}} \tag{4}$$

In order to determine U. Here

$$1 - \frac{\gamma_1 - 1}{2}\,\frac{U - u_1}{c_1} \geqslant 0, \quad 1 + \frac{\gamma_0 - 1}{2}\,\frac{U}{c_0} \geqslant 0. \tag{5}$$

Condition (5) notes the nonnegativeness of pressure at the boundary. Note that the equality signs in formula (5) can obtain only simultaneously and correspond to the separation of the gases. As we can see, in equation (4) $p_-(U)$ is a monotonically decreasing function of U, and $p_+(U)$ is monotonically increasing function of U. Where $U = 0$,

$$p_-(0) = p_1\left[1 + \frac{\gamma_1 - 1}{2}\,\frac{u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1 - 1}}, \quad p_+(0) = p_0. \tag{6}$$

From condition (1) therefore follows $p_-(0) < p_+(0)$. (7)

With a reduction in U, $p_-(U)$ will increase, and $p_+(U)$ will decrease. Thus, if $p_+(U)$ does not tend to zero, there exists the single root $U < 0$ of equation (4) satisfying the condition $U > -\frac{2}{\gamma_0 - 1} c_0$. (8)

Then equation (5) leads us to the inequality

$$u_1 > -\left(\frac{2}{\gamma_1 - 1} c_1 + \frac{2}{\gamma_0 - 1} c_0\right). \quad (9)$$

Thus, provided condition (9) is satisfied, there exists the root $U < 0$ of equation (4). We will show that when $u_1 < u_C$ configuration C is also compatible, i.e., the conditions $y_0 < y_1 \leq U \leq y_2 < y_3$. (10)

are satisfied. It is not difficult to see that

$$y_3 - y_2 = (u_0 + c_0) - (U + c_+) = -U - (c_+ - c_0) = -\frac{\gamma_0 + 1}{2} U > 0. \quad (11)$$

Inequalities

$$y_2 - U = (U + c_+) - U = c_+ > 0; \qquad U - y_1 = U - (U - c_-) = c_- > 0 \quad (12)$$

are self-evident. Finally,

$$y_1 - y_0 = U - u_1 + (c_1 - c_-) = \frac{\gamma_1 + 1}{2}(U - u_1). \quad (13)$$

But from equation (4) and conditions $p_1 > p_0$, $U < 0$ it follows that $U - u_1 > 0$. Hence $y_1 - y_0 > 0$. Thus, configuration C is compatible when $u_1 < u_C$ and provided that condition (9) is satisfied.

But if condition (9) is violated, equation (4) does not have the root U. In this case the gases **separate** from each other and equation (4) is replaced by two free boundary equations:

$$p_- = p_-(U_1) = p_1\left[1 - \frac{\gamma_1-1}{2}\,\frac{U_1-u_1}{c_1}\right]^{\frac{2\gamma_1}{\gamma_1-1}} = 0, \quad \text{i.e.,} \quad U_1 = u_1 + \frac{2}{\gamma_1-1}c_1, \qquad (14)$$

$$p_+ = p_+(U_0) = p_0\left[1 + \frac{\gamma_0-1}{2}\,\frac{U_0}{c_0}\right]^{\frac{2\gamma_0}{\gamma_0-1}} = 0, \quad \text{i.e.,} \quad U_0 = -\frac{2}{\gamma_0-1}c_0. \qquad (15)$$

Since it follows from the noncompliance of condition (9) that $U_1 < U$, the gases separate from each other and the solution is of the form shown in Figure 2.50. The regions $\Gamma_0 0 \Gamma_1$ and $\Gamma_2 0 \Gamma_3$ are regions of rarefaction waves, respectively, r = constant and s = constant, and the region $\Gamma_1 0 \Gamma_2$ is the vacuum region in which we put $\rho = 0$, $p = 0$, $c = 0$.

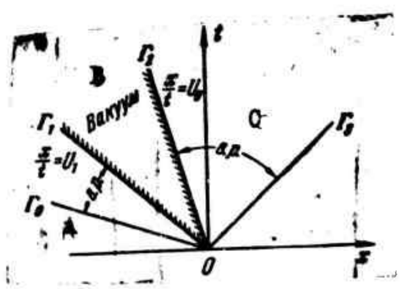

Figure 2.50

KEY:

A) Rarefaction wave

B) Vacuum

C) Rarefaction wave

5. Review of configurations. Gases with equal pressure. Let us now write the conditions for the existence of configurations A, B, and C on the presupposition

$p_1 > p_o$, $u_o = 0$. Configuration A is possible providing that the conditions

$$u_C < u_1 < u_B \tag{1}$$

are satisfied. Configuration B is possible providing that

$$u_1 > u_B, \tag{2}$$

and configuration C is possible providing that

$$u_1 < u_C, \tag{3}$$

where

$$u_C = \frac{2}{\gamma_1 - 1} c_1 \left[\left(\frac{p_0}{p_1} \right)^{\frac{\gamma_1 - 1}{2\gamma_1}} - 1 \right] < 0, \tag{4}$$

$$u_B = (1 - h_0) c_0 \sqrt{ \frac{\frac{p_1}{p_0} + h_0}{1 + h_0} + \frac{1 + h_0}{\frac{p_1}{p_0} + h_0} - 2 } > 0. \tag{5}$$

For the case $p_1 = p_o$, we have $u_C = u_B = 0$. (6)

Consequently, in this case when $u_1 > 0$ we have configuration B, but when u_1 0 -- configuration C.

Now let us note that conditions (1) - (6) bring into correspondence to any arbitrary discontinuity (ensemble of quantities V_1, p_1, ρ_1, c_1, u_1; V_o, p_o, ρ_o, c_o, u_o) one and only one configuration whose flow parameters are calculated uniquely, and where stability conditions are satisfied at the shock waves.

Therefore our consideration of the problem of the decay of an arbitrary discontinuity shows that it has one and only one stable self-modeling solution. Thus, we approved the theorem of the existence of uniqueness of the solution of the problem of discontinuity decay in the class of self-modeling solutions for polytropic gases.

However, the question rises: can the discontinuity decay problem have a stable, but not self-modeling solution?

A negative answer to this question can be obtained by two methods:

1) by the proof of the theorem of the uniqueness of the discontinuous solutions of gas dynamics equations, i.e., solutions with shock waves and with centered rarefaction waves;

2) by the direct proof of the self-modeling status of any stable solution to the discontinuity decay problem.

As for the first method, we must state that at the present time sufficiently general theorems of the uniqueness of discontinuous solutions to gas dynamics equations have not yet been obtained, and their derivation evidently involves great difficulties, though for polytropic (and normal) gases evidently no one doubts the uniqueness of this solution.

Pursuing the second method, we can actually demonstrate the self-modeling of the solution to the discontinuity decay problem by using certain concrete properties of any stable solution of this problem with piecewise-constant initial data.

However, we will not here deal with this proof, and as an example we will further refer to subsection 5 of section III in Chapter Four where a similar problem was solved for a system of two quasilinear equations of fairly general form.

6. Problem of discontinuity decay for an isothermal ideal gas. We will here understand by gases "1" and "0" two isothermal ideal gases whose equations of state are given in the form

$$p_1 = c_1^2\rho_1, \quad p_0 = c_0^2\rho_0, \quad c_1^2 = R_1T, \quad c_0^2 = R_0T. \tag{1}$$

An isothermal ideal gas can be considered formally as a polytropic gas with index $\gamma = 1$. The difference in our treatment lies in the fact that we omit the equation of conservation of energy and the third Hugoniot's condition by replacing it with the condition T = constant.

Since an analysis of critical configurations involved only the two first Hugoniot's conditions, all results of the preceding subsections can be directly applied to the isothermal case. In formulas of subsection 2-4 we must put $h_0 = h_1 = 0$, $\gamma_1 = \gamma_0 = 1$, reducing, wherever required, the indeterminacy. Let us consider expressions for u_C and u_B. Reducing the indeterminacy in formula (6.5.4) as $\gamma_1 \to 1$, we have

$$u_C = c_1 \ln \frac{p_0}{p_1} < 0 \quad (p_1 > p_0). \tag{2}$$

for u_B we have

$$u_B = c_0 \sqrt{\frac{p_1}{p_0} + \frac{p_0}{p_1} - 2} = c_0 \left[\sqrt{\frac{p_1}{p_0}} - \sqrt{\frac{p_0}{p_1}} \right]. \tag{3}$$

Conditions of configurations A, B, and C will be as before. Note also that where $\gamma_1 = \gamma_0 = 1$ conditions (6.4.9) are always satisfied, so that separation of gases and the formation of vacuum are impossible for isothermal gases.

Working formulas, after passing to the limit, are of the form:

Configuration A:

$$p_-(M_0) = p_1 \exp\left\{ \frac{u_1 - c_0\left(M_0 - \frac{1}{M_0}\right)}{c_1} \right\} = p_+(M_0) = p_0 M_0^2. \tag{4}$$

Formula (4) corresponds to (6.2.12).

Configuration B:

$$u_-(M_1) = u_1 - c_1\left(M_1 - \frac{1}{M_1}\right) = u_+(M_0) = c_0\left(M_0 - \frac{1}{M_0}\right) \tag{5}$$

and

$$M_1 = \sqrt{\frac{p_0}{p_1}} M_0. \tag{6}$$

so that finally:

$$u_1 - c_1\left[\sqrt{\frac{p_0}{p_1}}M_0 - \sqrt{\frac{p_1}{p_0}}\frac{1}{M_0}\right] = c_0\left[M_0 - \frac{1}{M_0}\right]. \qquad (7)$$

Formulas (5) and (6) correspond to (6.3.6) and (6.3.5).

Finally, for the case of configuration C

$$p_1 \exp\left\{-\frac{U-u_1}{c_1}\right\} = p_0 \exp\frac{U}{c_0}. \qquad (8)$$

Formula (8) corresponds to equation (6.4.4).

Note that all the written formulas can be readily obtained if we use Riemann invariants. We offer the reader the opportunity of carrying out these operations if he desires.

7. Problem of discontinuity decay for normal gases. In addition to the ordinary requirements I-V (cf subsection 3 of section IV), we will require additionally that the following property of Hugoniot's adiabat H be satisfied: at its upper branch the quantity $(p - p_0)(V_0 - V)$ must monotonically increase to ∞ simultaneously with increase in entropy S.

Then from relations (cf subsection 2 of section IV)

$$m^2 = \frac{p - p_0}{V_0 - V} > 0, \qquad (1)$$

$$(u - u_0)^2 = (p - p_0)(V_0 - V), \qquad (2)$$

where u_0, p_0, V_0 denote the parameters of flow ahead of the shock wave front, and u, p, V -- the parameters of flow behind the front, it follows that p and |u| are monotonically increasing functions*) of parameter |m|, or, which amounts to

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the same, $M_0 = \frac{|m|}{\rho_0 c_0}$. Recall also (cf subsection 2 of section III, as well as subsection 7 of section II) that in the traveling waves u and p are associated by the relation

$$S = \text{const}, \quad u - \Phi(S, p) = \text{const} \quad (\xi = \text{const}), \tag{3}$$
$$S = \text{const}, \quad u + \Phi(S, p) = \text{const} \quad (\eta = \text{const}), \tag{4}$$

where $\Phi(S, p)$ is defined by the formula

$$\Phi(S, p) = \int_{\bar{p}} \frac{dp}{a_m(S, p)}, \tag{5}$$

where $\bar{p}$ is the fixed limit of integration, and $a_m(S, p) = \rho c > 0$ (6)

can be assumed to be a function of pressure p and entropy S. From (5) and (6) it follows that $\Phi(S, p)$ is a monotonically increasing function of pressure p.

Function $\Phi(S, p)$ depends on the equation of state, therefore the functions $\Phi_0(S, p)$ and $\Phi_1(S, p)$ for gases "0" and "1" are, correspondingly, distinct, generally speaking.

Let us begin our examination of the problem of decay with configuration A by putting $u_0 = u_1 = 0, \quad p_1 > p_0,$ (7)

assuming that the equations of state of gases "1" and "0" are distinct and that for each of them conditions I-V and the condition of monotonic increase of u(M) are satisfied (cf (1) and (2)).

From assumption (7) follows configuration A (Figure 2.44). Invariant r is constant in region IV, therefore $u_1 + \Phi_1(S_1, p_1) = u_- + \Phi_1(S_1, p_-)$. (8)

In region II, $p_+ = p_0(M_0), \quad u_+ = u_0(M_0),$ (9)

*) The monotonic increase in $p(M_0)$, as shown in subsection 3 of section IV, is a consequence of conditions I-V.

where $p_o(M_o)$ and $u_o(M_o)$ denote the state ahead of the shock wave front in gas "C" for assigned M_o and increase to ∞ with increasing M_o.

Conditions for the continuity of velocity and pressure at the contact boundary Γ_2 lead to the equation for the determination of M_o:

$$u_-(M_0) = u_1 + \Phi_1(S_1, p_1) - \Phi_1(S_1, p_0(M_0)) = u_+(M_0) = u_0(M_0). \qquad (10)$$

The left side of (10) decreases monotonically, and the right increases monotonically up to ∞ with increasing M_o. When $M_o = 1$, according to (5) and (7) we have

$$\left. \begin{array}{c} u_-(1) = \Phi_1(S_1, p_1) - \Phi_1(S_1, p_0) > 0 \quad (u_1 = 0), \\ u_+(1) = u_0(1) = 0. \end{array} \right\} \qquad (11)$$

Hence, as before, it follows that equation (10) has one and only one root $M_o > 1$.

Fixing p_1 and p_o, we will vary u_1. The function

$$u_- = u_-(M_0, u_1) = u_1 + \Phi_1(S_1, p_1) - \Phi_1(S_1, p_0(M_0))$$

is a monotonically increasing function of u_1. Consequently, the root M_o of equation (10) is a monotonically increasing function of u_1.

Let us consider how the inequalities $y_o < y_1 < U < D$ (12)

vary with change in u_1. The inequality $D - U > 0$ is self-evident by virtue of relation (4.2.8):

$$(U - D)(u_0 - D) = D(D - U) = \frac{p_0(M_0) - p_0}{\rho_0(M_0) - \rho_0} > 0. \qquad (13)$$

Inequality $U - y_1 = U - (U - c_-) = c_- > 0$ is also self-evident.

Consider the difference

$$y_1 - y_0 = (U - c_-) - (u_1 - c_1) = (U - u_1) - (c_- - c_1). \qquad (14)$$

Since $u_- = U$, from (8) it follows that

$$U - u_1 - (c_- - c_1) = \Phi_1(S_1, p_1) - \Phi_1(S_1, p_-) + (c_1 - c_-). \qquad (15)$$

$p_- = p_0(M_0)$ and $c_- = c_-(S_1, p_-)$ increase with u_1; thus, differences $\Phi_1(S_1, p_1) - \Phi_1(S_1, p_-)$, and $c_1 - c_-$ are reduced. Thus, the difference $y_1 - y_0$ decreases with increase in u_1 and at some value $u_1 = u_B$ becomes equal to 0. The rarefaction wave region vanishes, and the solution will have the configuration AB (Figure 2.46).

When $u_1 = u_B$ $\qquad p_- = p_1,\ c_- = c_1,\ u_- = u_1,$ (16)

Therefore u_B is defined from the equations

$$\left.\begin{aligned} p_1 &= p_0(M_{0\,kp}), \\ u_B &= u_0(M_{0\,kp}) \end{aligned}\right\} \qquad (17)$$

in which $M_{c\ kp}$ appears as a parameter.

But if u_1 is reduced, then as we have seen M_0 will be reduced and when $u_1 = u_C$ will become equal to 1. When $u_1 = u_C$, the shock wave vanishes and the solution takes on the configuration AC (Figure 2.48).

The value u_C is found from the equation

$$u_C = \Phi_1(S_1, p_0) - \Phi_1(S_1, p_1) < 0, \qquad (18)$$

which follows from (10) when $M_0 = 1$.

When $\qquad u_1 > u_B \qquad (19)$

configuration B is always compatible (Figure 2.47).

Let us write the conditions for continuity of pressure and velocity at the contact boundary Γ_2:

$$p_-(M_1) = p_+(M_0), \qquad (20)$$
$$u_-(M_1) = u_+(M_0). \qquad (21)$$

The function $p_-(M_-)$ is a monotonically increasing function of the parameter M_1, therefore from equation (20) M_1 is determined monotonically increasing function of M_0 and equation (21) can be considered as the equation for the determination of $M_0 > M_{0\ kp} > 1$; $M_1 > 1$.

Noting now that the left side of (21) is a monotonically decreasing, and the right side -- a monotonically increasing function of M_0 and that

$$M_1(M_{0\,\text{кр}}) = 1,$$

$$u_+(M_{0\,\text{кр}}) = u_B, \quad u_-(1) = u_1 > u_B,$$

we conclude that equations (20) and (21) have one and only one root $M_0 > M_{0\ kp} > 1$; $M_1 > 1$.

Thus, the conditions for the compatibility of configuration B ($M_0 > 1, M_1 > 1$) are always satisfied when $u_1 > u_B$.

Finally, when

$$u_1 < u_C \tag{22}$$

we have configuration C (Figure 2.49).

The condition for the continuity of velocity and pressure at the contact boundary Γ_2 is of the form

$$\begin{aligned} u_- = u_1 + \Phi_1(S_1, p_1) - \Phi_1(S_1, p) = \\ = u_+ = \Phi_0(S_0, p) - \Phi_0(S_0, p_0) = U \end{aligned} \tag{23}$$

(p denotes pressure in the zones III and IV, $p_- = p_+ = p$). From it p and U are determined.

The left side of equation (23) decreases monotonically, and the right side increases monotonically -- with increase in p; when $p = p_C$

$$u_+ = u_+(p_0) = 0;$$

$$u_- = u_-(p_0) = u_1 + \Phi_1(S_1, p_1) - \Phi_1(S_1, p_0) = u_1 - u_B < 0.$$

Hence it follows that equation (23) has one and only one root $p = p_- = p_+ < p_0 < p_1$. The conditions for the compatibility of configuration C followed from the fact that $0 > U > u_1$. With further reduction in u_1 separation of gases becomes possible.

8. Solution of the problem of discontinuity decay in the plane of variables p, u (p, u-plot). In all cases of configurations A, B, C the state u_+, p_+, S_+ to the right of the contact boundary was associated with the state of $u_0 = 0$, p_0, S_0 in gas "O" at the initial instant either by the rarefaction wave relation

$$S_+ = S_0, \quad u_+ - \Phi_0(S_0, p_+) = u_0 - \Phi_0(S_0, p_0) = -\Phi_0(S_0, p_0). \tag{1}$$

or by the Hugoniot's relations, from which follows

$$p_+ = p_0(S_0, M_0), \quad u_+ = u_0(S_0, M_0). \tag{2}$$

where $p_0(S_0, M_0)$, $u_0(S_0, M_0)$ increase monotonically with increasing M_0, and $p_0(S_0, M_0) \to \infty$, $u_0(S_0, M_0) \to \infty$ provided $M_0 \to \infty$. From this it follows that we can cancel out the parameter M_0 from the functions (2) and obtain a new function: $u_+ = \Psi_0(S_0, p_+)$; (3)

at which $\Psi_0(S_0, p_+)$, just as $\Phi_0(S_0, p_+)$, is a monotonically increasing function of the variable p_+.

A necessary condition for the rarefaction wave $(y_3 - y_2 \geq 0)$ is the requirement

$$p_+ \leq p_0, \tag{4}$$

from whence follows that in determining u_+ we can employ only half of curve (1):

$$u_+ = \Phi_0(S_0, p_+) - \Phi_0(S_0, p_0). \tag{5}$$

assigned by the condition $p_+ \leq p_0$.

The condition for the stability of shock wave $(M_0 \geq 1)$, conversely, requires only that pressure p_+ be larger than p_0 $(p_+ > p_0)$; therefore when determining

u_+ we can also employ only half of curve (3) assigned by condition $p_+ \geqslant p_0$.
Therefore the curve

$$u_+ = \begin{cases} \Phi_0(S_0, p_+) - \Phi_0(S_0, p_0) & \text{where } p_+ \leqslant p_0, \\ \Psi_0(S_0, p_+) & \text{where } p_+ \geqslant p_0 \end{cases} \quad (6)$$

define the velocity u_+ at the contact boundary for the assigned pressure p_+.
The curve

$$u = g_0(S_0, p) = \begin{cases} \Phi_0(S_0, p) - \Phi_0(S_0, p_0) & \text{where } p \quad p_0, \\ \Psi_0(S_0, p) & \text{where } p \quad p_0 \end{cases} \quad (7)$$

describes in the plane of variables p, u a set of states which can be associated with the right state p_0, S_0, $u_0 = 0$ by means of the centered rarefaction wave or by a stable ($M_0 \geqslant 1$) shock wave.

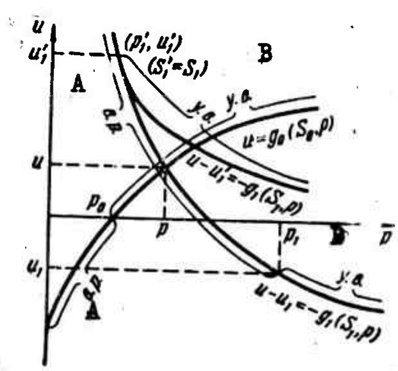

Figure 2.51

KEY:

A) Rarefaction wave B) Shock wave

We state that at the curve (7) a "state" is defined, even though at each of its points only two cf three hydrodynamic parameters are known -- p, u.

However, we can readily see that in the rarefaction wave ($p \leq p_o$) $S = S_o$, but when $p \geq p_o$ the entropy S is uniquely determined at each point on this curve from Hugoniot's conditions.

Stated briefly, curve (7) is the projection onto the plane of variables p, u of a curve located in the space of three variables (p, u, S) and describing a set of states (p, u, S), which can be associated with the right state p_o, $u_o = 0$, S_o by the rarefaction wave or by the shock wave.

We can readily see that the curve (7) passes through the point H and from the property of the second-order tangency at the point (p_o, $u_o = 0$) of Hugoniot's adiabat H and Poisson's adiabat A it follows that curve (7) exhibits at point (p_o, $u_o = 0$) and, therefore, everywhere a continuously differentiable tangent.

Figure 2.51 presents the approximate shape of curve (7); and some of its part corresponds to the shock wave (function (3)), and the lower -- to the rarefaction wave (function (5)).

Wholly analogously, the curve $u = u_1 - g_1(S_1, p)$, (8)

where

$$g_1(S_1, p) = \begin{cases} \Phi_1(S_1, p) - \Phi_1(S_1, p_1), & p < p_1, \\ \Psi_1(S_1, p), & p > p_1 \end{cases} \quad (9)$$

describes in the p, u-plane a family of states that can be associated with the state u_1, p_1, S_1, as with the left state, either by the rarefaction wave (r = constant) or by the shock wave ($m > 0$, $M_1 \geq 1$). We can readily see that $g_1(S, p)$ has a monotonically increasing function of p, so that curve (8) passes through the point (p_1, u_1) and has two continuous derivatives. The approximate shape of curve (8) is shown in Figure 2.51. Also shown in Figure 2.51 are the sections of curve (7) and (8) to which correspond the rarefaction wave (B.p.) and the shock waves (y. B.), respectively, in gases "0" and "1".

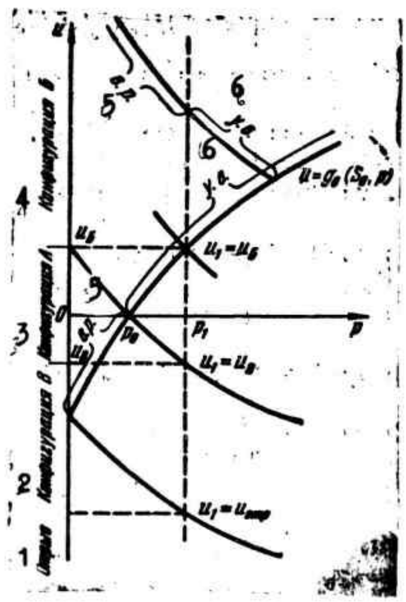

Figure 2.52

KEY:

1) Separation
2) Configuration C
3) Configuration A
4) Configuration B
5) Rarefaction wave
6) Shock wave

Since at the contact discontinuity it is always (save for the case of gas separation) that the continuity of velocity $u_- = u_+$ and pressure $p_- = p_+$, the solution of the discontinuity decay problem reduces to determining the point (p, u) $(p = p_- = p_+, u = u_- = u_+)$ of intersection of curves (7) and (8).

If the intersection point (p, u) of these two curves lies in the upper half $(p > p_0)$ of curve (7), then a shock wave is propagated in gas "0"; but if $p < p_0$, then a rarefaction wave propagates in gas "0". Similarly, if the point (p, u) of intersection of these two curves lies in the upper half $(p < p_1)$ of curve (8), then a rarefaction wave propagates in gas "1"; but if $p > p_1$, then a shock wave propagates in gas "1". Accordingly, Figure 2.52 presents the possible cases of intersection of these curves. Assigning, as always, condition $p_1 > p_0$ and noting that curves (8) for different u_1 differ from each other only by displacement, we present in Figure 2.52 the disposition of the possible configurations as a function of u_1, and also a graphical method of determining the quantities u_B, u_C, and u_{otp} [otp = separation of gases]. From Figures 2.51 and 2.52 also follows the graphical method of solving the problem of the decay of an arbitrary discontinuity*).

A similar consideration of the problem of discontinuity decay can be made also for the projection onto the plane (s, r) of Riemann invariants. In the plane (s, r) parts of the curves (7) and (8) corresponding to the rarefaction waves will be raised parallel to the coordinate axes, while another part will represent some smooth curve that smoothly (with two derivatives) is consistent with these rays (Figure 2.53). It is especially convenient to use the r,s-plot

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*) It must be borne in mind that curve (8), just as curve (7), depends parametrically on entropy $S_1(S_0)$, therefore the assignment of only the point (p_1, u_1) still does not determine it. If, however, we assume that for all u_1, $p_1 \geq p_0$ entropy S_1 is fixed, then curves $u - u_1 = -g_1(S_1, p)$ and $u - u_1' = -g_1(S_1, p)$ coincide when $u_1' - u_1 = g_1(p_1', S_1)$, $p \leq \min(p_1, p_1')$. On this basis, in Figure 2.52 are indicated the regions of values p_1, u_1 when $S = S_1 = \text{constant}$, in which configuration of the solution of the discontinuity decay problem is preserved

in solving the problem of discontinuity decay for isothermal gases.

In concluding this subsection, let us determine the function $g_0(S_0, p)$ for the case of polytropic gases.

For a polytropic gas

$$p = \frac{A^2(S)\rho^\gamma}{\gamma}, \qquad (10)$$

$$\Phi_0(S_0, p) - \Phi_0(S_0, p_0) = \frac{2}{\gamma_0 - 1}\left[\left(\frac{p}{p_0}\right)^{\frac{\gamma_0-1}{2\gamma_0}} - 1\right] c_0(S_0, p_0). \qquad (11)$$

In the case of a shock wave ($p > p_0$), let us express M_0 from (4.5.13):

$$M_0 = \sqrt{\frac{p + h_0 p_0}{p_0(1 + h_0)}}, \qquad (12)$$

after which in formula (4.5.16) we find $\Psi_0(S_0, p)$:

$$\Psi_0(S_0, p) = (1 - h_0)\, c_0(S_0, p_0)\left[\sqrt{\frac{p + h_0 p_0}{p_0(1 + h_0)}} - \sqrt{\frac{p_0(1 + h_0)}{p + h_0 p_0}}\right]. \qquad (13)$$

Thus, for a polytropic gas equation (7) is written in the form

$$u = g_0(S_0, p) = \begin{cases} \frac{2}{\gamma_0 - 1} c_0(S_0, p_0)\left[\left(\frac{p}{p_0}\right)^{\frac{\gamma_0-1}{2\gamma_0}} - 1\right] & \text{where } p \leq p_0, \\ (1 - h_0)\, c_0(S_0, p_0)\left[\sqrt{\frac{p + h_0 p_0}{p_0(1 + h_0)}} - \sqrt{\frac{p_0(1 + h_0)}{p + h_0 p_0}}\right] & \text{where } p \geq p_0. \end{cases} \qquad (14)$$

9. Linearized formulas of discontinuity decay in the case of polytropic gas. It is easy to see that the quantity $\Phi_0(S_0, p) - \Phi_0(S_0, p_0)$ and $\Psi_0(S_0, p)$

when $\frac{p}{p_0} = 1 + \varepsilon$ coincide, with an accuracy up to **terms** of the order of ε^3. The same conclusion follows from an analysis of weak shock waves made in subsection 7 of the section IV. Therefore, by carrying out in formula (6.8.14) an expansion in powers of the small quantity $\frac{p - p_0}{p_0}$ and limiting ourselves only to first-order terms, we get

$$u = g_0(S_0, p) = \frac{c_0}{\gamma_0} \frac{p - p_0}{p_0} = \frac{p - p_0}{\rho_0 c_0}. \quad (1)$$

Similarly, for weak waves

$$u_1 - u = g_1(S_1, p) = \frac{c_1}{\gamma_1} \frac{p - p_1}{p_1} = \frac{p - p_1}{\rho_1 c_1}. \quad (2)$$

Therefore we obtain the result that for the decay of a discontinuity with small amplitudes the values of the pressure p and velocity u at the contact discontinuity, independently of configuration, are expressed by the same formulas:

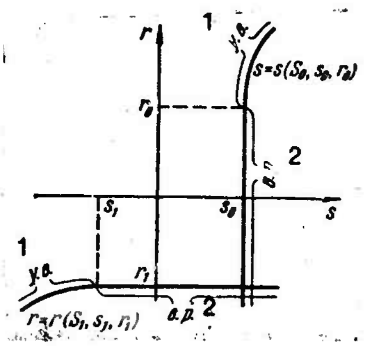

Figure 2.53
KEY:
1) Shock wave
2) Rarefaction wave

$$p = \frac{u_1 - u_0}{\frac{1}{\rho_1 c_1} + \frac{1}{\rho_0 c_0}} + \frac{\frac{p_1}{\rho_1 c_1} + \frac{p_0}{\rho_0 c_0}}{\frac{1}{\rho_1 c_1} + \frac{1}{\rho_0 c_0}}, \quad (3)$$

$$u = \frac{p_1 - p_0}{\rho_0 c_0 + \rho_1 c_1} + \frac{\rho_0 c_0 u_0 + \rho_1 c_1 u_1}{\rho_0 c_0 + \rho_1 c_1}, \quad (4)$$

where we replace the quantity u_1 by the difference $u_1 - u_c$. Formulas (3) and (4) describe the solution of the problem of the decay of a discontinuity in the acoustic approximation, i.e., for infinitely weak waves.

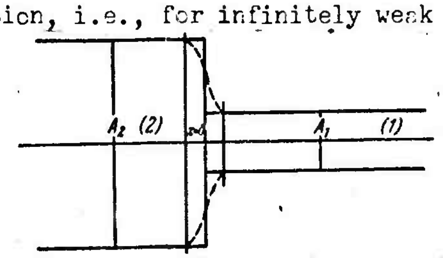

Figure 2.53a

10. Decay of a discontinuity in a variable-section channel. We will consider two semiinfinite cylindrical tubes with cross section layer as A_1 and A_2 abutting at the plane $x = 0$ (Figure 2.53a) and filled with gases characterized at time instant $t = 0$ by the parameters u_1, ρ_1, p_1, S_1, and, respectively, by u_2, ρ_2, p_2, S_2.

The gas pressure arising at $t > 0$ is two-dimensional; however, it can be assumed that the waves propagating in each tube as $t \to \infty$, $|x| \to \infty$ become close to one-dimensional. The approximate flow pattern (we will call this flow the decay of discontinuity at the section discontinuity) is based on the assumption that the asymptotes are established instantaneously and the flow decaysinto two one-dimensional flows divided by an infinitely thin transition zone enclosed between the planes $x = -\varepsilon$, $x = +\varepsilon$. The motion in the region of the transition is a steady flow, so that the quantity $f = (u, \rho, p)$ to the right and to the left of the transition zone (we will designate them by f^- and f^+, respectively) are associated by the relations

$$A_2(\rho u)^- = A_1(\rho u)^+, \qquad (1)$$

$$\left[\varepsilon(p, \rho) + \frac{p}{\rho} + \frac{u^2}{2}\right]^- = \left[\varepsilon(p, \rho) + \frac{p}{\rho} + \frac{u^2}{2}\right]^+. \qquad (2)$$

The first expresses the law of conservation of mass, and the second Bernoulli's law. To this we add a fair relation, which has a different form for different decay models. We will confine ourselves to the adiabat model in which the conservation of entropy in the transition zone is assumed*). For a polytropic gas, therefore, we have

$$\left(\frac{p}{\rho^\gamma}\right)^- = \left(\frac{p}{\rho^\gamma}\right)^+. \qquad (3)$$

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Let us consider the simplest case of discontinuity decay -- the impinging of a shock wave traveling with respect to the quiescent gas with parameters ρ_1, p_1, u_1 and exiting from the wide section of the tube into the narrow. We will assume that the discontinuity in the section is small, i.e., that the condition

$$[illegible]\ \delta < 1 \qquad (4)$$

is satisfied. Then we can assume the disturbance in the shock wave to be small and we can linearize the working formulas. After the shock wave passes through the section discontinuity, we will have the following configuration of discontinuities: to the right travels shock wave D_1, behind it at the point $x = 0$ occurs a discontinuity subject to relations (1)-(3), behind travels the reflected waves; between the shock wave that has gone by and section $x = 0$ lies the contact boundary. Figure 2.53b reflects the configuration of discontinuities in the x, t-plane. The line Γ_4 is the trajectory of the shock wave entering the narrow tube, Γ_1 is a trajectory of the reflected shock wave, Γ_2 is a transition zone, and Γ_3 is the trajectory of the contact boundary. At the lines Γ_1, Γ_2, Γ_3 the flow parameters experience discontinuities of small amplitude, and the corresponding abutment conditions can be linearized. Let Δp, Δu represent the total changes in p, u in the transition from states (2) to state (3), and let $\Delta_i p$ and $\Delta_i u$ stand for the changes in p, u for the transition across Γ_i ($i = 1, 2, 3$). The following linearized relations are valid:

a) at Γ_1: $\dfrac{\Delta_1 p}{p_2} + \dfrac{1+h}{1-h}\dfrac{\Delta_1 u}{c_2} = 0,$

b) at Γ_2: $-\delta + \dfrac{\Delta_2 u}{u_2} + \dfrac{\Delta_2 p}{\gamma p_2} = 0,$ (5)

c) $\dfrac{\Delta_2 p}{\rho_2} + u_2 \Delta_2 u = 0,$

d) at Γ_3: $\Delta_3 p = \Delta_3 u = 0.$

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*) A complete analysis of the discontinuity decay problem at the jump section is found in the papers of V. G. Dulov [11] and I. K. Yakushev [12].

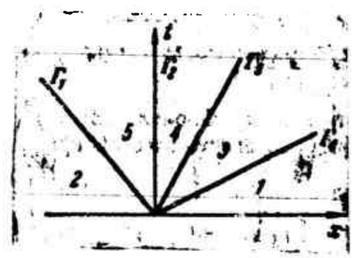


Figure 2.53b

Total changes Δu, Δp are caused by change in the force M of the shock wave as it makes its transition from the wide section of the tube into the narrow. Using Hugoniot's condition for a polytropic gas

$$\begin{aligned}
&a) \quad \frac{p_2}{p_1} = (1+h)M^2 - h, \\
&b) \quad \frac{u_2}{c_1} = (1-h)\left(M - \frac{1}{M}\right), \\
&c) \quad \frac{c_2}{c_1} = \sqrt{\left[(1+h) - \frac{h}{M^2}\right][(1-h) + hM^2]}, \\
&d) \quad \frac{\rho_2}{\rho_1} = \frac{M^2}{1-h+hM^2},
\end{aligned} \qquad (6)$$

we have

$$\begin{aligned}
&a) \quad \frac{\Delta p}{p_1} = \frac{\Delta_1 p + \Delta_2 p}{p_1} = 2(1+h)M\Delta M, \\
&b) \quad \frac{\Delta u}{c_1} = \frac{\Delta_1 u + \Delta_2 u}{c_1} = (1-h)\left(1 + \frac{1}{M^2}\right)\Delta M.
\end{aligned} \qquad (7)$$

Relation (6) enables us to express quantities with subscript 2 in terms of quantities with subscript 1 and known functions of M, and then the coefficients of equations (5a), (5b), and (5c) will be expressed in terms of M and quantities with subscript 1. Equations (5a), (5b), (5c), (7a) and (7b) yield a system of five equations relative to four quantities: $\Delta_1 u$, $\Delta_2 u$, $\Delta_1 p$, $\Delta_2 p$.

The condition of algebraic compatibility of equation (7) leads to an equation that was first obtained by Chester [13]:

$$\frac{dA}{A} = -\frac{2M\,dM}{(M^2-1)\,k(M)}, \tag{8}$$

where

$$k(M) = 2\left[\left(1 + (1-h)\frac{1-g^2}{g}\right)\left(2g + 1 + \frac{1}{M^2}\right)\right]^{-1}, \tag{9}$$

$$g^2(M) = \frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)}. \tag{10}$$

Equation (8) yields the relationship between variations in intensity of shock waves and the two cross sections. Chisnell [14] proposed using formula (8) for the decay of a nonsteady shock front traveling in a channel with continuously variable cross section. In the Chisnell theory a channel with a continuously variable cross section is approximated by a sequence of cylindrical channels (Figure 2.53c) abutting one another, and the transition of the shock wave from one cylindrical section to another is given above by formula (8).

By integrating equation (8), we find $Af(M)$ = constant, (11)

where

$$f = z^{\frac{1}{\gamma}}(z-1)(z+h)^{-\frac{1}{2}}\left[\frac{1+R}{1-R}\right]^{\sqrt{\frac{\gamma}{2(\gamma-1)}}}\left[\frac{R-\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}}{R+\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}}\right]\times$$

$$\times\exp\left[\left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}}\operatorname{arctg}\left\{\frac{2\gamma^{\frac{1}{2}}R}{\gamma-1}\right\}\right], \tag{12}$$

$$R = \left(1 + \frac{1}{hz}\right)^{-\frac{1}{2}}, \tag{13}$$

$$z = \frac{2\gamma}{\gamma+1}M^2 - h. \tag{14}$$

The quantity k(M) is a slowly changing function of M. Thus, for

If we assume k is a constant, integral (11) is simplified and leads to the form $A^k(M^2 - 1) = \text{constant.}$ (15)

For weak shock waves an arbitrary γ, $k(M) \rightarrow \frac{1}{2}$ and formula (15) becomes

$$M - 1 \sim A^{-\frac{1}{2}}. \qquad (16)$$

For strong shock waves

$$k(M)_{M\to\infty} \rightarrow k(\infty) = \frac{2}{(1+k_1)(1+k_2)}, \qquad (17)$$

$$k_1 = \sqrt{\frac{2}{\gamma(\gamma-1)}}, \quad k_2 = \sqrt{\frac{2(\gamma-1)}{\gamma}}.$$

Then from equation (15) we have $M \sim A^{-\frac{k_\infty}{2}}$. (18)

Estimate (18) was used by Chisnell to establish the asymptotes of a strong shock wave for the case of cylindrical and spherical symmetry.

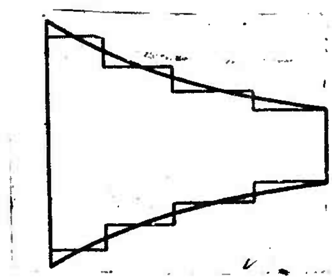


Figure 2.53c

From (18) follows $M \sim x^{-n}$, $n = \frac{k_\infty}{2}$ (19)

for a cylindrical shock wave and $M \sim x^{-n}$, $n = k_\infty$, (20)

for a spherical shock wave (x represents distances up to the axis or, correspondingly, the symmetry points).

A comparison of estimates (19), (20) with the self-modeling solution of Guderley for a convergent shock wave (cf section IX, subsection 5) shows excellent agreement. Since, for $V = \frac{5}{3}, \frac{7}{3}$, we have the following comparison of indexes n:

	a Цилиндрическая волна		d Сферическая волна	
	b Чиснелл	c Гудерлей	b Чиснелл	c Гудерлей
$\gamma = \frac{5}{3}$	0,2254	0,2260	0,4508	0,4527
$\gamma = \frac{7}{5}$	0,1971	0,1973	0,3941	0,3944

KEY:

a) Cylindrical wave

b) Chisnell

c) Guderley

d) Spherical wave

The theory outlined by Chisnell does not make allowance for the additional influence of secondary waves reflected from the channel walls and overtaking the shock wave. Corrections were introduced into formula (18) in the works of Chisnell [14] and [15], which incidentally proved to be unessential.

Withan [16] gave a simple interpretation of equation (8). As we know (cf [28]), the flow in the channel is described, in a one-dimensional approximation, by the equations

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t}+\frac{1}{A}\frac{\partial}{\partial x}(\rho u A)=0,\\ &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\\ &\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+u\frac{\partial}{\partial x}\left(\frac{p}{\rho^{\gamma}}\right)=0.\end{aligned}\right\} \qquad (21)$$

where A(x) is the cross section area of the channel.

Equations (21) can be rewritten in characteristic form:

$$\left.\begin{array}{ll} \text{a)} & \frac{1}{\rho c}\frac{dp}{dt_1}-\frac{du}{dt_1}+\frac{cu}{u-c}\frac{d\ln A}{dt_1}=0, \\ \text{b)} & \frac{d}{dt_2}\left(\frac{p}{\rho^{\gamma}}\right)=0, \\ \text{c)} & \frac{1}{\rho c}\frac{dp}{dt_3}+\frac{du}{dt_3}+\frac{cu}{u+c}\frac{d\ln A}{dt_3}=0, \end{array}\right\} \qquad (22)$$

where

$$\frac{d}{dt_1}=\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x},\quad \frac{d}{dt_2}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x},\quad \frac{d}{dt_3}=\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}. \qquad (23)$$

Assuming for definiteness that the shock wave travels from the left to the right and that the difference in the slopes of the shock wave trajectories and the overtaking characteristic is small, we can approximately assume that the relation

$$d\bar{r}=\frac{1}{\rho c}dp+du+\frac{cu}{u+c}d\ln A=dr+\frac{cu}{u+c}d\ln A=0 \qquad (24)$$

is satisfied not along the r-characteristic, but at the shock wave trajectory. From (6a) and (6b) we have

$$dr=du+\frac{1}{\rho_2 c_2}dp=c_1\left[(1-h)\left(1+\frac{1}{M^2}\right)+\frac{2p_1}{\rho_2 c_2 c_1}(1+h)M\right]dM. \qquad (25)$$

Substituting (25) into (24), we arrive at the relation

$$\left[(1-h)\left(1+\frac{1}{M^2}\right)+\frac{2p_1}{\rho_2 c_2 c_1}(1+h)M\right]dM+\frac{c_2 u_2}{c_1(u_1+c_1)}d\ln A=0, \qquad (26)$$

which by virtue of relations (6) is equivalent to (8).

Since the assumption (24) is valid not only for weak waves, but also for strong waves entering in the center, this explains the good agreement of Chisnell's theory with the self-modeling solution for the problem of a convergent shock wave.

Different approximations have been developed for flows possessing the property that the inclination of the characteristic overtaking the shock wave is close to these slopes of the shock front, and these approximations satisfactorily describe the behavior of flow near the front (Poincare-Lighthill-Go method [17] and the method of shock waves [18]).

Section VII. Interaction of Strong Discontinuities.

Based on the analysis of the problem of the decay of an arbitrary discontinuity, in **this section we** consider several problems of interaction of strong discontinuities, such as shock waves and contact boundaries.

In the merging of strong discontinuities piecewise-constant flows emerge, which can be taken as initial data for the solution of the problem of the decay of arbitrary continuity. Therefore the problem of the interaction of strong discontinuities reduces to the problem of discontinuity decay.

We consider in this section interactions of strong discontinuities in the following order:

1) the impinging of a shock wave at the contact boundary.

2) the union of shock waves traveling directly into each other relative to the gas.

3) The union of shock waves moving in the gas in the same direction.

All possible interactions of strong discontinuities are exhausted in the three combinations. The main problem will be to establish the configuration of the slope formed as a function of the parameters characterizing the interacting discontinuities.

We first make a general examination by the method of the p, u diagram, and then consider the case of the polytropic gas.

1. Impinging of a shock wave at the interface of two media. Suppose that a shock wave traveling in the medium x = 0 in the direction from left to right impinges against the boundary $x < 0$ of two media characterized by parameters $\bar{p}_0 = \rho_0, \bar{S}_0, \bar{u}_0 = 0$ (to the left of x = 0) and p_0, S_0, $u_0 = 0$ (at the right) at time instants t = 0. Thus, at the instant t = 0 the initial discontinuity with parameters p_1, S_1, u_1 (gas "1", left) and p_0, S_0, $u_0 = 0$ (gas "0", right) is formed. Here the point (p_1, u_1) must lie in the upper part of the curve (6.8.7) calculated for the gas initially lying at $x < 0$ and passing through the point $(p_0, 0)$, i.e.

$$u_1 = \Psi_0(\bar{S}_0, p_1), \quad p_1 > p_0, \quad u_1 > 0. \tag{1}$$

Thus, the set of possible states (p_1, u_1) of gas "1" is mapped in the plane (p, u) by a curve described by equation (1) (Figure 2.54).

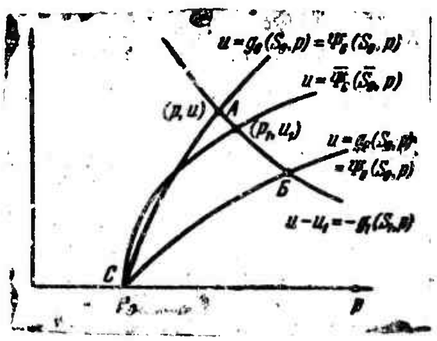


Figure 2.54

Suppose that the initial shock wave is given. Then also given is the point (p_1, u_1) lying on the curve (1) (Figure 2.54). In accordance with subsection 8 of section VI, the solution of the problem of the interaction of a shock wave with the contact boundary reduces to the determination of the point (p, u) of intersection with the curve (6.8.7) $u = g_0(S_0, p)$ (2)

for the gas "0" passing through the point (p_0, u_0) and the curve (6.8.8)

$$u = u_1 - g_1(S_1, p) \tag{3}$$

for gas "1" passing through the point (p_1, u_1).

Obviously, curve (2) can intersect curve (3) only in the separate branch, i.e., when $p > p_0$. Consequently, the induced flow has either configuration A (corresponds to point A of the intersection by curve (2) (CA) and by curve (3) (AB) in Figure 2.54), or configuration B (corresponds to point B of intersection by curve (2) (CA) and by curve (3) (AB)).

Thus, a shock wave is always propagated in gas "0", and a rarefaction wave or a shock wave is propagated in gas "1", depending on the mutual disposition of curves (1) and (2).

Let us consider the case of polytropic gas. According to (6.8.13)

$$\overline{\Psi}_0(\overline{S}_0, p) = (1 - \overline{h}_1)\overline{c}_0\left[\sqrt{\frac{p + \overline{h}_1 p_0}{p_0(1 + \overline{h}_1)}} - \sqrt{\frac{p_0(1 + \overline{h}_1)}{p + \overline{h}_1 p_0}}\right], \tag{4}$$

$$\Psi_0(S_0, p) = (1 - h_0)c_0\left[\sqrt{\frac{p + h_0 p_0}{p_0(1 + h_0)}} - \sqrt{\frac{p_0(1 + h_0)}{p + h_0 p_0}}\right]. \tag{5}$$

We introduce into our consideration the quantity

$$K(\theta) = \frac{\overline{\Psi}_0^2(\overline{S}_0, p)}{\Psi_0^2(S_0, p)} = \frac{(1 - h_1)^2}{(1 - h_0)^2}\frac{\overline{c}_0^2}{c_0^2}\frac{1 + h_0}{1 + h_1}\frac{\theta + h_0}{\theta + h_1}\frac{(\theta - 1)^2}{(\theta - 1)^2} = \frac{(1 - h_1)^2}{(1 - h_0)^2}\frac{(1 + h_0)}{(1 + h_1)}\frac{\theta + h_0}{\theta + h_1}\left(\frac{\overline{c}_0}{c_0}\right)^2, \tag{6}$$

where

$$\theta = \frac{p}{p_0}. \tag{7}$$

From this we have

$$K(1) = \frac{(1 + h_0)^2}{(1 - h_0)^2}\frac{(1 - h_1)^2}{(1 + h_1)^2}\left(\frac{\overline{c}_0}{c_0}\right)^2 = \left(\frac{\gamma_0}{\gamma_1}\right)^2\left(\frac{\overline{c}_0}{c_0}\right)^2, \tag{8}$$

$$K(\infty) = \frac{(1 - h_1)^2}{(1 - h_0)^2}\frac{(1 + h_0)}{(1 + h_1)}\left(\frac{\overline{c}_0}{c_0}\right)^2 = \frac{1 + h_1}{1 + h_0}K(1). \tag{9}$$

If $h_0 < h_1$ ($\gamma_0 < \gamma_1$), then $K(\infty) > K(1)$ and $K(\theta)$ is a monotonically increasing function of θ. But if $h_0 > h_1$ ($\gamma_0 > \gamma_1$), then $K(\infty) < K(1)$ and $K(\theta)$ is a monotonically decreasing function as $0 < \theta < \infty$.

Let us consider the following cases:

1) $K(1) \geqslant 1$, $K(\infty) \geqslant 1$. Then $K(\theta) \geqslant 1$ and $\bar{\Psi}_0(\bar{S}_0, p) \geqslant \Psi_0(S_0, p)$ for all $p \geqslant p_0$. In this case curve $u = \Psi_0(S_0, p)$ lies beneath the curve $u = \bar{\Psi}_0(\bar{S}_0, p)$ (Figure 2.54) and for any shock wave intensity, as a result of interaction with the contact boundary configuration B is induced, i.e., when $t > 0$ two shock waves propagate to different sites from the contact boundary (Figure 2.55).

2) $K(1) \leqslant 1$, $K(\infty) \leqslant 1$. Then $\bar{\Psi}_0(\bar{S}_0, p) \leqslant \Psi_0(S_0, p)$ and for any shock wave intensity the flow when $t > 0$ has configuration A, i.e., a shock wave propagates in gas "0" (as, it does, incidentally, always), and a centered rarefaction wave propagates in gas "1".

3) If $K(1) > 1$, $K(\infty) < 1$ ($\gamma_0 > \gamma_1$), then for some $p = P$ the curves $u = \bar{\Psi}_0(\bar{S}_0, p)$ and $u = \Psi_0(S_0, p)$ intersect each other. When $p_0 \leqslant p_1 \leqslant P$, we have configuration B; when $p_1 > P$, i.e., for sufficiently strong shock waves, we have configuration A.

4) $K(1) < 1$, $K(\infty) > 1$ ($\gamma_0 < \gamma_1$). Then curves $u = \bar{\Psi}_0(\bar{S}_0, p)$ and $u = \Psi_0(S_0, p)$ intersect each other at $p = P$; in the region $p_0 \leqslant p_1 \leqslant P$ the flow has configuration A; when $p_1 > P$ it has configuration B.

Let us consider the special case of identical gases ($\gamma_0 = \gamma_1 = \gamma$). Then

$$K(1) = K(\infty) = \left(\frac{\bar{c}_0}{c_0}\right)^2. \tag{10}$$

Considering relation $c^2 = \frac{\gamma p}{\rho}$, let us write (10) in the form

$$K(1) = K(\infty) = \frac{\rho_0}{\bar{\rho}_0}. \tag{11}$$

From this it follows that when the shock wave travels from a less dense gas into a more dense gas ($K > 1$), then the shock wave is reflected in the less dense gas (we have configuration B). But if the shock wave travels from a more dense gas into a less dense ($K < 1$), then a rarefaction wave is reflected (configuration A).

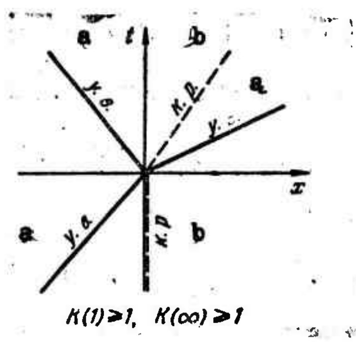


Figure 2.55

KEY:

a) Shock wave b) Contact discontinuity

This result can be compared with the conclusion in subsection 5 of section III, where the reflection of a traveling wave at the contact boundary was studied. There we obtain the conclusion that a wave of the same type as the incident wave is reflected from a more dense medium. Since the weak shock wave can be regarded as a weak Riemann compression wave, then we see that qualitatively the reflection of shock waves and Riemann waves at contact boundaries is identical in nature.

2. Encounter of two shock waves. Two shock waves travel toward each other in a substance with parameters $\bar{u}_0 = 0$, $\bar{p}_0$ and $\bar{S}_0$, leaving behind them states

$u_1 = \bar{\Psi}_0(\bar{S}_0, p_1)$, p_1, S_1 behind the left wave and $u_0 = -\bar{\Psi}_0(\bar{S}_0, p_0)$, p_0, S_0 behind the right (Figure 2.56). At the instant $t = 0$ the shock wavesencounter at the point $x = 0$ forming an initial discontinuity with parameters $u_1 = \bar{\Psi}_0(\bar{S}_0, p)$, p_1, S_1 (left) and $u_0 = -\bar{\Psi}_0(S_0, p_0)$, p_0, S_0 (right). For definiteness we suppose that

$$p_1 \geqslant p_0. \qquad (1)$$

Here two cases are possible:

a) Curve $u - u_0 = \Psi_0(S_0, p)$ is always below curve $u = \bar{\Psi}_0(\bar{S}_0, p)$ (Figure 2.57, a).

b) Curve $u - u_0 = \Psi_0(S_0, p)$ intersects the curve $u = \bar{\Psi}_0(\bar{S}_0, p)$ at some point (P, U) (Figure 2.57, b).

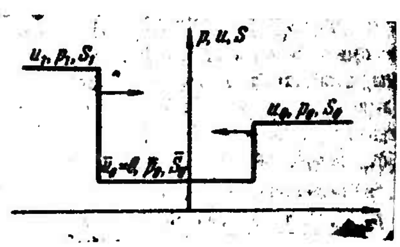
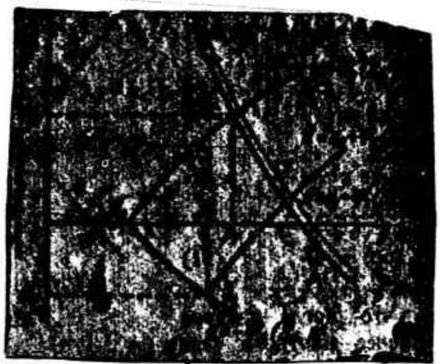
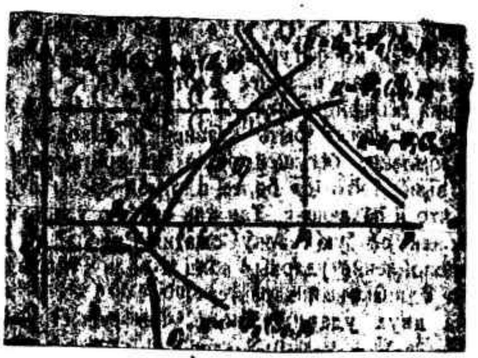


Figure 2.56

In the first case only configuration B is possible, in the second, both B and A, however in any case a shock wave will be propagated in gas "O".



a)



b)

Figure 2.57

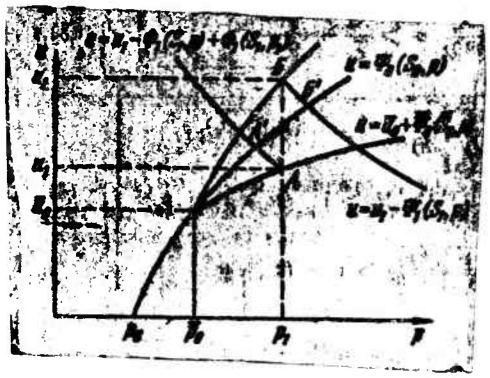


Figure 2.58

Let us consider more detail the case of the polytropic gas for which the intersection of the curves $u = \bar{\Psi}_0(\bar{S}_0, p)$ and $u = u_0 + \Psi_0(S_0, p) - \bar{\Psi}_0(S_0, p_0)$ is impossible. Thus we must prove the inequality

$$\bar{\Psi}_0(\bar{S}_0, p) + \bar{\Psi}_0(S_0, \bar{p}_0) > \Psi_0(S_0, p) \quad \text{where } p \geqslant p_0 > \bar{p}_0. \tag{2}$$

Considering formula (6.8.13) for $\Psi_0(S_0, p)$ and formula (6.8.12) we reduce inequality (2) to the form

$$\frac{M_1^2 - 1}{M_1} + \frac{M_0^2 - 1}{M_0} > \frac{c_0}{\bar{c}_0} \frac{M^2 - 1}{M} \tag{3}$$

where

$$M_0 = -\frac{D_1}{c_0} = \sqrt{\frac{p_0 + h\bar{p}_0}{\bar{p}_0(1+h)}} > 1 \quad (p_0 > \bar{p}_0),$$
$$M_1 = \frac{D_1}{c_0} = \sqrt{\frac{p + hp_0}{p_0(1+h)}} > M_0 > 1 \quad (p > p_0 > \bar{p}_0),$$
$$M = \frac{|D - u_0|}{c_0} = \sqrt{\frac{p + hp_0}{p_0(1+h)}} > 1 \quad (p > p_0). \tag{4}$$

From this follows the equality

$$(1+h)M^2 - h = \frac{(1+h)M_1^2 - h}{(1+h)M_0^2 - h}. \tag{5}$$

Allowing for relation (4.5.15):

$$\frac{c^2}{c_0^2} = \frac{[(1+h)M_0^2 - h][(1-h) + hM_0^2]}{M_0^2} \tag{6}$$

and also consequence (5):

$$M^2 - 1 = \frac{M_1^2 - M_0^2}{(1+h)M_0^2 - h}, \tag{7}$$

let us transform inequality (3) to the form

$$\frac{(M_1M_0 - 1)^2}{M_1^2} > \frac{1}{M^2}\frac{(M_1 - M_0)^2(1 - h + hM_0^2)}{(1+h)M_0^2 - h}. \tag{8}$$

Canceling out of inequality (8) the quantity M by using (5), we arrive at the inequality

$$\frac{(M_1M_0 - 1)^2}{M_1^2} > \frac{(M_1 - M_0)^2[1 - h + hM_0^2]}{M_1^2 - h + hM_0^2}. \tag{9}$$

When $M_1 = M_0$, inequality (9) will obviously be satisfied (this case corresponds to the encounter of equal-intensity shock waves). Let us show that inequality is satisfied for any $M_1 \geq M_0$. To do this, let us transform it to become

$$(M_1^2 - 1)(M_0^2 - 1)\{(1 - h)M_1^2 + 2hM_0M_1 - h] > 0,$$

from whence it readily follows that it always is satisfied when $M_1 \geq M_0 > 1$. Thus, in a polytropic gas only configuration B is produced in the encounter of two shock waves, i.e., when two shock waves meet each of them as it were "passes" through the other.

3. Union of shock waves traveling in the same direction. In a gas with parameters p_0, S_0, $u_0 = 0$ a shock wave travels from left to right with constant velocity $D_0 > 0$, leaving behind the state $\bar{u}_0, \bar{p}_0, \bar{S}_0$ located at the curve

$$u = \Psi_0(S_0, p). \qquad (1)$$

Behind the first shock wave, also with constant velocity D_1, $D_1 > D_0 > \bar{u}_0$, travels the second shock wave, leaving behind the state u_1, p_1, S_1 located at the curve

$$u - \bar{u}_0 = \bar{\Psi}_0(\bar{S}_0, p), \qquad (2)$$

passing through the point $(\bar{p}_0, \Psi_0(S_0, \bar{p}_0))$ (Figure 2.58). At the instant of union of the shock waves an initial discontinuity is induced, which on decaying yields a flow with configuration B, if the point $(p_1, \bar{u}_0 + \bar{\Psi}_0(\bar{S}_0, p_1))$ lies above the curve (1) (in Figure 2.58 this case is shown by point B), and also induced is a flow of configuration A if the point $(p_1, \bar{u}_0 + \Psi_0(\bar{S}_0, p_1))$ lies below the curve (1) (the corresponding solutions are noted by the points B' and A' in the figure). A shock wave will always propagate, however, in this case toward the side of gas "0".

Let us consider again in greater detail the case of an ideal gas. The mutual disposition of curves (1) and (2) is determined by the sign of the difference

$$\Delta = \Psi_0(S_0, \bar{p}_0) + \bar{\Psi}_0(\bar{S}_0, p) - \Psi_0(S_0, p) \qquad (3)$$

for the assigned **restrictions** $p_0 < \bar{p}_0 < p$. (4)

Let us introduce the notations:

$$\frac{D_0}{c_0} = M_0, \quad \frac{D_1 - \bar{u}_0}{\bar{c}_0} = \bar{M}, \quad \frac{D}{c_0} = M. \tag{5}$$

Here M_0 corresponds to the first shock wave traveling with respect to the background $(0, p_0)$ and leaving behind it the state $(\bar{u}_0, \bar{p}_0)$; $\bar{M}$ is the second shock wave traveling relative to the background $(\bar{u}_0, \bar{p}_0)$ leaving behind it the state (u_1, p_1); M is the possible shock wave traveling relative to the background $(0, p_0)$ and leaving behind it the state (u, p). Then the difference (3) is represented in the form

$$\Delta = (1-h)\,c_0 \frac{M_0^2 - 1}{M_0} + (1-h)\,\bar{c}_0 \frac{\bar{M}^2 - 1}{\bar{M}} - (1-h)\,c_0 \frac{M^2 - 1}{M}. \tag{6}$$

From relations

$$\frac{\bar{p}_0}{p_0} = (1+h)\,M_0^2 - h, \quad \frac{p_1}{\bar{p}_0} = (1+h)\,\bar{M}^2 - h, \quad \frac{p_1}{p_0} = (1+h)\,M^2 - h \tag{7}$$

follow

$$(1+h)\,\bar{M}^2 - h = \frac{(1+h)\,M^2 - h}{(1+h)\,M_0^2 - h}, \quad \text{or} \quad \bar{M}^2 = \frac{M^2 + hM_0^2 - h}{(1+h)\,M_0^2 - h}. \tag{8}$$

Substituting (8) into (6), we get

$$\Delta = (1-h)\,c_0 \left[\frac{M_0^2 - 1}{M_0} - \frac{M^2 - 1}{M} + \frac{\bar{c}_0}{c_0} \frac{M^2 - M_0^2}{(1+h)\,M_0^2 - h} \frac{1}{\bar{M}} \right] = \\ = (1-h)\,c_0 \left[\frac{(M_0 - M)(MM_0 + 1)}{MM_0} + \frac{\bar{c}_0}{c_0} \frac{M^2 - M_0^2}{(1+h)\,M_0^2 - h} \frac{1}{\bar{M}} \right]. \tag{9}$$

Using relation (4.5.15) (cf Figure (7.2.6)), which in this case can be written as

$$\frac{\bar{c}_0^2}{c_0^2} = \frac{\left[(1+h)\,M_0^2 - h\right]\left[(1-h) + hM_0^2\right]}{M_0^2}, \qquad (10)$$

we obtain from (9)

$$\Delta = (1-h)\,c_0\left[\frac{(M_0-M)(MM_0+1)}{MM_0} + \frac{\sqrt{(1-h)+hM_0^2}\,(M^2-M_0^2)}{M_0\sqrt{(1+h)\,M_0^2-h}}\,\frac{1}{\bar{M}}\right] =$$

$$= (1-h)\,c_0(M-M_0)\left[\frac{\sqrt{1-h+hM_0^2}\,(M+M_0)}{M_0\bar{M}\sqrt{(1+h)\,M_0^2-h}} - \frac{(MM_0+1)}{MM_0}\right]. \qquad (11)$$

Since, by (4), $M > M_0 > 1$, then we see that the sign of Δ coincide with the sign of the expression

$$\frac{(1-h+hM_0^2)(M+M_0)^2}{M_0^2\bar{M}^2\left[(1+h)\,M_0^2-h\right]} - \frac{(MM_0+1)^2}{M^2M_0^2}. \qquad (12)$$

Finally, here substituting $\bar{M}^2$ from formula (8), we get

$$\operatorname{sign}\Delta = \operatorname{sign}\left\{\frac{\left[1-h+hM_0^2\right]}{M^2+hM_0^2-h}\,\frac{(M+M_0)^2}{M_0^2} - \frac{(MM_0+1)^2}{M^2M_0^2}\right\} =$$

$$= \operatorname{sign}\left\{M^2(M+M_0)^2\left[1+h(M_0^2-1)\right] - (MM_0+1)^2\left[M^2+h(M_0^2-1)\right]\right\} =$$

$$= -\operatorname{sign}(M^2-1)(M_0^2-1)\left[(1-h)\,M^2 - 2hMM_0 - h\right]. \qquad (13)$$

and so, if

$$(1-h)\,M^2 - 2hMM_0 - h > 0. \qquad (14)$$

then $\Delta < 0$, and we have configuration A; but if

$$(1-h)\bar{M}^2 - 2hMM_0 - h < 0, \tag{15}$$

then we have configuration B.

It is not difficult to see that when the second shock wave is sufficiently strong ($\bar{M} \gg 1$, $M \gg 1$), inequality (14) is satisfied and we have configuration A. In order for the equation

$$(1-h)M^2 - 2hMM_0 - h = 0 \tag{16}$$

to have the root $M (M > M_0 > 1)$, it is necessary and sufficient that the inequalities

$$1 < M_0^2 < \frac{h}{1-3h} \tag{17}$$

or $\qquad h \geqslant 1/3$, i.e., $\gamma \geqslant 2$.

to be satisfied. If

$$(1 - 3h) \geqslant h, \text{ i.e., } h \leqslant \tfrac{1}{4},\ \gamma \leqslant 5/3. \tag{18}$$

then condition (15) cannot be satisfied for any $M_0 > 1$, and we will lower the configuration A. But if $\gamma > 5/3$, and we may also have configuration B. More exactly, when $M \leqslant M_{kp}$, where M_{kp} is larger than M_0, the root of equation (16), or what amounts to the same, when $\bar{M}^2 \leqslant \frac{M_{kp}^2 + hM_0^2 - h}{(1+h)M_0^2 - h}$,

we have configuration B, otherwise -- A.

4. Interaction of strong discontinuities in an isothermal gas. For the case of an isothermal gas, analysis of the direction of discontinuities is simplified. Actually, an isothermal gas can be considered as a polytropic gas where $\gamma = 1$.

For isothermal gas we have

$$\Psi_0(p) = c_0\left[\sqrt{\frac{p}{p_0}} - \sqrt{\frac{p_0}{p}}\right], \tag{1}$$

$$\Phi_0(p) = c_0 \ln \frac{p}{p_0}. \tag{2}$$

Let us consider the problem of **the incidence of a** shock wave against a **discontinuity contact** (cf Figure 2.54). As we show in subsection 1, the choice between configuration A and B is determined by the sign of the difference $\Delta = \bar{\Psi}_0(p) - \Psi_0(p)$ given the condition $p > p_0$. When $\Delta < 0$, we have configuration A, and when $\Delta > 0$ -- configuration B.

Taking (1) into account, we have

$$\Delta = (\bar{c}_0 - c_0)\left[\sqrt{\frac{p}{p_0}} - \sqrt{\frac{p_0}{p}}\right] \tag{3}$$

Since $p/p_0 > 1$, then configuration A obtains where $\bar{c}_0 < c_0$, and configuration B -- where $\bar{c}_0 > c_0$.

Taking the relation $p_0 = c_0^2 \rho_0 = \bar{c}_0^2 \bar{\rho}_0$. (4)

into account, we get the result that we have configuration A where $\bar{\rho}_0 < \rho_0$, and configuration B -- where $\bar{\rho}_0 > \rho_0$. Its result also follows directly from the result of subsection 1.

When solving problems of the interaction of shock waves, we must bear in mind that $\bar{c}_0 = c_0$.

Let us initially consider the intersection of shock waves traveling headon toward each other (cf Figures 2.56 and 2.57). Using the notations in subsection 2, putting $h = 0$, we have

$$\frac{p_2}{p_0} = M_0^2, \quad \frac{p_1}{p_0} = M_1^2, \quad \frac{p_1}{p_2} = M^2. \tag{5}$$

$$M^2 = \frac{M_1^2}{M_0^2}, \quad M = \frac{M_1}{M_0}. \tag{6}$$

The condition for configuration B is of the form (cf (7.2.3))

$$\frac{M_1^2-1}{M_1}+\frac{M_0^2-1}{M_0}>\frac{\left(\frac{M_1}{M_0}\right)^2-1}{\left(\frac{M_1}{M_0}\right)}=\frac{M_1}{M_0}-\frac{M_0}{M_1}. \qquad (7)$$

from whence we have $\quad (M_1 + M_0)(M_1 + 1)(M_0 - 1) > 0. \qquad (8)$

Thus, we also have configuration B.

From the relations

$$\frac{p_1}{p_0}=M_1^2, \quad \frac{p_0}{p_0}=M_0^2, \quad \frac{p}{p_1}=\overline{M}_0^2, \quad \frac{p}{p_0}=\overline{M}_1^2 \qquad (9)$$

we have

$$M_1\overline{M}_0=M_0\overline{M}_1, \quad \frac{\overline{M}_0}{M_0}=\frac{\overline{M}_1}{M_1}=K. \qquad (10)$$

From the relations

$$u=\bar{c}\left[M_1-\frac{1}{M_1}-\left(\overline{M}_0-\frac{1}{\overline{M}_0}\right)\right], \qquad (11)$$

$$u=\bar{c}\left[\overline{M}_1-\frac{1}{\overline{M}_1}-\left(M_0-\frac{1}{M_0}\right)\right], \qquad (12)$$

and taking (10) into account, we have

$$(M_0+M_1)(1-K)\left[1+\frac{1}{KM_0M_1}\right]=0. \qquad (13)$$

Hence it follows that $\quad K = 1. \qquad (14)$

Equality (14) signifies that after the wave encounter, each of them retains its intensity: $\quad \overline{M}_0 = M_0, \quad \overline{M}_1 = M_1; \qquad (15)$

and the relative velocities of the shock waves are preserved:

$$D_1=D_n-\bar{u}_0=D+|u_0|, \quad D_0=D_s-u_1. \qquad (16)$$

where D_Π [Π= right] is the velocity of the shock wave traveling to the right after the encounter, and D_Λ [Λ= left] is the velocity of the shock wave traveling to the left after the encounter.

Thus, working formulas for the interaction of colliding waves are of the form

$$p = M_0^2 M_1^2 \bar{p}_0, \quad u = \bar{c}_0\left[\left(M_1 - \frac{1}{M_1}\right) - \left(M_0 - \frac{1}{M_0}\right)\right]. \qquad (17)$$

where p and u is the pressure and velocity between the fronts of the divergent waves. Their velocities are determined by formula (16).

Let us consider the conclusion of problem of the union of waves traveling in the same direction. Condition (7.3.14) is satisfied when **h** = 0. Consequently, for the case of isothermal gas when the shock wave traveling in the same direction merge, we always have configuration A.

VIII. Interaction of Shock Waves With Traveling Waves

If the shock wave traveling at a constant velocity with respect to a constant background enters a traveling wave, its intensity is changed. In the case of a polytropic gas the entropy behind the wave front becomes variable, which complicates analytic examination. Therefore we will confine ourselves to a consideration of barotropic polytropic gases, in particular, an isothermal gas. Let us initially make the observation that is valid for all barotropic polytropic gases. Suppose that a shock wave is traveling with constant velocity with respect to a constant background (p_0, ρ_0, u_0) from left to right, leaving behind it a constant background (p_1, ρ_1, u_1) associated with (p_0, ρ_0, u_0) by Hugoniot's conditions. At some instant t_1 the shock wave enters the region of distrubed motion, which can be either a traveling wave, or a region of interference of traveling waves. Then the motion we just formed behind the shock wave is a traveling wave until a new shock wave is formed in it (Figure 2.59).

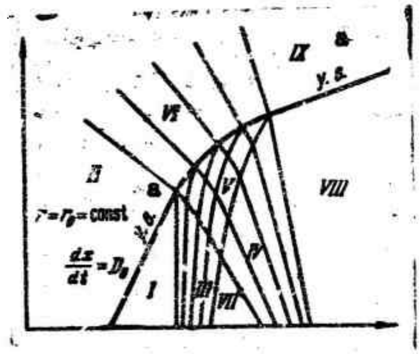


Figure 2.59

KEY:

a) Shock wave

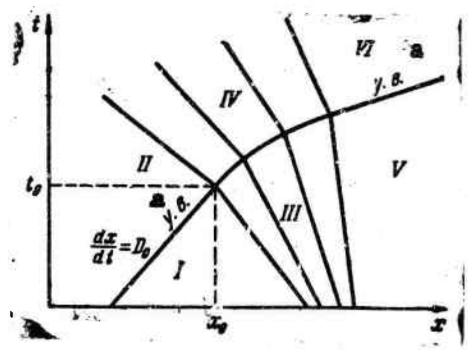


Figure 2.60

KEY:

a) Shock wave

1. Interaction of a shock wave with traveling wave in an isothermal gas. Suppose a constant flow (p_0, $u_0 = 0$) adjoins a rarefaction s-wave

$$r = r_0 = u + c \ln \rho = \text{constant} \qquad (1)$$

A shock wave travels at constant velocity D_0 with respect to a constant background(p_0, $u_0 = 0$) from left to right, leaving behind it the state

$$p_1 = c^2 \rho_1, \quad \rho_1 = \rho_0 M_0^2, \quad M_0 = \frac{D_0}{c}. \tag{2}$$

At the instant $t = t_0$ the shock wave enters region III of the traveling wave (Figure 2.60) where pressure and velocity increase in opposite direction, i.e.,

In region III of the traveling wave, Hugoniot's conditions are of the form (cf Section IV, formula (4.6.13) and (4.6.14))

$$r^- - r^+ = \varphi(M), \quad s^- - s^+ = \psi(M). \tag{3}$$

$$M = \frac{D - u^+}{c}, \quad \varphi(x) = c\left(x - \frac{1}{x} + \ln x^2\right), \tag{4}$$
$$\psi(x) = c\left(x - \frac{1}{x} - \ln x^2\right),$$

where D is the velocity of the shock wave; the quantities with the sign "+" denote the quantities in the wave III ahead of the shock wave front, and those with the sign "-" -- the variables behind the front of the shock wave (region IV).

In the case under consideration the s-wave $r^- = r_1$, $r^+ = r_0$. (5)

From equations (3) and (5) it follows that M is constant and equal to M_0. Thus, the amplitude of the shock wave is invariant and the quantities

$$\frac{p_+}{p_-}, \quad \frac{\rho_+}{\rho_-}, \quad u^+ - u^- \tag{6}$$

retain a constant value at the wave front.

In the region IV we have a rarefaction (compression) s-wave, if in region III the Riemann wave is a rarefaction (compression) wave. Actually, since $u^- - u^+$ is constant at wave front, then

$$\frac{\partial u^-}{\partial t} + D\frac{\partial u^-}{\partial x} = \frac{\partial u^+}{\partial t} + D\frac{\partial u^+}{\partial x},$$

or

$$\left(1 - \frac{u^- - c}{D}\right)\frac{\partial u^-}{\partial x} = \left(1 - \frac{u^+ - c}{D}\right)\frac{\partial u^+}{\partial x}, \quad (7)$$

such that the derivatives $\partial u^-/\partial x$ and $\partial u^+/\partial x$ (in the regions III and IV) are of the same sign. Since in the s-wave with constant invariant r (in zone III) the Riemann relation $x - (u - c)t = f(u)$ (8) is valid, where f(u) is some function of u and at the shock wave the relation

$$\frac{dx}{dt} = D = u + cM_0 = u^+ + D_0, \quad (9)$$

is valid, then by comparing (8) and (9), we obtain a differential equation for the trajectory of the shock wave:

$$x - \left[\frac{dx}{dt} - c(M_0 + 1)\right]t = f\left(\frac{dx}{dt} - cM_0\right). \quad (10)$$

In the particular case of a centered rarefaction wave when the following relation is valid:

$$\frac{x - x_1}{t - t_1} = u - c, \quad (11)$$

where (x_1, t_1) is the wave center, we can readily obtain the integral equation

$$\frac{dx}{dt} = \frac{x - x_1}{t - t_1} + D_0 + c. \quad (12)$$

Clearly, a similar analysis was obtained also for the case of shock wave traveling from right to left and encountering a Riemann r-wave.

For the case of a rarefaction wave ($\frac{\partial u}{\partial x} > 0, \frac{\partial p}{\partial x} < 0$), the shock wave proceeds "under the peak," by accelerating; for the case of a compression wave ($\frac{\partial u}{\partial x} < 0, \frac{\partial p}{\partial x} > 0$) the shock wave travels "into the peak", slowing down. This pattern of interaction can be realized in the problem with two pistons; first the right piston retreats from the gas, forming a rarefaction wave (r = constant), then the left piston advances into the gas, forming a shock wave that travels into the rarefaction wave.

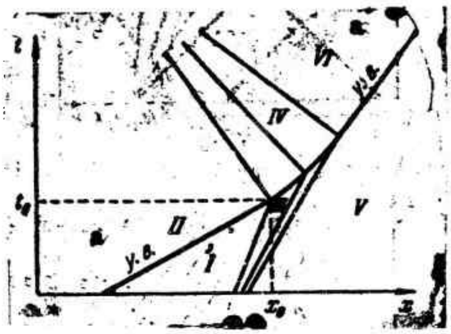


Figure 2.61

KEY:

a) Shock wave

Let us now consider the problem of the interaction of a shock wave travel ing to the right with a Riemann r-wave ($s = s_0$ = constant) (Figure 2.61). In this case the pressure and velocity ahead of the shock wave increase in the same direction ($\frac{\partial u}{\partial x}\frac{\partial p}{\partial x} > 0$).

In formula (3) we must put $r^- = r$ = constant, $s^+ = s_0$. (13)

Then for the case of rarefaction(compression) wave in region III, we have

$$\left.\begin{aligned} \frac{\partial u^+}{\partial x} > 0 \quad \left(\frac{\partial u^+}{\partial x} < 0\right), \\ \frac{\partial r^+}{\partial x} > 0 \quad \left(\frac{\partial r^+}{\partial x} < 0\right) \end{aligned}\right\} \tag{14}$$

and from relation (3) we find

$$\left.\begin{array}{ll}\frac{\partial M}{\partial x}<0 & \left(\frac{\partial M}{\partial x}>0\right), \\ \frac{\partial s^-}{\partial x}<0 & \left(\frac{\partial s^-}{\partial x}>0\right), \\ \frac{\partial u^-}{\partial x}<0 & \left(\frac{\partial u^-}{\partial x}>0\right).\end{array}\right\} \quad (15)$$

This means that for rarefaction wave III, the reflected wave IV is a compression wave, for compression III wave IV is a rarefaction wave. In the first case the intensity M of the wave is reduced, and the second -- is increased.

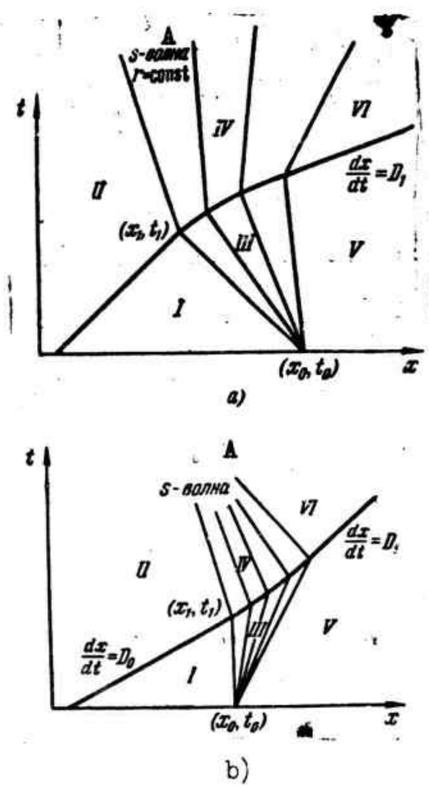


Figure 2.62

/KEY on following page/

KEY:

A) s-wave

The trajectory of a shock wave is found on analogy with the **foregoing.** This pattern can be realized in the problem with one piston: the piston initially retreats, forming a rarefaction wave, then advancing into the gas, forms a compression wave (shock wave).

2. Asymptotes of interaction of a shock wave and a centered rarefaction wave. As a result of interaction of a shock wave with a rarefaction wave III, after departing from it against the constant background V, the shock wave acquires instead of its initial velocity D_0, the velocity D_1. Clearly, the discontinuity in velocity $D_1 - D_0$ of a shock wave does not depend on the entry point (x_1, t_1) if there is smooth flow in the region IV. Suppose the entry point (x_1, t_1) draws nearer the center (x_0, t_0) of the wave.

Then the difference $D_1 - D_0$ remains unchanged. If $x_1 = x_0$, then we can speak of instantaneous interaction of shock wave with concentrated rarefaction wave. In this case we have a longitudinal discontinuity characterized by the states (II, V). We set up the problem: will the velocity D_1 of the shock wave in the problem of discontinuity decay coincide with velocity D_1 after the exiting of the shock wave from the rarefaction wave III? Let us first consider the case of the centered rarefaction s-wave (Figure 2.62, a).

States II and V are defined by the formulas

$$\left.\begin{aligned} u_2 = u_1 + c\left(M_0 - \frac{1}{M_0}\right), \quad p_2 = p_1 M_0^2, \\ u_2 - u_1 = c\left[\sqrt{\frac{p_2}{p_1}} - \sqrt{\frac{p_1}{p_2}}\right]. \end{aligned}\right\} \quad (1)$$

$$u_5 = u_1 + c \ln \frac{p_1}{p_5}. \quad (2)$$

Let us calculate the discontinuity decay (p_2, u_2), (p_5, u_5) using the (p, u)-diagram (Figure 2.63). In this case we have configuration A. The quantities u and p are found as the result of solving the solution

$$M - \frac{1}{M} - \ln\frac{p_2}{p_1} = \\ = M_0 - \frac{1}{M_0} - \ln\frac{p}{p_2}, \tag{3}$$

where we put

$$M^2 = \frac{p}{p_2}, \quad M_0^2 = \frac{p_2}{p_1}. \tag{4}$$

After uncomplicated transformation, from formula (3) we have

$$\frac{\varphi(M)}{c} = M - \frac{1}{M} + \ln M^2 = \frac{\varphi(M_0)}{c} = M_0 - \frac{1}{M_0} + \ln M_0^2. \tag{5}$$

From whence

$$M = M_0. \tag{6}$$

Thus, after the interaction of a shock wave traveling to the right with an s-wave, it acquires the same velocity as resulting from instantaneous interaction the shock wave and a concentrated rarefaction zone calculated by the discontinuity decay formulas. This agreement stems from the fact that reflected wave IV is a rarefaction wave and as $t \to \infty$ no singularities are induced in the motion behind the wave front. A similar pattern obtains in the interaction of a shock wave traveling to the left with a rarefaction r-wave.

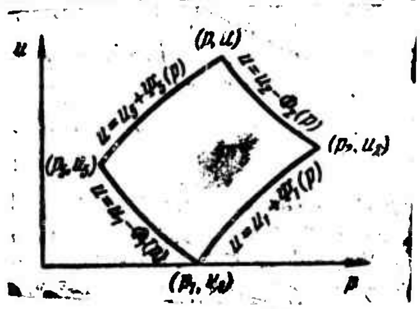


Figure 2.63

Here let us consider (cf /44/) the interaction of a shock wave traveling to the right with a r-wave (Figure 2.62,b).

In this case shock wave passing through region III acquires a velocity that is different from that resulting from instantaneous interaction of a shock wave with a concentrated rarefaction r-wave considered as the decay of an arbitrary discontinuity. The reason for this is the initiation of the shock wave traveling to the left from compression wave IV and then of a rarefaction wave traveling to the right and overtaking the shock wave traveling toward the right.

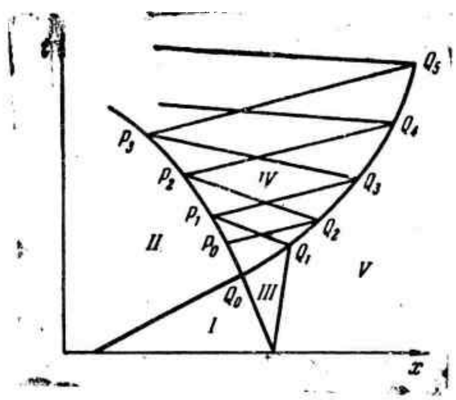


Figure 2.64

The shock wave appearing, by varying its velocity, changes the invariant r. These changes are transported along the r-characteristic and, arriving at the front of the shock wave traveling to the right change the velocity of the latter by inducing a change in the invariant s.

Changes in invariant s at the right wave along the s-characteristic are transported to the left shock wave, causing a change in the invariant r. From interaction of the rear fronts of the right and left shock waves a certain symptotic regime is generated. The limiting configurations and flow coincide with the configurations and flow resulting from the decay of discontinuity II and V

(Figure 2.64).

Let us consider front interaction in detail. We introduce the notations:

r_I, s_I are the values of the invariants in I;

r_{III}m s_{III} are the values of the invariants at the leading front of the wave in III;

r_V, s_V are the values of the invariants in V;

r_{II}, s_{II} -- values of the invariants in II;

r, s -- values of the invariants at the rear front of the right wave;

R, S -- values of the invariants at the rear front of the left wave;

D -- velocity of the wave traveling to the right;

$\bar{D}$ -- velocity of the wave traveling to the left;

$$M=\frac{D-u_0}{c_0}>1, \quad \bar{M}=\left|\frac{\bar{D}-\bar{u}}{c}\right|>1;$$

u_o, c_c -- quantities ahead of the front of the right wave;

$\bar{u}$, $\bar{c}$ -- quantities ahead of the front of the left wave;

r_i, s_i, M_i -- values of r, s, M at points Q_i;

R_i, S_i, $\bar{M}_i$ -- values of R, S, $\bar{M}$ at points P_i.

The following relations are valid at the front (Q_c, Q_1, Q_2, ...), (P_1, P_2, P_3, P_4, ...):

$$s-s_V=\psi(M),\ r-r_{III}=\varphi(M),\ s-s_{III}=\psi(M),\ r-r_V=\varphi(M), \tag{7}$$

and correspondingly $R-r_{II}=-\psi(\bar{M}),\ \ S-s_{II}=-\varphi(\bar{M})$.

The arcs $(P_1, P_2), (P_3, P_4), \ldots, (P_{2l-1}, P_{2l}), (Q_1, Q_2), (Q_3, Q_4), \ldots, (Q_{2l-1}, Q_{2l})$ correspond to the sections of the trajectories of the shock waves when they are traveling at constant velocity; the arcs $(P_0, P_1), (P_2, P_3), \ldots, (P_{2l}, P_{2l+1}), (Q_0, Q_1), (Q_2, Q_3), \ldots, (Q_{2l}, Q_{2l+1})$

correspond to the sections of variable velocity.

The following relations are valid:

$$S_i = s_i, \quad r_{i+2} = R_i. \tag{8}$$

At section Q_0Q_1, we have

$$M_1 < M_0, \quad s_1 < s_0. \tag{9}$$

Taking (7), (8), and (9) into account, we have

$$S_1 < s_0, \quad \bar{M}_1 > \bar{M}_0, \quad R_1 < R_0. \tag{10}$$

Thus, the left wave is strengthened at the section P_0P_1, and R and S are weakened. By virtue of (8), at section Q_2Q_3 we have

$$r_3 < r_2, \quad M_3 < M_2, \quad s_3 < s_2, \tag{11}$$

i.e., at section Q_2Q_3 the right wave becomes weaker. In the following the pattern is repeated: a wave traveling to the left is intensified, and that traveling to the right is attenuated; M_i monotonically decreases, $\bar{M}_i$ monotonically increases.

Let M_∞, $\bar{M}_\infty$ denote the limits M_i corresponding to $\bar{M}_i$ as $i \to \infty$. By virtue of (7) and (8), the following relations are valid:

$$r - r_V = \varphi(M_\infty), \quad s - s_V = \psi(M_\infty), \quad R - r_{II} = -\varphi(\bar{M}_\infty), \quad S - s_{II} = -\psi(M_\infty). \tag{12}$$

Formulas (12) are formulas for discontinuity decay for the states (II, V). Thus, we approve the coincidence of the asymptotic regime of interaction of a shock wave with a rarefaction wave and a flow derived from the discontinuity decay (II, V).

Note that for sufficiently large amplitude of the rarefaction wave III, the shock wave entering into it can be converted into a rarefaction wave, and configuration B can be replaced by configuration A.

3. Interaction of shock waves with waves traveling in barotropic polytropic gases. In the case of barotropic polytropic gases, Hugoniot's third condition is replaced by the condition of entropy constancy.

Hugoniot's conditions for a shock wave traveling to the right become

$$u_1 - u_0 = (1-h)\, c_0 \left(M_0 - \frac{1}{M_0}\right), \qquad M_0 = \frac{D - u_0}{c_0}, \tag{1}$$

$$\frac{p_1}{p_0} = (1+h) M_0^2 - h, \tag{2}$$

$$\frac{\rho_1^\gamma}{\rho_0^\gamma} = \frac{p_1}{p_0} = (1+h) M_0^2 - h, \tag{3}$$

$$\frac{c_1}{c_0} = \left(\frac{p_1}{\rho_1}\frac{\rho_0}{p_0}\right)^{1/2} = \left(\frac{\rho_1}{\rho_0}\right)^{\frac{\gamma-1}{2}} = \left[(1+h) M_0^2 - h\right]^{\frac{\gamma-1}{2\gamma}}. \tag{4}$$

Let us initially consider the impinging of a shock wave traveling from left to right at a rarefaction s-wave. From conditions (1)-(4) we have

$$r_1 - r_0 = c_0\left[(1-h)\left(M_0 - \frac{1}{M_0}\right) + \frac{2}{\gamma-1}\left(\frac{c_1}{c_0} - 1\right)\right] = c_0 F(M_0), \tag{5}$$

$$s_1 - s_0 = c_0\left[(1-h)\left(M_0 - \frac{1}{M_0}\right) - \frac{2}{\gamma-1}\left(\frac{c_1}{c_0} - 1\right)\right] = c_0 G(M_0). \tag{6}$$

Since $F(M_0)$ in formula (5) is a monotonically increasing function of M_0, and since c_0 satisfies the condition

$$\frac{\partial c_0}{\partial x} = -\frac{\gamma-1}{2}\frac{\partial u_0}{\partial x} < 0 \qquad (r = r_0 = \text{const}) \tag{7}$$

and

$$F(M_0)\frac{\partial c_0}{\partial x} + F'(M_0)\, c_0 \frac{\partial M_0}{\partial x} = 0,$$

then

$$\frac{\partial M_0}{\partial x} > 0. \tag{8}$$

i.e., the intensity of the shock wave becomes greater.

Clearly, behind the front of the shock wave we have a Riemann s-wave ($r = r_0$). We establish the sign of $\partial u/\partial x$ or, which amounts to the same thing, the sign of $\partial s/\partial x$ behind the shock wave front.

From relations (1), (5), and (7) we have

$$\frac{\partial u_1}{\partial x} = \left[1 + h\left(1 + \frac{1}{M_0^2}\right)\frac{F(M_0)}{F'(M_0)} - h\left(M_0 - \frac{1}{M_0}\right)\right]\frac{\partial u_0}{\partial x}. \qquad (9)$$

Since the approximation of the barotropic gases is valid for M_0 close to 1, the expression in the brackets is close to 1. Hence we have $\partial u_1/\partial x > 0$. (10) Thus, where IV is a rarefaction wave (Figure 2.62, a).

Let us now consider the impinging of a shock wave against the rarefaction r-wave. Then in (5) and (6) r_1 = constant, s_0 = constant. From (5) we have

$$r_1 = r_0 + c_0 F(M_0) = s_0 + c_0\left[\frac{4}{\gamma - 1} + F(M_0)\right]. \qquad (11)$$

Since $\partial c_0/\partial x > 0$, then from (11) there follows $\partial M_0/\partial x < 0$, (12)

i.e., the amplitude of the shock wave becomes weaker. Let us determine the sign of $\frac{\partial u_1}{\partial x} = \frac{1}{2}\frac{\partial s_1}{\partial x}$ behind the front of the shock wave.

For sufficiently small $|M_0 - 1|$, $G(M_0)$ is a monotonically increasing function of M_0. From (11) it follows that $\partial s_1/\partial x < 0$, i.e., the reflected wave IV is a compression wave.

Thus, we have seen that in the barotropic approximation, i.e., for sufficiently small $|M_0 - 1|$, the qualitative pattern of interaction is the same as for an isothermal gas.

Section IX. Analytic Solutions of One-Dimensional Gas Dynamics

1. General integral of an isentropic one-dimensional plane flow. A one-dimensional plane isentropic flow is described, as we know, by equations in the invariants (cf section II, subsection 7)

$$\left.\begin{aligned}\frac{\partial r}{\partial t}+\left[\frac{r+s}{2}+K(r-s)\right]\frac{\partial r}{\partial x}=0,\\ \frac{\partial s}{\partial t}+\left[\frac{r+s}{2}-K(r-s)\right]\frac{\partial s}{\partial x}=0,\end{aligned}\right\} \quad (1)$$

where

$$r=u+\int c\,d\ln\rho, \qquad s=u-\int c\,d\ln\rho, \quad (2)$$

and x is an Eulerian coordinate.

The function $K(r - s) = c$ is associated with the equation of state

$$p=p(S,\rho)=F(\rho) \quad (3)$$

by the relation

$$K(r-s)=g\left[h^{-1}(r-s)\right], \quad (4)$$

where

$$g=\sqrt{F'(\rho)}, \qquad h=2\int\sqrt{F'(\rho)}\frac{d\rho}{\rho}. \quad (5)$$

By the hodograph transformation, system (1) reduces to the linear:

$$\left.\begin{aligned}\frac{\partial x}{\partial s}-\left[\frac{r+s}{2}+K(r-s)\right]\frac{\partial t}{\partial s}=0,\\ \frac{\partial x}{\partial r}-\left[\frac{r+s}{2}-K(r-s)\right]\frac{\partial t}{\partial r}=0.\end{aligned}\right\} \quad (6)$$

Let us rewrite system (6) in the form

$$\left.\begin{aligned}\frac{\partial}{\partial s}\left[x-\left(\frac{r+s}{2}+K\right)t\right]&=-t\left(\frac{1}{2}-K'\right),\\ \frac{\partial}{\partial r}\left[x-\left(\frac{r+s}{2}-K\right)t\right]&=-t\left(\frac{1}{2}-K'\right).\end{aligned}\right\} \qquad (7)$$

From equation (7) it follows that the expression

$$dW=\left[x-\left(\frac{r+s}{2}+K\right)t\right]dr+\left[x-\left(\frac{r+s}{2}-K\right)t\right]ds \qquad (8)$$

is the exact differential of some function W, which we will call the potential function. From the expression for the exact differential we have

$$\left.\begin{aligned}x-\left(\frac{r+s}{2}+K\right)t&=\frac{\partial W}{\partial r},\\ x-\left(\frac{r+s}{2}-K\right)t&=\frac{\partial W}{\partial s}.\end{aligned}\right\} \qquad (9)$$

From equations (9), x and t can be expressed in terms of W_r, W_s:

$$t=-\frac{\frac{\partial W}{\partial r}-\frac{\partial W}{\partial s}}{2K(r-s)},\qquad x=-\frac{\left(\frac{r+s}{2}-K\right)\frac{\partial W}{\partial r}-\left(\frac{r+s}{2}+K\right)\frac{\partial W}{\partial s}}{2K(r-s)}. \qquad (10)$$

By virtue of relation (7) and (10), W satisfies the third-order equation

$$\frac{\partial^2 W}{\partial r\,\partial s}=\frac{\left(\frac{1}{2}-K'\right)}{2K(r-s)}\left(\frac{\partial W}{\partial r}-\frac{\partial W}{\partial s}\right). \qquad (11)$$

Thus, the potential function satisfies the special Darboux equation (cf Chapter One, Section XII, subsection 3)

$$\frac{\partial^2 W}{\partial x_1 \partial x_2} - f(x_1 + x_2)\left(\frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial x_2}\right) = 0, \quad (12)$$

where we put

$$x_1 = r, \quad x_2 = -s, \quad f = \frac{K' - \frac{1}{2}}{2K}. \quad (13)$$

It is not difficult to see that equation (12), by the substitution of

$$W = \varphi v, \ \varphi = ce^{\int f(\theta)\, d\theta}, \quad \theta = x_1 + x_2, \quad (14)$$

can be reduced to the form

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} = F(x_1 + x_2) v, \quad F = f^2 - f'. \quad (15)$$

Now let us consider those equations of state for which the total integral and the Riemann function are represented in closed form. Let us begin with polytropic gases for which p is an exponential function of ρ:

$$p = a^2 \rho^\gamma, \quad \gamma = \frac{c_p}{c_V} > 1. \quad (16)$$

In this case

$$K(r-s) = c = \frac{\gamma - 1}{4}(r - s). \quad (17)$$

From whence

$$f=\frac{-\frac{1}{2}+\frac{\gamma-1}{4}}{2\cdot\frac{\gamma-1}{4}(r-s)}=\frac{-\frac{1-\frac{\gamma-1}{2}}{\gamma-1}}{r-s}=\frac{\gamma-3}{2(\gamma-1)}\frac{1}{x_1+x_2}=-\frac{m}{x_1+x_2}. \tag{18}$$

The following property of the reduction is established by Darboux and enabling us to advance from one m value to another is valid for equation (12) with $f = -\frac{m}{x_1 + x_2}$

If W is the solution of equation (12) when $f = -\frac{m}{x_1 + x_2}$, then

$$v=LW, \quad L=\frac{1}{x_1+x_2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right), \tag{19}$$

is the solution of equation (12) when

$$f=-\frac{m'}{x_1+x_2}, \quad m'=m+1. \tag{20}$$

When $\gamma = +3$, m = 0 and W satisfies the equation

$$\frac{\partial^2 W}{\partial x_1 \partial x_2}=0, \tag{21}$$

which has the familiar general integral (D'Alembert's integral)

$$W=F(x_1)+G(x_2). \tag{22}$$

The following γ values correspond to integral positive values of m:

$$\gamma=\frac{2m+3}{2m+1}; \tag{23}$$

where m = 1, γ = 5/3 (monoatomic gas), when m = 2, γ = 7/5 (diatomic gas), and so on. By virtue of the property of reduction, corresponding to these m and γ values is the general integral

$$W(x_1, x_2) = L^m [F(x_1) + G(x_2)], \tag{24}$$

where the operator L is given by equality (19).

Let us show that the expression (24) for the general integral can be transformed to become

$$W(x_1, x_2) = \frac{\partial^{m-1}}{\partial x_1^{m-1}} \frac{F'(x_1)}{(x_1+x_2)^m} + \frac{\partial^{m-1}}{\partial x_2^{m-1}} \frac{G'(x_2)}{(x_1+x_2)^m}. \tag{25}$$

Let us assume in the following that $F'(x_1) = \Phi(x_1)$, $G'(x_2) = \Psi(x_2)$. (26)

Representation (25) obviously follows from (24) when $m = 1$. By induction, let us prove the equivalence of (24) and (25) for any m. Suppose

$$L^m F(x_1) = \frac{\partial^{m-1}}{\partial x_1^{m-1}} \frac{\Phi(x_1)}{(x_1+x_2)^m}. \tag{27}$$

Let us show that

$$L^{m+1} F(x_1) = \frac{\partial^m}{\partial x_1^m} \frac{\Phi(x_1)}{(x_1+x_2)^{m+1}}. \tag{28}$$

Using the assumption (27), we have

$$L^{m+1}F(x_1) = LL^m F(x_1) = L\left[\frac{\partial^{m-1}}{\partial x_1^{m-1}} \frac{\Phi(x_1)}{(x_1+x_2)^m}\right] =$$

$$= \frac{1}{x_1+x_2} \frac{\partial^m}{\partial x_1^m} \frac{\Phi(x_1)}{(x_1+x_2)^m} - \frac{m}{x_1+x_2} \frac{\partial^{m-1}}{\partial x_1^{m-1}} \frac{\Phi(x_1)}{(x_1+x_2)^{m+1}}. \tag{29}$$

After this it is not difficult to see that equality (28) is equivalent to the following:

$$(\theta H)^{(m)} = \theta H^{(m)} + mH^{(m-1)}, \tag{30}$$

where we assume, that for fixed x_2,

$$\theta = x_1 + x_2, \quad \frac{\partial}{\partial x_1} = \frac{d}{d\theta}, \quad f^{(m)} = \frac{d^m}{d\theta^m} f, \quad H = \frac{\sigma}{\theta^{m+1}} \tag{31}$$

Thus, we have shown that the general integral for equation (12) where $f = -\frac{m}{x_1 + x_2}$ is of the form

$$W(x_1 + x_2) = \frac{\partial^{m-1}}{\partial x_1^{m-1}}\left[\frac{\Phi(x_1)}{(x_1 + x_2)^m}\right] + \frac{\partial^{m-1}}{\partial x_2^{m-1}}\left[\frac{\Psi(x_2)}{(x_1 + x_2)^m}\right] \tag{32}$$

If we take x_1, x_2 to be complex numbers, and W, Φ, Ψ to be analytic functions of their variables, then by employing the familiar Cauchy's representation for derivatives of analytic functions, formula (32) can be represented as (cf [19])

$$\begin{aligned} W(x_1, x_2) &= D_1^{m-1}\frac{\Phi(x_1)}{(x_1 + x_2)^m} + D_2^{m-1}\frac{\Psi(x_2)}{(x_1 + x_2)^m} = \\ &= \frac{(m-1)!}{2\pi i}\left[\oint_{C_1} \frac{\Phi(t)}{(t + x_2)^m (t - x_1)^m} dt + \oint_{C_2} \frac{\Psi(t)}{(x_1 + t)^m (t - x_2)^m} dt\right]. \end{aligned} \tag{33}$$

In formula (33) $D_i = \partial/\partial x_i$; the contour C_1 is taken in the plane of the complex variable x_1; x_2 is considered as a complex parameter; the contour C_2 is taken in the x_2 plane; x_1 is the parameter.

Formula (33) is meaningful also for fractional m, if in it we replace (m - 1)! by $\Gamma(m)$; then operators D_1^{m-1}, D_2^{m-1} take on the significance of fractional derivatives first introduced by Riemann [20] and L..ouville [21].

Let us now proceed to the problem of defining the Riemann function for equation (12) when $f = -\frac{m}{x_1 + x_2}$.

We will employ equation (15), which at this f value becomes

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} = \frac{a}{\theta^2} v, \quad \theta = x_1 + x_2, \quad a = m(m-1). \tag{34}$$

For the Riemann function $R(\xi_1, \xi_2; x_1, x_2)$ of equation (34), the following representation is valid (cf Chapter One, Section XIII, Subsection 4):

$$R(\xi_1, \xi_2; x_1, x_2) = (1-\xi)^m F(m, m, 1, \xi). \qquad (35)$$

where

$$\xi = \frac{(x_1-\xi_1)(x_2+\xi_2)}{(x_1-\xi_2)(x_2+\xi_1)} \qquad (36)$$

and

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1\gamma}x + \dots + \frac{\alpha(\alpha+1)\dots(\alpha+k-1)\beta(\beta+1)\dots(\beta+k-1)}{k!\gamma(\gamma+1)\dots(\gamma+k-1)}x^k + \dots \qquad (37)$$

is the familiar Gauss hypergeometric series.

Note that for integral m series (37) becomes a polynomial. We can show that for integral m equation (34) admits of an m-th order differential relationship and, conversely, if any equation of the form (15) admits of an m-th order differential relationship, then $F(x_1 + x_2)$ satisfies the equation among whose solutions we have the function $\frac{m(m-1)}{(x_1+x_2)^2}$. Let clarify this assertion with an example. Let us consider equation (15) admitting of a second-order relationship. Then, as was shown in Chapter One, Section XII, $F(x_1 + x_2)$ satisfies the equation

$$\frac{\partial^2 \ln F}{\partial x_1 \partial x_2} = F, \qquad (38)$$

which in our case becomes $FF'' - F'^2 = F^3$, (39)

where the stroke denotes a derivative with respect to $\theta = x_1 + x_2$.

The function

$$F(\theta)=\frac{m(m-1)}{\theta^2}$$

satisfies equation (39) when $m = 2$. The general integral of equation (38) is of the form

$$\theta+C_2=\begin{cases}\frac{1}{\sqrt{C_1}}\ln\frac{\sqrt{C_1+2F}-\sqrt{C_1}}{\sqrt{C_1+2F}+\sqrt{C_1}}, & C_1>0,\\ \frac{2}{\sqrt{-C_1}}\operatorname{arctg}\sqrt{\frac{C_1+2F}{-C_1}}, & C_1<0.\end{cases} \quad (40)$$

Using the arbitrary constants C_1, C_2, using function (40) we can approximate functions $F(\theta)$ and obtain good approximations. Thus, G. A. Dombrovskiy obtained approximations of Darboux's equation (cf [22]).

In the next subsection we show how a knowledge of the general integral of the equation enables us to solve several problems in gas dynamics that lead to the interpretation of simple waves.

2. Problems of the interaction of elementary solutions. We consider two problems:

a) the interaction of two Riemann waves; and

b) the incidence of a Riemann wave against the interface of two media.

Suppose at instant $t = t_0$, a centered r-wave begins to propagate from the point $x = x_0$, and at the instant $t = t_1$ a centered s-wave from the point $x = x_1$. We can assume that the centered waves result from the departure of a piston from a gas with constant velocity (Figure 2.65a).

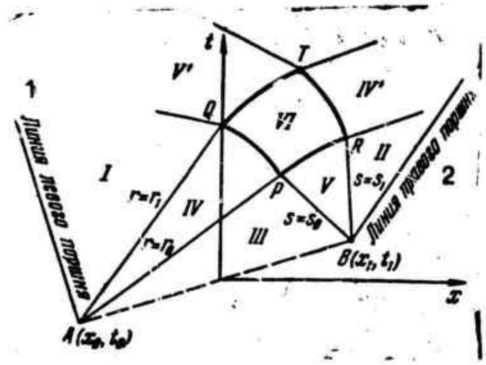


Figure 2.65a

KEY:

1) Line of left piston 2) Line of right piston

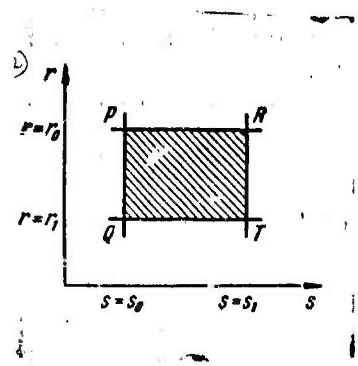


Figure 2.65b

In zone I-V the solution is of the form:

$$I:\quad r = r_1, \quad s = s_0, \quad r_1 < r_0; \tag{1}$$

$$II:\quad r = r_0, \quad s = s_1, \quad s_1 > s_0; \tag{2}$$

$$III:\quad u = u_0 = 0, \quad c = c_0, \quad r = r_0 = \frac{2}{\gamma - 1} c_0, \quad s = s_0 = -\frac{2}{\gamma - 1} c_0; \tag{3}$$

$$IV:\quad \frac{dx}{dt} = \frac{x - x_0}{t - t_0} = \frac{x + h}{t + \tau} = \alpha r + \beta s_0; \tag{4}$$

$$V:\quad \frac{dx}{dt} = \frac{x - x_1}{t - t_1} = \frac{x - h}{t - \tau} = \alpha s + \beta r_0. \tag{5}$$

In zone VI, which is the zone of interference and is mapped in the quadrangle PQTR of the r, s-plane (Figure 2.65b), the following representation is valid:

$$\left.\begin{aligned} x-(\alpha r+\beta s)t &= \frac{\partial W}{\partial r}, \\ x-(\alpha s+\beta r)t &= \frac{\partial W}{\partial s}. \end{aligned}\right\} \qquad (6)$$

where W satisfies the equation

$$\frac{\partial^2 W}{\partial r\,\partial s}=\frac{\kappa}{r-s}\left(\frac{\partial W}{\partial r}-\frac{\partial W}{\partial s}\right), \quad \kappa=\frac{3-\gamma}{2(\gamma-1)}. \qquad (7)$$

Using (4) and (5), we can impose boundary conditions constituting the Goursat's problem for W in the quadrangle PQTR at the size PR and PQ:

$$\left.\begin{aligned} PR:\ & r=r_0, & \frac{\partial W}{\partial s} &= k-\tau(\alpha s+\beta r_0); \\ PQ:\ & s=s_0, & \frac{\partial W}{\partial r} &= -k+\tau(\alpha r+\beta s_0). \end{aligned}\right\} \qquad (8)$$

Let us solve this problem for $\gamma = 5/3$. The general integral of equation (7) becomes

$$W=\frac{f(r)+g(s)}{r-s}. \qquad (9)$$

Boundary conditions (8) lead to the relations

$$\left.\begin{aligned} \frac{(r_0-s)g'(s)+g(s)+f(r_0)}{(r_0-s)^2} &= k-\tau(\alpha s+\beta r_0), \\ \frac{f'(r)(r-s_0)-f(r)-g(s_0)}{(r-s_0)^2} &= -k+\tau(\alpha r+\beta s_0). \end{aligned}\right\} \qquad (10)$$

Let us suggest for simplicity that $\tau = 0$, i.e., the retreat of the pistons occurs simultaneously. Then relations (10) become

$$\begin{aligned} (r_0-s)g'(s)+g(s)+f(r_0) &= k(r_0-s)^2, \\ (r-s_0)f'(r)-f(r)-g(s_0) &= -k(r-s_0)^2. \end{aligned} \qquad (11)$$

i.e., $f(r)$ and $g(s)$ are second-order polynomials of their variables. Thus, considering that W defined with them accuracy up to the additive constant, we can put

$$W = \frac{Ar^2 + Bs^2 + C(r+s) + D}{r-s}. \tag{12}$$

Satisfying boundary conditions (8), we find A, B, C, D, obtaining finally for W the expression

$$W = \frac{-k(r^2+s^2) + [illegible]}{r-s}. \tag{13}$$

Formulas (6) and (13) define the solution zone VI. In the zones IV and V we have traveling Riemann waves, already encountered, which are difficult to calculate.

Now let us consider the interaction of a traveling wave with contact boundary. From the point $x = 0$, $t = 0$ a centered Riemann's s-wave is propagated, which strikes the contact boundary at $t = 0$ at the point $x = -h$ (Figure 2.66a). We can consider the case when both media are polytropic gases with $\gamma = 5/3$. The pattern of motion is described by the formulas:

region I: $r = r_1,\ s = s_1 = -r_1,\ c = c_1,\ S = S_1,\ u = u_1 = 0;$ (14)

region II: $r = r_1,\ \alpha s + \beta r_1 = \xi = \frac{x}{t};$ (15)

region III: $r = r_1,\ s = s_2;$ (16)

region IV: $r = r_2,\ s = s_2 = -r_2,\ c = c_2,\ S = S_2.$ (17)

IV is the region of interference of the incident and reflected waves, regions II and V are traveling waves, and regions VI and VII are uniform flow waves.

Let us first of all determine the flow in region IV (Figure 2.66b).

Using formulas (6), we can impose for the function $W(r, s)$ the boundary conditions at the line AC:

$$[illegible] \qquad (18)$$

Hence, by using the formula for the general integral (9), we obtain

$$g'(s)(r_0 - s) + f(r_0) + g(s) = 0. \qquad (19)$$

From equation (19) it follows that $g(s)$ is a linear function of s. Since W is defined with an accuracy up to the additive constant, the sum $f(r) + g(s)$ can contain the term $C(r - s)$ with arbitrary constant C.

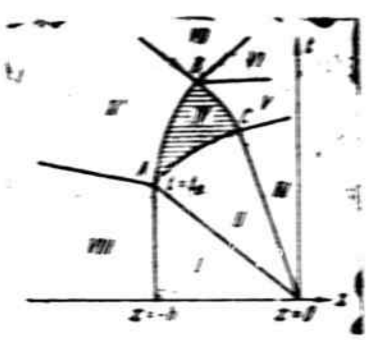


Figure 2.66a

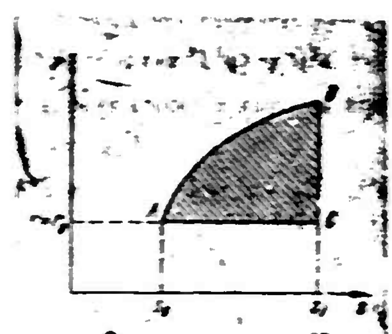


Figure 2.66b

By selecting the constant C, we can put $g(s)$ = constant = B. Transposing B in $f(r)$, we finally get

$$W(r, s) = \frac{f(r)}{r - s}, \quad f(r_0) = 0. \qquad (20)$$

Now let us consider the boundary condition of the contact boundary AB. We initially determine a line AB in the hodograph plane. Noting the constancy of invariant $r = r_0$ to the left of boundary, from the conditions at the contact boundary we find that the quantities r, s at the right side of the boundary are associated by the linear relation

$$s = \frac{1+x}{1-x} r - \frac{2x}{1-x} r_1, \quad x = \left(\frac{A_0}{A_1}\right)^{\frac{1}{\gamma}}, \tag{21}$$

where A_0, A_1 are entropy constants of the gases to the right, and to the left, respectively of the boundary.

The boundary condition at AB in the x, t-plane is of the form

$$dx/dt = (r - s)/2. \tag{22}$$

Using formulas (6) and (21), we get

$$[(\beta - \alpha)(r - s)(\Psi_{rr} + \Psi_{rs}) + \Psi_r - \Psi_s] dr + $$
$$+ [(\beta - \alpha)(r - s)(\Psi_{rs} + \Psi_{ss}) + \Psi_r - \Psi_s] ds = 0. \tag{23}$$

$$ds = \frac{1+x}{1-x} dr. \tag{24}$$

Hence, noting that from (21) it follows that

$$r - s = \frac{2x}{x-1}(r - r_1), \tag{25}$$

we obtain an equation for $f(r)$:

$$x^2 (r - r_1)^2 f'' + x(3 - x)(r - r_1) f' + 3(1 - x) f = 0, \tag{26}$$

whose general integral is of the form

$$f(r) = C_1 (r - r_1)^{\lambda_1} + C_2 (r - r_1)^{\lambda_2}, \tag{27}$$

where C_1, C_2 are arbitrary constants, and λ_1, λ_2 are the roots of the characteristic equation

$$x^2 \lambda^2 + x(3 - 2x)\lambda + 3(1 - x) = 0. \tag{28}$$

The constants C_1, C_2 are determined from the conditions

$$f(r_0) = C_1(r_0 - r_1)^{\lambda_1} + C_2(r_0 - r_1)^{\lambda_2} = 0. \tag{29}$$

$$f'(r_0) = C_1\lambda_1(r_0 - r_1)^{\lambda_1 - 1} + C_2\lambda_2(r_0 - r_1)^{\lambda_2 - 1} = -2r_0(k_0 + c_0 l_0). \tag{30}$$

We note in conclusion that the passing wave II' is a traveling rarefaction wave; the wave V can be either a rarefaction wave, or a compression wave.

For the case $r_B < r_C$ we have a rarefaction wave, and in the case $r_B > r_C$ -- a compression wave.

As shown by the investigations in Section III, subsection 5, the reflected wave V will be a rarefaction wave when $\mathcal{H} > 1$, and a compression wave when $\mathcal{H} < 1$. For the case $\mathcal{H} > 1$ the regions II' and V will be two rarefaction waves and the resulting solution as $h \to 0$ will tend to the solution of the corresponding problem of discontinuity decay (cf Section VIII, subsection 2).

This solution was shown us by N. N. Anuchina.

The general solution to the problem of the incidence of a Riemann wave against the interface of two media is to be found in the work by Laut [23].

3. Plane one-dimensional flows with variable entropy. The Martin method. Martin (cf [24]) succeeded in reducing the equations of gas dynamics to the Monge-Ampere equation and then was able to employ the method of the intermediate integral (cf Chapter One, Section XII) and to obtain in this way a generalization of Riemann invariants for the case of flow with variable entropy.

Further investigations of Martin [25], Ludford [26], and Yu. S. Zav'yalov [27] afforded a total determination of the class of equations of state and entropy distribution functions for which the method of the intermediate integral is applicable.

We will begin from equations in Lagrangian coordinates

$$\partial u/\partial t + \partial p/\partial q = 0;$$

$$a)\ \partial V/\partial t - \partial u/\partial q = 0, \qquad b)\ \partial S/\partial t = 0. \tag{1}$$

Corresponding to equations (1a) and (1b) are the potentials φ_1 and φ_2 (cf Chapter One, Section V, subsection 3):

$$V\,dq + u\,dt = d\varphi_1. \tag{2}$$

$$u\,dq - p\,dt = d\varphi_2. \tag{3}$$

Martin introduces the potential function ξ by means of the equality

$$d\xi = u\,dq + t\,dp = d\varphi_2 + d(pt). \tag{4}$$

From equality (4) we have

$$\frac{\partial \xi}{\partial q} = u, \qquad \frac{\partial \xi}{\partial p} = t. \tag{5}$$

$$du = \xi_{qp}\,dp + \xi_{qq}\,dq, \qquad dt = \xi_{pp}\,dp + \xi_{pq}\,dq. \tag{6}$$

$$\frac{\partial u}{\partial p} = \xi_{qp}, \qquad \frac{\partial u}{\partial q} = \xi_{qq}. \tag{7}$$

$$\frac{\partial t}{\partial p} = \xi_{pp}, \qquad \frac{\partial t}{\partial q} = \xi_{pq}. \tag{8}$$

Substituting (6) into (2), we find

$$d\varphi_1 = (V + u\xi_{pq})\,dq + u\xi_{pp}\,dp. \tag{9}$$

The condition of the exact differential for equation (9), with (7) taken into account, leads to the Monge-Ampere equation for the function of ξ:

$$\xi_{qq}\xi_{pp} - \xi_{pq}^2 = \frac{\partial V(p, S)}{\partial p}. \tag{10}$$

For a certain equation of state $V = V(p, S)$ and for assigned entropy distribution $S = S(q)$ in the right side of equation (10), we have the specific function

$$\frac{1}{\rho^2} = f(p, q) = -\frac{\partial V(p, S(q))}{\partial p} \tag{11}$$

and equation (10) takes on the form of the equation

$$\xi_{\psi\psi}\xi_{tt} - \xi_{\psi t}^2 + f(p, q) = 0 \tag{12}$$

which we studied in Chapter One, Section XII.

Note that the trace of p and q as impending variables signifies that p is not a function of q or, which amounts to the same, of S, i.e., the thermodynamic flow parameters constitute a two-dimensional manifold.

The exceptional case when p = p(S) will be treated by us later.

Martin's method consists in seeking a family of solutions of equation (12) with a one-functional arbitrary choice satisfying the additional relation

$$\Phi(p, q, t, \xi_\psi, \xi_t) = \varphi(p, q, \xi, u, t) = \text{const.} \tag{13}$$

Relation (13) is the intermediate integral of the Monge-Ampere equation (12).

Clearly, equations (12) and (13) are compatible only for these specific function f(p, q). A complete analysis of compatibility made by Martin [25], Ludford [26], and independently by Yu. S. Zav'yalov [27] showed that intermediate integral exists for the following functions f(p, q):

$$f(p, q) = \sqrt{F'(a_1 p + a_2 q)}, \tag{14}$$

$$f(p, q) = \frac{1}{(q + a_2)^2}\sqrt{F'\left(\frac{p + a_1}{q + a_2}\right)}. \tag{15}$$

In the first case

$$\varphi = a_2 t - a_1 u \pm g(a_1 p + a_2 q). \tag{16}$$

In the second case

$$\varphi = \xi - t(p + a_1) - u(q + a_2) \pm g\left(\frac{p + a_1}{q + a_2}\right). \tag{17}$$

In both cases $g(\theta)$ is associated with $F(\theta)$ by the relation

$$g(\theta)=\sqrt{F(\theta)}. \qquad (18)$$

Let us consider the case of the polytropic gas when

$$V = A^{\frac{1}{\gamma}} p^{-\frac{1}{\gamma}}, \quad A^2 = A^2(S), \qquad (19)$$

$$f=\sqrt{-\left(\frac{\partial V}{\partial p}\right)_S} = A^{\frac{1}{2\gamma}} \frac{1}{\sqrt{\gamma}} p^{-\frac{\gamma+1}{2\gamma}}. \qquad (20)$$

If entropy S is constant and f is a function solely of p, then we can use the representation (14) when $\alpha_2 = 0$. Comparing (14) and (20) we find:

$$\alpha_1 = 1, \quad F = A^{\frac{1}{\gamma}} p^{-\frac{1}{\gamma}}, \quad g = \frac{2}{\gamma-1}\sqrt{\gamma A^{\frac{1}{\gamma}}}\, p^{\frac{\gamma-1}{2\gamma}} = \frac{2}{\gamma-1} c. \qquad (21)$$

Then we have

$$\varphi = -u \pm \frac{2}{\gamma-1} c, \qquad (22)$$

i.e., the intermediate integral is none other than a Riemann invariant. This justifies the term "generalized Riemann invariant" introduced by Martin and Ludford for intermediate integral (13) of equation (10).

Now let us consider the case of variable entropy. We will show that among the flows with constant Riemann generalized invariant there are flows adjoining the quiescent region across the shock wave. Let us use representation (15), (17) for this purpose. We will assume that the shock wave travels with respect to a zero background ($\rho_0 = 1$, $u_0 = 0$) and is strong, i.e., formulas (4.7.1) - (4.7.4) are valid for quantities behind the wave front:

$$\left.\begin{aligned} p &= \frac{2}{\gamma+1}\rho_0 D^2 = \frac{2}{\gamma+1} D^2, \\ \rho_1 &= \frac{\gamma+1}{\gamma-1}\rho_0 = \frac{\gamma+1}{\gamma-1}, \\ u &= \frac{2}{\gamma+1} D, \end{aligned}\right\} \qquad (23)$$

where

$$D = \frac{dx}{dt} = \frac{dq}{dt}. \qquad (24)$$

Using (18) and (19), for A(S) and g(θ) we obtain the expressions

$$A^{\frac{2}{\gamma}} = c^2 \gamma q^{\frac{1-3\gamma}{\gamma}} \qquad (25)$$

$$g = \frac{2\gamma}{\gamma-1} c \left(\frac{p}{q}\right)^{\frac{\gamma-1}{2\gamma}}, \qquad (26)$$

where c is some constant, α_1, α_2 are assumed equal to zero. Suppose that at the instants t = 0, a shock wave begins to be propagated in the quiescent gas traveling according to the exponential law $q = Ct^{\alpha}$. (27)

Let us find the index α to which corresponds the constant value of the function φ from (17) at the wave front. Then φ will be identically constant behind the wave front and will also have a constant generalized Riemann invariant.

From (17), taken (3) into account, we find

$$d\varphi = s\,dq - p\,dt = d(sq) \mp dg\left(\frac{p}{q}\right), \qquad (28)$$

whence it follows that

$$q\,ds + p\,dt \mp dg\left(\frac{p}{q}\right) = 0. \qquad (29)$$

By virtue of assumption (27), relation (29) at the rear front of the wave, after dividing by dt, becomes

$$\left(\frac{2}{\gamma+1}\alpha(\alpha-1)C^2 + \frac{2}{\gamma+1}C^2\alpha^2\right)t^{2\alpha-2} + Bt^{\frac{\gamma-1}{2\gamma}(\alpha-2)-1} = 0, \qquad (30)$$

where

$$B = \mp c\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma-1}{2\gamma}} (C\alpha^2)^{\frac{\gamma-1}{2\gamma}} (\alpha-2). \qquad (31)$$

Hence it follows that

$$\alpha = \frac{2}{3\gamma+1}. \qquad (32)$$

Generalized Riemann waves can be considered as solutions with differential relationship (Chapter One, Section XII)

$$\Phi\ (q,\ t,\ \varphi_1,\ \varphi_2,\ u,\ p,\ S(q)) = 0, \tag{33}$$

where φ_1, φ_2 are the potentials of system (1).

That was indicated, the definition of the function Φ reduces to integration of a linear homogeneous system. Since here the independent variables are q and t, the differential relation (33) of equation (1) also has the case excluded from consideration by Martin, where $p = p(S)$. This case will be treated in the next subsection.

4. Equation of hydrodynamic surface. For the case of ideal gas equations (9.3.1) will become

$$\frac{\partial u}{\partial t}+\frac{\partial p}{\partial q}=0, \quad \psi p^{-\varkappa}\frac{\partial p}{\partial t}+\frac{\partial u}{\partial q}=0, \quad \frac{\partial \psi}{\partial t}=0, \tag{1}$$

$$\varkappa=\frac{\gamma+1}{\gamma}, \quad \psi=\frac{1}{\gamma}A^{\frac{2}{\gamma}}. \tag{2}$$

We obtain an equation in the hodograph space u, p, ψ for system (1). Employing the algorithm given in Section XII of Chapter One for the surface

$$\psi = \psi(p,\ u), \tag{3}$$

we arrive at the following equation:

$$\left.\begin{aligned} &\frac{\partial q}{\partial p}=\frac{\partial \psi}{\partial p}\mu, \quad \frac{\partial q}{\partial u}=\frac{\partial \psi}{\partial u}\mu, \\ &\frac{\partial t}{\partial p}=-\psi p^{-\varkappa}\frac{\partial \psi}{\partial u}\mu, \quad \frac{\partial t}{\partial u}=-\frac{\partial \psi}{\partial p}\mu, \end{aligned}\right\} \tag{4}$$

where μ is an undetermined cofactor.

Conditions for the integrability of equations (4) reduce to equations for μ:

$$\left.\begin{aligned}\psi p^{-\varkappa}\frac{\partial\psi}{\partial u}\frac{\partial\ln\mu}{\partial u}-\frac{\partial\psi}{\partial p}\frac{\partial\ln\mu}{\partial p}=\frac{\partial^2\psi}{\partial p^2}-\frac{\partial\left(\psi p^{-\varkappa}\frac{\partial\psi}{\partial u}\right)}{\partial u}=-\Phi,\\ \frac{\partial\psi}{\partial p}\frac{\partial\ln\mu}{\partial u}-\frac{\partial\psi}{\partial u}\frac{\partial\ln\mu}{\partial p}=0.\end{aligned}\right\} \tag{5}$$

Solving (5) relative to $\frac{\partial\ln\mu}{\partial u}$ and $\frac{\partial\ln\mu}{\partial p}$, we find

$$\frac{\partial\ln\mu}{\partial u}=\frac{\partial\psi}{\partial u}\varphi,\qquad \frac{\partial\ln\mu}{\partial p}=\frac{\partial\psi}{\partial p}\varphi. \tag{6}$$

where

$$\varphi=\frac{\Phi}{\Delta},\quad \Delta=\left(\frac{\partial\psi}{\partial p}\right)^2-\psi p^{-\varkappa}\left(\frac{\partial\psi}{\partial u}\right)^2. \tag{7}$$

Finally, the conditions for the integrability of equations (6) yield the following third-order equation:

$$\begin{vmatrix}\frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial p}\\ \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial p}\end{vmatrix}=0. \tag{8}$$

Integrating equation (8), we arrive at the second-order equation

$$\varphi=\frac{\frac{\partial\left(\psi p^{-\varkappa}\frac{\partial\psi}{\partial u}\right)}{\partial u}-\frac{\partial^2\psi}{\partial p^2}}{\left(\frac{\partial\psi}{\partial p}\right)^2-\psi p^{-\varkappa}\left(\frac{\partial\psi}{\partial u}\right)^2}=f(\psi), \tag{9}$$

where $f(\psi)$ is an arbitrary function of ψ.

It is not difficult to see that equation (9) admits of the following integrals:

$$\psi = g(p), \tag{10}$$

$$u = h(\psi), \tag{11}$$

where g and h are arbitrary functions of their arguments. The solutions corresponding to them are of the form

$$u = Ct + C_1, \quad p = -Cq + C_2, \quad \psi = g(p). \tag{12}$$

$$\left.\begin{aligned} \int \frac{1}{\psi} h'(\psi)\, d\psi &= Cq + C_1, \\ \frac{1}{1-\varkappa} p^{1-\varkappa} &= -Ct + C_2, \quad u = h(\psi). \end{aligned}\right\} \tag{13}$$

Relation (10) gives at once the exceptional case when transformation to the Monge-Ampere equation is impossible.

Solutions of the form (12), (13) will obtain by K. P. Stanyukovich [28].

If $\psi = \psi(u, p)$ is the solution of equation (9), the solution of the initial equation is obtained by successive integration of the wholly integrable systems (6) and (4).

The formulas restoring the solution are of the form

$$\ln \mu = \int f(\psi)\, d\psi, \quad \mu = e^{\int f(\psi)\, d\psi} = F(\psi). \tag{14}$$

$$q = \int \mu \left(\frac{\partial \psi}{\partial p} dp + \frac{\partial \psi}{\partial u} du \right) = \int \mu\, d\psi = \int F(\psi)\, d\psi. \tag{15}$$

$$t = -\int F(\psi) \left[\left(\psi p^{-\varkappa} \frac{\partial \psi}{\partial u} \right) dp + \frac{\partial \psi}{\partial p} du \right]. \tag{16}$$

Corresponding to the given surface $\Phi(\psi, p, u) = 0$, by virtue of formulas (14)-(16), is the solution of equations (1), uniquely with an accuracy up to the constants. The uniqueness is violated in the event of the function ψ = constant, to it corresponds a family of solutions dependent on two arbitrary functions of the same argument.

5. Solution with arbitrary constant choice of symmetrical one-dimensional equations in gas dynamics. We will use general equations of one-dimensional flows

$$\left.\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}&=0,\\ \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{\nu\rho u}{x}&=0,\\ \frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}&=0, \end{aligned}\right\} \qquad (1)$$

where the parameter ν is equal to 0 in the plane, 1 in the cylindrical, and 2 in the spherical cases.

Below we will assume that the equation of state is of the form

$$p = a^2(S)\rho^{\gamma} \qquad (2)$$

In this case system (1) can be transformed to become

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{p}{\rho}\frac{\partial \ln p}{\partial x}&=0,\\ \frac{\partial \ln \rho}{\partial t}+u\frac{\partial \ln \rho}{\partial x}+\frac{\partial u}{\partial x}+\frac{\nu u}{x}&=0,\\ \frac{\partial \ln p}{\partial t}+u\frac{\partial \ln p}{\partial x}+\gamma\left[\frac{\partial u}{\partial x}+\frac{\nu u}{x}\right]&=0. \end{aligned}\right\} \qquad (3)$$

In the plane case ($\nu = 0$) of system (1), (3) we have coefficients dependent only on unknown functions, and we can admit **similitude** and **shear** transformations with respect to the independent variables x and t. In the cylindrical and spherical cases, only **similitude** transformation with respect to x, t is possible. Converting to logarithmic scales with respect to x, t and to the dimensionless velocity

$$\tau=\ln t,\quad \lambda=\ln x,\quad u=\frac{x}{t}U,\quad c=\frac{x}{t}C, \qquad (4)$$

system (1) can be written in the form*)

$$\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial \lambda}+C^2\frac{\partial \varphi}{\partial \lambda}+C^2\frac{\partial \psi}{\partial \lambda}+U^2-U=0, \qquad (5)$$

$$\frac{\partial \varphi}{\partial \tau}+U\frac{\partial \varphi}{\partial \lambda}+\frac{\partial U}{\partial \lambda}+(\nu+1)U=0, \qquad (6)$$

$$\frac{\partial \psi}{\partial \tau}+U\frac{\partial \psi}{\partial \lambda}=0. \qquad (7)$$

/*) on following page/

where $\varphi = \ln \rho, \quad \psi = \frac{1}{\gamma} \ln a^2.$

In system (5)-(7), for any γ the coefficients do not depend on the arguments τ and λ, and consequently, as shown by the analysis made in Chapter One, Section XII, it has simple waves

$$U = U(\theta), \quad \varphi = \varphi(\theta), \quad \psi = \psi(\theta),$$
$$\theta = \theta_0 + a_1\lambda + a_2\tau. \tag{8}$$

Since the coefficients of system (5)-(7) do not depend on φ, ψ, then solutions of the form

$$U = U(\theta), \quad \varphi = \varphi(\theta) + l\tau, \quad \psi = \psi(\theta) - \frac{\gamma - 1}{\gamma} l\tau, \tag{9}$$

are possible, where l is an arbitrary constant.

Converting to the variables u, ρ, p, we obtain the representation

$$u = \frac{x}{t} U(\xi) = t^{-\left(\frac{a_2}{a_1}+1\right)} \left(\frac{\xi}{\xi_0}\right)^{\frac{1}{a_1}} U(\xi), \quad p = \rho_0 \left(\frac{x}{t}\right)^2 t^l P(\xi) =$$
$$= \rho_0 t^{l-2\left(\frac{a_2}{a_1}+1\right)} \left(\frac{\xi}{\xi_0}\right)^{\frac{2}{a_1}} P(\xi), \quad \rho = \rho_0 t^l R(\xi), \tag{10}$$

where

$$\xi = e^{\theta} = \xi_0 x^{a_1} t^{a_2}, \quad \xi_0 = e^{\theta_0}. \tag{11}$$

Solutions of the form (10) are called self-model (g**)

Clearly, the indexes a_1 and a_2 are determined with an accuracy up to the multiplier, so that only their ratio $a_1/a_2 = -k$ is the essential parameter.

The system of equations for $U(\xi)$, $R(\xi)$, $P(\xi)$ corresponding to system (3) is of the form

---

*) Equations of gas dynamics were considered in this form by K. P. Stanyukovich [28].

[**) on following page]

$$
\left.
\begin{aligned}
&\text{a)}\quad (a_1U + a_2)\frac{d \ln \dots}{d\xi} + \dots \text{[illegible]} \\
&\text{b)}\quad (a_1U + a_2)\frac{d\ln R}{d\xi} + a_1\frac{dU}{d\xi} = -k(\nu+1)U \dots \\
&\text{c)}\quad (a_1U + a_2)\frac{d\ln P}{d\xi} + \gamma a_1\frac{dU}{d\xi} = -k(\nu+1)\gamma U + 2(U - l + 2)
\end{aligned}
\right\}
\qquad (12)
$$

A linear combination of equations (12b) and (12c) leads to an integral (so-called integral of adiabaticity)

$$
\begin{gathered}
\ln P - \varkappa \ln R + (\gamma - \varkappa)\ln\left(\frac{1}{k} - U\right) + [(\nu+1)(\gamma-\varkappa)+2]\frac{\theta}{\dots} = \ln C_1 \\
\varkappa = \frac{(\nu+1)\gamma + 2 + k(l-2)}{k + (\nu+1)},
\end{gathered}
\qquad (13)
$$

which enables us for any k and l to reduce the number of equations (12) to two.

Let us show that for certain k and l system (12) admits of yet another integral.

Equations (1) have as the law of conservation the relation

$$
\frac{\partial}{\partial t}\left[\rho x^{\nu}\left(\frac{1}{\gamma-1}\frac{p}{\rho} + \frac{1}{2}u^2\right)\right] + \frac{\partial}{\partial x}\left[\rho x^{\nu} u\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{1}{2}u^2\right)\right] = 0, \qquad (14)
$$

expressing the law of conservation of energy. To corresponse the self-modeling equation

$$
\frac{dA}{d\xi} + [\nu + 3 + k(l-3)]B - k\xi\frac{dB}{d\xi} = 0, \qquad (15)
$$

where

$$
\left.
\begin{aligned}
A &= \xi^{l-2}\left[\frac{P}{\gamma-1} + \frac{PU^2}{2}\right], \\
B &= \xi^{l-3}U\left[\frac{\gamma}{\gamma-1}P + \frac{PU^2}{2}\right], \quad \xi = t x^{-k}.
\end{aligned}
\right\}
\qquad (16)
$$

**) Self-modeling solutions of one-dimensional gas dynamics as solutions that are invariant relative to certain transformation groups were first considered by Bechert [illegible].

Given the condition

$$[\text{illegible}] \tag{17}$$

equation (15) admits of the integral (energy integral)

$$A - k\,(\xi B) = \text{constant.} \tag{18}$$

When the energy integral is present, system (12) reduces to a single equation.

We give a brief review of certain problems leading to self-modeling solutions. Fundamentally these solutions describe flows adjoining the quiescent region through the shock wave or through a weak discontinuity. Here the characteristic circumstance is the zero-equality of the pressure in the quiescent region. Then the shock wave can be considered as infinitely strong, and Hugoniot's conditions acquires a homogeneous form, which then enables us to find the solution in the self-modeling form.

As we know (cf Section IV, subsection 7), Hugoniot's conditions at a strong shock wave are of the form

$$\frac{\rho}{\rho_0} = \frac{\gamma+1}{\gamma-1}, \quad u = \frac{2}{\gamma+1} D, \quad p = \frac{2}{\gamma+1} \rho_0 D^2, \tag{19}$$

where D = dx/dt is the shock wave velocity.

We will assume that the trajectory of the shock wave is the **ξ-line** (i.e., a line whose equation is ξ = constant). Then from (11) follows

$$D = \frac{dx}{dt} = -\frac{a_2}{a_1}\frac{x}{t} = \frac{1}{k}\frac{x}{t}. \tag{20}$$

Hugoniot's conditions (19) for dimensionless quantities U, R, P become

$$U(\xi) = \frac{2}{\gamma+1}\frac{1}{k}, \quad R(\xi) = \frac{\gamma+1}{\gamma-1}\xi^{-l}, \quad P(\xi) = \frac{2}{\gamma+1}\frac{1}{k^2}\xi^{-l}. \tag{21}$$

Hence it follows that l = 0. After these preliminary remarks, let us consider several self-modeling solutions.

1. Convergent shock wave. This problem was posed and solved by Guderley [30] and independently by K. P. Stanyukobich [28] for the case $\gamma = 7/5$ and fully detailed for all γ by a group of Soviet mathematicians (cf review [31]). The problem can be set up thusly. In a quiescent gas with parameters p_o, ρ_o, $u_o = 0$, a shock wave is in motion, accelerating with respect to the direction toward the center $x = 0$. At the instant $t = 0$ it arrives with infinite velocity at the center, which is a singular point of the system of equations (1). The analytic character of the solution's singularity at the point $x = 0$, $t = 0$ is highly complex. It is assumed that the solution in the neighborhood of the center is represented by analytic functions of the fractional-exponential argument $\xi = \xi_o x^{a_1} t^{a_2}$ and therefore is self-modeling in nature. For convenience in formulating the boundary conditions, we put in (11)

$$a_2 = 1, \; a_1 = -k. \tag{22}$$

Since the shock wave is infinitely strong, then conditions (19) and (20) are valid. We select ξ_o so that at the shock wave $\xi = 1$. Then conditions (21) become

$$U(1) = \frac{2}{\gamma+1}\frac{1}{k}, \quad R(1) = \frac{\gamma+1}{\gamma-1}, \quad P(1) = \frac{2}{\gamma+1}\frac{1}{k^2}. \tag{23}$$

The constant C_1 in integral (13) is defined from conditions (23). Using integral (13) and introducing the new variables (cf [31])

$$y = 1 - kU, \quad z = \frac{k^2 C^2}{1-kU} = \frac{k^2 \gamma P}{R(1-kU)}, \tag{24}$$

system (12) can be reduced to two equations:

$$\left.\begin{aligned} &\text{a)} \quad k\xi \frac{dy}{d\xi} = \frac{q(y, z)}{z-y}, \\ &\text{b)} \quad k\xi \frac{dz}{d\xi} = \frac{z}{y}\frac{p(y, z)}{z-y}, \end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned} p(y,z) &= (3-y)z-(2\gamma-1)y^2+ \\ &\quad +[2\gamma-3+(k-1)(\gamma-2)]y-\gamma(k-1), \\ q(y,z) &= (1-y)(y-3z)-(1-k)\left(\frac{2}{\gamma}z+1-y\right). \end{aligned}\right\} \quad (26)$$

On conditions (23) we find

$$y(1)=\frac{\gamma-1}{\gamma+1}, \quad z(1)=\frac{2\gamma}{\gamma+1} \quad (27)$$

When t = 0, the solution $u(x,t)$, $\rho(x, t)$, $p(x,t)$ must be bounded. Hence it follows that $U(0) = 0$, $P(0) = 0$, (28)

Which in variables y and z signifies $y(0) = 1$, $x(0) = 0$. (29)

Thus, we reduce the problem of seeking the self-modeling solution to the boundary problem (27), (29) toward the system (25).

Dividing equation (25b) by (25a), we find

$$\frac{dz}{dy}=\frac{z}{y}\frac{p}{q} \quad (30)$$

We arrive at the equivalent formulation. Find the solution of equation (30) passing through the points $M_1\left(\frac{\nu-1}{\nu+1}, \frac{2\nu}{\nu+1}\right)$, $M_2(1, 0)$ in the y, z-plane.

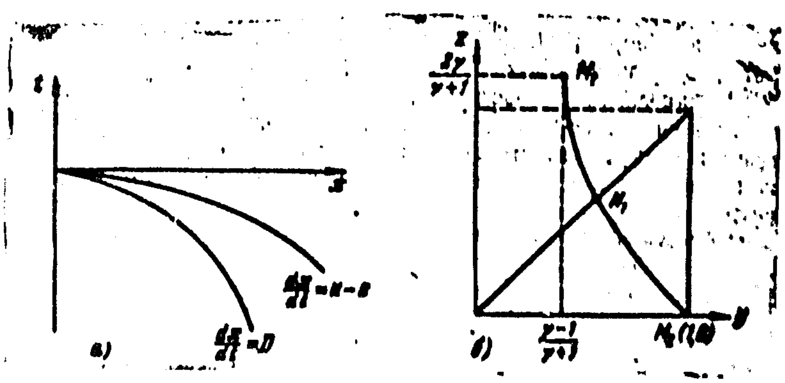


Figure 2.67

The problem posed is overdetermined and is solvable only for certain k values. It is not difficult to verify that the characteristic

$$dx/dt = u - c. \qquad (31)$$

arriving at the center is the ξ-line and is mapped in the y, z-plane by the straight line (Figure 2.67)

$$y - z = 0. \qquad (32)$$

The points M_1, M_2 lie on different sites of the straight line $z - y = 0$ and, therefore, the desired integral curve $y = y(\xi)$, $z = z(\xi)$ must intersect the straight line at some point M corresponding to some value of the parameter ξ_M.

If when the curve $y(\xi)$, $z(\xi)$ passes through the point M: $\{y = y(\xi_M),$ $z = y(\xi_M)\}$, the functions $p(y, z)$ and $q(y, z)$ do not change sign, then the right side of equations (25) do change sign and the functions $y(\xi)$ and $z(\xi)$ become nonunique, which is impossible. Hence it follows that at the point M: $\{y(\xi_M), z(\xi_M)\}$ the quantities $p(y, z)$ and $q(y, z)$ also tend to zero, and it is a singular point. Conditions $y - z = 0$, $p = 0$ lead to the equation

$$z^2 - \left[1 - \frac{1-k}{2\nu}(1 + k)\right]z - \frac{1-k}{2} = 0 \qquad (33)$$

Thus, we have two singular points N_1 and N_2 at the straight line $y - z = 0$. The position of these points, and also the nature of these singularities depend on the two parameters ν and k. In the case $\nu = 7/5$ (investigated by Guderley), the point N_1 is a saddle point and a unique integral curve connecting the points M_1 and M_2 exists. It passes through the point M_1 (cf Figure 2.67, b). We will further provide the reader for detailed analysis to the review [31]. Let us indicate only that there are domains of the parameters ν and k for which the problem is uniquely solvable.

2. The problem of the collapse of a spherical cavity is similarly investigated. A mass of isentropic gas escape into a vacuum so that the boundary of the

gas with the vacuum travels with acceleration toward the center and at the instant t = 0 arrives with unconnected velocity at the center x = 0.

For the dimensionless invariants

$$r = k\left(u + \frac{2}{\gamma - 1} c\right), \quad s = k\left(u - \frac{2}{\gamma - 1} c\right) \tag{34}$$

we obtain the system of equations

$$\xi \frac{dr}{d\xi} = \frac{p(s, r)}{q(r, s)}; \quad \xi \frac{ds}{d\xi} = \frac{p(r, s)}{q(s, r)}, \tag{35}$$

where

$$\left\{\begin{aligned} p(r, s) &= \frac{4}{\gamma - 1} ks - 2\left(\frac{1}{\gamma - 1} + 1\right) s^2 - \left(\frac{2}{\gamma - 1} - 1\right) rs + r^2, \\ q(r, s) &= \frac{4}{\gamma - 1} - \left(\frac{2}{\gamma - 1} + 1\right) r - \left(\frac{2}{\gamma - 1} - 1\right) s. \end{aligned}\right\} \tag{36}$$

The boundary with the vacuum is a ξ-line. ξ_0 is selected so that $\xi = 1$. Then the boundary conditions for system (35) are of the form

$$r(1) = s(1) = 1, \tag{37}$$

$$r(0) = s(0) = 0. \tag{38}$$

We arrive at a boundary problem in the r, s-plane.

Find the integral curve of the equation

$$\frac{ds}{dr} = \frac{p(r, s)\, q(r, s)}{p(s, r)\, q(s, r)}, \tag{39}$$

passing through the points $M_1(1, 1)$ and $M_2(0, 0)$.

We can similarly demonstrate that the desired integral curve must pass through singular points equation (39). Detailed analysis shows that there is no unique analytic solution (cf [31]).

3. The point of explosion problem was solved by L. I. Sedov [32] and by D. I. Taylor [42].

At the point x = 0 and instant t = 0, a finite amount of energy E is instantaneously released, which is converted to the energy of the moving gas separated from the quiescent medium by a shock wave. Since at t values the energy concentration and the pressure are large, then we can neglect the pressure p_0 of the quiescent gas, which makes possible the self-modeling of approximation. For convenience in formulating the boundary conditions, we put in (11) $a_1 = 1$, $a_2 = -1/k$ and neglect ξ_0 so that $\xi = 1$ at the shock wave. Then the flow resulting from the point exploding is described by that solution of system (12) which satisfies conditions (23) and, moreover, the symmetry condition

$$\lim \xi U(\xi) \rightarrow 0, \quad \xi \rightarrow 0. \tag{40}$$

which signifies that $u(0, t) = 0$.

Let us show that in addition to the adiabaticity integral (13), we also have the energy integral. By virtue of the assumption $p_0 = 0$, the energy flow across the shock wave is equal to zero, and the energy of the moving gas is identically equal to E:

$$E = \int_0^x \rho\left(\varepsilon + \frac{u^2}{2}\right) x^\nu dx. \tag{41}$$

Converting to the quantities P, R, U, we obtain the relation

$$E = \frac{\rho_0}{\xi_0^{\nu+3}} t^{\frac{\nu+3}{k} - 2} \int_0^1 \left(\frac{1}{\gamma-1} P + \frac{1}{2} RU^2\right) \xi^{\nu+2} d\xi. \tag{42}$$

Hence it follows that $k = \frac{\nu + 3}{2}$. But this, given the condition $l = 0$, is necessary and sufficient for the existence of the energy integral. From the conditions at the shock wave it follows that the constant of integration in (18) is equal to 0, and the energy integral becomes

$$aU\left(\frac{\gamma}{\gamma-1}P+\frac{1}{2}RU^2\right)=\frac{1}{\gamma-1}P+\frac{1}{2}RU^2. \qquad (43)$$

Considering that 1 = 0, we write integral (13) thusly:

$$\gamma\ln P-(\gamma-1)\ln R+\ln\left(\frac{1}{k}-U\right)+(\nu+3)\theta=\ln C_1. \qquad (44)$$

The constant C_1 in (44) is determined from the conditions at the shock wave:

$$C_1=\frac{2a^2(\gamma-1)^\gamma}{(\gamma+1)^{\gamma+1}}, \quad a=\frac{1}{k}. \qquad (45)$$

We transform equation (43) to become

$$\ln P-\ln R+\ln\left(\gamma U-\frac{1}{k}\right)-\ln\left(\frac{1}{k}-U\right)-2\ln U=\ln\frac{\gamma-1}{2}. \qquad (46)$$

Differentiating (44) and (46) with respect to θ and canceling out the quantities $\frac{d\ln P}{d\theta}$ and $\frac{d\ln R}{d\theta}$ from equations (44), (46), and (12b), we arrive at the equation for U:

$$\frac{dU}{d\theta}=U\frac{(2-bU)(\gamma U-a)}{\gamma(1+\gamma)U^2-2a(\gamma+1)U+2a^2}, \qquad (47)$$

where

$$b=(\nu+3)-(2-\gamma)(\nu+1)>0. \qquad (48)$$

Integrating (47), we find

$$C\xi=U^{-a}(\alpha_1-U)^{-\beta_1}(U-\alpha_2)^{\beta_2}, \qquad (49)$$

where

$$\left.\begin{aligned}&\alpha_1=\frac{2}{b}, \quad \alpha_2=\frac{a}{\gamma};\\&\beta_1=\frac{a^2b^2+2(\gamma+1)(\gamma-ab)}{b(2\gamma-ab)}, \quad \beta_2=\frac{(\gamma-1)a}{2\gamma-ab}.\end{aligned}\right\} \qquad (50)$$

Note that the quantities α_1, α_2, $\alpha_1 - \alpha_2$, β_2 are positive for any γ and ν, and the quantity β_1 is positive for any $\gamma > 1$ and for all ν.

The constant C in (49) is determined from the conditions of shock wave ($\xi = 1$) and is equal to

$$C=\left(\frac{2}{\gamma+1}a\right)^{-\alpha}\left(\alpha_1-\frac{2}{\gamma+1}a\right)^{-\beta_1}\left(\frac{2}{\gamma+1}a-\alpha_2\right)^{\beta_2}. \qquad (51)$$

If next we use energy integral (43) and integral (44), (45), we get the following expressions for R and P:

$$R=\left[4C^2\frac{(\gamma-1)^{\gamma-1}}{(\gamma+1)^{\gamma+1}}\right]^{\frac{1}{2-\gamma}}\left[\frac{\gamma U-a}{(a-U)^2U^2}\right]^{\frac{1}{2-\gamma}}\xi^{\frac{\nu+3}{\gamma-2}}, \qquad (52)$$

$$P=\frac{\gamma-1}{2}RU^2\frac{a-U}{\gamma U-a}. \qquad (53)$$

From formula (49) it follows that as $\xi \to 0$, $U \to \alpha_2$, hence it follows that the symmetry condition (10) is satisfied.

A detailed analysis of the formulas of the solution (cf [32]) shows that U attains the values α_2 as $\xi \to 0$, for $\gamma < 7$ when $\nu = 2$ and for all γ when $\nu = 0, 1$.

The point of explosion for $\gamma > 7$, $\nu = 2$ is accompanied by the formation of a cavity, whose boundary in the x, t-plane is mapped by the ξ-line.

The point of explosion problem was considered by us in the adiabatic approximation, without allowing for thermal conductivity. Clearly, for sufficiently small t the energy concentration is high and we must take into account thermal conductivity.

The point of explosion problem with thermal conductivity taken into account was considered in the works of V. P. Korobeynikov [33] and V. Ye. Neuvazhayev [34].

A problem that is very similar in formulation to the point explosion problem is the problem of motion induced under the action of the piston with a spherical (cylindrical, plane) surface.

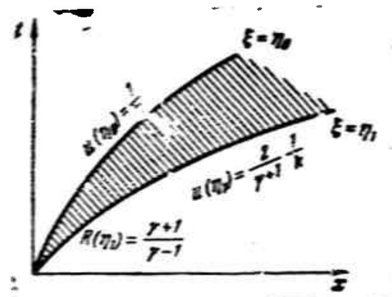


Figure 2.68

If we assume that the piston velocity varies according to the exponential law

$$u_L = \text{constant } t^{\alpha}, \qquad (54)$$

then the flow induced is described by the self-modeling equations (12). In contrast to the point of explosion problem, here we have only one adiabaticity integral (13) and there is no energy integral. Therefore system (12) from which p is canceled out by means of integral (13) reduces to two equations for $u(\xi)$ and $R(\xi)$.

This flow is bounded by two ξ- lines: By the line of the shock wave and by the piston line (Figure 2.68). Hence it follows that

$$\alpha = 1/k, \qquad 1 = 0. \qquad (55)$$

The following boundary conditions are imposed on $u(\xi)$ and $R(\xi)$: at the shock wave, when $\xi = \eta_1$

$u(\eta_1) =$ [illegible], $R(\eta_1) =$ [illegible] (56)

at the piston line, where $\xi = \eta_0$, $u(\eta_0) = 1/k$. (57)

Integration of the system for $u(\xi)$ and $R(\xi)$ is carried out from the value $\xi = \eta_1$ to the value $\xi = \eta_0$ at which condition (57) is met. Since the energy integral is absent, the index k is arbitrary.

The piston problem was investigated in detail in the works of N. L. Krasheninnikov, N. S. Mel'nikova, and N. N. Kochina.

We will further rear to the monograph [33] which gives a detailed analyeis of the piston problem and the explosion problem, together with references.

A highly interesting solution to yet another self-modeling problem -- the instantaneous shock problem -- was given by Ya. B. Zel'dovich [35] and further investigated by V. B. Adamskiy [36], A. I. Zhukov, and Ya. M. Kazhdan [37].

6. Self-modeling solutions in Lagrangian coordinates. Equations (9.5.1) after conversion to the mass Lagrangian coordinate

$$q(x, t) = \int_0^x \rho(x, t)\, x^\nu\, dx \qquad (1)$$

become:

$$\left.\begin{array}{ll} \text{a)} & \dfrac{\partial u}{\partial t} + x^\nu \dfrac{\partial p}{\partial q} = 0, \\ \text{b)} & \dfrac{\partial v}{\partial t} - \dfrac{\partial (u x^\nu)}{\partial q} = 0, \\ \text{c)} & \dfrac{\partial S}{\partial t} = 0, \\ \text{d)} & \dfrac{\partial x}{\partial t} = u. \end{array}\right\} \qquad (2)$$

Using representation (9.5.10), from (1) we find

$$q = \frac{\rho_0 t^{1+\frac{\nu+1}{\delta}}}{\xi_0^{\nu+1}} \int_0^{\xi} R(\xi)\xi^{\nu} d\xi. \tag{3}$$

Hence it follows that the quantity

$$\eta = \eta_0 q t^{-\left[1+\frac{\nu+1}{\delta}\right]} = \frac{\xi_0^{\nu+1}}{\rho_0} q t^{-\left[1+\frac{\nu+1}{\delta}\right]} \tag{4}$$

is a function of ξ .

Taking (9.5.10) into account, we find that the quantities f: (u, v, S, x) can be represented as

$$f = f_0 t^{\beta} F(\eta), \tag{5}$$

which in fact proves the self-modeling status in Lagrangian coordinates.

7. Flows with linear velocity profile. An extremely interesting class of solutions exhibiting functional arbitrary choice was treated by L. I. Sedov [38]. This class is wholly defined by the presence of the linear velocity profile*)

$$u = A(t)x. \tag{1}$$

Differentiating equality (1) with respect to t and taking (9.6.2d) into account, we find

$$\frac{\partial u}{\partial t} = \left(\frac{\dot{A}}{A} + A\right) u, \quad \dot{A} = \frac{dA}{dt}. \tag{2}$$

Integrating (2) with respect to t, we get

$$u = B(t)U(q), \quad B(t) = e^{\int \left(\frac{\dot{A}}{A} + A\right) dt}, \tag{3}$$

where U(q) is an arbitrary function of q.

---

*) The case of inhomogenous coupling u = A(t)x + B(t) contradicts the symmetry condition u(0, t) = 0. This kind of flow was considered also by A. I. Zhukov (private communication), Hafele [39], Horner [40] and Keller [41].

Integrating equation (9.6.2d) with respect to t, we have

$$s = C(t)U(q), \quad [illegible] \tag{4}$$

Equalities (1), (3), and (4) are noncontradictory, since we can verify without difficulty that $B = AC$. From equation (9.6.2b) we determine v:

$$v = \frac{C^{\nu+1}(t)}{\nu+1}\,\frac{dU^{\nu+1}(q)}{dq} \tag{5}$$

Hence, using the equation of state (9.5.2), we obtain an expression for p:

$$p = \psi(q)v^{-\gamma} = (\nu+1)^{\gamma}C(t)^{-(\nu+1)\gamma}\psi(q)\left[\frac{dU^{\nu+1}(q)}{dq}\right]^{-\gamma} \tag{6}$$

where

$$\psi(q) = \psi[S(q)]. \tag{7}$$

Equation (9.6.2a) enables us to define the functions C(t) and to harmonize the arbitrary functions U(q) and $\psi(q)$. Substituting (3) and (6) into (9.6.2a), we find after separation of variables

$$C^{\gamma(\nu+1)-\nu}\ddot{C}(t) = -(\nu+1)^{\gamma}\mu, \tag{8}$$

$$U^{\nu-1}\frac{d}{dq}\left\{\psi(q)\left[\frac{dU^{\nu+1}}{dq}\right]^{-\gamma}\right\} = \mu, \tag{9}$$

where μ is a constant.

If C(t) half-size equation (8), and functions U(q) and $\psi(q)$ are harmonized by means of equation (9), then equalities (3)-(6) define a solution dependent on a single arbitrary function. The solution (3)-(6) can be realized as the escape into a vacuum of a spherical volume of gas and the flow of gas behind the convergent spherical wave. A detailed consideration of these flows is given in the monograph [38].

CHAPTER THREE DIFFERENCE METHODS OF SOLVING EQUATIONS IN GAS DYNAMICS

Section I. Fundamental Concepts of the Theory of Difference Schemes

In this section we expound the fundamental concepts and facts of the theory of difference schemes, referring the reader for more extensive details to monographs and journal articles (cf list of literature for this chapter).

The theory of difference schemes has two fundamental aspects:

1) methods of constructing difference schemes; and

2) validation of the selected difference schemes, i.e., investigation of the convergence of the corresponding computational algorithm.

In this section we dwell mainly on studying the convergence of difference schemes, and in the next section -- on methods of constructing them. First let us recall several facts from functional analysis that are essential for our further exposition [1-3].

1. Linear operators in normed spaces. A finite-measure unitary space U_N is the term given to the complex space X_N of vectors x in which for each element $x = (x_1, \ldots, x_n) \in X_N$ the norm $\|x\|$ is introduced by the rule

$$\|x\| = \sum_{i=1}^{N} x_i \bar{x}_i, \qquad (1)$$

where x_i are components of vector x, and $\bar{x}_i$ are complexly conjugate to x_i.

Suppose A is a linear operator in U_N. The Hermitian norm $\|A\|$ of operator A is determined given to the upper bound of the quantity $\|A_x\|/\|x\|$ where x is an element of U_N, and $\|x\|$, $\|Ax\|$ are understood in the sense of (1).

The linear functional space $X = \{u\}$ is called normed if for each element (function) $u \in X$ some negative number $\|u\|$ is defined, called the norm of u such that the following requirements are met:

1) $\|u\| > 0$ for any element $u \in X$ that is not a zero element; the norm of a zero element is 0;

2) $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$ (triangle inequality); and

3) $\|cu\| = |c| \; \|u\|$.

Introducing the concept of the norm enables us to define the passage to the limit in the space X. By definition $u = \lim_{i \to \infty} u_i$ if

$$\|u - u_i\| \to 0, \quad i \to \infty, \quad u_i \in X, \; u \in X.$$

The sequence $\{u_i\}$ is called fundamental if

$$\|u_i - u_j\| \leqslant \varepsilon(N), \quad i, j \geqslant N,$$

and $\varepsilon(N) \to 0$ as $N \to \infty$. Normed X is called complete or Banach if any fundamental sequence $\{u_i\}$ converges to some element $u \in X$. In the following, if a specific norm is not indicated, we will denote complete normed spaces by B.

Suppose $U \subset B$ is a certain class of function. Let us form the set $\bar{U}$ (closure of U) as follows: $u \in \bar{U}$ if u is the limit of the sequence $\{u_i\}$, $u_i \in U$. Clearly, $U \subseteq \bar{U}$, and $\bar{U}$ can be defined as the complement U by the limit elements.

The class $U \subset B$ is called dense in B if $\bar{U} = B$.

Let us consider examples of certain functional spaces. The linear space of the functions u(x) continuous over segment $[a, b]$ together with all the derivatives up to order p inclusively becomes a Banach space if we introduce the norm

We will denote this norm by $C_r(a, b)$. In particular, the space of continuous functions $u(x)$ with norm $\|u\| = \max_x |u(x)|$ will be denoted by $C_0(a, b)$ or simply by $C(a, b)$.

We will let $L_2(a, b)$ stand for the space of functions summable by quadrature over the segment [a, b] in which the norm $\|u\|^2 = \int_a^b u^2(x)\,dx$ is introduced.

The relation (Bunyakovski-Schwartz inequality) $\|w\|^2 \leq \|u\| \, \|v\|$ where $w(x) = |u(x)v(x)|^{\frac{1}{2}}$ is valid for the norm in L_2. The aggregate of polynomials $P_n(x) = a_\alpha x^\alpha$ ($\alpha = 0, 1, \ldots, n$) is dense in $C(a, b)$ (Weierstrass theorem). For the aggregate of trigonometric polynomials $T_n(x) = a_\alpha e^{i\alpha x}$ ($\alpha = -n, \ldots, 0, \ldots, n$) is dense in the space $C(a, b)$ of continuous periodic functions when $b-a \leq 2\pi$. $C(a, b)$ is dense in $L_2(a, b)$.

Suppose A is a linear operator defined for some dense class $U \subset B$ and transferring the function $u \in U$ to the function $v \in B$.

We will refer to the quantity $\|A\| = \sup \frac{\|Au\|}{\|u\|}$, $u \in U$. as the norm $\|A\|$ of operator A.

We will call operator A bounded if $\|A\| < \infty$, and unbounded, otherwise. Bounded operator A exhibits the properties:

a) $Au = \lim_{i \to \infty} Au_i$, if $u = \lim_{i \to \infty} u_i$, $u, u_i \in U$; and

b) if $\{u_i\}$ is a fundamental sequence, then Au_i is also a fundamental sequence quence.

If A is a bounded operator, then the domain U of its definition can be enlarged to the entire space B (extension of operator A) (cf [1-3]). Let $\bar{A}$ stand for the extended operator. By definition, $\bar{A}u = \lim_{i \to \infty} Au_i$, $u = \lim_{i \to \infty} u_i$.

It is not difficult to show that $\bar{A}u = Au$, $u \in U$, $\|\bar{A}\| = \|A\|$. An example of the extension of operator (nonlinear) is given in section VIII of Chapter One. We will consider the space B of Lipschitz-continuous functions $u(x)$ with the norm

$$\|u\| = \sup_{x,\xi} \max\left\{|u(x)|, \frac{|u(x)-u(\xi)|}{|x-\xi|}\right\}.$$

Differentiable functions $u \in C_1$ form a dense class in the space. We defined operator S of the solution of Cauchy's problem for quasilinear equations in section VIII of Chapter One in the space C_1, and here the solution $u = Su_c$ is bounded in the norm B. Therefore operator S can be extended to the class B of Lipschitz-continuous functions. Extended operator $\bar{S}$ brings into correspondence to each $u_c \in B$ a solution $u = \bar{S}u_c$ bounded in B and which is a generalized solution of the system of equations.

The totality of operators A defined in B forms the linear set X_A. This set becomes normed if we introduce as the norm of operator A considered as the element X_A, its norm as the norm of the operator B (induced norm).

Then we can define the proximity of bounded operators, and we will state that a family of operators $A(\tau)$ dependent on parameter τ reduces to A in the sense of uniform topology, if $\|A(\tau) - A\| \to 0$, $\tau \to 0$. We can speak of the convergence of operators $A(\tau)$ to A in the sense of a strong topology if $\|[A(\tau) - A]u\| \to 0$ as $\tau \to 0$ for arbitrary $u \in B$.

Finally, if the family of operators $A(\tau)$ is unbounded in the totality, then the proximity of operators A can be estimated for some functional class $U \subset B$. For the case when $\|[A(\tau) - A]u\| \to 0$ as $\tau \to 0$ for an arbitrary $u \in U$, then we will state that the family $A(\tau)$ approximates operator A and we will denote this by the symbol $\sim$: $A(\tau) \sim A$.

In the following we will briefly state that operator $A(\tau)$ dependent on parameter τ approximates operator A.

Let us now consider several spaces associated with difference schemes.

Stated briefly, difference methods of integrating systems of differential, integral-differential, and integral equations of mathematical physics consists in converting from derivative to difference relations and from integrals to sums.

In practical terms, this means converting from a finite-measure space of functions of a continuous argument to a finite-measure space of grid functions and reducing equations for functions of a continuous argument to algebraic relations. This approach, convenient in practice, gives rise to difficulties when proving convergence, since a grid function and the function of a continuous argument it approximates are defined in different spaces with different norms. Moreover, the norm of a grid function depends on grid parameters and changes together with them.

Therefore in a theoretical investigation it is also convenient to consider difference operators in the same functional space as that for the operators they approximate. Under this method of consideration we assume that difference equations are satisfied by functions of a continuous argument at each point of the domain in question. As we will see, such an approach is not always possible.

Let us illustrate both methods of examination by a simple example. We stipulate for the thermal conductivity

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \quad a=\text{const}, \tag{2}$$

for the mixed Cauchy's problem:

$$u(x,0)=u_0(x), \qquad 0<x<l; \tag{3}$$

$$u(0,t)=u(l,t)=0, \qquad 0\leqslant t\leqslant t_0. \tag{4}$$

Let us introduce a grid in the x, t-plane, assuming

$$\left.\begin{array}{ll} x_i=ih \quad (i=0,1,\ldots,N+1), & N+1=\frac{l}{h}, \\ t_m=m\tau \quad (m=0,1,\ldots,M), & M=\frac{t_0}{\tau}. \end{array}\right\} \tag{5}$$

Let us define the grid function u_i^m at the points of grid (5), denoting by u_i^m the value of the function at the point $x_i = ih$, $t_m = m\tau$. Let us replace relations (2)-(4) by the algebraic relations

$$\frac{u_i^{m+1}-u_i^m}{\tau} = a^2\frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2},$$
$$u_i^0 = u_0(ih), \quad u_0^m = 0, \quad u_{N+1}^m = 0.$$

For a fixed m the quantities u_i^m ($i = 1, \ldots, N$) are components of the N-dimensional vector u^m for which a given norm can be defined, for example:

$$\|u^m\| = \max_i |u_i^m| \quad (i = 1, \ldots, N)$$

or

$$\|u^m\| = \sqrt{h\sum_{i=1}^{N}(u_i^m)^2}.$$

Ordinarily the grid function u_i^m is extended in some fashion, for example by interpolation, to the entire domain under consideration

$$0 \leq x \leq \ell, \quad 0 \leq t \leq t_0.$$

A proof of the convergence of u_i^m to $u(x, t)$ can be obtained by demonstrating the convergence of the function $\tilde{u}(x, t)$ obtained by interpolation to $u(x, t)$ or by proving that $\{u_i^m\} \to \{u(ih, m\tau)\}$ for all $\tau, h \to 0$, $0 \leq i \leq \ell/h$, $0 \leq m \leq t_0/\tau$.

We can readily see that the solution of the mixed problem (2)-(4) reduces to the solution of Cauchy's problem for equation (2) with initial condition

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty, \tag{6}$$

imposed on the entire straight line $t = 0$ if the function $u_0(x)$ is periodic with period 2ℓ and is odd at the integral $[-\ell, \ell]$. Actually, in this case the initial function $u_0(x)$ and the solution $u(x, t)$ of problem (2), (6) are represented by the series:

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l}, \quad u(x, t) = \sum_{k=1}^{\infty} a_k(t) \sin\frac{k\pi x}{l}, \quad a_k(0) = a_k$$

and, therefore, initial data (3) and boundary conditions (4) are satisfied.

Here, the solution u(x, t) of problem (2), (6) will be a periodic function of the variable x with period 2ℓ, which in the strip $0 \leq x \leq \ell$ coincides with the solution of the mixed problem (2)-(4).

Let us replace Cauchy's problem (2), (6) with periodic function $u_c(x)$ by the following difference problem:

$$\frac{\tilde{u}(x, t+\tau) - \tilde{u}(x, t)}{\tau} = a^2 \frac{\tilde{u}(x+h, t) - 2\tilde{u}(x, t) + \tilde{u}(x-h, t)}{h^2}, \quad (7)$$

$$\tilde{u}(x, 0) = u_0(x), \quad -\infty < x < \infty, \quad (8)$$

for the function $\tilde{u}(x, t)$.

Under this consideration, boundary conditions vanish, and initial condition (8) in combination with difference equation (7) enable us to define the function $\tilde{u}(x, t)$ at straight lines t = k (k = 1, 2,...).

Thus, for a fixed $t = k\tau$ the functions u(x, t) and $\tilde{u}(x, t)$ are defined over the entire straight line $-\infty < x < \infty$. Proof of the convergence of $\tilde{u}(x,t)$ to u(x, t) reduces to proving the convergence of the functions of one variable at this straight line t = constant. Clearly, this examination is not possible for any boundary conditions (4).

In those cases when we must consider problems associated with boundary conditions, we will convert to grid functions. In the following subsections we give a comparative analysis of Cauchy's problems for difference and differential equations.

In concluding this subsection, let us agree on the following notation. If the function $u(x, t)$ for arbitrarily fixed t belongs, as a function of x, to Banach space B, we can regard it as a one-parametric family of elements of the space and denote it with $u(t) \in B$. In particular, $u(t) \in C_q(a, b)$ signifies that $u(x, t)$ for fixed t has at the interval [a, b] q continuous derivatives with respect to x; $u(t) \in L_2(-\ell, \ell)$ signifies that $\int_{-\ell}^{\ell} u^2(x, t)dx < \infty$. Sometimes for brevity we will omit the domain of definition of the functions and simply write C_q, L_2, and so on.

2. Correctness of Cauchy's problem in Banach space for linear systems of differential equations. The theory of generalized solutions of differential equations began to develop fairly recently, since the 1940s. Referring the reader desiring a closer treatment to the monographs [4-7], we present in brief a more specialized theory that does not require the concept of the generalized function and derivative (cf [8, 9]). In the strip G: $|x| < \infty$, $0 \leq t \leq \bar{t}$ let us consider the system $\partial u/\partial t = L(D)u$, (1)

where $u = \{u_1(x, t), \ldots, u_n(x, t)\}$ is a vector function of x, t; L(D) is a differential-matrical operator whose coefficients depend on x, t:

$$L(D) = a_\alpha(x, t)D^\alpha, \quad \alpha = 1, \ldots, p, \quad D = \frac{\partial}{\partial x}, \tag{2}$$

$$a_\alpha(x, t) = \| a^i_{\alpha j}(x, t) \| \quad (i, j = 1, \ldots, n; \ \alpha = 1, \ldots, p). \tag{3}$$

The solution of system (1) is the term given to the function $u(x, t)$ that has continuous derivatives appearing in (1) and satisfying equation (1). It is required that the solution $u(x, t)$ thus exhibit differentiability with respect to t and that a continuous derivative $\frac{\partial^p u(x, t)}{\partial x^p}$ exists for any t, i.e., $u(t) \in C_p$. The initial data

$$u(x, t_0) = u_0(x), \quad 0 \leq t_0 \leq \bar{t}. \tag{4}$$

can be imposed on system (1). We will presuppose the periodicity of the coefficient of system (1) and initial function $u_0(x)$ with respect to x with period $2l$. As for system (1), we will assume, further, that for an arbitrary $t_0 \in [0, \bar{t}]$ an arbitrary function $u_0(x) \in C_q$ ($q \geqslant p$) there exists a unique solution $u(x, t)$ defined in the strip $t_0 \leqslant t \leqslant \bar{t}$.

The correspondence $u(t_0) \rightarrow u(t)$ ($t_0 \leqslant t \leqslant \bar{t}$), which we can write as

$$u(t) = S(t, t_0)u_0(x) = S(t, t_0)u(t_0), \quad (5)$$

defines the transform operator $S(t, t_0)$. If for any $u_0(x) \in C_q$ $u(t) \in C_q$, $t_0 \leqslant t \leqslant \bar{t}$, then we can state about the system (1) that it exhibits the property of extencibility in C_q. In this case the family of $S(t, t_0)$ exhibits the property of composition in C_q, i.e.,

$$S(t_2, t_0) = S(t_2, t_1)S(t_1, t_0), \quad 0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant \bar{t}. \quad (6)$$

Equality (6) signifies that multiple use of the transform operator does not remove the function $u(t)$ from the space C_q. If in this case

$$\|S(t_2, t_1)\|_{C_q} \leqslant M(\bar{t}), \quad 0 \leqslant t_1 \leqslant t_2 \leqslant \bar{t}, \quad (7)$$

then we can call Cauchy's problem (1), (4) correct in C_q. If

$$\|S(t+\tau, t)\|_{C_q} \leqslant 1 + C(\bar{t})\tau, \quad 0 \leqslant t < t+\tau \leqslant \bar{t}, \quad (8)$$

then we will call the problem (1), (4) uniformly correct in C_q. We will also call operator $S(t_2, t_1)$ the solution operator of Cauchy's problem, and we will also call operator S the solution operator of Cauchy's problem and call operator $S(t + \tau, t)$ -- the step operator.

If the initial data do not belong to C_q or if system (1) does not have the property of extensibility in C_q, it becomes necessary to define the generalized solution, i.e., a solution belonging to a more extensive space than the space C_q.

We will assume that there is a Banach space B containing C_q as a dense class and such that the operator $S(t_2, t_1)$ is bounded in the norm of B for the class u(t) C_q. Then the operator $S(t_2, t_1)$ can be extended in B with norm preserved. Equality (5), where $S(t, t_o)$ refers to an extended operator, and u(x, t) refers to a function of B, defines the generalized solution u(x, t) of Cauchy's problem (1), (4) that is correst in B providing that (7) is satisfied and is uniformly correct in B when (8) is satisfied. Clearly, in this case system (1) exhibits the property of extensibility in B.

Let us clarify the concepts we have introduced with a number of examples.

Define the displacement operator T(h) by means of the equality

$$T(h)u(x) = u(x+h). \tag{9}$$

We can readily see that in the space C_q and L_p of periodic function

$$\|T(h)\| = 1. \tag{10}$$

Consider the equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a = \text{const} > 0, \tag{11}$$

with initial data

$$u(x, 0) = u_o(x). \tag{12}$$

If $u_o \in C_1$, then Cauchy's problem (11), (12) has a solution. The transform operator can be represented as $S(t_2, t_1) = T[-a(t_2 - t_1)]$. (13)

Suppose $u_o \in L_2$, $u_o \in C_1$. Then the solution of the problem (11), (12) does not exist in C_1, but the equality $u(t) = T(-at)u_o$ (14)

is physically meaningful and defines the generalized solution of Cauchy's problem (11), (12) in L_2. An analogous approach is possible also for equations of acoustics

$$\frac{\partial u}{\partial t} - a^2\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad a = \text{const}, \tag{15}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \tag{16}$$

If $u_0 \in C_1$, $v_0 \in C_1$, then $u(t) \in C_1$, $v(t) \in C_1$ and equations (15) are equivalent to a system in invariants

$$\frac{\partial r}{\partial t} + a\frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} - a\frac{\partial s}{\partial x} = 0, \qquad (17)$$

where $r = u - av$, $s = u + av$ are Riemann invariants. But if $u_0 \in L_2$, $v_0 \in L_2$, then $r_0 \in L_2$, $s_0 \in L_2$ and the equalities

$$r(t) = T(-at) r_0, \quad s(t) = T(at) s_0 \qquad (18)$$

define the generalized solution in the space L_2. Here we have in the norm L_2

$$\|r(t)\| = \|r_0\|, \quad \|s(t)\| = \|s_0\|, \qquad (19)$$

if the initial functions are r_0, s_0(u_0, v_0) are periodic with period $2l$. If we define the norm as the vector function $f = \{u, v\}$ by means of the equality

$$\|f\|^2 = \int_{-l}^{l} (u^2 + a^2v^2)\, dx, \qquad (20)$$

then from equalities (18) and (19) follows

$$\|f(t)\|^2 = \frac{1}{2}\int_{-l}^{l} [r^2(x, t) + s^2(x, t)]\, dx = \frac{1}{2}\int_{-l}^{l} [r^2(x, 0) + s^2(x, 0)]\, dx = \|f(0)\|^2, \qquad (21)$$

i.e., the generalized solution is extensible in norm (20). From equalities (18) it follows that Cauchy's problem (15), (16) is uniformly correct as $0 < t < \infty$ in the C_1 class, and from equality (21) it follows that it is uniformly correct also in the class L_2.

3. Fourier's method. A solution operator can be effectively constructed for the system (1.2.1) with coefficients $a_\alpha(t)$ dependent only on time by means of Fourier's method*). Suppose $u_0 \in C_q$ is a periodic vector-function with period 2π. The vector-function $u_0(x)$ is represented by the Fourier series

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$$u_0(x) = \sum_{k=-\infty}^{\infty} C_0(k) e^{ikx}, \quad (1)$$

where

$$C_0(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(x) e^{-ikx} dx, \quad (2)$$

which when $q = 1$ converges to $u_0(x)$ absolutely and uniformly. Assuming q to be sufficiently large, we will seek the solution of problem (1.2.1) and (1.2.4) in the form of the Fourier series $u(x, t) = \sum_{k=-\infty}^{\infty} C(k, t) e^{ikx}$, (3)

i.e., as the superposition of the functions $v(x, t) = C(k, t)e^{ikx}$, (4)

We will call these functions harmonics.

Let us establish the condition under which a harmonic is a solution of (1.2.1). Substituting (4) into (1.2.1), we find

$$\left[\frac{\partial}{\partial t} - L(D)\right] v = \left[\frac{\partial C}{\partial t} - L(ik, t) C\right] e^{ikx}, \quad (5)$$

where $L(ik, t)$ is the matrix $L(ik, t) = a_\alpha(t)(ik)^\alpha$, $\alpha = 1, \ldots, p$. (6)

From (5) it follows that harmonic (4) is a solution of (1.2.1) if the vector $C(k, t)$ satisfies the differential equation

$$\frac{\partial C(k, t)}{\partial t} - L(ik, t) C(k, t) = 0. \quad (7)$$

Consequently, for series (3) to be a solution of Cauchy's problem (1.2.1), (1.2.4), $C(k, t)$ must satisfy, when $k = 0, \pm 1, \pm 2, \ldots$, equation (7) and the initial conditions $C(k, t_0) = C_0(k)$, $k = 0, \pm 1, \pm 2, \ldots$, (8)

*) Cf monographs [10-12] for a treatment of Fourier series theory.

so that when $t = t_o$ series (3) must transform into (1). If series (3) belongs to C_r, $r \geqslant p$, then it is a solution of Cauchy's problem (1.2.1), (1.2.4).

Let us estimate the smoothness of solution (3) as a function of the smoothness of the initial data. As we know, the following estimate of its Fourier coefficients is valid for $u_o(x) \in C_q$:

$$|C_0(k)| = \frac{|\lambda_k|}{|k|^q}, \quad \sum_{k=-\infty}^{\infty} |\lambda_k|^2 < \infty, \tag{9}$$

where λ_k of the Fourier coefficients of the function $\frac{d^q u_o(x)}{dx^q}$. Suppose $S(k, t_2, t_1)$ is the **transform** operator of system (7), corresponding to an arbitrarily fixed k. We will call the operator $S(k, t_2, t_1)$ the spectral image or the Fourier-image of operator $S(t_2, t_1)$. By definition of $S(k, t_2, t_1)$, we have

$$C(k, t_2) = S(k, t_2, t_1) C(k, t_1). \tag{10}$$

We will assume that in the interval $[0, \bar{t}]$ system (7) of ordinary differential equations is a uniformly stable system with respect to k, i.e.,

$$\sup_k \| S(k, t_2, t_1) \| = N(t_2, t_1) < \infty, \tag{11}$$

where $\| S(k, t_2, t_1) \|$ is the Hermitian form of operator $S(k, t_2, t_1)$ in the space U_n of components $C(k, t)$. If $u(t_o) \in C_q$, then by virtue of (9) and (11) the following estimate is valid:

$$|C(k, t)| < N \frac{|\lambda_k|}{|k|^q},$$

from which it follows that $u(t) \in C_{q-1}$. Consequently, if $q \geqslant p + 1$, then there exists a solution of problem (1.2.1), (1.2.4) which is not necessarily extensible in C_q.

Now let us consider the generalized solution of the problem (1.2.1), (1.2.4) in $L_2(-\pi, \pi)$. If $u(t) \in L_2(-\pi, \pi)$, then

$$\|u(t)\|^2 = 2\pi \sum_{k=-\infty}^{\infty} |C(k, t)|^2. \tag{12}$$

Hence, taking (10) into account, we have

$$\|S(t_2, t_1)\|_{L_2} = \sup \|S(k, t_2, t_1)\|_{U_n} = N(t_2, t_1) < \infty. \tag{13}$$

Consequently, a generalized solution exists in L_2. Condition (13) thus is the condition for the correctness of Cauchy's problem (1.2.1), (1.2.4) in $L_2(-\pi, \pi)$. Let us give an explicit expression for the transform operator $S(t_2, t_1)$ in terms of $S(k, t_2, t_1)$. From equalities (3) and (2), we have

$$u(x, t_2) = \sum_{k=-\infty}^{\infty} S(k, t_2, t_1) C(k, t_1) e^{ikx} =$$
$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} S(k, t_2, t_1) \left(\int_{-\pi}^{\pi} u(s, t_1) e^{-iks} ds \right) e^{ikx} =$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} K(t_2, t_1, x-s) u(s, t_1) ds, \tag{14}$$

where

$$K(t_2, t_1, x-s) = \sum_{k=-\infty}^{\infty} S(k, t_2, t_1) e^{ik(x-s)}. \tag{15}$$

When the order of integration and summation was changed, we used the theorem of the possibility of termwise integration of the Fourier series.

Thus, in this particular case the transform operator $S(t_2, t_1)$ is a convolution type integral operator. Let us consider a special case when equation (1.2.1) is an equation with constant coefficients. The matrix L(ik, t) = L(ik) does not depend on t, and the solution of problem (7), (8) is of the form (cf [9])

$$C(k, t) = e^{L(ik)(t-t_0)} C_0(k). \quad (16)$$

Here $e^{L(ik)t}$ is the matrix (exponential) which can be represented by the series

$$e^{L(ik)t} = E + \sum_{m=1}^{\infty} \frac{L^m(ik) t^m}{m!}. \quad (17)$$

From (16) follows

$$S(k, t_2, t_1) = e^{L(ik)(t_2-t_1)}, \quad (18)$$

$$S(k, t_2, t_1) = S(k, t_2-t_1, 0). \quad (19)$$

We obtain the estimate

$$\| S(t_2, t_1) \| = \sup_k \| e^{L(ik)(t_2-t_1)} \|_{U_n} \leqslant \sup_k e^{\| L(ik)(t_2-t_1) \|_{U_n}}. \quad (20)$$

for the transform operator $S(t_2, t_1)$. Thus, Cauchy's problem with constant coefficients is correct if

$$\sup_k \| L(ik)(t_2-t_1) \|_{U_n} \leqslant N(\bar{t}), \quad t_1 \leqslant t_2 \leqslant \bar{t}. \quad (21)$$

We can interpret condition (21) in the following manner.

Let us consider the harmonic solution to system (1.2.1):

$$u(x, t) = u_0 e^{\omega t + ikx}, \quad (22)$$

where u_0 is a constant vector. If (22) is the solution of (1.2.1), then

$$\mathrm{Det} \| \omega E - L(ik) \| = 0. \quad (23)$$

Clearly, for the correctness of system (1.2.1) it is necessary and sufficient that

$$\mathrm{Re}\,\omega \leq \mu, \tag{24}$$

where μ is a constant not dependent on k.

Condition (24) signifies that any harmonic solution increases an amplitude not more strongly than $e^{\mu t}$.

In other words, solutions of system (1.2.1) are of the same order of growth as the solution of a certain system of ordinary differential equations. This signifies that system (1.2.1) can be majorized by a system of ordinary differential equations.

Let us consider that the system

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}, \quad \mu > 0, \tag{25}$$

with constant matrix A. If system (25) for $\mu = 0$ is hyperbolic, then by linear transformation in the space of components $u_1, \ldots, u_n$ we can reduce it to the canonical form

$$\frac{\partial r_j}{\partial t} + \xi_j \frac{\partial r_j}{\partial x} = \mu \frac{\partial^2 r_j}{\partial x^2}, \quad j = 1, \ldots, n. \tag{26}$$

The Fourier transform of system (26) leads to a system of equations for Fourier coefficients $C_j(k, t)$ of functions $r_j(x, t)$:

$$\frac{\partial C_j}{\partial t} + (i\xi_j k + \mu k^2) C_j = 0 \quad (j = 1, \ldots, n). \tag{27}$$

Hence

$$C_j(k, t) = e^{-(\mu k^2 + i\xi_j k) t} C_j(k, 0), \tag{28}$$

$$S(k, t_2, t_1) = \| e^{-(\mu k^2 + i\xi_j k)(t_2 - t_1)} \delta_{jl} \| \quad (j, l = 1, \ldots, n), \tag{29}$$

$$\| S(t_2, t_1) \| = \sup_k \| S(k, t_2, t_1) \| = \sup_k | e^{-(\mu k^2 + i\xi_j k)(t_2 - t_1)} | = 1. \tag{30}$$

System (25) is correct in L_2 when $\mu \geq 0$.

The Fourier method is an effective method of estimating the norms of operators and analyzing properties of solutions in L_2. It is directly transferrable to the theory of difference equations with coefficients dependent on t.

The investigation of correctness is severely complicated for equations with coefficients dependent on x and t.

With the example of equations of acoustics with variable speed of sound

$$\frac{\partial u}{\partial t} - a\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad 0 < a_0 \leqslant a(x, t) \leqslant a_1 < \infty, \quad (31)$$

we will show the application of another method of studying correctness -- the method of energy inequalities, or the energy integral method. Multiplying the first equation in (31) by 2u, the second by 2av, and carrying out the manipulations, we obtain

$$\frac{\partial u^2}{\partial t} + \frac{\partial (av^2)}{\partial t} = 2\frac{\partial}{\partial x}(auv) + \frac{\partial a}{\partial t} v^2 - 2\frac{\partial a}{\partial x} uv. \quad (32)$$

Hence we arrive at the inequality

$$\frac{\partial}{\partial t}(u^2 + av^2) \leqslant 2\frac{\partial}{\partial x}(auv) + b(u^2 + av^2), \quad (33)$$

where

$$b = \max\left|\frac{\partial \ln a}{\partial t}\right| + \max\left|\frac{1}{\sqrt{a}}\frac{\partial a}{\partial x}\right|. \quad (34)$$

Suppose functions a(x, t), u(x, t), v(x, t) are periodic relative to x with period 2π. Then by integrating (33) with respect to x over the interval $[-\pi, \pi]$, we get

$$\frac{\partial}{\partial t}\|f\|^2 \leqslant b\|f\|^2, \quad (35)$$

where

$$f(t) = \{u(t), v(t)\}, \quad \|f\|^2 = \int_{-\pi}^{\pi} (u^2 + av^2)\,dx. \tag{36}$$

From (35) follows

$$\|f(t)\|^2 < e^{ct} \|f(0)\|^2. \tag{37}$$

Thus, Cauchy's problem for equations (31) is correct in the norm (36).

The dependence of norm (36) on t is inessential: the solution $f(t) = \{u(t), v(t)\}$ of system (31) with variable coefficient $a(x, t)$ is correct in the norm (36) with constant a, $a_0 \leq a \leq a_1$.

In concluding the subsection, we will dwell on a local analysis of the correctness of equations with variable coefficients, which is also called the method of "freezing" of coordinates. Let us associate with each point $P = (\tilde{x}, \tilde{t})$ a system with constant coefficients

$$\frac{\partial u}{\partial t} = L(P, D)u, \quad L(P, D) = a_\alpha(\tilde{x}, \tilde{t}) D^\alpha \quad (\alpha = 1, \ldots, p). \tag{38}$$

where $\tilde{x}$, $\tilde{t}$ are considered as parameters.

The harmonic solution (22) of system (38) corresponding to some point $P = (\tilde{x}, \tilde{t})$ for sufficiently large k is a strongly oscillating function of the variable x. In a sufficiently small region G_ε : $\tilde{x} - \varepsilon \leq x \leq \tilde{x} + \varepsilon$, $\tilde{t} - \varepsilon \leq t \leq \tilde{t} + \varepsilon$ the coefficients $a_i(x, t)$ of system (1.2.1), which we assume to be continuous and smooth, can be approximately assumed as constant, while the harmonic solution (22) varies quite strongly. Clearly, the harmonic (22), being a solution to system (38) with constant coefficients, is at the same time an approximation of the solution of system (1.2.1) in the domain G_ε with a high degree of accuracy. Therefore the behavior of (22) in the domain G_ε affords a concept of the properties of the correctness of system (1.2.1).

A hypothesis exists that is valid for many equations, to the effect that a necessary and sufficient condition for the correctness of system (1.2.1) in L_2 is the correctness of the local system (38) in L_2 for any point $P = (\tilde{x}, \tilde{t})$ (hypothesis of local correctness).

This examination becomes more exact, the larger k is (high-frequency harmonics); therefore this criterion of correctness will also be called the criterion of asymptotic correctness.

Local analysis of solution stability is employed also for nonlinear systems. Let us at once emphasize the substantial difference between linear and nonlinear equations from the viewpoint of their stability properties.

The property of superposition is valid for linear homogeneous equations. This signifies that if $u_1(x, t)$, $u_2(x, t)$ are solutions of the linear homogeneous equation

$$\partial u/\partial t = Lu, \tag{39}$$

then $u(x, t) = C_1u_1(x, t) + C_2u_2(x, t)$ is also a solution of (39). Thus, a space of solutions of (39) is linear. In this case we can speak about the stability of the solution of system (39) not only relative to small perturbations, but also relative to arbitrary perturbations belonging to some normed linear space. For the case of nonlinear system, the property of superposition is not obtained, therefore we must speak about the stability of a concrete solution relative to sufficiently small perturbations.

The procedure of investigating this stability of nonlinear solutions adopted in practice consists of a chain of simplifying the assumptions, which ultimately culminates in the harmonic analysis of stability. First of all, for a given nonlinear system and its specific solution a linear equation is constructed in variations (equation for small perturbations).

Suppose

$$\frac{\partial v}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \qquad (40)$$

is a quasilinear system. If u(x, t) is a solution of system (40), and $\bar{u}(x, t)$ is some other solution, then the vector $v = \bar{u} - u$ satisfies the system of equations

$$\frac{\partial v}{\partial t} + A(u)\frac{\partial v}{\partial x} + [A(\bar{u}) - A(u)]\frac{\partial u}{\partial x} + [A(\bar{u}) - A(u)]\frac{\partial v}{\partial x} = 0. \qquad (41)$$

Equation (41) can be transformed to become

$$\frac{\partial v}{\partial t} + A(u)\frac{\partial v}{\partial x} + B^{*}v + C^{*}v = 0, \qquad (42)$$

where the matrixes B* and C* are formed from the three-indicial matrix

$$\dot{A}(u^{*}) = \left\{\frac{\partial a_{ij}(u^{*})}{\partial u^{*}_{k}}\right\} \qquad (43)$$

as follows:

$$B^{*} = \left\{\frac{\partial a_{i\alpha}(u^{*})}{\partial u_{j}}\frac{\partial u_{\alpha}}{\partial x}\right\}, \qquad C^{*} = \left\{\frac{\partial a_{i\alpha}}{\partial u_{j}}\frac{\partial v_{\alpha}}{\partial x}\right\}, \qquad (44)$$

and u* is an intermediate value between u and $\bar{u}$.

To the first approximation we put u* = u, and then we obtain a nonlinear equation in variation

$$\frac{\partial v}{\partial t} + A\frac{\partial v}{\partial x} + Bv + \left(\dot{A}\frac{\partial v}{\partial x}\right)v = 0. \qquad (45)$$

Neglecting quantity $(\dot{A}\frac{\partial v}{\partial x})v$, we obtain a linear equation in variation

$$\frac{\partial v}{\partial t} + A\frac{\partial v}{\partial x} + Bv = 0. \qquad (46)$$

Secondly, let us use as our working hypothesis the assumption that the correctness of linear system (46) is a necessary condition for this stability of the solution of quasilinear system (40). Finally, the correctness of (46) is investigated by the method of local harmonic analysis, which is based in turn on the hypothesis of local correctness. This analysis is employed when investigating the stability of many physical processes (cf for example [13 & 14]).

Clearly, this procedure of investigating stability based on two hypotheses -- the hypothesis of correctness of the system in variations and the hypothesis of local correctness -- is a rough estimate in many cases. Let us clarify this procedure with a simple example.

Let us consider in the strip $0 \leq t \leq \bar{t}$, $|x| < \infty$, the Cauchy's problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad u(x, 0) = u_0(x). \tag{47}$$

The linear system in variations is of the form

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x}v. \tag{48}$$

Suppose u(x, t) is a solution of (47) corresponding to the initial data

$$u_o(x) = a = \text{constant}.$$

Then $u(x, t) \equiv a$ and equation (48) take on the form

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = 0. \tag{49}$$

Equation (49) is correct in C_p in the strip $|x| < \infty$, $0 \leq t \leq \bar{t}$, where $\bar{t}$ is as large as we wish. Harmonic analysis applied to (49) confirms this. At the same time the actual problem (47) is correct in C_1 only in the narrow strip $0 \leq t \leq \bar{t}^*$ where $\bar{t}^*$ depends on initial data. We assume, for example,

$$u_0(x) = a + \varepsilon \sin kx,$$

Then Cauchy's problem (47) reduces to the gradient catastrophe at the time instant $t = \bar{t}^* = \frac{1}{\varepsilon k}$, which tends to 0 as $k \to \infty$. This example indicates that the test of local asymptotic correctness is applicable only to linearized system (46), and not to initial system (40). Let us indicate one further detail in the investigation of the stability of nonlinear systems.

If we construct for the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\mu(u)\frac{\partial u}{\partial x}\right] \quad (50)$$

at once a local equation (with "frozen" coefficients), and then vary it, we get the equation

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = \mu(u)\frac{\partial^2 v}{\partial x^2}. \quad (51)$$

But if we initially construct a linear equation in variations, then "freeze" the coefficients, then we arrive at the equation

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = \mu(u)\frac{\partial^2 v}{\partial x^2} + 2\left[\mu'(u)\frac{\partial u}{\partial x}\right]\frac{\partial v}{\partial x} + \left[\mu''(u)\left(\frac{\partial u}{\partial x}\right)^2 + \mu'(u)\frac{\partial^2 u}{\partial x^2}\right]v. \quad (52)$$

If solution $u(x, t)$ of equation (50) is identically equal to a constant $u(x, t) \equiv a$ = constant, equations (51) and (52) coincide; if the solution of $u(x, t)$ has large gradients, then (51) and (52) differ widely.

A similar procedure was employed also for difference nonlinear equations in the work by Neumann and Richtmyer [15] (cf subsection 4 of section II).

4. Cauchy's difference problem. Let us formulate Cauchy's problem (1.2.1), (1.2.4) in correspondence with Cauchy's difference problem:

$$\frac{u^{m+1}(x) - u^m(x)}{\tau_{m+1}} = \Lambda_1 u^{m+1}(x) + \Lambda_0 u^m(x). \quad (1)$$

$$u^0(x) = u_0(x). \quad (2)$$

where Λ_1, Λ_0 are operators defining B, dependent on t_m, τ_{m+1}, and also, generally speaking, on other parameters, and

$$t_m = \tau_1 + \dots + \tau_m, \quad 0 \leq t_m \leq t, \quad 0 < \tau_i$$

(we have assumed at the outset (1.2.1), (1.2.4) $t_0 = 0$).

The correctness of the problem (1), (2) is defined analogously to the preceding.

Let us rewrite relations (1) as $A_m u^{m+1} = B_m u^m$, (4)

where $A_m = E - \tau_{m+1}\Lambda_1$, $B_m = E + \tau_{m+1}\Lambda_0$. We will assume that operator A_m is invertible in B for all values of the problem parameters. Then system (4) can be solved:

$$u^{m+1} = C_{m+1}u^m, \quad C_{m+1} = A_m^{-1}B_m. \quad (5)$$

Since operator C_m is defined in B, then the system of recursion relations (5) or a system of relations (4) equivalent to it define the sequence of functions $u^m(x) \in B$, if and only if $u^0(x) \in B$. Thus, the problem (1), (2) is formulated in the space B.

From (5) it follows that $u^m = C_{m,k}u^k$, (6)

where $C_{m,k} = C_m C_{m-1} \dots C_{k+1}$.

We will call operator $C_{m,k}$ the transform operator, operator $C_m = C_{m,m-1}$ -- the step operator, and operator $C_{m,0}$ -- the operator of the solution of problem (1), (2).

Definition. Problem (1), (2) is correct in B if there exists a number of $M(\bar{t}) > 0$ such that $\|C_{m,k}\|_B \leq M(\bar{t})$ (7)

for all $0 \leq k \leq m$, $t_m \leq \bar{t}$ and for sufficiently small $\tau_k > 0$; it is uniformly correct if there exists $N > 0$ such that

$$\|C_{m,m-1}\| < 1 + N\tau_m \qquad (8)$$

for all m of (3) and for sufficiently small τ_m.

Regarding system (1) as an algorithm for the determination of $u^m(x)$ with respect to the initial function $u^0(x)$, we will call relation (1) a difference scheme. If Cauchy's problem (1), (2) is correct, we will call scheme (1) stable.

Let us give a definition of an approximation.

Definition. Problem (1), (2) approximates the problem (1.2.1), (1.2.4) in the space B if

$$\|(S_m - C_m)u(t_{m-1})\|_B = \tau_m \varepsilon_m(\tau_m), \qquad (9)$$

where $S_m = S(t_m, t_{m-1}), u(t) = S(t, 0)u_0(x)$ is the generalized solution of the problem (1.2.1), (1.2.4), $u_0(x)$ is an arbitrary element of B, and $\longrightarrow 0$ is uniform relative to m or $\tau = \max \tau_m \longrightarrow 0$.

Definition. Solution $u^m(x)$ of problem (1), (2) converges in B to the solution u(x, t) a problem (1.2.1), (1.2.4), if for arbitrary initial function $u_0 \in B$

$$\max_m \|u^m - u(t_m)\|_B = \max_m \|[C_{m,0} - S(t_m, 0)]u_0\|_B \to 0 \qquad (10)$$

as $\tau \to 0$.

First theorem of convergence. If

1) problem (1), (2) and problem (1.2.1), (1.2.4) are correct in B, and if

2) problem (1), (2) approximates problem (1.2.1), (1.2.4) then the solution $u^m(x)$ of problem (1), (2) converges in B to the solution u(x,t) of the problem (1.2.1), (1.2.4).

Proof. Let us use the representation

$$S(t_m, 0) = S_m S_{m-1} \dots S_1, \quad C_{m,0} = C_m C_{m-1} \dots C_1. \quad (11)$$

to estimate the quantity (10). Hence it follows that

$$C_{m,0} - S(t_m, 0) = C_m C_{m-1} \dots C_1 - S_m S_{m-1} \dots S_1 = \sum_{k=1}^{m} C_m \dots C_{k+1}(C_k - S_k) S_{k-1} \dots S_1 = \sum_{k=1}^{m} C_{m,k}(C_k - S_k) S_{k-1,0}. \quad (12)$$

Using (12), we obtain

$$\|u^m - u(t_m)\| = \|[C_{m,0} - S(t_{m,0})] u_0\| \leqslant \sum_{k=1}^{m} \|C_{m,k}\| \|(C_k - S_k) S(t_{k-1,0}) u_0\| = \sum_{k=1}^{m} \|C_{m,k}\| \|(C_k - S_k) u(t_{k-1})\|. \quad (13)$$

Employing the correctness estimate (7) and the approximation estimate (9), we find

$$\|u^m - u(t_m)\| \leqslant M(\bar{t}) \sum_{k=1}^{m} \tau_k e_k(\tau_k) \leqslant M(\bar{t})\, t \max_k \varepsilon_k(\tau_k). \quad (14)$$

Hence follows the convergence of

$$\|u^m - u(t_m)\| \to 0 \quad \text{when} \quad \tau \to 0, \quad (15)$$

and the theorem is proven.

Convergence theorems of this type were formulated in the works of V. S. Ryaben'kiy [16], N. N. Meyman [17], P. Lax and R. Richtmyer [18]. Our proof scheme belongs to Lax and Richtmyer.

We defined approximation in terms of bounded operators S_L, C_Γ. Actually, it is more convenient to define approximation in terms of unbounded operators L, Λ_1, Λ_0. In this case the estimate of proximity is made not for the generalized solution $u(t) \in B$, but for the solutions of the problem (1.2.1), (1.2.4) $u(t) \in C_q$.

Definition. Problem (1), (2) approximates the problem (1.2.1), (1.2.4) in the class C_q in the norm B if

$$\max \| R_{m+1} \|_B = \max \left\| \frac{u(t_{m+1}) - u(t_m)}{\tau_{m+1}} - \Lambda_1 u(t_{m+1}) - \Lambda_0 u(t_m) \right\|_B \to 0 \qquad (16)$$

as $\tau = \max \tau_m \to 0$ for an arbitrary fixed solution $u(t) \in C_q$ of problem (1.2.1), (1.2.4). We call the quantity R_Γ the remainder of the difference equation (1).

Second convergence theorem. If

1) problems (1), (2) and (1.2.1), (1.2.4) are correcting B, and

2) $\| A_m^{-1} \| = \| (E - \tau_{m+1} \Lambda_1)^{-1} \|_B \leqslant N(\bar{t})$, and

3) problem (1), (2) approximates the problem (1.2.1), (1.2.4) in the sense (16), then the solution of problem (1), (2) converges to the solution of problem (1.2.1), (1.2.4).

Proof. The solution of the inhomogeneous difference problem

$$u^{m+1} = C_{m+1} u^m + F_{m+1} \qquad (17)$$

is of the form

$$u^m = C_{m,0} u^0 + C_{m,\alpha} F_\alpha \quad (\alpha = 1, \ldots, m). \qquad (18)$$

This can be verified inductively without difficulty.

The difference $v^m = u^m - u(t_m)$ satisfies the difference equation

$$\frac{v^{m+1}-v^m}{\tau_{m+1}} = \Lambda_1 v^{m+1} + \Lambda_0 v^m - R_{m+1}. \qquad (19)$$

By virtue of the invertibility of A_m, relations (19) are transformed to become (cf (4), (5))

$$v^{m+1} = C_{m+1} v^m + F_{m+1}, \qquad F_{m+1} = -\tau_{m+1} A_m^{-1} R_{m+1}. \qquad (20)$$

Using for v^m formula (18) and noting that $v^0 = 0$, we find

$$v^m = C_{m\alpha} F_\alpha \qquad (\alpha = 1, \ldots, m). \qquad (21)$$

Hence follows the estimate

$$\| v^m \| \leqslant M(\bar{t}) N(\bar{t}) t \max_\alpha \| R_\alpha \|_B. \qquad (22)$$

denoting the convergence

$$\| v^m \|_B \rightarrow 0 \quad \text{where} \quad \tau \rightarrow 0, \text{ and so on.} \qquad (23)$$

Now we can establish the structure of solutions of difference and differential Cauchy's problems.

Let us consider the linear space X of functions u(x, t) of two independent variables x, t defined in the closed domain G: $t \in [t_1, t_2]$, $x \in [-\ell, \ell]$ where $[t_1, t_2] \subset [0, t]$, where the functions are periodic relative to variable x with period 2ℓ. If for any fixed $t \in [t_1, t_2]$ $u(t) = u(x, t) \in B$, then we can state that $u(x, t) \in X$.

Introducing the norm B_1 into X: $\| u \|_{B_1} = \sup_t \| u(t) \|_B$. (24)

Taking account of the domain of determinacy G of the functions u(x, t), the space X with norm B_1 will be denoted by $B_1(G)$. By virtue of the correctness of Cauchy's problem (1.2.1), (1.2.4), the solution u(x, t) of this problem belongs to the space $B_1(G_0)$, where G_0 is the domain $[0, t] \times [-1, 1]$.

Suppose A is an operator effective in B. Obviously, operator A is meaningful also in the space B_1: $[Au(x, t)]_B$. The converse, of course, is invalid: an operator defined in B_1 generally speaking is not defined in B. Let us define in B_1 the displacement operator $T(h, \tau)$ by the equality

$$v(x, t) = T(h, \tau)u(x, t) = u(x + h, t + \tau). \qquad (25)$$

Obviously, the operators $T(h, 0)$, $\partial/\partial x = D$, exist simultaneously in the spaces B and B_1, but the operators $\partial/\partial t = D_0$, $T(0, \tau)(\tau \neq 0)$, defined in B_1 are not defined in B.

Let us denote for brevity

$$\left.\begin{aligned} T(h, 0) &= T_1, & T(-h, 0) &= T_{-1} = T_1^{-1}, \\ T(0, \tau) &= T_0, & T(0, -\tau) &= T_{-0} = T_0^{-1}. \end{aligned}\right\} \qquad (26)$$

Definition. The matrical operator defined in B_1:

$$\Lambda(T) = b_{\beta_0\beta_1}(x, t, \tau, h) T_0^{\beta_0} T_1^{\beta_1}, \qquad (27)$$
$$\beta_0 = -q_0, -q_0 + 1, \ldots, q_0, \quad \beta_1 = -q_1, \ldots, q_1,$$

where the matrices $b_{\beta_0 \beta_1}$ act in the space $\{u_n\}$ of components of vector-function $u(x, t)$ will be called a difference operator.

Operator Λ is defined for the function $u(x, t) \in B_1(G)$, where $G = [q_0 \tau, \bar{t} - q_0 \tau] \times [-\ell, \ell]$.

Operator (27) is called finite if $q_0, q_1 \leq Q$, where Q does not depend on τ, h.

Clearly, the sum and product of finite operators is again a finite operator. The operator that is the reciprocal of a finite operator is not, generally speaking, finite.

Suppose

$$\Omega = a_{\alpha_0\alpha_1} D_0^{\alpha_0} D_1^{\alpha_1}; \quad \alpha_0 = 0, 1, \ldots, p_0, \quad \alpha_1 = 0, 1, \ldots, p_1, \qquad (28)$$

is a matrical-differential operator. Let us define the approximation $\Lambda \sim \Omega$, where Λ is operator (27).

Let us consider the function

$$R(x, t, \tau, h) = R(\tau, h)u = [\Lambda(\tau, h, T) - \Omega(D)]u(x, t), \qquad (29)$$

where $u(x, t) \in C_q \subset B_1$, operator $\Lambda(\tau, h, T)$, $\Omega(D)$, $R(\tau, h) = \Lambda(\tau, h, T) - \Omega(D)$ are considered in B_1. If for any $u(x, t) \in C_q \subset B_1$ providing

$$\tau^2 + h^2 \leq \tau_0^2, \qquad (30)$$

where τ_0 depends on the choice of $u(x, t)$ the following estimate is valid:

$$\| R(x, t, \tau, h) \|_{B_1} = 0(\tau^\alpha) + 0(h^\beta), \quad \alpha > 0, \quad \beta > 0, \qquad (31)$$

then we will state that operator Λ absolutely approximates operator Ω in norm B_1 for class C_q with order α with respect to t and with order β with respect to x. But if relation (31) is violated, that is, if it does not hold for arbitrary τ, h from the neighborhood (30), but the following estimate is valid for a certain law of passage to the limit $h = h(\tau)$:

$$\| R(x, t, \tau, h(\tau)) \|_{B_1} = 0(\tau^\alpha), \quad \alpha > 0, \qquad (32)$$

then we speak of the conditional approximation of operator Ω by operator Λ of order α. For the case of conditional approximation, the exponent α depends on the law of the limit process $h = h(\tau)$. As a rule, estimates of the form

$$\| R(x, t, \tau, h) \|_{B_1} = 0(\tau^\alpha) + 0(h^\beta) + 0(\tau^\nu h^\delta),$$

where ν and δ can be even negative are valid for a conditionally approximating operator in the neighborhood (30).

The equality

$$u(x+h, t+\tau) = T(h, \tau)u(x, \tau) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} \frac{\partial^{\alpha+\beta}u(x, t)}{\partial x^\alpha \partial t^\beta} \frac{h^\alpha}{\alpha!} \frac{\tau^\beta}{\beta!} =$$
$$= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\left(\frac{h^\alpha D_1^\alpha}{\alpha!} \frac{\tau^\beta D_0^\beta}{\beta!}\right)u(x, t) = e^{hD_1+\tau D_0}u(x, t). \qquad (33)$$

is valid for the analytic function $u(x, t)$. Thus, the relation

$$T(h, \tau) = e^{hD_1 + \tau D_0} \qquad (34)$$

is valid for the class of analytic functions. We will use relation (34) also for nonanalytic functions, interrupting the expansion of the function $e^{hD_1+\tau D_0}$ in series at the corresponding term.

Then the determination of approximation $\Lambda \sim \Omega$ can be carried out as follows. Expression*)

$$R = \Lambda - \Omega = b_{\beta_0\beta_1} T_0^{\beta_0} T_1^{\beta_1} - a_{\alpha_0\alpha_1} D_0^{\alpha_0} D_1^{\alpha_1} = \\ = b_{\beta_0\beta_1}(x, t, \tau, h)\, e^{\beta_0 \tau D_0 + \beta_1 h D_1} - a_{\alpha_0\alpha_1}(x, t)\, D_0^{\alpha_0} D_1^{\alpha_1} \qquad (35)$$

is expanded in exponential series with respect to τ and h. Then we verify the condition for the absolute approximation (31) and the orders α and β. If we are interested in approximation for specific limit processes, then we insert into this series $h = C\tau^{\nu}$ $(\nu > 0)$, after which R becomes

$$R = \Lambda - \Omega = P(\tau, D_0, D_1), \qquad (36)$$

where $P(\tau, D_0, D_1)$ is a fractional-exponential series in τ and exponential in D_0, D_1. Then the smallest power of τ in (36) determines the order of approximation for the law of the limit process $h = C\tau^{\nu}$. The order of approximation $\Lambda \sim \Omega$ depends on the class U of the comparison. If $U_1 \subset U$, the order of approximation in class U_1 can only be increased compared with class U.

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*) In equality (35) the expressions for R, Λ, Ω are operators. However, in several cases we will signify by the symbols D_i, T_i not operators, but quantities $D_i u$, $T_i u$ defined for sufficiently smooth functions. In this sense algebraic relations can exist between the expressions for D_0 and D_1.

Let us consider several examples that illustrate the concept of order of approximation, absolute and provisional approximation.

Let us define the order of approximation $\Lambda \sim \Omega$ for the operators

$$\Omega = D_0 + aD_1, \quad \Lambda = \frac{T_0 - E}{\tau} + a\,\frac{E - T_{-1}}{h}, \quad a > 0. \tag{37}$$

Following the general algorithm (34), (35), we find

$$\Lambda - \Omega = \frac{1}{2!}\tau D_0^2 + \frac{1}{3!}\tau^2 D_0^3 + \ldots - \frac{a}{2!} hD_1^2 + \frac{a}{3!} h^2 D_1^3 - \ldots \tag{38}$$

From (38) it follows that for the class C_m ($m \geq 2$) the approximation is absolute and is of the first order with respect to τ and h.

Let us consider the approximation $\Lambda \sim \Omega$ for the class U of analytic solutions of the equation $\Omega u = 0$. Here we must formally put $D_0 = -aD_1$ and equality (38) becomes

$$R = \Lambda - \Omega = \Big[\frac{ah}{2!} D_1^2 (\varkappa - 1) - \frac{ah^2}{3!} D_1^3 (\varkappa^2 - 1) + \ldots \\ \ldots + (-1)^m \frac{ah^{m-1}}{m!} D_1^m (\varkappa^{m-1} - 1) + \ldots\Big], \tag{39}$$

where $\varkappa = a\tau/h$.

When $\varkappa = 1$, the approximation $\Lambda \sim \Omega$ for the analytic solutions of equation $\Omega u = 0$ is of infinite order.

It is convenient to establish the approximation in harmonic polynomials

$$P_N = \sum_{k=-N}^{N} C_k e^{\omega t + ikx}. \tag{40}$$

for equations with constant coefficients in the norm L_2. In this case we conveniently convert to spectral images $\bar{\Lambda}$, $\bar{\Omega}$ of the operators Λ, Ω, which are defined by the equality

$$\begin{aligned} \Lambda(C_k e^{\omega t + ikx}) &= (\bar{\Lambda} C_k)\, e^{\omega t + ikx}, \\ \Omega(C_k e^{\omega t + ikx}) &= (\bar{\Omega} C_k)\, e^{\omega t + ikx}. \end{aligned} \tag{41}$$

Matrices $\bar{\Lambda}$, $\bar{\Omega}$ and Λ, Ω operate in the space U_n of components C_k and are derived from the operator-matrices $\Lambda(T_0, T_1)$, $\Omega(D_0, D_1)$ by the formal substitution

$$T_0 \to e^{\omega\tau}, \quad T_1 \to e^{ikh}, \quad D_0 \to \omega, \quad D_1 \to ik \tag{42}$$

Then to equality (36) there corresponds the equality

$$\tilde{R} = \tilde{\Lambda} - \tilde{\Omega} = P(\tau, \omega, ik). \tag{43}$$

Equations (36) and (43) are equivalent by virtue of (42), however the representation (43) is valid for functions expandable in Fourier series, while (36) is valid only for analytic functions. Returning with this point of view to our example (37), we see that an infinite order of approximation $\Lambda \sim \Omega$ obtains also when $\mathcal{H} = 1$ and for discontinuous functions represented by the Fourier series. It is not difficult to see that this is associated with the coincidence of operators of the solution of the equations

$$\Lambda u = 0 \quad \text{and} \quad \Omega u = 0.$$

Actually, when $\mathcal{H} = 1$ we have

$$S(t + \tau, t) = C_{m+1, m} = T_1(-a\tau), \quad t = m\tau.$$

Now let us show an example of a scheme with a provisional approximation.

We put the equation

$$\Omega u = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = (D_0 + aD_1)u = 0, \quad a = \text{const}. \tag{44}$$

into correspondence with the difference equation (Lax scheme)

$$\Lambda u = \left[\frac{T_0 - \frac{T_1 + T_{-1}}{2}}{\tau} + a\frac{T_1 - T_{-1}}{2h}\right]u = 0. \tag{45}$$

Let us represent operator Λ as

$$\Lambda = \frac{T_0 - E}{\tau} + a\frac{T_1 - T_{-1}}{2h} - \frac{h^2}{2\tau}\frac{T_1 - 2E + T_{-1}}{h^2}. \tag{46}$$

Using this scheme for verifying the approximation, we see that equation (45) under the law of the limit process h = constant $\cdot\, \tau$ approximates equation (44):

under the law of the limit process $h^2 = 2\mu\tau$, it approximates the parabolic equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}. \qquad (47)$$

This example shows that in the case of conditional [provisional] approximation the difference operator can approximate different differential operators for different laws of the limit process.

Now we can give a third definition for the approximation of Cauchy's problem (1.2.1), (1.2.4) by Cauchy's difference problem (1), (2).

Cauchy's difference problem (1), (2) characterizes Cauchy's problem (1.2.1), (1.2.4), if the operator

$$\Lambda = \frac{T_0 - E}{\tau} - \Lambda_1 T_0 - \Lambda_0 \qquad (48)$$

approximate*) as $\to 0$ the differential operator $\Omega = \partial/\partial t - L(D)$ (49) from any smooth solution $u(x, t) \in C_q$ of the problem (1.2.1), (1.2.4). Now let us compare different definitions of approximation. The third definition is the most formalized and most effective. It is not difficult to see that from it stems the second solution.

Actually, if $u(x, t) \in C_q$ is a solution of (1.2.1), (1.2.4), then as shown above, $u(x, t) \in B_1$.

Let us form the quantity (remainder of the scheme)

$$R(\tau, x, t) = R(\tau)u(x, t), \qquad R(\tau) = \Lambda(\tau) - \Omega.$$

If $u(x, t)$ is a sufficiently smooth solution, then $R(\tau, x, t) \in B_1$. Noting that $R(\tau_{m+1}, x, t_m) = R(\tau_{m+1})u(x, t_m)u(x, t_m) = R_{m+1}$, from (16) we have

*) Since the dependence of operators Λ_1 and Λ_0 on parameters h, τ is not indicated here, the approximation $\Lambda \sim \Omega$ can be both absolute as well as provisional.

$$\max_m \| R_{m+1} \|_B < \max_\theta \| R(\theta, x, t) \|_B, \quad \theta < \tau = \max \tau_m.$$

from whence follows our assertion.

Let us show that if the operator $A = E - \tau\Lambda_1$ is invertible, then from the second definition of approximation (formula (16)) follows the first (formula (9)) in the class $C_q \subset B$ in which (16) holds.

Suppose $\| A^{-1} \|_B \leq N(\bar{t})$. From (16) there follows

$$\tau R_{m+1} = A [u(t_{m+1}) - A^{-1}(E + \tau\Lambda_0) u(t_m)] = A [(S_{m+1} - C_{m+1}) u(t_m)].$$

Hence

$$\tau (S_{m+1} - C_{m+1}) u(t_m) = \tau A^{-1} R_{m+1},$$

$$\| (S_{m+1} - C_{m+1}) u(t_m) \|_B < N(\bar{t}) \tau \| R_{m+1} \|_B.$$

and equality (9) is satisfied, since $\max_m \| R_{m+1} \|_B \to 0$ as $\tau \to 0$.

Third theorem of convergence. If

1) problems (1), (2) and (1.2.1), (1.2.4) are correct in B,

2) $\|A^{-1}\|_B = \| (E - \tau\Lambda_1)^{-1} \|_B \leq N(\bar{t})$,

3) $\Lambda_1 + \Lambda_0 \sim L$ in B_1.

then solution of (1), (2) converges to solution (1.2.1), (1.2.4) in B_1

Proof. Actually, in B we have

$$\frac{T_0 - E}{\tau} \sim \frac{\partial}{\partial t}, \quad \Lambda_1 T_0 + \Lambda \sim \Lambda_1 + \Lambda_0 \sim L.$$

From whence it follows that $\Lambda = \left(\frac{T_0 - E}{\tau} - \Lambda_1 T_0 - \Lambda_0\right) \sim \Omega = \frac{\partial}{\partial t} - L(D).$

By what has been proven, (16) is satisfied, and we are under the conditions of the second convergence theorem.

5. Stability criteria of difference schemes. The theorems formulated in subsection 4 enables us to establish the convergence of a difference scheme as soon as we have established approximation and stability. Approximation criteria

are relatively simple, are local in character, and for the most part are reducible to a Taylor series expansion. Stability criteria are much more complicated. In this respect we present brief review of various stability criteria.

If the scheme is uniformly stable, then of course it is stable. Thus, uniform stability is a sufficient condition of stability. This enables us in most cases to reduce stability to the definition of the norm of the step operator.

Let us first consider equations with coefficients dependent on t. In this case the Fourier transform is possible for scheme (1.4.1). If

$$u^m(x) = \sum_{k=-\infty}^{\infty} C^m(k) e^{ikx} \qquad (1)$$

is the representation of $u^m(x)$ by a Fourier series, then for $C^m(k)$ we obtain a system of difference equations

$$\frac{C^{m+1}(k) - C^m(k)}{\tau} = \bar{\Lambda}_1 C^{m+1} + \bar{\Lambda}_0 C^m, \qquad (2)$$

where $\bar{\Lambda}_1$, $\bar{\Lambda}_0$ are spectral images of operators Λ_1, Λ_0.

The Fourier transform places the step operator $(E - \tau\bar{\Lambda}_1)^{-1}(E + \tau\bar{\Lambda}_0)$ of scheme (2) into correspondence to the step operator $(E - \tau\Lambda_1)^{-1}(E + \tau\Lambda_0)$ of scheme (1.4.1). On analogy with formula (1.3.13), we have

$$\|C_{m,l}\|_{L_2} = \sup_k \|C_{m,l}(k)\|_{U_n}, \qquad (3)$$

where $C_{m,\ell}(k)$ is the transform operator of the difference scheme (2) transforming $C^\ell(k)$ to $C^m(k)$.

If the estimate

$$\|C_{m+1,m}(k)\|_{U_n} \leqslant 1 + C(\bar{t})\tau, \qquad (4)$$

is valid for the step operator $C_{m+1,m}(k)$ of system (2), then by virtue of (3) scheme (1.4.1) is stable in L_2.

Thus, equality (3) enables us to reduce the stability problem to a purely algebraic problem of defining the norm of the step operator $C_{m+1,m}(k)$.

We know that the spectral radius of a matrix, that is, the maximum modulus of its characteristic root, does not exceed the matrix norm. Hence we obtain the necessary stability criterium (Neumann criterion):

For the scheme (1.4.1) to be uniformly stable, it is necessary that the inequality

$$R_\lambda (C_{m+1,m}(\kappa)) \leqslant 1 + M(\bar{t})\tau , \qquad (5)$$

be valid, where R_λ (C) is the spectral radius of matrix C.

For the case when $C_{m+1,m}(k)$ is a normal matrix, that is, interchangeable with its adjoint counterpart, its norm coincides with the spectral radius and criterion (5) is a necessary and sufficient condition for uniform stability of scheme (1.4.1). A detailed analysis of different estimates of matrical norms can be found in [9].

The stability estimate for schemes with coefficients dependent on x, t is much more involved. Here we use the following methods of estimating the correctness of difference problem (1.4.1), (1.4.2):

1. method of majorant or a priori estimate

2. local algebraic method.

The simplest majorant estimate is an estimate for schemes with positive coefficients. Let us consider for the equation

$$\frac{\partial r}{\partial t} + \xi(x, t)\frac{\partial r}{\partial x} = g(x, t) r, \quad \xi(x, t) > 0, \qquad (6)$$

the following difference scheme:

$$\frac{r^{m+1}(x) - r^m(x)}{\tau} + \xi^m(x)\frac{r^m(x) - r^m(x-h)}{h} = g^m(x) r^m(x). \qquad (7)$$

We rewrite scheme (7) in the form

$$r^{m+1}(x)=[1-\varkappa^m(x)]r^m(x)+\varkappa^m(x)r^m(x-h)+\tau g^m(x)r^m(x), \quad (8)$$

where $\varkappa^m(x)=\xi^m(x)\frac{\tau}{h}$.

Under the condition $0 \leq 1 - \varkappa^m(x) \leq 1$ (9)

the estimate

$$|r^{m+1}(x)| \leqslant [1-\varkappa^m(x)]|r^m(x)|+\varkappa^m(x)|r^m(x-h)|+ +\tau|g^m(x)||r^m(x)| \quad (10)$$

is valid. Selecting as the norm of the solution the quantity

$$\|r^m\|=\max_x|r^m(x)|, \quad (11)$$

we find from (10)

$$\|r^{m+1}\| \leqslant (1+g\tau)\|r^m\|, \quad (12)$$

where we denote

$$g=\max_{m,x}|g^m(x)|. \quad (13)$$

Hence follows uniform stability of schemes (7) and the space $C(-\ell, \ell)$ with norm (11).

Let us show uniform stability of (7) in $L_2(-\ell, \ell)$*). Multiplying equality (8) by $r^{m+1}(x)$ and using (9), we have

$$[r^{m+1}(x)]^2=[1-\varkappa^m(x)]r^m(x)r^{m+1}(x)+\varkappa^m(x)r^m(x-h)r^{m+1}(x)+ +\tau g^m(x)r^m(x)r^{m+1}(x) \leqslant \frac{1-\varkappa^m(x)}{2}[r^m(x)]^2+ +\frac{1-\varkappa^m(x)}{2}[r^{m+1}(x)]^2+\frac{\varkappa^m(x)}{2}[r^m(x-h)]^2+ +\frac{\varkappa^m(x)}{2}[r^{m+1}(x)]^2+\tau g\frac{[r^m(x)]^2+[r^{m+1}(x)]^2}{2}= =\frac{1}{2}[r^{m+1}(x)]^2+\frac{1-\varkappa^m(x)}{2}[r^m(x)]^2+\frac{\varkappa^m(x)}{2}[r^m(x-h)]^2+ +\tau g\frac{[r^m(x)]^2+[r^{m+1}(x)]^2}{2}. \quad (14)$$

*) Stability in $L_2(-\ell, \ell)$ follows from the stability in $C(-\ell, \ell)$ only for finite ℓ.

From whence

$$\left(\frac{1}{2}-g\frac{\tau}{2}\right)[r^{m+1}(x)]^2 \leqslant \frac{1-\varkappa^m(x)}{2}[r^m(x)]^2+\frac{\varkappa^m(x)}{2}[r^m(x-h)]^2$$
$$+\frac{g\tau}{2}[r^m(x)]^2=\frac{1-\varkappa^m(x)}{2}[r^m(x)]^2+\frac{\varkappa^m(x-h)}{2}[r^m(x-h)]^2$$
$$+\frac{g\tau}{2}[r^m(x)]^2+\frac{\varkappa^m(x)-\varkappa^m(x-h)}{2}[r^m(x-h)]^2. \qquad (15)$$

Integrating inequality (15) with respect to x within the limits from $-\ell$ to ℓ,

$$\|r^{m+1}\| \leqslant \frac{1+g\tau+C\tau}{1-g\tau}\|r^m\|^2=[1+O(\tau)]\|r^m\|^2, \qquad (16)$$

where we put

$$C=\sup_{h,x,m}\frac{h}{\tau}\left|\frac{\xi^m(x)-\xi^m(x-h)}{h}\right|, \quad \|r^m\|^2=\int_{-l}^{l}[r^m(x)]^2\,dx. \qquad (17)$$

Whence follows the uniform stability of scheme (7) in L_2 if condition (9) is satisfied and $\xi^m(x)$ is Lipschitz as continuous.

These estimates are transformed without making changes to schemes approximating systems of equations in invariants

$$\frac{\partial r_k}{\partial t}+\xi_k\frac{\partial r_k}{\partial x}=g_k^a r_a, \quad a, k=1, \ldots, n. \qquad (18)$$

K. O. Friedrichs (cf [19]) introduced general concept of positive schemes — difference schemes with positive matrices — and established for them the sufficiency criterion of correctness in L_2. We formulate the Friedrichs criterion, limiting ourselves to the case of a single space variable.

Suppose the linear system $\partial u/\partial t = A(x, t)\,\partial u/\partial x$ (19)

is approximated by the explicit difference scheme

$$u^{m+1}(x)=\sum_{\alpha=-e}^{e_1}B_\alpha(x, t, \tau, h)\,u^m(x+\alpha h). \qquad (20)$$

Here $A = \|A_{ij}\|$, $B_\alpha = \|B^\alpha_{ij}\|$ are matrices in the space of the components of vector u.

Scheme (20) can be also written as

$$u^{m+1} = C_\alpha u^m, \quad C_\alpha = B_\alpha T_\alpha, \quad \alpha = -q_1, \dots, q_1. \tag{21}$$

Suppose scheme (21) satisfies the condition $\sum_{\alpha=-q_1}^{q_1} B_\alpha = E$, (22)

which signifies that the constant vector $u^m(x) \equiv u$ = constant is a solution of (21). Then scheme (20) approximate equation (19) given the condition

$$\frac{h}{\tau} \sum_{\alpha=-q_1}^{q_1} \alpha B_\alpha = \bar{A}. \tag{23}$$

The Friedrichs criterion is formulated thusly:

Scheme (21) is correct in L_2 if matrices B_α are symmetric, positive, and Lipschitz-continuous relative to x, so that the condition

$$\frac{h}{\tau} \left\| \frac{B_\alpha(x+h) - B_\alpha(x)}{h} \right\| < b. \tag{24}$$

is satisfied.

Multiplying equality (20) scalarly*) by u^{m+1}, and taking condition (22) and inequalities

$$(B_\alpha u, v) \leqslant \sqrt{(B_\alpha u, u)} \sqrt{(B_\alpha v, v)} \leqslant \frac{(B_\alpha u, u) + (B_\alpha v, v)}{2}, \tag{25}$$

to account, we have

$$
\begin{aligned}
(u^{m+1}, u^{m+1}) &= \sum_{\alpha=-q_1}^{q_1} (B_\alpha(x) u^m(x+\alpha h), u^{m+1}(x)) \leqslant \\
&\leqslant \frac{1}{2}\left(\sum_{\alpha=-q_1}^{q_1} B_\alpha(x) u^{m+1}(x), u^{m+1}(x) \right) + \\
&+ \frac{1}{2} \sum_{\alpha=-q_1}^{q_1} (B_\alpha(x) u^m(x+\alpha h), u^m(x+\alpha h)) = \\
&= \frac{1}{2}(u^{m+1}(x), u^{m+1}(x)) + \\
&+ \frac{1}{2} \sum_{\alpha=-q_1}^{q_1} (B_\alpha(x+\alpha h) u^m(x+\alpha h), u^m(x+\alpha h)) - \\
&- \frac{1}{2} \sum_{\alpha=-q_1}^{q_1} ([B_\alpha(x+\alpha h) - B_\alpha(x)] u^m(x-\alpha h), u^m(x+\alpha h)).
\end{aligned} \tag{26}
$$

[*) on following page]

Integrating inequality (26) within the limits from $-\ell$ to ℓ, we find

$$\|u^{m+1}\|^2 \leqslant \frac{1}{2}\|u^{m+1}\|^2 + \frac{1}{2}b\tau\|u^m\|^2 + \frac{1}{2}\int_{-l}^{l}\sum_{\alpha=-q_1}^{q_1}(B_\alpha(x+\alpha h)u^m(x+\alpha h),\ u^m(x+\alpha h))\,dx, \qquad (27)$$

where we assume

$$\|u\|^2 = \int_{-l}^{l}(u(x),\ u(x))\,dx.$$

Taking the periodicity of function u(x, t) and condition (22) into account, we make the following transformation:

$$\int_{-l}^{l}\sum_{\alpha=-q_1}^{q_1}(B_\alpha(x+\alpha h)u^m(x+\alpha h),\ u^m(x+\alpha h))\,dx =$$

$$= \sum_{\alpha=-q_1}^{q_1}\int_{-l}^{l}(B_\alpha(x+\alpha h)u^m(x+\alpha h),\ u^m(x+\alpha h))\,dx =$$

$$= \sum_{\alpha=-q_1}^{q_1}\int_{-l}^{l}(B_\alpha(x)u^m(x),\ u^m(x))\,dx = \int_{-l}^{l}\sum_{\alpha=-q_1}^{q_1}(B_\alpha u^m(x),\ u^m(x))\,dx =$$

$$= \int_{-l}^{l}\left(\left(\sum_{\alpha=-q_1}^{q_1}B_\alpha\right)u^m(x),\ u^m(x)\right)dx = \|u^m\|^2.$$

Finally we have

$$\|u^{m+1}\|^2 \leqslant (1+b\tau)\|u^m\|^2. \qquad (28)$$

The assertions stand proven.

Schemes with positive coefficients and matrices represent a bounded, though extremely important class of difference schemes. As a rule, these are

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*) We refer to the scalar product in the space of components $u_1, \ldots, u_n$: $(u, v) = u_\alpha v_\alpha$ $(\alpha = 1, \ldots, n)$.

schemes of first-order accuracy in which the derivatives are approximated by one-sided difference relations. For approximations of higher orders of accuracy when centered differences are taken, as a rule we do not obtain positive coefficients. In this case mojorant estimates of stability are complicated. This kind of estimate is called apriori estimate.

The apriori estimate method for difference schemes is analogous to the corresponding method for differential equations, wherein the difference case its realization encounters major difficulties. This is obviously associated with the specific details of difference analysis in which many relations of ordinary analyses then do not hold or take a more cumbersome form.

Let us consider the apriori estimate method for the example of equations of acoustics (1.3.31), in whose integration will use the implicit scheme:

$$\frac{u^{m+1}-u^m}{\tau}=a^{m+1}\frac{\Delta_1}{h}v^{m+1}, \tag{29}$$

$$\frac{v^{m+1}-v^m}{\tau}=\frac{\Delta_{-1}}{h}u^{m+1}, \quad \Delta_1=T_1-E, \quad \Delta_{-1}=E-T_{-1}. \tag{30}$$

The apriori estimate for this scheme is analogous to the energy inequality for the system (1.3.31) establishing subsection 3. Multiplying (29) by $2u^{m+1}$, (30) by $2a^{m+1}v^{m+1}$, and manipulating, we obtain after the uncomplicated transformation

$$\begin{aligned}&[(u^{m+1})^2+a^{m+1}(v^{m+1})^2]-[(u^m)^2+a^m(v^m)^2]=\\ &=-(u^{m+1}-u^m)^2-a^{m+1}(v^{m+1}-v^m)^2+(a^{m+1}-a^m)(v^m)^2+\\ &\quad+\frac{2\tau}{h}a^{m+1}(u^{m+1}\Delta_1v^{m+1}+v^{m+1}\Delta_{-1}u^{m+1}).\end{aligned} \tag{31}$$

Taking formulas of the difference differentiation of the product

$$\Delta_1(fg)=(\Delta_1f)g+(T_1f)\Delta_1g=\Delta_1fT_1g+f\Delta_1g,$$
$$\Delta_{-1}(fg)=(\Delta_{-1}f)g+(T_{-1}f)\Delta_{-1}g=\Delta_{-1}fT_{-1}g+f\Delta_{-1}g.$$

into account, let us transform the expression $a^{m+1}(u^{m+1}\Delta_1 v^{m+1} + v^{m+1}\Delta_{-1}u^{m+1})$ to the following form:

$$a^{m+1}[u^{m+1}\Delta_1 v^{m+1} + v^{m+1}\Delta_{-1}u^{m+1}] = a^{m+1}[\Delta_1(u^{m+1}v^{m+1}) - T_1 v^{m+1}\Delta_1 u^{m+1} + v^{m+1}\Delta_{-1}u^{m+1}] = a^{m+1}\Delta_1[u^{m+1}v^{m+1} - v^{m+1}\Delta_{-1}u^{m+1}] = a^{m+1}\Delta_1[v^{m+1}T_{-1}u^{m+1}] = \Delta_1[a^{m+1}v^{m+1}T_{-1}u^{m+1}] - \Delta_1 a^{m+1}T_1[v^{m+1}T_{-1}u^{m+1}] = \Delta_1[a^{m+1}v^{m+1}T_{-1}u^{m+1}] - u^{m+1}T_1 v^{m+1}\Delta_1 a^{m+1}. \quad (32)$$

Then, from (31) we can obtain without difficulty the equality

$$[(u^{m+1})^2 + a^{m+1}(v^{m+1})^2] - [(u^m)^2 + a^m(v^m)^2] \leqslant \frac{2\tau}{h}\Delta_1(a^{m+1}v^{m+1}T_{-1}u^{m+1}) + b_1\tau[(u^m)^2 + a^m(v^m)^2] + b_2\tau[(u^{m+1})^2 + T_1 a^{m+1}(T_1 v^{m+1})^2], \quad (33)$$

where

$$b_1 = \sup_{x,m,\tau}\left|\frac{a^{m+1} - a^m}{a^m\tau}\right|, \quad b_2 = \sup_{x,m,h}\left\{\frac{1}{\sqrt{T_1 a^{m+1}}}\left|\frac{\Delta_1 a^{m+1}}{h}\right|\right\}. \quad (34)$$

Integrating (33) relative to x, we obtain

$$\|\Phi^{m+1}\|^2 - \|\Phi^m\|^2 \leqslant b_1\tau\|\Phi^m\|^2 + b_2\tau\|\Phi^{m+1}\|^2, \quad (35)$$

where

$$\|\Phi\|^2 = \int_{-\pi}^{\pi}(u^2 + av^2)\,dx. \quad (36)$$

Hence follows the estimate

$$\|\Phi^{m+1}\|^2 \leqslant \frac{1+b_1\tau}{1-b_2\tau}\|\Phi^m\|^2. \quad (37)$$

which proves the uniform correctness of the scheme (29), (30) for the case of Lipschitz-continuous function a(x, t).

Similar estimates are established for the scheme in which the quantities in the upper layer appear with weight α and in lower layer with weight β :

$$\left.\begin{aligned} &\frac{u^{m+1}-u^{m}}{\tau}=\alpha_1 a^{m+1}\frac{\Delta_1}{h}v^{m+1}+\beta_1 a^{m}\frac{\Delta_1}{h}v^{m},\\ &\frac{v^{m+1}-v^{m}}{\tau}=\frac{\Delta_{-1}}{h}[\alpha_2 u^{m+1}+\beta_2 u^{m}],\\ &\alpha_i+\beta_i=1,\quad \alpha_i>0,\quad \beta_i>0 \qquad (i=1,2). \end{aligned}\right\} \tag{38}$$

Now let us proceed to the local algebraic method of investigating correctness. Underlining this method is the study of the properties of a local difference operator.

Suppose

$$\Lambda=b_{\beta_0\beta_1}(x,t,\tau,h)T_0^{\beta_0}T_1^{\beta_1},\quad \beta_0=-q_0,\ldots,q_0,\quad \beta_1=-q_1,\ldots,q_1,$$

is a difference operator with variable coefficients. Then the operator with constant coefficients $\Lambda(\bar{x},\bar{t}) = b_{\beta_0\beta_1}(\bar{x},\bar{t},\tau,h)T_0^{\beta_0}T_1^{\beta_1}$, corresponding to the fixed value $x = \bar{x}$, $t = \bar{t}$, will be called the local operator corresponding to operator Λ at the point $x = \bar{x}$, $t = \bar{t}$. Local criteria of the stability of a scheme enables us to establish this stability of the difference schemes, based on the stability properties of the local difference operator of the scheme. Local stability criteria are thus a generalization of the method of freezing coefficients employed in the theory of differential equations (cf subsection 3).

We consider the local algebraic method for the example of the scheme for equations of acoustics with variable speed of sound. Let us show the equivalence of system (38) to a second-order difference equation.

Write system (38) in operator form:

$$\frac{\Delta_0}{\tau} u^m = (\alpha_1 a^{m+1} T_0 + \beta_1 a^m E) \frac{\Delta_1}{h} v^m. \quad (39)$$

$$\frac{\Delta_0}{\tau} v^m = (\alpha_2 T_0 + \beta_2 E) \frac{\Delta_{-1}}{h} u^m, \quad \Delta_0 = T_0 - E. \quad (40)$$

Multiplying equation (39) operator by operator by $(\alpha_2 T_0 + \beta_2 E)\frac{\Delta_{-1}}{h}$, equation (40) by Δ_0/τ , and manipulating, we cancel out u^m and obtain for v^m the equation

$$\left(\frac{\Delta_0}{\tau}\right)^2 v^m = \frac{v^{m+2} - 2v^{m+1} + v^m}{\tau^2} =$$
$$= (\alpha_2 T_0 + \beta_2 E) \frac{\Delta_{-1}}{h} (\alpha_1 a^{m+1} T_0 + \beta_1 a^m E) \frac{\Delta_1}{h} v^m =$$
$$= \frac{\Delta_{-1}}{h} \left[\alpha_1\alpha_2 a^{m+2} \frac{\Delta_1}{h} v^{m+2} + \left(\alpha_1\beta_2 + \alpha_2\beta_1\right) a^{m+1} \frac{\Delta_1}{h} v^{m+1} + \beta_1\beta_2 a^m \frac{\Delta_1}{h} v^m\right].$$

Shifting the time index by 1 and denoting

$$\alpha_1\alpha_2 = \gamma_1, \quad \alpha_1\beta_2 + \alpha_2\beta_1 = \gamma_0, \quad \beta_1\beta_2 = \gamma_{-1}, \quad (41)$$

we obtain a three-layer scheme

$$\frac{v^{m+1} - 2v^m + v^{m-1}}{\tau^2} = \frac{\Delta_{-1}}{h} \sum_{s=-1}^{1} \gamma_s a^{m+s} \frac{\Delta_1}{h} v^{m+s}. \quad (42)$$

Clearly, γ_{-1}, γ_0, γ_1 satisfy the condition $\gamma_{-1} + \gamma_0 + \gamma_1 = 1$. (43)

For further analysis it is more convenient to us to convert to the space of grid functions defined over the grid

$$\left.\begin{array}{ll} x = lh & (l = 0, 1, 2, \ldots, N+1); \\ t = m\tau & (m = 0, 1, \ldots, M), \quad (N+1)h = \pi, \quad M\tau = \bar{t}. \end{array}\right\} \quad (44)$$

For equation (42), which we will rewrite in indicial form, let us formulate the mixed problem:

$$\frac{v_l^{m+1}-2v_l^m+v_l^{m-1}}{\tau^2}=\frac{\sum_{s=-1}^{1}\gamma_s a_{l+\frac{1}{2}}^{m+s}\left(v_{l+1}^{m+s}-v_l^{m+s}\right)}{h^2}-\frac{\sum_{s=-1}^{1}\gamma_s a_{l-\frac{1}{2}}^{m+s}\left(v_l^{m+s}-v_{l-1}^{m+s}\right)}{h^2} \tag{45}$$

$$\left(a_{l+\frac{1}{2}}^{m+s}=a\left[\left(l+\frac{1}{2}\right)h,\ (m+s)\tau\right];\ l=1,2,\ldots,N\right).$$

$$\left.\begin{array}{l} v_l^0=\varphi(lh)=\varphi_l,\quad \dfrac{v_l^1-v_l^0}{\tau}=\psi(lh)=\psi_l \\ (l=1,\ \ldots,N), \\ v_0^m=0,\quad v_{N+1}^m=0. \end{array}\right\} \tag{46}$$

At each step $t = t_m$ the solution v^m of the difference problems (45) and (46) can be given by the N-dimensional vector

$$v^m=\{v_0^m=0,\ v_1^m,\ \ldots,\ v_N^m,\ v_{N+1}^m=0\}.$$

We will consider the case when $a(x, t) = b(t)a(x)$. (47)

Let us formulate for equation (42) the difference boundary problem for the eigenvalues: find the eigenfunction $\overset{k}{w} = \{\overset{k}{w}_0, \ldots, \overset{k}{w}_{N+1}\}$ and the eigenvalue λ_k of the equation

$$\Lambda\overset{k}{w}=\left(\frac{\Delta_{-1}}{h}a(x)\frac{\Delta_1}{h}\right)\overset{k}{w}=\lambda_k\overset{k}{w}, \tag{48}$$

if

$$\overset{k}{w}_0=\overset{k}{w}_{N+1}=0. \tag{49}$$

This problem is the difference analog of the Sturm-Liouville problem for a self-adjoint second-order differential equation. The indicial notation of equations (48) is of the form

$$\frac{a_{l+\frac{1}{2}}(\overset{k}{w}_{l+1}-\overset{k}{w}_l)-a_{l-\frac{1}{2}}(\overset{k}{w}_l-\overset{k}{w}_{l-1})}{h^2}=\lambda_k\overset{k}{w}_l \tag{50}$$

$$(k=1,\ \ldots,N;\quad l=1,2,\ \ldots,N).$$

The matrix of equations (49) and (50) is a symmetrical three-diagonal Jacobian matrix. As we know (cf for example [47]), this matrix is simple in structure and when $a(x) > 0$ its eigenvalues λ_k are real and negative. To each of these eigenvalues λ_k there corresponds the function $\overset{k}{w}$ of the problem (48), (49), defined with an accuracy up to the arbitrary multiplier. The normed system of N eigenfunctions $\overset{1}{w}, \ldots, \overset{N}{w}$ of problems (48), (49) forms an orthonormed basis in the space E_N of vectors with N components. Therefore the vector $v^m = \{0, v_1^m, \ldots, v_N^m, 0\}$, being the solution of the problem (45), (46), can be uniquely represented if the form of a combination of vectors $\overset{k}{w}$:

$$v_i^m = \rho_\alpha^m \overset{\alpha}{w}_i \, {}^{*)} \qquad (\alpha = 1, \ldots, N). \tag{51}$$

Substituting representation (51) into difference equation (45), taking (48) and (47) into account, and noting the linear dependence of vectors $\overset{k}{w}$, we obtain

$$\frac{\rho_k^{m+1} - 2\rho_k^m + \rho_k^{m-1}}{\tau^2} = \left[\gamma_1 b^{m+1}\rho_k^{m+1} + \gamma_0 b^m \rho_k^m + \gamma_{-1} b^{m-1}\rho_k^{m-1}\right]\lambda_k. \tag{52}$$

where $b^m = b(m\tau)$.

The problem of defining the stability of difference equation (42) or, which amounts to the same thing, system (39), (40) reduces to defining the stability of equation (52) where $k = 1, 2, \ldots, N$.

Difference scheme (42) or (39), (40) is stable providing condition (47) is met, if for sufficiently small $\tau \leq \tau_0$ the estimate

$$|\rho_k^m| \leqslant K(\tau_0). \quad k = 1, 2, \ldots, N; \quad m \leqslant \frac{\bar{t}}{\tau} = M. \tag{53}$$

is valid. The problem is solved especially simply when $a(x, t)$ does not depend

---

*) Recall that summation is to be carried out with respect to the repeating Greek subscripts.

on t. Then we can put $b(t) = 1$ and system (52) becomes

$$\frac{\rho_k^{n+1} - 2\rho_k^n + \rho_k^{n-1}}{\tau^2} = (\gamma_1 \rho_k^{n+1} + \gamma_0 \rho_k^n + \gamma_{-1} \rho_k^{n-1}) \lambda_k \tag{54}$$

The solution ρ_k^n of system (54) is given by the formula

$$\rho_k^n = C_k^1 (\overset{1}{z}_k)^n + C_k^2 (\overset{2}{z}_k)^n. \tag{55}$$

where $\overset{1}{z}_k$, $\overset{2}{z}_k$ are the roots of the characteristic equation*) (multiplication coefficients)

$$\frac{z_k^2 - 2z_k + 1}{\tau^2} = (\gamma_1 z_k^2 + \gamma_0 z_k + \gamma_{-1}) \lambda_k \tag{56}$$

In this case the criterion on uniform stability is of the form

$$|\overset{s}{z}_k| \leqslant 1 + C\tau \quad (k = 1, 2, \ldots, N;\ s = 1, 2), \tag{57}$$

where $C > 0$ is a constant not dependent on τ, h, k.

This stability criterion is not effective enough, since for arbitrary $a(x)$ λ_k are unknowns. We can formulate the following effective stability criterion:

Suppose $\overset{1}{z}_k(\bar{x})$, $\overset{2}{z}_k(\bar{x})$ are the multiplication coefficients and $\lambda_k(\bar{x})$ are the eigenvalues corresponding to local operators $\Lambda(\bar{x})$. If the estimates

$$|\overset{s}{z}_k(\bar{x})| \leqslant 1 + C\tau \quad (k = 1, 2, \ldots, N;\ s = 1, 2) \tag{58}$$

hold for all $\bar{x}$ and if the constant C does not depend on $\bar{x}$, k, then difference scheme (54) is correct.

This local criterion of stability reduces the problem of determining the stability of operator Λ with variable coefficient $a(x)$ to a purely algebraic

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*) Investigations of stability were similarly pursued in the works [20], [21]. For difference boundary problems, cf also [22].

problem of determining the stability of the local operator $\Lambda(\bar{x})$.

Actually, for local operator $\Lambda(\bar{x})$ equation (50) becomes

$$\varepsilon \frac{v^k_{i+1} - 2v^k_i + v^k_{i-1}}{h^2} = \lambda_k v^k_i, \quad \varepsilon = \varepsilon(\bar{x}).$$

Hence follows that eigenfunction of the problem (48), (49) for a local operator $v^k = \sin kx$ $(x = ih,\ i,\ k = 1, \dots, N)$

$$\lambda_k = -\frac{4\varepsilon \sin^2 \frac{kh}{2}}{h^2}.$$

In the particular case when $\nu_1 = \nu_{-1} = 0$, $\nu_0 = 1$ ("cross" scheme, cf example 3 following subsection), conditions (57), (58) are satisfied,

$$x(\bar{x}) = \frac{\sqrt{\varepsilon(\bar{x})}\,\tau}{h} < 1. \qquad (59)$$

If condition (59) is satisfied at each point $\bar{x} = jh$, $j = 1, \dots, N$, then scheme (42) providing $\nu_1 = \nu_{-1} = 1$ will be stable. Condition (59) is the local Courant criterion*).

Let us prove, following the work of Yu. Ye. Boyarintsev [24], the formulated stability criterion (58). This criterion follows from the property of monotonicity of the characteristic roots of the Jacobian matrices.

We will consider the Jacobian symmetrical matrices $A = \{a_{ij}\}$, $i, j = 1, \dots, N$, of the form

$$a_{ij} = a_{i-1/2}\delta_{i-1}^{j} - (a_{i-1/2} + a_{i+1/2})\delta_i^j + a_{i+1/2}\delta_i^{j+1}. \qquad (60)$$

corresponding to positive functions $a(x)$. Suppose $A_s = \{a^s_{ij}\}$, where A_1 corresponds to $a_1(x)$, A_2 - $a_2(x)$, $a_s(x)$ 0.

Let us consider the pencil of matrices $B = A_1 + (A_2 - A_1)$. (61)

*) Criterion (59) was first formulated as the condition for the convergence of the "cross" difference scheme for the weighed equation with the constant speed of sound a in the work by R. Courant, K. Friedrichs, and G. Lewy [25].

We have for the normed eigenvector $x(\alpha)$ of matrices $B(\alpha)$ and the eigenvalue $\lambda(\alpha)$ corresponding to it

$$Bx(\alpha) = \lambda(\alpha)\, x(\alpha), \quad (62)$$

$$\frac{dB}{d\alpha} x(\alpha) + B\frac{dx}{d\alpha} = \frac{d\lambda}{d\alpha} x(\alpha) + \lambda \frac{dx}{d\alpha}, \quad (63)$$

$$\frac{dB}{d\alpha} = (A_2 - A_1). \quad (64)$$

Multiplying equality (63) scalarly by x, and taking the orthogonality of x and $dx/d\alpha$ into account, along with equality (64), we obtain

$$((A_2 - A_1)x,\ x) = \frac{d\lambda}{d\alpha}. \quad (65)$$

If $a_2(x) \geq a_1(x) > 0$, then $((A_2 - A_1)x,\ x) \leq 0$, and from equality (65) follows

$$\lambda_{2k} \leqslant \lambda_{1k} < 0, \quad |\lambda_{2k}| \geqslant |\lambda_{1k}|, \quad (66)$$

where λ_{sk} are the eigenvalues of matrix A_s.

Thus, we have proven the monotonic dependence of the roots λ_k on the function $a(x)$.

Let us now consider the dependence of the roots z_k of equation (56) on $_k$. We assume for simplicity $\nu_1 = \nu_{-1} = 0$, $\nu_0 = 1$. Then for z_k we have the expression

$$z_k = 1 - \frac{1}{2}\tau^2|\lambda_k| \pm \sqrt{\left(1 - \frac{1}{2}\tau^2|\lambda_k|\right)^2 - 1}.$$

If the roots (56) are conjugate, then they both equal modulus 1, since their product equals 1. If the roots z_k are real, then the following estimate is valid:

$$\max\{|\overset{1}{z}_k|, |\overset{2}{z}_k|\} = \frac{1}{2}\tau^2|\lambda_k| + \frac{1}{2}\sqrt{|\lambda_k|\tau(|\lambda_k|\tau^2 - 4)} - 1.$$

Hence it follows that corresponding to the larger value $|\lambda_k|$ is the larger value $\max\limits_s |\overset{s}{z}_k|$. These properties of monotonicity of λ_k, $\max\limits_s |\overset{s}{z}_1|$ enables us to prove the local stability criterion. Actually, if all local schemes are stable, then in particular the scheme with $a_o = \max a(x)$ is stable. But then, by virtue of the property of the monotonicity of λ_k, the eigenvalues λ_k corresponding to $a(x) \leq a_o$ are smaller moduluswise than the corresponding eigenvalues of operator Λ with constant coefficient a_o. Based on the property of monotonicity, the quantity $\max\limits_s |z_k^s|$ for the equation with coefficient $a(x)$ will also be less than the corresponding quantity for the equation with $a(x) = a_o$. Hence follows the stability of the scheme with variable coefficient $a(x)$, which is thus proven.

For the case of explicit schemes for the system of hyperbolic equation

$$\partial u/\partial t + A\,\partial u/\partial x = 0$$

Lax [23] proved the validity of the local Neumann criterion. Yu. Ye. Boyarintsev proved the local criterion for the implicit scheme (42) of the general form, when $a(x, t)$ is not of the form (47) and the local operators Λ depend on x, t, and are noncommutative.

The local criterion of Yu. Ye. Boyarintsev is founded on the following theorem of comparison:

If the difference scheme (42) is stable for $a(x, t) = a_2(x, t)$, then it is stable also for $a(x, t) = a_1(x, t) \leq a_2(x, t)$. If the difference scheme (42) is unstable for $a(x, t) = a_1(x, t)$, then it is unstable also for $a(x, t) = a_2(x, t) \geq a_1(x, t)$.

Hence at once must follow the local criterion of stability for (42): if all local schemes (42) are stable (unstable), then scheme (42) is stable (unstable) as well. We refer the reader to the work [24] for the proof of these theorems.

The local stability criterion ......

[missing pages 359-370 in text]

.... a similar pattern is also observed in equations with partial derivatives. For example, for the equation

$$\frac{\partial r}{\partial t} + a\frac{\partial r}{\partial x} + br = 0, \qquad (45)$$

where a and b are positive constants, the explicit scheme of running computation

$$\frac{r^{m+1}-r^{m}}{\tau} + a\frac{\Delta_{-}r^{m}}{h} + br^{m} = 0$$

is stable where $\mathcal{H} = \frac{a\tau}{h} \leq 1$, but is not strongly stable. Actually, when $r_0(x)$ = constant, we have

$$r^{m} = (1-b\tau)r^{m-1} = \ldots = (1-b\tau)^{m} r_0.$$

When $b\tau > 2$, we get the solution of r^m, which for sufficiently large m deviates from the solution of equation (45). Note that for sufficiently large b the restriction $\tau < 2/b$ can be stronger than the restriction due to Courant's criterion $a\tau/h \leq 1$. The implicit scheme

$$\frac{r^{m+1}-r^{m}}{\tau} + a\frac{\Delta_{-}r^{m+1}}{h} + br^{m} = 0$$

is also not strongly stable.

The implicit scheme

$$\frac{r^{m+1}-r^{m}}{\tau} + a\frac{\Delta_{-}r^{m+1}}{h} + br^{m+1} = 0$$

is strongly stable.

7. Dispersion analysis of difference schemes. Approximation viscosity. Suppose the difference equation

$$\Lambda u = b_{\beta_0\beta_1} T_0^{\beta_0} T_1^{\beta_1} u = 0, \quad \beta_0 = -q_0, \ldots, q_0; \ \beta_1 = -q_1, \ldots, q_1. \qquad (1)$$

approximates the differential equation

$$\Omega u = a_{\alpha_0\alpha_1} D_0^{\alpha_0} D_1^{\alpha_1} u = 0, \quad \alpha_0 = 0, \ldots, p_0; \ \alpha_1 = 0, \ldots, p_1. \qquad (2)$$

If the matrices $a_{\alpha_0\alpha_1}$ and $b_{\beta_0\beta_1}$ have constant coefficients, then we employ a Fourier analysis to investigate both stability and approximation.

For the harmonic $u = u_0 e^{\omega t+ikx}$ to be the solution of equation (1) or else of equation (2), ω and k must satisfy the condition Det $\|\bar{\Lambda}\| = 0$ (3) or, correspondingly, Det $\|\bar{\Omega}\| = 0$, (4) where the matrices

$$\bar{\Lambda} = b_{\beta_0\beta_1} e^{\beta_0\omega\tau} e^{ik\beta_1 h}, \quad \bar{\Omega} = a_{\alpha_0\alpha_1}\omega^{\alpha_0}(ik)^{\alpha_1} \qquad (5)$$

are spectral images of operators Λ and Ω.

Equations (3) and (4) are called dispersion relations, corresponding to equations (1) and (2).

The solution $\omega = \omega(\tau, h, k)$ ($\omega = \omega(k)$) of equation (3) (and those of (4)) yields complete information about the properties of solutions of equation (1) (and those of (2)).

Thus, the approximation condition $\Lambda \sim \Omega$ is of the form $\omega(\tau, h, k) \to \omega(k)$ (6) for an arbitrarily thick k and τ, $h \to 0$. If relation (6) is valid for arbitrary approach of τ and h to zero, then the approximation is absolute, otherwise it is conditional.

Equation (1) defines a stable scheme if for sufficiently small τ and h $\mathrm{Re}\,\omega(\tau, h, k) \leq \mu$, (7) and the constant μ does not depend on k*).

The scheme is absolutely stable if estimate (7) holds when $0 < \tau^2 + h^2 \leq \tau_0^2$, and it is conditionally stable or unstable otherwise.

We will limit ourselves to considering the hyperbolic system

$$\partial u/\partial t + A\partial u/\partial x = 0, \qquad (8)$$

where A is a constant matrix with real distinct eigenvalues $\xi = \xi_i$ (i = 1, ..., n).

Proceeding to invariants, we get the system

$$\frac{\partial r_i}{\partial t} + \xi_i \frac{\partial r_i}{\partial x} = 0 \quad (i = 1, \ldots, n). \qquad (9)$$

The dispersion equation is of the form

$$(\omega + \xi_1 ik)(\omega + \xi_2 ik) \ldots (\omega + \xi_n ik) = 0. \qquad (10)$$

Thus, for the case of a hyperbolic system we have

$$\mathrm{Re}\,\omega = 0, \quad |\rho| = 1, \quad \rho = e^{\omega t}. \qquad (11)$$

Suppose

$$\Lambda u = \left(\frac{T_0 - E}{\tau} + \Lambda_1 T_0 + \Lambda_0\right) u = 0 \qquad (12)$$

is a difference scheme corresponding to (8).

---

*) Condition (7) is the familiar Neumann condition which, generally speaking, is only a necessary condition for the stability of scheme (1). In several cases it is also a sufficient condition (cf [9]).

We will state that scheme (12) exhibits approximational viscosity if

$$\left.\begin{array}{lll} |\rho| < 1 & \text{for } k \neq 0, & \\ |\rho| = 1 & \text{for } k = 0, & \rho = e^{\omega(\tau, h, k)}, \end{array}\right\} \qquad (13)$$

where $\omega(\tau, h, k)$ is the solution of the dispersion equation corresponding to equation (12).

For example, for equations of acoustics let us compare approximations of the running computation type (1.6.24):

$$\Lambda\begin{pmatrix} u \\ v \end{pmatrix} = \begin{Vmatrix} \frac{T_0 - E}{\tau} - a\frac{h}{2}\frac{\Delta_1\Delta_{-1}}{h^2} & -a^2\frac{\Delta_1 + \Delta_{-1}}{2h} \\ -\frac{\Delta_1 + \Delta_{-1}}{2h} & \frac{T_0 - E}{\tau} - \frac{ah}{2}\frac{\Delta_1\Delta_{-1}}{h^2} \end{Vmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad (14)$$

and of the "cross" type (1.6.25):

$$\Lambda\begin{pmatrix} u \\ v \end{pmatrix} = \begin{Vmatrix} \frac{T_0 - E}{\tau} & -a^2\frac{\Delta_{-1}}{h} \\ -\frac{\Delta_1 T_0}{h} & \frac{T_0 - E}{\tau} \end{Vmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \qquad (15)$$

For equation (14) we have

$$\rho_{1,2} = 1 - 2\varkappa\sin\frac{kh}{2}\left(\sin\frac{kh}{2} \pm i\cos\frac{kh}{2}\right),$$

$$|\rho_1|^2 = |\rho_2|^2 = 1 - 4\varkappa(1 - \varkappa)\sin^2\frac{kh}{2}.$$

Whence it follows that

$$\left.\begin{array}{lll} |\rho_1| = |\rho_2| < 1, & k \neq 0, & \varkappa < 1, \\ |\rho_1| = |\rho_2| = 1, & k = 0, & \varkappa < 1, \\ |\rho_1| = |\rho_2| = 1, & k = \text{any number}, & \varkappa = 1. \end{array}\right\} \qquad (16)$$

For the "cross" scheme we have

$$\rho_{1,2} = 1 - 2\varkappa^2\sin^2\frac{kh}{2} \pm \sqrt{\left(1 - 2\varkappa^2\sin\frac{kh}{2}\right)^2 - 1}, \qquad (17)$$

$$|\rho_1| = |\rho_2| = 1, \quad \varkappa \leqslant 1, \quad k = \text{any number}$$

From (16) and (17) it follows that the running computation scheme exhibits approximational viscosity where $\varkappa < 1$, the "cross" scheme exhibits this property where $\varkappa \leq 1$, and the running computation scheme does not exhibit approximational viscosity when $\varkappa = 1$.

Let us dwell in more detail on this fact. Inequalities (13) signify that the amplitude of a harmonic solution decays with time. Parabolic differential equations exhibit a similar property.

Thus, the difference equation approximating a hyperbolic equation, providing that condition (13) is met, exhibits the properties of a parabolic equation.

Let us consider scheme (14). As was shown in section VI, it is equivalent to the running computation scheme

$$\frac{r^{m+1}-r^m}{\tau}+a\frac{\Delta_{-1}r^m}{h}=0, \qquad (18)$$

$$\frac{s^{m+1}-s^m}{\tau}-a\frac{\Delta_1 s^m}{h}=0. \qquad (19)$$

Each of these equation approximates the corresponding equation in invariants:

$$\frac{\partial r}{\partial t}+a\frac{\partial r}{\partial x}=0, \quad \frac{\partial s}{\partial t}-a\frac{\partial s}{\partial x}=0,$$

where $r = u - av$, $s = u + av$.

Let us limit ourselves to analyzing one of the difference equations (18). We will show that solution (18) behaves emphatically as a solution of some parabolic equation. We obtain the most characteristic pattern for the case of discrete initial data. Suppose function $r_0(x)$ is of the form

$$r_0(x)=\begin{cases}1, & x\leqslant 0,\\ 0, & x>0.\end{cases} \qquad (20)$$

We will show that when the Courant criterion $\varkappa = a\tau/h \leqslant 1$ is satisfied, scheme (18) exhibits the property of monotonicity (cf [28]), that is, the monotonic profile $r^m(x)$ converts to the monotonic profile $r^{m+1}(x)$. Actually, suppose

$$\Delta_{-1}r_i^m = r_i^m - r_{i-1}^m \leqslant 0 \quad \text{for all } i.$$

From (18) we have

$$r_i^{m+1}=(1-\varkappa)r_i^m+\varkappa r_{i-1}^m,$$

$$\Delta_{-1}r_i^{m+1}=r_i^{m+1}-r_{i-1}^{m+1}=(1-\varkappa)\Delta_{-1}r_i^m+\varkappa\Delta_{-1}r_{i-1}^m\leqslant 0$$

where $\varkappa \leqslant 1$, which in fact proves the assertion.

Let us follow the variation in the profile of $r^m(x)$, which initially had the form of a "shelf" (20).

From equality (18) we have

$$r^{m+1} = Cr^m, \quad C = \alpha E + \beta T_{-1}, \quad \alpha = 1 - \varkappa, \ \beta = \varkappa. \tag{21}$$

$$r^m = C^m r^0 = (\alpha E + \beta T_{-1})^m r^0 = \sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k} \beta^k T_{-k} r^0. \tag{22}$$

Converting to indicial notation, we find

$$r_i^m = \sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k} \beta^k r^0_{i-k}. \tag{23}$$

From formula (23) it follows that where $r_i^m = 1$ when $i \leqslant 0$; $r_i^m \leqslant 1$ when $i \leqslant m$; $r_i^m = 0$ where $i \geqslant m + 1$, and the values of r_i^m $(i = 1, 2, \ldots, m)$ depend monotonically on i, varying from 1 to 0.

Thus, at the m-th step the profile of r_i^m has the form of a curve shown in Figure 3.6.

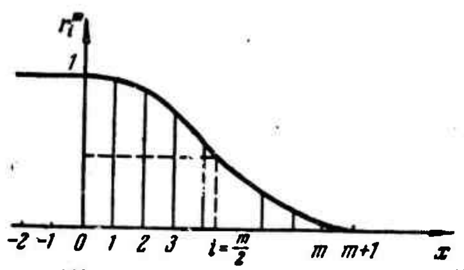

Figure 3.6

For example, where $\varkappa = \frac{1}{2}$, $\alpha = \beta = \frac{1}{2}$ and for an even number of steps $m = 2k$ the mean value $r = \frac{1}{2}$ will lie at the point $k = m/2$. The profile of r_i^m will be symmetric relative to the point $i = k$, $r = \frac{1}{2}$.

The mean value of r will travel over the grid x with velocity a, while the profile becomes symmetrically smoothed relative to the central point. Here we have a complete analogy with the smoothing of the initial discontinuity in Cauchy's problem for the equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}.$$

Clearly, this smoothing of the discontinuity is the result of the operation of approximational viscosity.

An even more graphic example of the operation of approximational viscosity is given by the analysis of the propagation of unit error by means of the so-called ε-scheme (cf [28, 29]).

Suppose that at the time instant t = 0, the initial data of r_i^0 are of the form

$$r_i^0 = \frac{1}{h}\delta_i^0 = \begin{cases} 0, & i \neq 0, \\ \frac{1}{h}, & i = 0. \end{cases} \qquad (24)$$

Then formula (23) yields

$$r_i^m = \frac{1}{h}\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}\beta^k\delta_{i-k}^0 = \frac{1}{h}\binom{m}{i}\alpha^{m-i}\beta^i. \qquad (25)$$

Here the profile of r_i^m is of the form of the binomial law of distribution, and the profile maximum is shifted at the rate dx/dt = a.

Using the analogy with probability theory*), it is not difficult to show the limit profile to which the function tends for suitable norming x = ih and r_i^m.

Using Stirling's formula, we have

$$\binom{m}{i}\alpha^i\beta^{m-i} = \frac{1}{\sqrt{2\pi m\alpha\beta}}e^{-\frac{1}{2}\left(\frac{i-m\alpha}{\sqrt{m\alpha\beta}}\right)^2} + \theta\frac{c_1}{m}, \qquad (26)$$

where c_1 is a quantity dependent on α, but not dependent on i, and $|\theta| < 1$. For large m

$$r_i^m \approx \frac{1}{h\sqrt{2\pi m\alpha\beta}}e^{-\frac{1}{2}\left(\frac{m-i-m\alpha}{\sqrt{m\alpha\beta}}\right)^2} = \frac{1}{h\sqrt{2\pi}}\frac{1}{\sqrt{m\alpha\beta}}e^{-\frac{1}{2}\left(\frac{m\beta-i}{\sqrt{m\alpha\beta}}\right)^2}. \qquad (27)$$

Let us introduce the quantities

$$R^m = r^m\sqrt{m\alpha\beta}, \quad y = \frac{i-m\alpha}{\sqrt{m\alpha\beta}}. \qquad (28)$$

Then

$$R^m(y) \to \frac{1}{h\sqrt{2\pi}}e^{-\frac{y^2}{2}}, \qquad (29)$$

---

*) A. I. Zhukov [30] pointed to the analogy of asymptotic properties of difference decisions with limit theorem of probability theory. Our analysis mainly follows that of A. I. Zhukov.

if m $\to \infty$, and i varies so that y is finite.

We will show that asymptotic properties of the profile of r^m are associated with the asymptotic properties of the operator of solution C from (21). Let S = $T(-a\tau)$ denote the step operator of Cauchy's problem

$$(D_0+aD_1)r=\frac{\partial r}{\partial t}+a\frac{\partial r}{\partial x}=0, \quad r(x,\ 0)=r_0(x), \qquad (30)$$

and let $r(x, m\tau) = \bar{r}^m(x)$ stand for the solution (20), and $r^m(x)$ -- the solution (22). Then

$$\left.\begin{aligned} r^m=C^m r^0, \quad \bar{r}^m=S^m r^0, \\ r^m=C^m S^{-m}\bar{r}^m=(CS^{-1})^m\bar{r}^m. \end{aligned}\right\} \qquad (31)$$

The operator CS^{-1} characterizes the deviation by one step of the difference solution from the exact. Let us convert to spectral images of the operators C, S, CS^{-1}. The corresponding multiplication coefficients are of the form

$$\left.\begin{aligned} &\rho_C=1-\varkappa(1-e^{-ikh})=(1-\varkappa)+\varkappa e^{-ikh}, \quad \rho_S=e^{-ia\tau k}, \\ &\rho_{CS^{-1}}=\rho_C\rho_S^{-1}=(1-\varkappa+\varkappa e^{-ikh})\,e^{iak\tau}= \\ &\qquad =1-\frac{1-\varkappa}{\varkappa}\frac{a^2\tau^2}{2}k^2+O(k^3\tau^3). \end{aligned}\right\} \qquad (32)$$

Now let us consider asymptotic properties of the deviation operator $(CS^{-1})^m$. If we fix k, then as m $\to \infty$, $\tau \to 0$, we have

$$\rho_{(CS^{-1})^m}=\left[1-\frac{1-\varkappa}{\varkappa}\frac{a^2\tau^2}{2}k^2+O(\tau^3k^3)\right]^m\to e^{-b^2k^2t}, \qquad (33)$$
$$b^2=\frac{1-\varkappa}{\varkappa}\frac{a^2\tau}{2}.$$

Thus, the operator of the solution $\Sigma(t)$ of difference equation (18) is asymptotically represented as $\Sigma(t) = S(t)\Omega(t)$, where S(t) is the operator of the solution of differential equation (30), and $\Omega(t)$ is the operator whose spectral images yielded by formula (33).

We can show that the operator of solution $\Sigma(t)$ asymptotically coincides with the operator of the solution of the differential equation

$$\frac{\partial r}{\partial t}+a\frac{\partial r}{\partial x}=b^2\frac{\partial^2 r}{\partial x^2}. \qquad (34)$$

Let us pose for equation (34) a problem with initial data of the δ-function type:

$$r(x, 0) = \begin{cases} 0, & |x| > \frac{h}{2}, \\ \frac{1}{h}, & |x| \leq \frac{h}{2}. \end{cases} \qquad (35)$$

The solution of the Cauchy's problem (34), (35) is of the form

$$r(x, t) \sim \frac{1}{2\sqrt{\pi b^2 t}} e^{-\frac{(x-at)^2}{4b^2 t}}. \qquad (36)$$

as $h \to 0$, the approximation formula (36) becomes exact.

Setting $i = x/h$, $m = t/\tau$, we see that expressions (27) and (36) coincide.

Equation (34) can be derived from equation (18) as follows. The operator

$$\Lambda = \frac{T_0 - E}{\tau} + a\frac{\Delta_{-1}}{h} = \frac{e^{\tau D_0} - E}{\tau} + a\,\frac{E - e^{-hD_1}}{h}$$

is expanded in a Taylor series relative to τ and h with an accuracy up to second-order terms

$$\Lambda \simeq \Omega = D_0 + aD_1 + \frac{1}{2}\tau D_0^2 - \frac{1}{2}ahD_1^2. \qquad (37)$$

Next, expressing D_0 from equation (30): $D_0 = -aD_1$, and inserting in (37), we obtain the differential equation $\frac{\partial r}{\partial t} + a\frac{\partial r}{\partial x} - b^2\frac{\partial^2 r}{\partial x^2} = 0$,

where $b^2 = \frac{1}{2}ah(1 - \kappa)$, which coincides with equation (34).

We will call equation (34) the first differential approximation of difference equation (18).

We can readily note that difference equation (18) approximates equation (34) for solutions $u \in C_3$, and does so for equations (30) with an accuracy of the quantities of the order τ^2 and h^2. Therefore with an accuracy up to quantities of the second order we can state that the difference scheme (18) "adds" to equation (30) the "viscosity" $b^2 \frac{\partial^2 r}{\partial x^2}$.

This algorithm of the first differential approximation of the difference scheme is due to A. I. Zhukov.

The method of the first differential approximation can be extended to the symmetrical difference scheme

$$\left.\begin{aligned} &\frac{u^{m+1}-u^m}{\tau}-a^2\frac{\Delta_1+\Delta_{-1}}{2h}v^m=\nu\frac{\Delta_1\Delta_{-1}}{h^2}u^m, \\ &\frac{v^{m+1}-v^m}{\tau}-\frac{\Delta_1+\Delta_{-1}}{2h}u^m=\nu\frac{\Delta_1\Delta_{-1}}{h^2}v^m, \end{aligned}\right\} \quad (38)$$

which, when $\nu = h^2/2\tau$, is the Lax scheme, where $\nu = a^2\tau/2$ -- it is the predictor-corrector scheme, and when $\nu = ah/2$ -- it is the running computation scheme.

After converting to invariants $r = u - av$, $s = u + av$, scheme (38) decomposed into equations for each of the invariants:

$$\left.\begin{aligned} &\frac{r^{m+1}-r^m}{\tau}+a\frac{\Delta_1+\Delta_{-1}}{2h}r^m=\nu\frac{\Delta_1\Delta_{-1}}{h^2}r^m, \\ &\frac{s^{m+1}-s^m}{\tau}-\frac{\Delta_1+\Delta_{-1}}{2h}s^m=\nu\frac{\Delta_1\Delta_{-1}}{h^2}s^m, \end{aligned}\right\} \quad (39)$$

and asymptotic analysis by means of the first differential approximation is wholly applicable.

Let us note that symmetric schemes (38) are asymptotically monotonic and only one of them -- the running computation scheme -- is strictly monotonic.

In the general case of nonsymmetric approximation, operators of the conversion of difference and differential equations are noncommutative operators and asymptotic analysis becomes complicated. In any case it is clear that properties of monotonicity of nonsymmetric schemes are poorer than for symmetric.

Thus, for the "cross" scheme with viscosity

$$\left.\begin{aligned} &\frac{u^{m+1}-u^m}{\tau}-a^2\frac{\Delta_{-1}u^m}{h}=\nu\frac{\Delta_1\Delta_{-1}}{h^2}u^m=\nu_0 ah^\alpha\frac{\Delta_1\Delta_{-1}}{h^2}u^m, \\ &\frac{v^{m+1}-v^m}{\tau}-\frac{\Delta_1 u^{m+1}}{h}=0 \end{aligned}\right\} \quad (40)$$

the scheme in invariants is of the form

$$
\left.
\begin{aligned}
&\frac{r^{m+1}-r^m}{\tau}+a\frac{\Delta_{-1}}{h}r^m=\frac{ah}{2}\frac{\Delta_1\Delta_{-1}}{h^2}\left[\left(\nu_0h^{\alpha-1}-1-\varkappa\right)s^m+\right.\\
&\qquad\left.+\left(\nu_0h^{\alpha-1}-1+\varkappa\right)r^m\right]-\frac{\nu_0h^\alpha a^2\tau}{2}\frac{\Delta_1^2\Delta_{-1}}{h^3}(r^m+s^m),\\
&\frac{s^{m+1}-s^m}{\tau}-a\frac{\Delta_1}{h}s^m=ah\frac{\Delta_1\Delta_{-1}}{h^2}\left[\frac{\nu_0h^{\alpha-1}-1+\varkappa}{2}s^m+\right.\\
&\qquad\left.+\frac{\nu_0h^{\alpha-1}+1-\varkappa}{2}r^m\right]+\frac{\nu_0h^\alpha a^2}{2}\tau\frac{\Delta_1^2\Delta_{-1}}{h^3}(r^m+s^m).
\end{aligned}
\right\}
\quad (41)
$$

We can readily see that the solution (41) with constant invariant r or s does not exist, whence follows the nonmonotonicity of the difference profile of the acoustic shock wave.

Section II. Method of Constructing Difference Schemes for Equations in Gas Dynamics

1. Methods of describing gas dynamic flows and construction of difference schemes. The nature of the schemes of integration used for equations in gas dynamics depends heavily on the method of describing hydrodynamic flow. In the preceding chapters we employ the following three ways of describing flow:

First method. The region G of the phase plane x, t in which motion is considered is partitioned by strong and weak discontinuities into regions G_i of smooth flow in which equations of gas dynamics are satisfied, while compatibility conditions are satisfied at the discontinuities. In this consideration the generalized solution is a set of smooth solutions defined in the regions G_i and adjoining each other across discontinuity lines with the observance of compatibility conditions. In this description it becomes necessary to numerically integrate the equations of gas dynamics in the region G_i with conditions for contingency at the discontinuity lines being satisfied. Here any identity transformations of the equations are admissible in each of the regions G_i, as is an arbitrary and sufficiently exact difference approximation of equations and contingency conditions.

The best known is the difference method corresponding to the first technique of description, namely the method of characteristics. Actually, among the lines of separation we have weak discontinuities and contact boundaries, which as the characteristics which make the characteristic difference scheme convenient.

A total detailing of the description of a flow, which is a positive feature of the method of characteristics, at the same time, impedes its realization on a computer owing to the complicated logic calculating singularities and constructing of the calculation front.

Of course, the method of characteristics is not the only difference method that can be employed within the framework of a detailed description of flow.

Second method. A generalized solution is defined by integral laws of conservation in Eulerian or Lagrangian coordinates. This description is unique, since both equations in gas dynamics and the compatibility conditions are consequences of the laws of conservation.

Difference schemes corresponding to the second technique of description are obtained by the unique approximation of the laws of conservation independently of the nature of the flow and therefore are called homogeneous schemes or schemes of continuous computation*).

Third method. The generalized solution is defined as the limit of the classical solution of some system of quasilinear parabolic equations with small parameters at the leading derivatives.

If

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=f(u) \qquad (1)$$

is the initial system of equations in gas dynamics written as laws of conservation, then the corresponding parabolic system is of the form

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=f(u)+\frac{\partial}{\partial x}\left(\mu B(u)\frac{\partial u}{\partial x}\right). \qquad (2)$$

*) The concept of homogeneous schemes was introduced and studied -- in a somewhat different sense -- in studies by A. N. Tikhonov and A. A. Samarskiy (cf, for example, [42]).

Here $u = u(x, t)$ is the vector-function describing flow; $f(u), \varphi(u)$ are certain functions of vector argument u; B(u) is a square matrix; and μ is a small parameter*).

The matrix B(u) must be chosen in such a way that the solution u(x, t) of system (2) exhibits sufficient smoothness and as $\mu \to 0$ approximates in some sense to the solution of system (1).

Difference schemes based on the third method of consideration thus partake of the nature of continuous computation schemes. In several cases the second and third approaches lead to identical schemes. We will begin our consideration of schemes with the method of characteristics.

2. Method of characteristics. The method of characteristics is one of the most widespread of the methods of interpreting systems of hyperbolic equations. Its typical feature compared with other difference methods is the minimum use of integration operators and the related maximum proximity of the region of dependence of the difference scheme and the region of dependence of the system of differential equations. Smoothing of profiles that is characteristic of difference schemes with a fixed grid is at a minimum in the method of characteristics, since the grid used in this method is constructed with allowance for the region of dependence of the system.

The method of characteristics has been presented in detail in the monographs [31, 32], therefore we will limit ourselves to a brief presentation of the features of this method and elements of the difference algorithm.

The method of characteristics begins from the approximation of a system of characteristic equations of gas dynamics in a characteristic grid. Here both Eulerian and Lagrangian coordinates can be used.

We will first consider the method of characteristics as applied to a system of quasilinear equations in invariants:

$$\left.\begin{aligned} \left(\frac{\partial s}{\partial t}\right)_1 &= \frac{\partial s}{\partial t} + \xi_1(r, s)\frac{\partial s}{\partial x} = F_1(r, s, x, t), \\ \left(\frac{\partial r}{\partial t}\right)_2 &= \frac{\partial r}{\partial t} + \xi_2(r, s)\frac{\partial r}{\partial x} = F_2(r, s, x, t). \end{aligned}\right\} \qquad (1)$$

*) In some schemes with artificial viscosity, μ becomes a function of $\partial u/\partial x$.

As we know, equations of gas dynamics reduce to a system of equations of the type (1) in the case of plane, cylindrical, and spherical symmetry and constant entropy (Eulerian coordinates) and in the case of plane symmetry and variable entropy (Lagrangian coordinates) (cf Chapter Two).

Suppose for system (1) a problem with the initial data

$$r(x, 0) = r_0(x), \quad s(x, 0) = s_0(x), \quad a \leq x \leq b, \qquad (2)$$

is posed, with a smooth solution in some region G containing the segment [a, b] of the x axis (Figure 3.7).

Let us decompose the segment [a, b] into the intervals $[x_i, x_{i+1}]$, $x_0 = a$, $x_{N+1} = b$. The points $(x_i, 0)$ form the first series of calculated points (series I). The next series of calculated points (series II) contains the points of intersection of r- and s-characteristics omitted from the series I points. If the m-th series of calculated points (x_i^m, t_i^m) is defined, then the next series (x_i^{m+1}, t_i^{m+1}) is defined by means of the formulas (first approximation)

$$\left.\begin{aligned} \frac{x_i^{m+1} - x_i^m}{t_i^{m+1} - t_i^m} &= \xi_{2i}^m, \\ \frac{x_i^{m+1} - x_{i+1}^m}{t_i^{m+1} - t_{i+1}^m} &= \xi_{1\,i+1}^m, \end{aligned}\right\} \quad (3)$$

where $\xi_{\alpha i}^m = \xi_\alpha(r_i^m, s_i^m, x_i^m, t_i^m)$, $\alpha = 1, 2$, and r_i^m, s_i^m are values of the invariants at the points x_i^m, t_i^m.

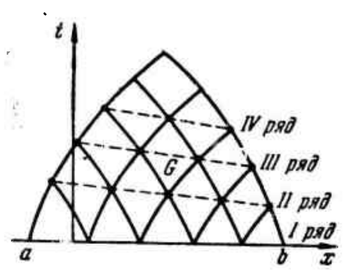

Figure 3.7

Next, the invariants r_i^m, s_{i+1}^m with the corresponding increments

$$\left.\begin{aligned} r_i^{m+1} &= r_i^m + \Delta r_i^m = r_i^m + F_{2i}^m (t_i^{m+1} - t_i^m), \\ s_i^{m+1} &= s_{i+1}^m + \Delta s_{i+1}^m = s_{i+1}^m + F_{1\,i+1}^m (t_i^{m+1} - t_{i+1}^m), \\ F_{\alpha i}^m &= F_\alpha (r_i^m, s_i^m, x_i^m, t_i^m), \quad \alpha = 1, 2. \end{aligned}\right\} \tag{4}$$

are shifted to the point (x_i^{m+1}, t_i^{m+1}). The first approximation scheme signifies the change of characteristics passing through the points of the lower series x_i^m, t_i^m with straight lines (3) and the approximation integration of equations (1) along characteristics by Euler's method.

To revise formulas (3) and (4), let us employ the recalculation of both points of the series (x_i^{m+1}, t_i^{m+1}), as well as the values r_i^{m+1}, s_i^{m+1}. In formulas (3) the right sides $\xi_{\alpha i}^m$ (slope of lines approximating the characteristics) are replaced by the half-sums

where

$$\frac{x_i^{m+1} - x_i^m}{t_i^{m+1} - t_i^m} = \frac{\xi_{2i}^m + \xi_{2i}^{m+1}}{2}; \quad \frac{x_i^{m+1} - x_{i+1}^m}{t_i^{m+1} - t_{i+1}^m} = \frac{\xi_{1\,i+1}^m + \xi_{1i}^{m+1}}{2}, \tag{5}$$

$$\xi_{\alpha i}^{m+1} = \xi_\alpha (r_i^{m+1}, s_i^{m+1}, x_i^{m+1}, t_i^{m+1}). \tag{6}$$

and $x_i^{m+1}, t_i^{m+1}, r_i^{m+1}, s_i^{m+1}$ in formula (6) are calculated by means of formulas (3), (4).

The invariants r_i^m, s_i^m with increments refined by the trapezoidal formula are translated to the points (x_i^{m+1}, t_i^{m+1}):

$$\left.\begin{aligned} r_i^{m+1} &= r_i^m + \Delta r_i^m = r_i^m + \frac{F_{2i}^m + F_{2i}^{m+1}}{2} (t_i^{m+1} - t_i^m), \\ s_i^{m+1} &= s_{i+1}^m + \Delta s_{i+1}^m = s_{i+1}^m + \frac{F_{1\,i+1}^m + F_{1i}^{m+1}}{2} (t_i^{m+1} - t_{i+1}^m). \end{aligned}\right\} \tag{7}$$

In (7) F^m_{1i}, F^m_{2i} retain their former values from (4), and F^{m+1}_{1i}, F^{m+1}_{2i} are defined by the formulas

$$F^{m+1}_{\alpha i} = F_\alpha(r^{m+1}_i, s^{m+1}_i, x^{m+1}_i, t^{m+1}_i) \quad (\alpha = 1, 2), \qquad (8)$$

where r^{m+1}_i, s^{m+1}_i have been calculated by (4), and x^{m+1}_i, t^{m+1}_i — by (5).

Reconversion by formulas (5) - (8) increases the order of accuracy of this scheme, but the additional conversion does not lead to the following increase in order of accuracy, and therefore it is sufficient to limit ourselves to a single order.

From a polytropic gas with $\gamma = 3$, flat symmetry, and constant entropy equations (1) become (cf Chapter Two, Section II, subsection 9)

$$\frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = 0. \qquad (9)$$

In this case even formulas (3) and (4) yield, without conversion, an exact solution of Cauchy's problem for (9), since the characteristics (9) are straight lines.

Scheme (3), (4), with the first order of exactness for smooth functions, in this case exhibits an infinitely large order of accuracy for the class of smooth solutions of system (9).

At the same time the scheme of any order of exactness with constant grid yields only an approximate solution of problem (9).

This example clearly illustrates the advantages of a characteristic grid which minimizes the difference of regions of dependence of the scheme and of the equation and thus of the residual term.

Construction of the calculation front can be carried out in a regular case also in a different fashion, not relative to spatially similar series, as indicated in Figure 3.7, but relative to characteristic lines. This calculational algorithm is extended to more general systems of equations. Suppose

$$l^k_\alpha\left(\frac{du_\alpha}{dt}\right)_k = l^k_\alpha\left(\frac{\partial u_\alpha}{\partial t} + \xi_k\frac{\partial u_\alpha}{\partial x}\right) = f_k \quad (k, \alpha = 1, \ldots, n) \quad (10)$$

is a hyperbolic system in characteristic form for which invariants exist.

The equations $dx/dt = \xi_k (x, t, u)$ define n one-parametric families of characteristics. In the general case any pair of families form a characteristic grid which does not coincide with the grid corresponding to the other pair. Suppose for definiteness that we have selected a pair of characteristics corresponding to the indices $k = 1$, $k = n$, M is the calculated point on the grid (Figure 3.8), A_k is the basis of the k-th characteristic descending from M at the line AB. The points A_1 and A_n are grid points, the points $A_2, \ldots, A_{n-1}$ are located between them, and the determination of u at the points $A_2, \ldots, A_{n-1}$ requires interpolation relative to the values $u(A_1)$ and $u(A_n)$. Thus, besides translation along the characteristic there appears the interpolation operator, which leads to smoothing effects that are inherent in ordinary difference methods.

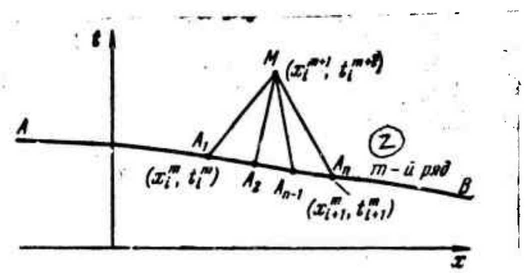

Figure 3.8

KEY: z. m-th series.

In the general case equations (10) do not have invariants, and we must convert to the more complex extended system (cf Chapter One, Section V). Nonetheless the method of characteristics in this case as well preserves with high accuracy the domain of dependence of the hyperbolic system.

This regular algorithm of the construction of a characteristic difference grid and the approximation integration of equations in invariants is possible in the domain of a smooth solution. If the neighborhood of a calculated point contains any singularity (shock wave, contact boundary, arbitrary discontinuity, site of initiation of a shock wave, and centered rarefaction wave), then the formulas are modified based on the nature of the singularity and the configuration of the grid.

Let us consider several typical configurations.

1) Contact boundary. Let us illustrate the algorithm with the example of a plane-symmetric piecewise-isentropic flow and the Lagrangian system of coordinates.

Now the calculated points must lie not only on the r- and s-characteristics, but also at the contact boundary, which is the coordinate line q = constant (Figure 3.9). The inhomogeneity of the calculation associated with boundary leads to two calculation fronts.

Suppose P_1 and Q_1 are the last calculated points at the contact boundary Γ, located on the left, and correspondingly right, side of it; M and K are points of the calculation fronts neighboring them. The points N and L are computed in regular fashion. In the first approximation, starting from the point N, we can calculate the point P_2 of intersection of the r-characteristic MN with boundary Γ and determine in it the value of the invariant $r_{\text{л}}$. Similarly, starting from the point M, we can define the point Q_2 and the invariant s in it. The value of $r_{\text{л}}$ is interpolated relative to the points P_1 and P_2 at Q_2.

The conditions for the continuity of p and u at the contact boundary, taken at the point Q_2 lead to the relations

$$\left.\begin{aligned} p &= P_{\text{л}}(S_{\text{л}}, r_{\text{л}} - s_{\text{л}}) = P_{\text{п}}(S_{\text{п}}, r_{\text{п}} - s_{\text{п}}), \\ u &= \frac{r_{\text{л}} + s_{\text{л}}}{2} = \frac{r_{\text{п}} + s_{\text{п}}}{2}. \end{aligned}\right\} \qquad (11)$$

$r_{\text{п}}$ and $s_{\text{л}}$ are defined at Q_2 from (11). Thus, the calculation front from the right advances by a single step, and in the first approximation we can find the points R and Q_3 and the invariant $s_{\text{п}}$ at the latter. This makes it possible to calculate in similar fashion to the preceding point P_2, after which the calculation cycle is completed in the first approximation. Second-approximation formulas can also be constructed, which are quite complex.

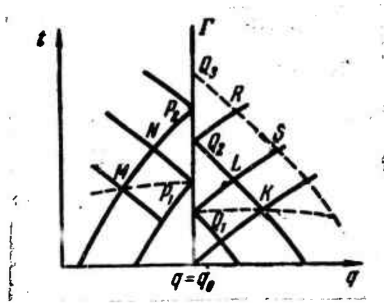

Figure 3.9

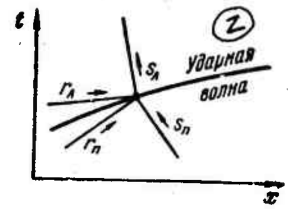

Figure 3.10

KEY:
Z — shock wave.

2) Shock wave. Again the presence of a discontinuity leads to two calculation fronts, however in contrast to the preceding case the discontinuity line no longer is a time-similar line. As follows from Champlin's theorem, the shock wave line will be spatially-similar at the forward front and time-similar at the rear (Figure 3.10).

A shock wave line "cuts" the grid of characteristics ahead of it, and all values of quantities at the forward front are known, since they are transported by characteristics from below.

At the same time only the invariant r (for the case of a wave travelling to the right) is translated relative to the characteristic at the rear front, while the invariant s is swept from the shock wave line, being defined at this line from Hugoniot's conditions. For simplicity of our treatment, let us limit ourselves to the case of an isothermal gas, when Hugoniot's conditions are written in invariants (cf Chapter Two, Section IV, Subsection 6):

$$r_л - r_п = a\varphi(M), \qquad (12)$$

$$s_л - s_п = a\psi(M), \qquad (13)$$

$$M = \frac{R - u_п}{a}, \quad \varphi(M) = M - \frac{1}{M} + \ln M^2, \quad \psi(M) = M - \frac{1}{M} - \ln M^2, \qquad (14)$$

where a denotes the isothermal speed of sound.

Knowing $r_{л}$, $r_{п}$, $u_{п}$ from (12) and (14), let us determine the shock wave velocity R, and from (13) determine $s_{л}$.

Suppose the shock wave line (Figure 3.11) intersects at points A and B the element (cell) LMNP of the characteristic grid already computed, and suppose that at point A we know the quantities $r_{л}$, $s_{л}$, $r_{п}$, $s_{п}$, and, therefore, R.

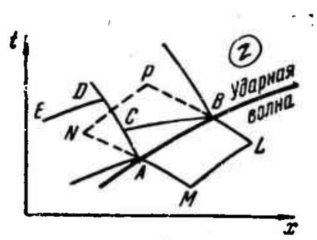

Figure 3.11

KEY:

Z — shock wave.

Then the characteristic triangle EAD to the left of the shock wave line is wholly defined. From point A let us draw a segment of the straight line AB

$$\frac{x_B - x_A}{t_B - t_A} = R_A,$$

approximating the trajectory of the shock wave in the neighborhood of A. At the characteristic PL we can integrate at the point B the values of $r_{п}$, $s_{п}$ by their values at the points P and L.

Since the segment AD has been wholly calculated, then on it we can find the point C such that in the first approximation the r-characteristic drawn through it passes to the point B. This problem is solved by linear interpolation of invariants r_D, s_D, r_A, s_A at point C given the condition that the segment of the straight line

$$\frac{x - x_C}{t - t_C} = \frac{r_C + s_C}{2} + a$$

will pass through the point B.

If we denote the parameter of linear interpolation along AD by θ, then the working formulas for the determination of θ will be of the form

$$\frac{x_B - x_C}{t_B - t_C} = \frac{x_B - [x_A + \theta(x_D - x_A)]}{t_B - [t_A + \theta(t_D - t_A)]} = (u_A)_z + \theta[u_D - (u_A)_z] + a.$$

Next, determine all quantities. After we determine $(r_B)_\pi$, we can calculate R_B, after which the point B is reconverted by the trapezoidal formula

$$\frac{x_B - x_A}{t_B - t_A} = \frac{R_A + R_B}{2}.$$

This enables us to recalculate point C, and also variants $(r_B)_\pi$, $(s_B)_\Pi$, $(r_B)_\Pi$, after which the point B is determined, and the calculation front to the left of the shock wave can advance by yet another step.

Similarly, with some complication, we perform the calculation for the case of flow with variable entropy when we must introduce trajectories into our consideration.

Of course, still other formulas of calculation and other configurations of the mutual arrangement of the shock wave line and the characteristic grid are possible.

Similar difficulties arise when considering singularities of the following types:

3) centered rarefaction wave;

4) breakdown of a discontinuity;

5) intersection of characteristics of one family with subsequent formation of a shock wave; and

6) a boundary with a vacuum when degeneration of the grid elements occurs.

In each specific situation, the problem of the advance of the calculation front by a single step in the neighborhood of a given singularity is elementary and reduces to interpolation and to the solution of problems and analytic geometry.

Difficulties of the method of characteristics lie in the construction of the calculation front when a large number of singularities of different types are present. Then the calculation becomes irregular and the determination of the possible configuration and the choice of the calculation formulas becomes the main problem. To this we can add the difficulties of memory distribution caused by bifurcation by the shock wave of the characteristic grid.

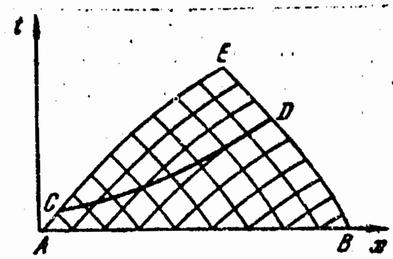

Figure 3.12

Let us consider this fact in greater detail.

As we have noted, the method of characteristics admits of the two-layer construction of a calculation in a regular domain, so that to determine the (m + 1)-th layer it is sufficient that we know the m-th layer and, perhaps, one point of the (m + 1)-th layer (second method). If after calculating the (m + 1)-th layer the results of the m-th layer are blurred, then when the characteristic grid is bifurcated by a shock wave the latter can also be initiated in the calculated and "blurred" elements of the grid (Figure 3.12), which makes extension of the calculation impossible.

In spite of major logical difficulties in realizing the method of characteristics on computers, programs permitting calculation with a high degree of accuracy of flows containing large numbers of singularities have been formulated in the USSR (cf [32]).

3. Explicit running calculation schemes. The closest to the method of characteristics are the running calculation schemes. Like the method of characteristics, they begin from equations in invariants or in characteristic form, but the difference grid is no longer a characteristic grid.

We will begin with a consideration of explicit running calculation schemes for a one-dimensional plane piecewise-isentropic gas flow not containing shock waves. As we know, this flow is described in Lagrangian coordinates by a system of equations in invariants (cf Chapter Two, Section III)

$$\frac{\partial r}{\partial t} + a(S, r - s)\frac{\partial r}{\partial q} = 0, \quad \frac{\partial s}{\partial t} - a(S, r - s)\frac{\partial s}{\partial q} = 0, \qquad (1)$$

where $a = c\rho$ is the mass speed of sound, and entropy S is a piecewise-constant function, whose sites of discontinuity are contact boundaries. In the following a will be considered as a function of the single argument $\theta = r - s$.

Let us formulate the problem with the following initial and boundary conditions for system (1)

$$r(q, 0) = r_0(q), \quad s(q, 0) = s_0(q), \quad 0 \leqslant q \leqslant Q, \qquad (2)$$
$$u(0, t) = f(t), \quad p(Q, t) = g(t). \qquad (3)$$

Boundary conditions (3) signify that the velocity of the left boundary and the pressure at the right are given.

Of course, we can select another combination of boundary conditions (for example, at both bounds velocities of pressures are given). The equation of state is given by the formula*) $p = P(S, r - s)$. (4)

When no shock waves are present in the flow, the entropy is constant at each line q = constant, including at the boundaries $q = 0$, $q = Q$. Therefore boundary conditions (3) can be written in the form

$$r(0, t) + s(0, t) = 2f(t),$$
$$r(Q, t) - s(Q, t) = k(t) = P^{-1}(p, S) = P^{-1}(g(t), S). \qquad (5)$$

The ordinary conditions given below are imposed on the internal contact boundaries $q = q_i$: $p_i^+ = p_i^-$, $u_i^+ = u_i^-$, (6)
which in terms of invariants, are of the form

*) We assume for simplicity that the contact boundaries separate gases which have the same equation of state (4), but with different entropy values. The problem is slightly complicated if the gases are assumed to be different.

$$\left.\begin{array}{c}P(S_i^+, r_i^+ - s_i^+) = P(s_i^-, r_i^- - s_i^-),\\ r_i^+ + s_i^+ = r_i^- + s_i^-.\end{array}\right\} \tag{7}$$

Here the sign "+" denotes quantities to the right, and the sign "-" denotes quantities to the left of the i-th boundary.

For the case of a polytropic gas when

$$P(S, r-s) = \mathcal{F}(S)(r-s)^{\alpha}, \tag{8}$$

where $\alpha = \frac{2\nu}{\nu - 1}$, and $\mathcal{F}(S)$ is expressed in terms of the entropy constant A(S) (Chapter Two), conditions (7) become linear:

$$r_i^+ - s_i^+ = \varkappa_i(r_i^- - s_i^-), \quad \varkappa_i = \left(\frac{\mathcal{F}_i^-}{\mathcal{F}_i^+}\right)^{\frac{1}{\alpha}}. \tag{9}$$

Converting to difference equations and assuming at first the absence of internal boundaries, let us construct a grid that is uniform relative to mass such that

$$\left.\begin{array}{c}h = \Delta q_i = q_{i+1} - q_i = \text{const} = \frac{Q}{N+1},\\ q_0 = 0, \quad q_1 = h, \ldots, q_{N+1} = (N+1)h = Q.\end{array}\right\} \tag{10}$$

To solve the problem formulated, we can propose the following explicit running calculation scheme:

$$r_i^{m+1} = (1 - \varkappa_i^m) r_i^m + \varkappa_i^m r_{i-1}^m, \tag{11}$$

$$s_i^{m+1} = (1 - \varkappa_i^m) s_i^m + \varkappa_i^m s_{i+1}^m \quad (i = 1, \ldots, N), \tag{12}$$

$$\varkappa_i^m = a_i^m \cdot \frac{\tau}{h},$$

to which the following initial data are added: $r_i^0 = r_{oi}$, $s_i^0 = s_{oi}$ as well as the boundary conditions

$$r_0^m + s_0^m = 2f^m, \tag{13}$$

$$r_{N+1}^m - s_{N+1}^m = k^m. \tag{14}$$

Using relations (11), r_i^{m+1} is determined successively from $i = 1$ to $i = N + 1$, and s_{N+1}^{m+1} is determined from (14).

Similarly, from (12) s_i^{m+1} $(i = N, \ldots, 0)$ is determined, and r_0^{m+1} is determined from (13). Next, the $(m + 1)$-th layer is calculated, and to determine the $(m + 2)$-th layer the procedures are repeated etc. Given the condition

$$\varkappa_i^m \leq 1 \qquad (15)$$

scheme (11), (12) is a scheme with positive coefficients, which ensures — as was shown in subsection 5 of Section I — its stability. Let us prove the convergence of the solution of the problem (11) - (14) to the solution of the problem (1), (3).

Let us assume that the problem (1) - (3) has the solution $r(q, t)$, $s(q, t)$, displaying continuous second derivatives relative to q, t.

Let δr_i^m and δs_i^m stand for the differences

$$\delta r_i^m = r_i^m - r(ih, m\tau), \quad \delta s_i^m = s_i^m - s(ih, m\tau),$$

where $r(ih, m\tau)$, $s(ih, m\tau)$ represent the exact solution of the problem (1) - (3) at the grid points $q = ih$, $t = m\tau$.

The quantities δr_i^m, δs_i^m satisfy the difference equations

$$\left.\begin{aligned} \delta r_i^{m+1} &= (1 - \varkappa_i^m)\,\delta r_i^m + \varkappa_i^m\,\delta r_{i-1}^m - \tau\,\{a\,(r_i^m - s_i^m) - \\ &- a\,[r(ih, m\tau) - s(ih, m\tau)]\}\,\frac{r(ih, m\tau) - r((i-1)h, m\tau)}{h} + R_{1i}^m \\ \delta s_i^{m+1} &= (1 - \varkappa_i^m)\,\delta s_i^m + \varkappa_i^m\,\delta s_{i+1}^m + \tau\,\{a\,(r_i^m - s_i^m) - \\ &- a\,[r(ih, m\tau) - s(ih, m\tau)]\}\,\frac{s((i+1)h, m\tau) - s(ih, m\tau)}{h} + R_{2i}^m \end{aligned}\right\} \qquad (16)$$

with the following initial and boundary conditions:

$$\left.\begin{aligned} \delta r_i^0 &= 0, & \delta s_i^0 &= 0, \\ \delta r_0^m + \delta s_0^m &= 0, & \delta r_{N+1}^m - \delta s_{N+1}^m &= 0. \end{aligned}\right\} \qquad (17)$$

the residual terms R_{1i}^m and R_{2i}^m are of the order $O(\tau^2)$.

Using the smoothness of the solution $r(q, t)$, $s(q, t)$, we can rewrite system (16) in the form

$$\left.\begin{aligned} \delta r_l^{m+1} &= (1-x_l^m)\,\delta r_l^m + x_l^m\,\delta r_{l-1}^m - \tau\alpha_l^m(\delta r_l^m - \delta s_l^m) + R_{1l}^m \\ \delta s_l^{m+1} &= (1-x_l^m)\,\delta s_l^m + x_l^m\,\delta s_{l+1}^m + \tau\beta_l^m(\delta r_l^m - \delta s_l^m) + R_{2l}^m \end{aligned}\right\} \quad (18)$$

where

$$\alpha_l^m = a'(\theta^*)\,\frac{r(lh, m\tau) - r((l-1)h, m\tau)}{h},$$

$$\beta_l^m = a'(\theta^*)\,\frac{s((l+1)h, m\tau) - s(lh, m\tau)}{h}$$

are bounded quantities, $\theta = r - s$, θ^* is an intermediate θ value in the finite-increment formula.

It is not difficult to see that given the condition (15), the following estimate is valid for the step operator C_{m+1} of the problem (17), (18):

$$\| C_{m+1} \| \leqslant 1 + 2A\tau,$$

where $A = \max_{i,m} \{|\alpha_i^m|, |\beta_i^m|\}$, and the norm of the vector $\{\delta r^m, \delta s^m\} = \{\delta r_i^m, \delta s_i^m\}$ is defined as $\max_i \max\{|\delta r_i^m|, |\delta s_i^m|\}$. Consequently, the following estimate (cf Section I, Subsection 4) is valid:

$$|\delta r_i^m|,\ |\delta s_i^m| \leqslant Ce^{2At} \max_{l,m} \max\left\{\left|\frac{R_{1l}^m}{\tau}\right|,\ \left|\frac{R_{2l}^m}{\tau}\right|\right\} = O(\tau),$$

from which follows the convergence $|\delta r_i^m| \to 0$, $|\delta s_i^m| \to 0$ as $\tau \to 0$ uniformly relative to i, m in the domain of existence of the solution $r(q, t)$, $s(q, t)$.

If contact boundaries are present, then relation (7) at the boundaries will also enter into the difference equations. For simplicity of our discussion, let us assume that there exists a single contact boundary located at the point $q = jh$ of the grid. Then the invariants r, s are discontinuous at the point $q = jh$. Equations (11), (12), which become

$$(r_j^{m+1})_- = (1 - x_j^m)_-(r_j^m)_- + (x_j^m)_-\,r_{j-1}^m,$$

$$(s_j^{m+1})_+ = (1 - x_j^m)_+(s_j^m)_+ + (x_j^m)_+\,s_{j+1}^m,$$

are valid for the left values $(r_j^{m+1})_-$ and the right values $(s_j^{m+1})_+$.

To these we must add relation (7):

$$P[(S_j)_-, (r_j^{m+1})_- - (s_j^{m+1})_-] = P[(S_j)_+, (r_j^{m+1})_+ - (s_j^{m+1})_+],$$
$$(r_j^{m+1})_+ + (s_j^{m+1})_+ = (r_j^{m+1})_- + (s_j^{m+1})_-,$$

which for known $(r_j^{m+1})_-$, $(s_j^{m+1})_+$ enables us to determine $(r_j^{m+1})_+$, $(s_j^{m+1})_-$, after which the computation is continued as usual.

When several contact boundaries are present, each of them is calculated by the indicated algorithm.

If shock waves are absent in the flow, but the entropy changes from element to element, the equations describing motion become inhomogeneous:

$$\frac{\partial s}{\partial t} - a(q, r-s)\frac{\partial s}{\partial q} = F_1(q, r-s),$$
$$\frac{\partial r}{\partial t} + a(q, r-s)\frac{\partial r}{\partial q} = F_2(q, r-s).$$

Then the left sides of the equations are approximated, just as in the homogeneous case, the right sides are taken from the m-th layer, and the convergence is proven analogously as under the same condition (15).

In the general case, the system of quasilinear equations, including also equations in hydrodynamics, is not reduced to invariants. However, in this case as well running calculation is possible, starting from the equations in the characteristics form. This scheme was proposed in the study [26].

Suppose that for the hyperbolic system

$$l_\alpha^k(u_1, \ldots, u_n)\left[\frac{\partial u_\alpha}{\partial t} + \xi_k(u_1, \ldots, u_n)\frac{\partial u_\alpha}{\partial x}\right] = f_k(u_1, \ldots, u_n) \quad (19)$$
$$(k, \alpha = 1, \ldots, n)$$

the following initial data are assigned: $u_\alpha(x, 0) = u_{\alpha 0}(x)$. (20)

Let us approximate Cauchy's problem (19), (20) by the Cauchy's difference problem

$$l_\alpha^k(u_1^m, \ldots, u_n^m)\left[\frac{u_\alpha^{m+1} - u_\alpha^m}{\tau} + \xi_k(u_1^m, \ldots, u_n^m)\frac{\Delta u_\alpha^m}{h}\right] = f_k(u_1^m, \ldots, u_n^m), \quad (21)$$
$$u_\alpha^0(x) = u_{\alpha 0}(x), \quad (22)$$

where $u^m(x)$ is the solution of (21), (22) defined at the instant $t = m\tau$, and

$$\begin{rcases} \Delta = \Delta_{-1} \quad \text{when} \quad \xi_k(u_1^m, \dots, u_k^m) > 0, \\ \Delta = \Delta_1 \quad \text{when} \quad \xi_k(u_1^m, \dots, u_k^m) < 0. \end{rcases} \quad (23)$$

We can rewrite the difference scheme (21) in the form

$$\begin{rcases} l_k^{km} u_k^{m+1} = l_k^{km} [(1 - \varkappa_k^m) E + \varkappa_k^m T_{-1}] u_k^m + f_k^m \tau, & \varkappa_k^m > 0, \\ l_k^{km} u_k^{m+1} = l_k^{km} [(1 + \varkappa_k^m) E - \varkappa_k^m T_1] u_k^m + f_k^m \tau, & \varkappa_k^m < 0. \end{rcases} \quad (24)$$

where it is assumed

$$l_k^{km} = l_k^k(u_1^m, \dots, u_k^m), \quad \varkappa_k^m = \frac{\xi_k^m \tau}{h}, \quad \xi_k^m = \xi_k(u_1^m, \dots, u_k^m). \quad (25)$$

When $|\varkappa_k^m| \leq 1$, the difference operators

$$(1 - \varkappa_k^m) E + \varkappa_k^m T_{-1} \quad (\varkappa_k^m > 0), \quad (1 + \varkappa_k^m) E - \varkappa_k^m T_1 \quad (\varkappa_k^m < 0)$$

become positive, and scheme (24) is analogous to the schemes that are positive in the Friedrichs sense. The work [26] showed that given the condition $|\varkappa_k^m| \leq 1$, the solution of $u^m(x)$ of the boundary problem (21), (22) converges in C to the solution of the problem (19), (20).

We have considered two variants of running calculation: a) in invariants and b) for a characteristic system. Relative to the formulation of the boundary conditions and relative to the simplicity algorithm, calculation in invariants is preferable [text pages 393 - 435 missing]

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CHAPTER FOUR GENERALIZED SOLUTIONS OF HYPERBOLIC SYSTEMS OF QUASILINEAR EQUATIONS

Section I. Formulation of Cauchy's Problem for the Class of Discontinuous Functions

1. General Remarks. In Chapter Two we saw that differential equations of gas dynamics are consequences of more general integral laws of conservations — of mass, momentum, and energy.

The transition from integral laws of conservation to differential is possible only for a certain smoothness of flow. If a smooth flow does not exist, then in order to define flow (discontinuous or not exhibiting the required smoothness), we must resort to integral laws of conservation.

The same approach was adopted in the theory of discontinuous (generalized) solutions of hyperbolic systems of quasilinear equations, conceived in recent decades.

We will consider the conservative system of quasilinear equations

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u, x, t)}{\partial x}=f(u, x, t) \qquad (1)$$

(u, φ, f are vectors with n components) as a consequence of the system of integral laws of conservation

$$\oint_C u\,dx-\varphi(u, x, t)\,dt+\iint_{\mathcal{G}_C} f(u, x, t)\,dx\,dt=0, \qquad (2)$$

which must be satisfied for any piecewise-smooth closed contour C and the domain $\mathcal{G}_C$ it bounds.

If the function $u(x, t)$ satisfies integral laws of conservation (2) for any closed contours C and domains $\mathcal{G}_C$, and $\varphi \in C_1$, $f \in C_0$, then from this it follows that the function $u(x, t)$ satisfies the system of differential equations (1).

It is also obvious that any solution $u(x, t)$ of system (1) ($u(x, t) \in C_1$) satisfies integral laws of conservation (2). The integral relations (2) serve to introduce the concept of the generalized solution of system (1).

We will confine our inquiry to the class K of functions $u(x, t)$ satisfying the following requirements:

First requirement. In any finite portion of the half-plane $t \geqslant 0$ there exists a finite number of discontinuity lines $x = x(t)$ and a finite number of discontinuity points; exterior to these lines and points the function $u(x, t)$ is continuous and has continuous first derivatives.

Second requirement. Left $u(x - 0, t)$ and right $u(x + 0, t)$ limiting values exist at the discontinuity lines $x = x(t)$; for definiteness we will assume that $u(x, t) = u(x + 0, t)$.

We will call the vector-function $u(x, t) \in K$ the generalized solution of system of equations (1) if for an arbitrary piecewise-smooth contour C and the domain bounded by it $\mathcal{G}_C$ integral laws of conservation (2) are satisfied.

Other definitions of the generalized solution of system (1) are also used.

Suppose $g(x, t) \in C_1$ is a finite function (it tends to zero outside the finite portion of the plane x, t).

Let us multiply each of the equations in system (1) by $g(x, t)$ and integrate the results of the multiplication over the half-plane $t \geqslant 0$.

Performing integration by parts, we get

$$\iint\limits_{t>0} \left[u_i \frac{\partial g}{\partial t} + \varphi_i(u, x, t) \frac{\partial g}{\partial x} + f_i(u, x, t)\, g(x, t)\right] dx\, dt + \\ + \int_{-\infty}^{\infty} g(x, 0)\, u_i(x, 0)\, dx = 0 \qquad (i = 1, 2, \ldots, n). \quad (3)$$

Equations (3), just as (2) do not contain arbitrary functions u(x, t) and are not meaningless for discontinuous u(x, t). We will call the function u(x, t) $\in$ K a generalized solution of system (1) if equalities (3) are satisfied for any finite function $g(x, t) \in C_1$.

Finally, using the concept of potential (Chapter One, Section V), let us introduce yet another definition of a generalized solution of system (1).

Suppose the vector-function $\Phi(x, t) = \{\Phi_1(x, t), \ldots, \Phi_n(x, t)\}$ is continuous and has first derivatives $\partial\Phi/\partial x \in K$. Then if at all points for which the derivatives $\partial\Phi/\partial x$, $\partial\Phi/\partial t$ exist, they satisfy the system of nonlinear integral-differential equations

$$\frac{\partial\Phi_i}{\partial t} + \varphi_i\left(\frac{\partial\Phi}{\partial x}, x, t\right) = \int_0^x f_i\left(\frac{\partial\Phi}{\partial\xi}, \xi, t\right) d\xi \qquad (i = 1, \ldots, n), \quad (4)$$

then the function $u(x, t) = \partial\Phi/\partial x$ (5)
will be called a generalized solution of system (1). We can easily note that each of these definitions naturally is generalized to broader classes of functions u(x, t). Nonetheless we will confine ourselves to the class K, since more general classes of generalized solutions have not yet been studied well enough.

Now let us note that the system of quasilinear equations admits sometimes of several different representations in the form of laws of conservation. For example, one equation $\partial u/\partial t + u\,\partial u/\partial x = 0$ (6)
can be represented both in the form $\partial u/\partial t + \partial/\partial x\,(u^2/2) = 0$, (7)
as well as

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) = 0. \quad (8)$$

Thus, any solution of equation (6) satisfies simultaneously the following integral laws of conservation:

$$\oint_C u\,dx - \frac{u^2}{2}\,dt = 0, \quad (9)$$

$$\oint_C \frac{u^2}{2}\,dx - \frac{u^3}{3}\,dt = 0. \quad (10)$$

We can easily, however, see that the discontinuous function u(x, t) can satisfy one of the equations (9), (10), but not satisfy the other.

This situation reflects an interesting fact, interpretable as follows: different processes can be described by the very same **differential** equations, but by different integral laws of conservation. Therefore the difference of the processes shows up only in the discontinuous solutions.

Introducing the concept of the generalized solution of conservative system (1), we uniquely fix the integral laws of conservation (2).

2. Hugoniot's conditions. Suppose $u(x, t) \in K$ and $x = x(t)$ is an equation of one of the discontinuity lines of the function $u(x, t)$. Let us denote

$$\left.\begin{aligned} &D = x'(t), \quad u_{\text{л}}(t) = u(x(t) - 0, t), \quad u_{\text{п}}(t) = u(x(t) + 0, t), \\ &[u] = [u(x, t)] = u(x + 0, t) - u(x - 0, t), \\ &[u(x(t), t)] = u_{\text{п}}(t) - u_{\text{л}}(t), \\ &[\varphi(u, x, t)] = \varphi(u(x + 0, t), x, t) - \varphi(u(x - 0, t), x, t). \end{aligned}\right\} \quad (1)$$

Just as Hugoniot's conditions at the discontinuity line of flow (Chapter Two, Section 4, Subsection 1) derive from the integral laws of conservation of mass, momentum, and energy, likewise from the integral laws of conservation (1.1.2) at the line $x = x(t)$ of discontinuity of the solution $u(x, t)$ there follows the satisfaction of the equalities $D[u] = [\varphi(u, x, t)]$ (2) or, in components, $D[u_i] = [\varphi_i(u, x, t)]$. Condition (2) relates the left and right limit values of the decision at the discontinuity line*).

On analogy with the case of gas dynamics, we will call these equations Hugoniot's conditions.

If the function $u(x, t) \in K$ satisfies the system of differential equations (1.1.1) everywhere except at the discontinuity lines, and if Hugoniot's conditions (2) are satisfied at the discontinuity lines, then obviously integral laws of conservation (1.1.2) are satisfied by any closed contour C. Therefore the function $u(x, t)$ in this case will be generalized solution of system (1.1.1).

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*) The same conditions follow for piecewise-continuous $u(x, t)$ and piecewise-smooth discontinuity lines also from definitions of generalized solutions (1.1.3), or (1.1.4), and (1.1.5).

With the simplest examples let us inspect the consequences which stem from Hugoniot's conditions.

Suppose the system of equations (1.1.1) is semilinear. Then

$$\frac{\partial \varphi_i(u, x, t)}{\partial u_k} = a_{ij}(x, t), \qquad (3)$$

therefore Hugoniot's conditions (2) are transformed to become

$$D[u_i] = [\varphi_i(u, x, t)] = \sum_{j=1}^{n} a_{ij}(x, t)[u_j]$$

or

$$\sum_{j=1}^{n} (a_{ij}(x, t) - \delta_{ij} D)[u_j] = 0 \quad (i = 1, 2, \ldots, n). \qquad (4)$$

If the solution $u(x, t)$ is discontinuous, then $\sum_{j=1}^{n} [u_j]^2 \neq 0$ and

$$\mathrm{Det}\,((a_{ij}(x, t) - D\delta_{ij})) = 0,$$

i.e., the quantity $D = x'(t)$ must coincide with the eigenvalue of the matrix $A = ((\partial \varphi_i / \partial u_j))$. Suppose

$$D = \xi_k(x, t), \qquad (5)$$

then

$$[u] = Ur^k(x, t) \qquad (6)$$

or, in components,

$$[u_i] = Ur_i^k(x, t). \qquad (7)$$

The right eigenvector of the matrix $A(x, t)$ corresponding to the eigenvalue $\xi_k(x, t)$ is denoted by $r^k(x, t) = \{r_i^k(x, t)\}$.

Thus, for the semilinear system, by (5) $\quad dx/dt = \xi_k(x, t), \quad (8)$
i.e., the discontinuity lines of the solution are characteristics of the system of equations (1.1.1).

Analogous to weak discontinuity, strong discontinuity of a solution of a semilinear system of equations is also propagated along characteristics of the system.

Note that a similar property is also exhibited by solutions of a weakly-nonlinear system of quasilinear equations (Chapter One, Section X). Actually, it is not difficult to verify that discontinuities of a generalized solution of weakly-nonlinear system of quasilinear equations can lie on the characteristics of this system.

For the case of a single quasilinear equation (n = 1) Hugoniot's conditions (2) are rewritten in the form

$$x'(t) = D = \frac{[\varphi(u, x, t)]}{[u]} = \frac{\varphi(u_п(t), x(t), t) - \varphi(u_л(t), x(t), t)}{u_п(t) - u_л(t)}. \qquad (9)$$

Equality (9) can be interpreted geometrically as follows. The velocity $D = x'(t)$ of a discontinuity line $x = x(t)$ is equal to the slope (tangent of angle α) of the chord AB to the u axis (Figure 4.1). The following inequalities are satisfied for the case shown in Figure 4.1: $\varphi'_u(u_п(t), x(t), t) < D < \varphi'_u(u_л(t), x(t), t)$.

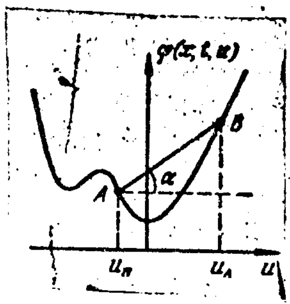

Figure 4.1

If $\varphi''_{uu}(u, x, t) \neq 0$, then by formula (9) the value of D is distinct both from $\varphi'_u(u_п(t), x(t), t)$ and from $\varphi'_u(u_л(t), x(t), t)$ if $u_п(t) \neq u_л(t)$. Thus, the discontinuity line $x = x(t)$ is no longer a characteristic for the nonlinear equation.

Note that different Hugonoit's conditions correspond to different integral laws of conservation which in turn correspond to the same system of quasilinear equations. Thus, from integral law (1.1.9) follows Hugoniot's condition

$$D = \frac{[u^2]}{2[u]} = \frac{u_п + u_л}{2}, \qquad (10)$$

and from (1.1.10)

$$D = \frac{2}{3}\frac{[u^3]}{[u^2]} = \frac{2}{3}\frac{u_п^2 + u_п u_л + u_л^2}{u_п + u_л}. \qquad (11)$$

It is not difficult to present an example of discontinuous functions $u(x, t)$ satisfying one of the Hugoniot's conditions (10) and (11) and not satisfying the other.

3. Stable and unstable discontinuities. Stability conditions. A solution (classical) of a system of quasilinear equations, as we saw in Chapter One, is uniquely defined in the domain of definition by its initial value at $t = 0$. It turns out, however, that satisfaction of integral laws of conservation and integral conditions does not at all guarantee the uniqueness of a discontinuous solution; rather, we can specify a set of essentially distinct discontinuous functions $u(x, t)$ satisfying both integral laws of conservation as well as initial conditions.

Let us confirm this by a very simple example. For the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \qquad (1)$$

the following initial condition is assigned:

$$u(0, x) = u_0(x) = \begin{cases} u^- & \text{for} \quad x < 0, \\ u^+ & \text{for} \quad x > 0. \end{cases} \qquad (2)$$

We will seek the bounded piecewise-continuous function $u(x, t)$ satisfying integral law of conservation (1.1.9) and the initial condition (2).

Hugoniot's condition (1.2.10) must be satisfied at the discontinuity lines, therefore the function

$$u_1(x, t) = \begin{cases} u^- & \text{for} \quad x < D_1 t, \\ u^+ & \text{for} \quad x > D_1 t, \end{cases} \qquad D_1 = \frac{u^- + u^+}{2} \qquad (3)$$

is the desired solution. This function is constant except at the discontinuity line $x = D_1 t$ at which the Hugoniot's condition (1.2.10) is satisfied, and takes on the initial values (2).

Suppose $u^- < u^+$. Let us construct another solution to the problem (1), (2) formula it is:

$$u_2(x, t) = \begin{cases} u^- & \text{for} \quad x \leq tu^-, \\ x/t & \text{for} \quad tu^- \leq x \leq tu^+, \\ u^+ & \text{for} \quad x \geq tu^+. \end{cases} \qquad (4)$$

The solution $u = u_2(x, t)$ is continuous when $t > 0$, continuously differentiable everywhere except the lines $x = tu^-$, $x = tu^+$, and satisfies equation (1) in the half-plane $t \geq 0$ in the large. We indicate two solutions $u = u_1(x, t)$, $u = u_2(x,t)$

of Cauchy's problem (1), (2). Each of the solutions satisfies integral law of conservation (1.1.9) and initial condition (2). Thus, here we meet the fact of the nonuniqueness of the solution. However, it seems natural that a rational formulation of Cauchy's problem in the class of discontinuous functions must lead to the uniqueness of the solution.

To single out the unique solution of problem, we make the following assumptions:

First. Any (classical) solution of a system of quasilinear equations, when it exists, is the "true" solution of the system also in the class of generalized (discontinuous) solutions.

Second. The limits of (classical) solutions of a system of quasilinear equations are the "two" solutions of integral laws of conservation in the case of discontinuous functions.

Let us clarify this in somewhat more detail.

The first requirement is a natural extension of the fact that the class of generalized solutions of Cauchy's problem for a system of quasilinear equations is an expansion of the class of classical solutions. If we rejected this, then classical solutions will not have practical importance when considering Cauchy's problem for the class of discontinuous functions*).

The second requirement is a natural consequence of the first requirement and the assumption of the continuous dependence of generalized solutions on the initial data of Cauchy's problem.

But the continuous dependence of solutions on initial data of Cauchy's problem is one of the conditions for the correctness of Cauchy's problem.

*) If the first requirement is rejected, Cauchy's problem for equation (1) with initial condition $u(0, x) \equiv 0$ has an infinite set of discontinuous solutions. For example,

$$u(x, t) = \begin{cases} 0 & \text{for} \quad x + t < 0, \\ -2 & \text{for} \quad -t < x < 0, \\ 2 & \text{for} \quad 0 < x < t, \\ 0 & \text{for} \quad x - t > 0 \end{cases}$$

will be the solution of this problem.

Thus, the ordinary concept of the correct formulation of Cauchy's problem leads us to the first and second requirements.

A fuller formulation of the requirements imposed on a generalized solution will be given below.

Thus far we are only making qualitative use of the first and second requirements for the further study of properties of continuous solutions of systems of quasilinear equations and, in particular, for deriving the unique ("true") solution $u = u(x, t)$ of Cauchy's problem (1), (2). Figures 4.2 and 4.3 give the behavior of the characteristics $x = x_0 + u(x, t)t$ for the solutions $u = u_1(x, t)$ and $u = u_2(x, t)$.

Let us consider the solution $u = u_\delta(x, t)$ of equation (1) satisfying the condition

$$u_\delta(0, x) = u^0_\delta(x), \qquad (5)$$

where $u^0(x)$ is a monotonically increasing function of variable x, coinciding outside the segment $|x| < \delta$ with $u_0(x)$ (Figure 4.4).

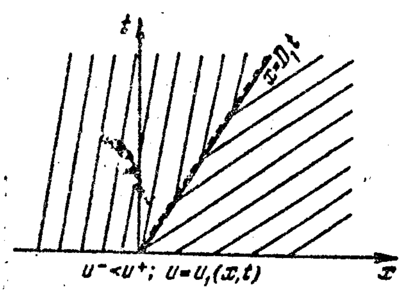

Figure 4.2

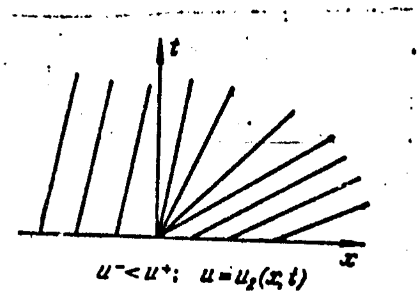

Figure 4.3

Figure 4.5 shows the characteristics for the solution $u\ (x, t)$. Comparing Figures 4.3 and 4.5 we note that

$$\lim_{\delta \to 0} u_\delta(x, t) = u_2(x, t), \qquad (6)$$

where equality (6) holds for any x, t, except the point (0, 0), where the limit $u_\delta(x, t)$ as $\delta \to 0$ does not exist. Therefore, according to the second requirement, the "true" solutions of Cauchy's problem (1), (2) must be regarded as $u = u_2(x, t)$. We will call the solution $u = u_1(x, t)$ the unstable discontinuous solution, since smoothing of the initial function for as small a section as we wish leads to a classical solution (true) far from $u_1(x, t)$.

From a comparison of Figures 4.2 and 4.3 we conclude that the cause of the instability of solution $u_1(x, t)$ is that at the discontinuity line $x = D_1 t$ of the solution $u_1(x, t)$ it is not the characteristics issuing from the points of the initial axis $t = 0$ (and thus, intrinsic to the initial data) that intersect each other, but the characteristics issuing from the points of the discontinuity line $x = D_1 t$.

In this sense we can state that the discontinuity of the solution shown in Figure 4.2 is "invented" and is not caused by the intersection of characteristics intrinsic to the initial values. We call this discontinuity an unstable discontinuity.

But if in the condition of Cauchy's problem (1), (2) $u^- > u^+$, then the solution $u_1(x, t)$ given by formula (3) will have characteristics shown in Figure 4.6.

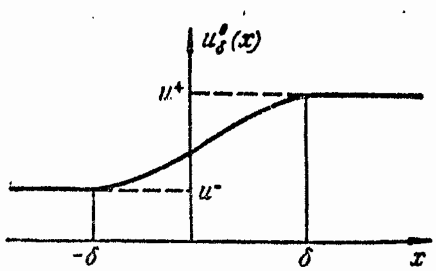

Figure 4.4

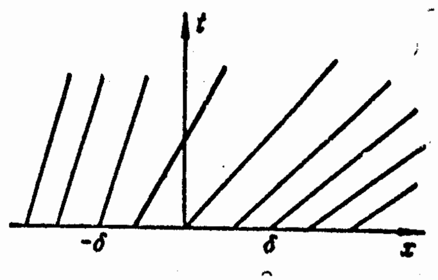

Figure 4.5

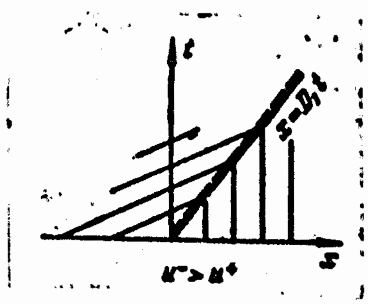

Figure 4.6

In this case characteristics intrinsic to initial values (arrival characteristics) intersect each other at the discontinuity line $x = D_1 t$. In this case no continuous solution of this Cauchy's problem exists, and the smoothed solution $u_\delta(x, t)$ is also discontinuous. Therefore the solution $u = u_1(x, t)$ and the discontinuity line $x = D_1 t$ where $u^- > u^+$ will be called stable.

For equation (1) the condition of the intersection at the discontinuity line of the arrival characteristics is described by the inequality is

$$u_{\text{л}}(t) > D > u_{\text{п}}(t), \quad D = x'(t). \qquad (7)$$

for the more general quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = f(u, x, t) \qquad (8)$$

the velocity of the characteristics is $\varphi'_u(u, x, t)$, therefore the condition for intersection at the line $x = x(t)$ of the arrival characteristics is described similar to (7):

$$\varphi'_u(u_\pi(t), x(t), t) > D > \varphi'_u(u_n(t), x(t), t); \quad D = x'(t). \qquad (9)$$

The discontinuity lines of the solutions are characteristics for the semilinear equation (8) ($\varphi''_{uu}(u, x, t) \equiv 0$). Thus, if we seek more general stability conditions that are also for linear equations, then it must be written in the form $\varphi'_u(u_\pi(t), x(t), t) \geqslant D \geqslant \varphi'_u(u\ \ (t), x(t), t)$. (10)
As we will see, conditions (10) ensure uniqueness and admit of the existence of a generalized solution of equation (8) for the case when $\varphi''_{uu}(u, x, t)$ does change in sign, i.e., for the case of a function $\varphi(u, x, t)$ that is convex relative to the variable u. We will call conditions (10) the conditions for the stability of the generalized solution in the case of sign-constancy of φ''_{uu}.

But if $\varphi''_{uu}(u, x, t)$ is a sign-variable function of u, the conditions (10) do not ensure either uniqueness or continuous dependence of the solution on the initial data, i.e.; conditions (10) are irregular conditions.

Actually, suppose $\varphi = \varphi(u)$, $f \equiv 0$ and suppose that the following initial condition is formulated for equation (8):

$$u(x, 0) = u_0(x) = \begin{cases} u^+ & \text{for } x < 0, \\ u^- & \text{for } x > 0. \end{cases} \qquad (11)$$

Suppose the graph of function $\varphi(u)$ and the position of the points u^- and u^+ are as shown in Figure 4.7. The function

$$u = u_1(x, t) = \begin{cases} u^- & \text{for } x < Dt, \\ u^+ & \text{for } x > Dt, \end{cases}$$

$$D = \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}, \qquad (12)$$

satisfies the integral law of conservation $\oint u\, dx - \varphi(u) dt = 0$ of equation (8), initial condition, and the proposed "stability condition" (10). The latter obviously can be inspected from Figure 4.7, since $\varphi'_u(u^{-1}) > D > \varphi'_u(u^+)$; D is the slope of the chord AB to the u axis.

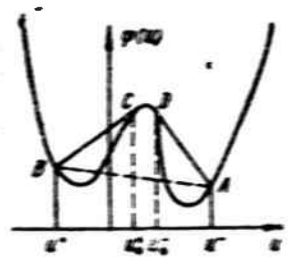

Figure 4.7

Let us, however, construct the second solution $u = u_2(x, t)$ of our Cauchy's problem. Let us given the function $u_2(x, t)$ as follows:

$$u_2(x, t) = \begin{cases} u^- & \text{for} \quad x < D^*_1 t, \\ f(x/t) & \text{for} \quad D^*_1 t < x < D^*_2 t, \\ u^+ & \text{for} \quad x > D^*_2 t. \end{cases} \tag{13}$$

where

$$D^*_1 = \frac{\varphi(u^-) - \varphi(u^-_*)}{u^- - u^-_*}, \quad D^*_2 = \frac{\varphi(u^+_*) - \varphi(u^+)}{u^+_* - u^+}; \tag{14}$$

$f(\xi)$ is determined from the equation $\xi = \varphi'_u[f(\xi)]$. (15)
and the points u^+_*, u^-_* (abscissae of the points C and D) — from the condition that the straight line BC is tangent to the graph of the function $\varphi(u)$ at the point C, and the line AD — at the point D; additionally, it is assumed that $\varphi''_{uu}(u) \neq 0$ for $u^+_* \leq u \leq u^-_*$ (cf Figure 4.7).

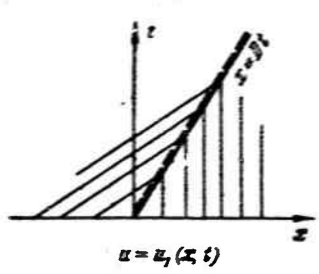

Figure 4.8

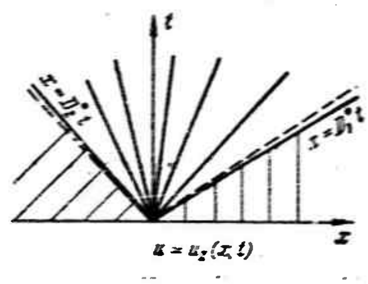

Figure 4.9

The solution $u = u_2(x, t)$ given by formulas (13) - (15) satisfies the integral law of conservation for equation (8) and for conditions (10). Thus, if $\varphi''_{uu}(u, x, t)$ is a sign-variable function of argument u, then the "stability conditions" (10) satisfy two solutions $u_1(x, t)$ and $u_2(x, t)$.

Figures 4.8 and 4.9 show the field of characteristics for the solutions $u = u_1(x, t)$ and $u = u_2(x, t)$.

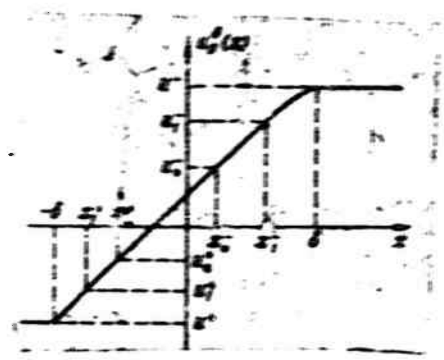

Figure 4.10

Incidentally let us note one interesting property of solution $u_2(x, t)$. The discontinuity line $x = u_2(x, t)$ is a characteristic of solution $u_2(x, t)$, calculated by the rules of limit values of the solution $u_n(t)$ on it; similarly, the discontinuity line $x = D^*_1 t$ is a characteristic calculated from the values $u_\pi(t)$ at the line $x = D^*_1(t)$.

To give preference to one of these two solutions, let us again employ the method of verifying the stability of the solutions by smoothing the initial function.

Let us smooth the initial function $u_c(x)$ by using a monotonic function $u_c(x)$ which is shown in Figure 4.10. The points (x^-_*, u^-_*), (x^+_*, u^+_*) correspond to the points D and C in Figure 4.7, and the points u^-_1 and u^+_1 are the points of inflection of the graph of the function $\varphi(u)$, i.e., $\varphi''_{uu}(u_1^-) = \varphi''_{uu}(u_1^+) = 0$.

Let us represent the characteristics of the solution $u = u_\delta(x, t)$ in Figure 4.11.

As we can see from Figure 4.7, $\varphi'_u(u)$ on the interval $[u^-, u_1^-]$ is a monotonically increasing function of variable u and, therefore, the slope of the characteristics $\varphi'_u(u^o_\delta(x))$ at the initial axis monotonically declines over the segment $[-\delta, x_1^-]$; similarly, $\varphi'_u(u^o_\delta(x))$ monotonically increases over the segment $[x_1^-, x_1^+]$ and decreases over the segment $[x_1^+, \delta]$.

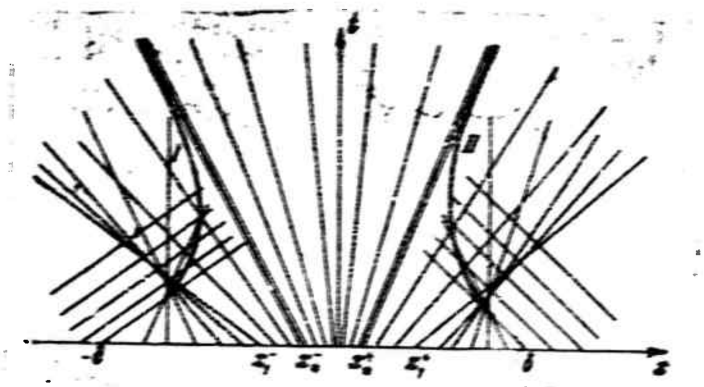

Figure 4.11

Careful inspection of the pattern of characteristics (Figure 4.11) leads us to the conclusion that the discontinuity of the solution at which the left value is the quantity u^- and the right u^+ cannot arise for a smooth initial function $u^o(x)$.

Actually, we can see from Figure 4.11, two discontinuities must arise for this to be true, of which the discontinuity I located to the left and corresponding to the values u^- and u^-_* will have velocity $D^*_1 = \varphi'_u(u^-_*) < \varphi'_u(u^+_*)$, and discontinuity II corresponding to the values u^+_* and u^+ will have the velocity $D^*_2 = \varphi'_u(u^+_*)$. Thus, the lines of discontinuities which are formed given the initial function $u^o(x)$ (they are shown in Figure 4.11 by the bold lines and are denoted by the numbers I and II) will behave as follows. Discontinuity line I will never intersect the characteristic issuing from the point x^-_* of the initial axis and will approach it asymptotically as $t \to \infty$. This obviously derives from the inequality

$$\frac{\varphi(u)-\varphi(u^-)}{u-u^-} < \varphi'_u(u^-_*) \qquad \text{for} \quad u^-_* < u < u^-_1.$$

Quite analogously we conclude that the discontinuity line II will asymptotically approach the characteristic issuing from the point x^+_* of the initial axis and will never intersect it.

It will now become obvious that if the quantity δ tends to zero, the solution $u_\delta(x, t)$ will tend to $u_2(x, t)$.

Thus, the solution $u_1(x, t)$ and the discontinuity line $x = D_1 t$ are unstable.

And so, condition (10) does not guarantee this stability of the generalized discontinuous solution of quasilinear equation (8) for the case when $\varphi''_{uu}(u, x, t)$ is sign-variable.

A correct generalization of stability conditions for a discontinuous solution of equation (8) for the case when φ''_{uu} is sign-variable can be obtained upon careful inspection of the solutions which are derived when discontinuities are smoothed. Thus, a solution of the type $u_1(x, t)$ (Figure 4.8) could be stable if the discontinuity lines I and II (Figure 4.11) would overtake one another, i.e., the pattern of motion of discontinuities I and II would be schematically of the following form (Figure 4.12).

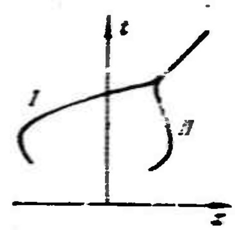

Figure 4.12

These considerations lead us to the following stability condition.

Suppose at the discontinuity line $x = x(t)$ of the solution of equation (8) $u_л(t) = u^-$, $u_п(t) = u^+$ and suppose $x'(t) = D$. The discontinuities called stable if the inequalities

$$\frac{\varphi(u^*, x, t) - \varphi(u^-, x, t)}{u^* - u^-} \geqslant D \geqslant \frac{\varphi(u^*, x, t) - \varphi(u^+, x, t)}{u^* - u^+}, \quad (16)$$

$$u^* \in (u^-, u^+), \quad x = x(t).$$

are satisfied for any value of u* from the interval (u^-, u^+).

We can easily note that the discontinuity solution $u_1(x, t)$ does not satisfy condition (16). Actually, if we select, for example, as u* any number from the interval (u^+_*, u^-_*), then inequalities (16) are violated. But discontinuities of solution $u_2(x, t)$ obviously satisfy these inequalities, which is clear from Figure 4.7.

Finally, we note that conditions (9) automatically derive from (16) if we assume that $\varphi''_{uu}(u, x, t) \neq 0$. Thus, the stability condition (16) contains stability conditions (9) for the convex function $\varphi(u, x, t)$ as a special case.

The question of the stability of discontinuous solutions even more involved for a hyperbolic system of quasilinear equations. The point is that for complex systems we actually have no graphical methods of constructing discontinuous solutions. Therefore an elucidation of the correctness of any given stability conditions for a solution is extremely laborious.

When formulating stability conditions we will rely on the analogy with cases of a single quasilinear equation and the system of equations in gas dynamics (Chapter Two).

Let us limit ourselves to considering the systems that are hyperbolic in the small, i.e., we would assume that

$$\xi_1(u, x, t) < \xi_2(u, x, t) < \ldots < \xi_n(u, x, t) \qquad (17)$$

Just as in the case of a single equation, the stability equations must require that the arrival characteristics of a single family intersect each other at the discontinuity line $x = x(t)$. Suppose, for example, that the arrival characteristics of the k-th family intersect at the line $x = x(t)$, then

$$\begin{aligned} &\xi_k(u_1(t), x(t), t) > D > \\ &> \xi_k(u_2(t), x(t), t), \quad D = x'(t). \end{aligned} \qquad (18)$$

If nothing more is required, then it is possible that the arrival characteristics of other families as well will intersect each other at the discontinuity line $x = x(t)$, and for several families the arrival characteristics will be absent at the discontinuity line.

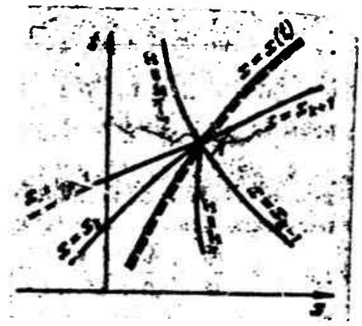

Figure 4.13

Therefore we can add the following two additional inequalities to inequalities (18):

$$\xi_{k-1}(u_s(t), x(t), t) < D < \xi_{k+1}(u_n(t), x(t), t), \qquad (19)$$

which with reference to (17) lead, in the neighborhood of the discontinuity line $x = x(t)$, to the pattern of characteristics shown in Figure 4.13.

In Figure 4.13 integral curves of the following equations

$$\frac{dx_i}{dt} = \xi_i(u(x, t), x, t) \qquad (20)$$

are drawn through the point A of the discontinuity line for the numbers $i = k - 1, k, k + 1$. By conditions (17), (18), (19) $n + 1$ arrival characteristics (at the rate of one from each family and two from the k-th family) reach the point A and $n - 1$ departing characteristics (at the rate of one from each family, except for the k-th) depart from point A.

And thus, we will state that stability conditions are satisfied at the discontinuity line $x = x(t)$ of the piecewise-continuous solution $u(x, t)$ of the system of quasilinear equations hyperbolic in the small, if the following inequalities are satisfied:

$$\left.\begin{aligned} &\xi_k(u_s(t), x(t), t) > D > \xi_k(u_n(t), x(t), t), \\ &\xi_{k-1}(u_s(t), x(t), t) < D < \xi_{k+1}(u_n(t), x(t), t). \end{aligned}\right\} \qquad (21)$$

Conditions (21) were published in the work [38]. The number k for which conditions (21) are satisfied is called the index of the discontinuity line.

On analogy with the case of a single equation, it is clear that stability condition (21) can ensure uniqueness also for the continuous dependence of solutions on initial data only in a certain, possibly narrow, class of systems of quasilinear equations. However, this class has also not been found. It is possible that conditions (21) guarantee the uniqueness of the discontinuous solution for systems of quasilinear equations satisfying the requirement

$$r^k(u, x, t)\,\mathrm{grad}_u\,\xi_k(u, x, t) \neq 0 \quad (k = 1, 2, \ldots, n) \tag{22}$$

(compare Section X of Chapter One) where $r^k(u, x, t)$ is the right eigenvector of the matrix $A = ((\partial\varphi_i/\partial u_j))$.

4. Irreversibility of processes described by discontinuous solution of systems of quasilinear equations. Suppose when $0 \leq t \leq t_1$, the solution $u = u_1(x, t)$ of the Cauchy's problem

$$\frac{\partial u}{\partial t} + \frac{\partial\varphi(u, x, t)}{\partial x} = f(u, x, t), \quad u(x, 0) = u_0(x). \tag{1}$$

has been constructed. We will state that the solution $u = u_1(x, t)$ describes a reversible process if the solution of the inverse Cauchy's problem with the following initial condition formulated at $t = t_1$,

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} + \dfrac{\partial\varphi(u, x, t)}{\partial x} = f(u, x, t), \\ u(x, t_1) = u_1(x, t_1), \quad 0 < t < t_1, \end{array}\right\} \tag{2}$$

coincides in the strip $0 \leq t \leq t_1$ with $u_1(x, t)$; but if the solution of problem (2) is distinct from $u_1(x, t)$ then we will state $u_1(x, t)$ describes an irreversible process.

If $u_1(x, t)$ is a classical solution of problem (1), then obviously it describes a reversible process. Actually, the smooth function $u_1(x, t)$ is a unique smooth solution both of problem (1), as well as problem (2), as stems from the uniqueness theorem of the classical solution of a system of quasilinear equations proven in Chapter One.

Advancing to the problem of the reversibility of the process described by the discontinuous solution of problem (1), we must consider that we will regard only stable solutions as solutions of this problem. Suppose $u_1(x, t)$ is a stable discontinuous solution of problem (1) and that $x = x(t)$ is its discontinuity line. Therefore, the stability conditions

$$\xi_u(u_1(t), x(t), t) > D > \xi_u(u_2(t), x(t), t) \qquad (3)$$

are satisfied at the line $x = x(t)$.

We will also label as solutions of the inverse Cauchy's problem (2) only solutions that are stable with respect to variation and the initial data. We can easily see, however, that variation in the direction of motion at time t leads to stability conditions that are the inverse of (3), i.e.,

$$\xi_u(u_1(t), x(t), t) < D < \xi_u(u_2(t), x(t), t) \qquad (4)$$

since here the right and left positions are interchanged.

Thus, if $u_1(x, t)$ is a stable discontinuous solution of problem (1), then $u = u_1(x, t)$ is not a stable solution of the inverse Cauchy's problem (2), since it does not satisfy conditions (4). This means that the function $u = u_2(x, t)$, clearly distinct from $u_1(x, t)$ will be a stable solution of the inverse Cauchy's problem (2), and that the solution $u_1(x, t)$ describes an irreversible process.

Let us clarify our conclusions here with a very simple example. For the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = 0 \qquad (5)$$

with the initial condition

$$u(x, 0) = u_0(x) = \begin{cases} u^- & \text{for } x < 0, \\ u^+ & \text{for } x > 0. \end{cases} \qquad (6)$$

as we have seen above, for the case $u^- > u^+$ the following function is a stable discontinuous solution:

$$u_1(x, t) = \begin{cases} u^- & \text{for } x < Dt, \\ u^+ & \text{for } x > Dt, \end{cases} \qquad (7)$$

$$D = \frac{u^- + u^+}{2}$$

But if we select any instant $t = t_1 > 0$ and solve the inverse Cauchy's problem for equation (5), that is, the problem when the initial position

$$u(x, t_1) = u_1(x, t_1) = \begin{cases} u^- & \text{for } x < x_1, \\ u^+ & \text{for } x > x_1, \end{cases} \quad x_1 = Dt_1, \tag{8}$$

is assigned at $t = t_1$ and if we seek the solution where $0 \leq t \leq t_1$, then the following function will become the stable solution of this problem:

$$u_2(x, t) = \begin{cases} u^- & \text{for } x - x_1 \leq u^-(t_1 - t), \\ \frac{x - x_1}{t - t_1} & \text{for } u^-(t_1 - t) \leq x - x_1 \leq u^+(t_1 - t), \\ u^+ & \text{for } x - x_1 \geq u^+(t_1 - t). \end{cases} \tag{9}$$

Figure 4.14 presents a pattern of characteristics for the solutions $u_1(x,t)$ ar. $u_2(x, t)$. Characteristics of solution $u_1(x, t)$ are shown by solid lines, and for solution $u_2(x, t)$ — by dash lines. Thus, the solution $u_1(x, t)$ and $u_2(x, t)$ are distinct in zone I.

And thus, a discontinuous stable solution of a hyperbolic system of quasilinear equations describes an irreversible process. This conclusion is valid only for hyperbolic equations, since for equations of other types the inverse Cauchy's problem can prove to be incorrect, that is, the irreversibility of the process will have a different nature in this case; in particular, even smooth solutions can describe irreversible processes.

Finally, we wish to direct the attention also to the fact that this conclusion is valid only for essentially nonlinear hyperbolic systems of quasilinear equations.

Actually, discontinuous solutions are linear, semilinear, and weakly-nonlinear systems of equations of hyperbolic type describe irreversible processes. This stems from the fact that the uniqueness theorem of discontinuous solutions for the systems can be proven only on the basis of integral laws of conservation, that is, unstable solutions are nonexistent for these systems.

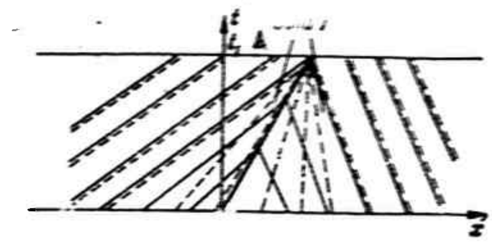

Figure 4.14

KEY:

A -- Zone I

Let us state, finally, several considerations on the nature of the continuous dependence of generalized solutions of a system of quasilinear equations on initial values.

From the examples considered above it is clear that the measure of proximity of generalized solutions cannot be found in the norm of space C. As we have seen in this chapter, for the case of a single quasilinear equation stable generalized solutions exhibit the property that

$$\sup_x |\Phi(x, t) - \overline{\Phi}(x, t)| \to 0, \qquad (10)$$

if

$$\sup_x |\Phi(x, 0) - \overline{\Phi}(x, 0)| \to 0. \qquad (11)$$

Here $\Phi(x, t)$, $\overline{\Phi}(x, t)$ are potentials of the generalized solutions $u(x, t)$ and $\bar{u}(x, t)$. We will state that solutions exhibiting this property are continuously dependent on initial data "in the potential metric."

This property of stable solutions of single equations enables us to assume that for systems of quasilinear equations as well the stability generalizes solutions denotes their continuous dependence on initial conditions "in the potential metric."

Section II. A Single Quasilinear Equation

1. Review of results. A single quasilinear equation is the simplest case

of a system of quasilinear equations in which there are essential simplifying details. Naturally, therefore, the first results in the study of discontinuous solutions of Cauchy's problem were obtained for a single quasilinear equation.

In 1950 E. Hopf published the classical work [63] which constructed a discontinuous solution of Cauchy's problem

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0, \quad u(x, 0)=u_0(x). \qquad (1)$$

The E. Hopf method consists of the following.

Instead of Cauchy's problem (1) he considers a different Cauchy's problem:

$$\frac{\partial u_\mu}{\partial t}+\frac{\partial}{\partial x}\frac{u_\mu^2}{2}=\mu\frac{\partial^2 u_\mu}{\partial x^2}, \quad u_\mu(x, 0)=u_0(x), \quad \mu>0. \qquad (2)$$

whose solution is written explicitly. The generalized solution of Cauchy's problem (1) is determined as the limit $u_\mu(x, t)$ as $\mu \to 0$:

$$u(x, t)=\lim_{\mu\to 0} u_\mu(x, t). \qquad (3)$$

An explicit expression for $u_\mu(x, t)$ and formula (3) for the generalized solution u(x, t) Cauchy's problem (1) made it possible to study in detail the properties of a discontinuous solution of problem (1).

Thus, the E. Hopf construction can be regarded as the first of result of establishing the existence and uniqueness of a generalized solution of Cauchy's problem for a single quasilinear equation in a sufficiently broad class of initial functions (it is sufficient to require only boundedness and measurability of the initial function $u_0(x)$ for the E. Hopf construction).

Let us note that Cauchy's problem (2) has been also considered by Borger [2, 3] and by Cole [23]. Germain and Bader [11] established that the solution of problem (1) is unique if at the discontinuity points u(x, t) the stability condition $u_Л > u_П$ is satisfied.

However, a vigorous validation of the solution of Cauchy's problem (1) was given first by E. Hopf. In particular, E. Hopf was the first to state that the function u(x, t) given by formula (3) satisfies equation (1) in the sense of satisfying the integral equality

$$\iint_{t>0} \left(u(x, t)\frac{\partial g}{\partial t} + \frac{u^2(x, t)}{2}\frac{\partial g}{\partial x} \right) dx\,dt + \int_{-\infty}^{\infty} g(x, 0)\,u_0(x)\,dx = 0 \quad (4)$$

for an arbitrary smooth finite function $g(x, t)$.

In 1954 O. A. Oleynik [40, 41] considered the following Cauchy's problem:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = 0, \quad (5)$$

$$u(x, 0) = u_0(x), \quad (6)$$

assuming that condition $\varphi''_{uu}(u, x, t) > 0$ ($\varphi''_{uu} < 0$) was satisfied in a class of piecewise-continuous and piecewise-differentiable solutions, and prove the existence and uniqueness of the solution. In 1956-1957 O. A. Oleynik [44, 46] extended this result also to the class of bounded measurable solutions $u(x, t)$. A. N. Tikhonov and A. A. Samarskiy [61] considered the following Cauchy's problem in 1954 for an inhomogeneous law of conservation:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = f(u, x, t) \quad (7)$$

in the class $u(x, t) \in K$, also assuming the convexity of the function $\varphi(u, x, t)$ ($\varphi''_{uu} \neq 0$).

In several subsequent works the properties of solutions of Cauchy's problem were refined for equation (7). Several new methods of solving Cauchy's problem were proposed for equations (5) and (7), including the "potential method" [27] and the "potential smoothing" method [56].

The works of I. M. Gel'fand, O. A. Oleynik, and A. S. Kalashnikov [10, 48, 50, 24] considered and in several cases solved Cauchy's problem for a nonconvex function $\varphi(u, x, t)$ when the quantity $\varphi''_{uu}(u, x, t)$ is sign-variable.

Although in this case the problem was solved in much less detail, nonetheless the main features of this case were clarified.

In this section we consider almost exclusively the case of the convex function $\varphi(u, x, t)$; the case of the nonconvex $\varphi(u)$ is analyzed only in order to demonstrate the complications that arise.

For simplicity in our presentation of results, we will limit ourselves, as a rule, to the class of solutions $u(x, t) \in K$, though most of the results are transferred without appreciable complications also to the class of bounded measurable solutions.

2. E. Hopf construction. E. Hopf considered the solution $u(x, t)$ of equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\frac{u^2}{2}=0 \tag{1}$$

with the initial condition $u(x, 0) = u_0(x)$ (2)

as a limit as $\mu \rightarrow 0$ of the solutions $u_\mu(x, t)$ of a different Cauchy's problem:

$$\frac{\partial u_\mu}{\partial t}+\frac{\partial}{\partial x}\frac{u_\mu^2}{2}=\mu\frac{\partial^2 u_\mu}{\partial x^2}, \quad \mu>0. \tag{3}$$

$$u_\mu(x, 0)=u_0(x). \tag{4}$$

We will assume the function $u_0(x)$ to be bounded over the entire axis $|x|<\infty$, exhibiting a piecewise-continuous first derivative, and with a first-order discontinuity point.

We will assume that we know the solution $u_\mu(x, t)$ of Cauchy's problem (3), (4) and that it is continuously differentiable when $t > 0$. Then, by equation (3), the curvilinear integral

$$\Phi_\mu(x, t)=\int_{(0,0)}^{(x,t)} u_\mu\,dx+\left[\mu\frac{\partial u_\mu}{\partial x}-\frac{u_\mu^2}{2}\right]dt \tag{5}$$

does not depend on the integration path; here

$$\frac{\partial\Phi_\mu}{\partial x}=u_\mu, \quad \frac{\partial\Phi_\mu}{\partial t}=\mu\frac{\partial u_\mu}{\partial x}-\frac{u_\mu^2}{2}. \tag{6}$$

Canceling function u_μ from (6), we obtain an equation which $\Phi_\mu(x, t)$ satisfies:

$$\frac{\partial\Phi_\mu}{\partial t}+\frac{1}{2}\left(\frac{\partial\Phi_\mu}{\partial x}\right)^2=\mu\frac{\partial^2\Phi_\mu}{\partial x^2}. \tag{7}$$

formulas (4) and (5) lead to the initial condition for Φ_μ:

$$\Phi_\mu(x, 0)=\Phi_0(x)=\int_0^x u_0(\xi)\,d\xi. \tag{8}$$

By our assumption, $u_0(x) = o(|x|)$ as $x \to \pm\infty$. (9)

Hence it follows that $\Phi_0(x) = o(x^2)$ as $x \to \pm\infty$. (10)

obviously, $\Phi_0(x)$ is a continuous function of variable x, varying the piecewise-continuous first derivative.

We now solve Cauchy's problem (7), (8) for Φ_μ: we find $u_\mu(x, t)$ by formula (6), and then, as $\mu \to 0$, we obtain at the limit the function $u(x, t)$, which we call the generalized solution of Cauchy's problem (1), (2). The substitution

$$\Phi_\mu(x, t) = -2\mu \ln \Psi_\mu(x, t) \tag{11}$$

reduces Cauchy's problem (7), (8) to Cauchy's problem for the equation of thermal conductivity

$$\frac{\partial \Psi_\mu}{\partial t} = \mu \frac{\partial^2 \Psi_\mu}{\partial x^2} \tag{12}$$

with the initial condition

$$\Psi_\mu(x, 0) = \Psi_\mu^0(x) = \exp\left\{-\frac{1}{2\mu}\Phi_0(x)\right\} = \exp\left\{-\frac{1}{2\mu}\int_0^x u_0(\xi)\,d\xi\right\}. \tag{13}$$

By formula (10) $\Psi_\mu^0(x) = o(e^{\frac{x^2}{2\mu}})$ as $x \to \pm\infty$, (14)

and therefore Cauchy's problem (12), (13) has a unique solution, which is yielded by the formula

$$\Psi_\mu(x, t) = \frac{1}{2\sqrt{\pi\mu t}}\int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\xi)^2}{4\mu t}\right\}\Psi_\mu^0(\xi)\,d\xi = \frac{1}{2\sqrt{\pi\mu t}}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}\left[\Phi_0(\xi) + \frac{(x-\xi)^2}{2t}\right]\right\}d\xi. \tag{15}$$

Now from formulas (11) and (6) we obtain an expression for $u_\mu(x, t)$:

$$u_\mu(x, t) = \frac{\int_{-\infty}^{\infty}\left(\frac{x-\xi}{t}\right)\exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\}d\xi}{\int_{-\infty}^{\infty}\exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\}d\xi}, \tag{16}$$

where

$$\lambda(t, x, \xi) = \Phi_0(\xi) + \frac{(x-\xi)^2}{2t}. \tag{17}$$

It follows from formula (17) that the function $\lambda(t, x, \xi)$ is a continuous function of all its variables as $t > 0$. Based on condition (10) we can assert that for any fixed x, $t > 0$ $\quad \lambda(t, x, \xi) \to +\infty$ as $\xi \to \pm\infty$. (18)

This means that the continuous function $\lambda(t, x, \xi)$ takes on — for fixed x, $t > 0$ — the smallest value of $\lambda_{min}(x, t)$ for some bounded set of values of the variable ξ. Let us denote this set by $m(x, t)$.

Let us introduce into our consideration function $v(t, x, \xi)$:

$$v(t, x, \xi) = \lambda(t, x, \xi) - \lambda_{min}(x, t) = \Phi_0(\xi) + \frac{(x-\xi)^2}{2t} - \lambda_{min}(x, t) \qquad (19)$$

By the definition of the quantity $\lambda_{min}(x, t)$ as the absolute minimum relative to the variable ξ of the function $\lambda(t, x, \xi)$, we obviously conclude that

$$v(t, x, \xi) \geq 0; \qquad (20)$$

and here the equality sign in (20) holds only for the case when ξ belongs to the set $m(x, t)$. Multiplying the numerator and denominator of formula (16) by $\exp\left\{+\frac{\lambda_{min}(x,t)}{2\mu}\right\}$, we give the following form:

$$u_\mu(x, t) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-\xi}{t}\right) \exp\left\{-\frac{v(t, x, \xi)}{2\mu}\right\} d\xi}{\int_{-\infty}^{\infty} \exp\left\{-\frac{v(t, x, \xi)}{2\mu}\right\} d\xi}. \qquad (21)$$

Suppose $\xi^-(x, t)$ and $\xi^+(x, t)$ are, respectively, the exact lower and exact upper bounds of the set $m(x, t)$:

$$\xi^-(x, t) = \inf m(x, t), \quad \xi^+(x, t) = \sup m(x, t), \quad \xi^-(x, t) \leq \xi^+(x, t). \qquad (22)$$

Let us graphically clarify the definition of the set $m(x, t)$ and its bounds ξ^-, ξ^+. To do this, let us introduce a new function $\eta(t, x, \xi)$:

$$\eta(t, x, \xi) = \lambda(t, x, \xi) - \frac{x^2}{2t} = \eta(t, 0, \xi) - \xi\frac{x}{t}. \qquad (23)$$

The functions $\lambda(t, x, \xi)$, $v(t, x, \xi)$, and $\eta(t, x, \xi)$ take on the smallest values for fixed x, $t > 0$ of the same points $\xi \in m(x, t)$.

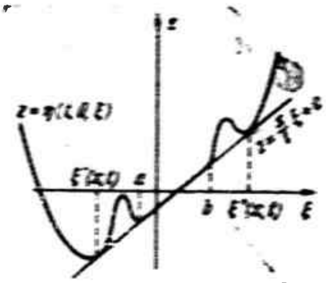

Figure 4.15

Figure 4.15 shows in the plane of variables ξ, z the curve $z = \eta(t, 0, \xi)$ and some straight line $z = x\xi/t + c$. The inclination of this straight line is given for fixed x, $t > 0$. Let us select such a number c so that this straight line is tangent to the curve $z = \eta(t, 0, \xi)$ from beneath, and nowhere intersects it. Then the set of points ξ in which $\xi x/t + c = \eta(t, o, \xi)$ constitutes the set m(x, t). Actually, if curve $z = \eta(t, 0, \xi)$ "rests" on the straight line $z = x\xi/t + c$, then this means that at $\eta(t, 0, \xi) - x\xi/t = \eta(t, x, \xi)$ at the points at which $\xi x/t + c = \eta(t, 0, \xi)$ takes on its smallest value. Let us note that the number c coincides with $\eta_{min}(x, t)$.

In Figure 4.15, the set m(x, t), except for the bounds $\xi^-(x, t)$, $\xi^+(x. t)$, contains further the segment $a \leq \xi \leq b$.

Let us now establish the following properties of the quantities ξ^-, ξ^+:

$$\xi^+(x, t) \leqslant \xi^-(x', t) \text{ when } x < x', \tag{24}$$

$$\xi^-(x-0, t) = \xi^-(x, t),\ \xi^+(x+0, t) = \xi^+(x, t), \tag{25}$$

$$\xi^-(+\infty, t) = +\infty, \quad \xi^+(-\infty, t) = -\infty. \tag{26}$$

To prove property (24), let us consider the difference

$$\eta(t, x+\Delta x, \xi) - \eta(t, x+\Delta x, \xi^+(x, t)) = \\ = \eta(t, x, \xi) - \eta(t, x, \xi^+(x, t)) - \frac{\Delta x}{t}[\xi - \xi^+(x, t)]. \tag{27}$$

By virtue of the definition of the upper bound

$$\eta(t, x, \xi) - \eta(t, x, \xi^+(x, t)) \quad \begin{cases} \geqslant 0 & \text{when } \xi \leq \xi^+(x, t), \\ > 0 & \text{when } \xi = \xi^+(x, t). \end{cases} \tag{28}$$

Suppose $\Delta x > 0$. It follows from formulas (27) and (28) that

$$\eta(t, x+\Delta x, \xi) - \eta(t, x+\Delta x, \xi^+(x, t)) \quad \begin{cases} > 0 & \text{when } \xi < \xi^+(x, t), \\ = 0 & \text{when } \xi = \xi^+(x, t). \end{cases} \tag{29}$$

By the definition of $\xi^-(x, t)$

$$\eta(t, x+\Delta x, \xi^-(x+\Delta x, t)) - \eta(t, x+\Delta x, \xi^+(x, t)) \leqslant 0 \quad (\Delta x > 0). \tag{30}$$

Comparing formula (30) with (29), we see that $\xi^-(x + \ x, t) \geqslant \xi^+(x, t)$. This then proves to be inequality (24).

The proof of inequality (24) is illustrated geometrically. Actually, if $x' > x$, then the slope of the reference line $z = \xi x'/t + c'$ is greater than the slope of the line $z = \xi x/t + c$ and from Figure 4.15 inequality (24) follows at once.

In the similar fashion we can readily establish the properties (25) and (26). Let us note that since the function $\lambda(t, x, \xi)$ is a continuous function of all its variables when $t > 0$, then its absolute minimum $\lambda_{min}(x, t)$ relative to the variable ξ is also a continuous function of the variable x, t.

Let us turn to formula (21) and show that for any $x, t > 0$

$$\frac{x - \xi^+(x, t)}{t} \leqslant \liminf_{\substack{\mu \to 0 \\ x' \to x \\ t' \to t}} u_\mu(x', t') \leqslant \limsup_{\substack{\mu \to 0 \\ x' \to x \\ t' \to t}} u_\mu(x', t') \leqslant \frac{x - \xi^-(x, t)}{t}. \tag{31}$$

From formulas (31), in particular it follows that if at the point $x, t (t > 0)$ the condition $\xi^-(x, t) = \xi^+(x, t)$ is satisfied, then at this point there exists the limit

$$\lim_{\substack{\mu \to 0 \\ x' \to x \\ t' \to t}} u_\mu(x', t') = \frac{x - \xi^-(x, t)}{t} = \frac{x - \xi^+(x, t)}{t}. \tag{32}$$

Proceeding to prove formula (31), for brevity we denote $\xi^+(x, t) = \xi^+$, $\xi^-(x, t) = \xi^-$. Suppose $\varepsilon > 0$ is an arbitrarily small positive number. Let us select positive numbers a and b that are sufficiently small that for all values ξ, x', t', satisfying the conditions

$$|x'-x|+|t'-t|<a, \quad \xi^- - 2b<\xi<\xi^+ + 2b, \quad 0<a\leq\frac{t}{2} \tag{33}$$

the following inequalities are satisfied:

$$l_1 = \frac{x-\xi^+}{t}-\varepsilon<\frac{x'-\xi}{t'}<\frac{x-\xi^-}{t}+\varepsilon=l. \tag{34}$$

The functions $\xi^-(x, t)$ and $\xi^+(x, t)$ are semicontinuous, respectively, to the left and to the right, and therefore if the value of a is small enough, then providing that inequalities (33) and (34) are satisfied, we can assume also that the following have been met:

$$\xi^- - b<\xi^-(x', t')<\xi^+(x', t')<\xi^+ + b. \tag{35}$$

Write formula (21) as applied to point x', t':

$$u_\mu(x', t')=\frac{\int\limits_{-\infty}^{\infty}\frac{x'-\xi}{t'}\exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\}d\xi}{\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\}d\xi}. \tag{36}$$

Let us estimate the numerator in formula (36):

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty}\frac{x'-\xi}{t'}\exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\}d\xi= \\
&=l\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{v}{2\mu}\right\}d\xi+\int\limits_{-\infty}^{\infty}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi> \\
&>l\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{v}{2\mu}\right\}d\xi+\int\limits_{-\infty}^{\xi^- - 2b}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi+ \\
&\qquad+\int\limits_{\xi^+ + 2b}^{\infty}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi= \\
&=\int\limits_{-\infty}^{\infty}\frac{x'-\xi}{t'}\exp\left\{-\frac{v}{2\mu}\right\}d\xi-\int\limits_{\xi^- - 2b}^{\xi^+ + 2b}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi.
\end{aligned} \tag{37}$$

In obtaining (37) we use the condition (34). Quite similarly, we obtain for the numerator the estimate from above

$$\int_{-\infty}^{\infty} \frac{x'-\xi}{t'} \exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\} d\xi <$$

$$< \ell \int_{-\infty}^{\infty} \exp\left\{-\frac{v}{2\mu}\right\} d\xi + \int_{-\infty}^{\xi^- - 2b} \left[\frac{x'-\xi}{t'} - \ell\right] \exp\left\{-\frac{v}{2\mu}\right\} d\xi +$$

$$+ \int_{\xi^+ + 2b}^{\infty} \left[\frac{x'-\xi}{t'} - \ell\right] \exp\left\{-\frac{v}{2\mu}\right\} d\xi \quad (38)$$

Dividing the estimate obtain in (37) by the denominator in formula (36), we obviously get

$$u_\mu(x', t') > \ell + \frac{\int_{-\infty}^{\xi^- - 2b} \left[\frac{x'-\xi}{t'} - \ell\right] \exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\} d\xi}{\int_{-\infty}^{\infty} \exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\} d\xi} +$$

$$+ \frac{\int_{\xi^+ + 2b}^{\infty} \left[\frac{x'-\xi}{t'} - \ell\right] \exp\left\{-\frac{v}{2\mu}\right\} d\xi}{\int_{-\infty}^{\infty} \exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\} d\xi} . \quad (39)$$

Let us show that the last two terms in inequality (39) tend to zero as $\mu \to 0$ uniformly relative to x', t', if the latter satisfies constraint (33). Noting that the linear function $\left[\frac{x'-\xi}{t'} - \ell\right]$ on the semi-intervals $(-\infty, \xi^- - 2b]$ and $[\xi^+ + 2b, +\infty)$ is estimated moduluswise, respectively, by the quantities $\omega(\xi^- - \xi)$ and $\omega(\xi - \xi^+)$ and the value of ω can be selected independently of x', t', if the latter satisfies (33), let us write:

$$\frac{\left|\int_{-\infty}^{\xi^- - 2b}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi\right|}{\int_{-\infty}^{\infty}\exp\left\{-\frac{v}{2\mu}\right\}d\xi} < \frac{\int_{-\infty}^{\xi^- - 2b}\omega(\xi^- - \xi)\exp\left\{-\frac{v}{2\mu}\right\}d\xi}{\int_{-\infty}^{\xi^-(x',t')}\exp\left\{-\frac{v}{2\mu}\right\}d\xi}, \quad (40)$$

$$\frac{\left|\int_{\xi^+ + 2b}^{\infty}\left[\frac{x'-\xi}{t'}-l\right]\exp\left\{-\frac{v}{2\mu}\right\}d\xi\right|}{\int_{-\infty}^{\infty}\exp\left\{-\frac{v}{2\mu}\right\}d\xi} < \frac{\int_{\xi^+ + 2b}^{\infty}\omega[\xi-\xi^+]\exp\left\{-\frac{v}{2\mu}\right\}d\xi}{\int_{\xi^+(x',t')}^{\infty}\exp\left\{-\frac{v}{2\mu}\right\}d\xi} \quad (41)$$

For all x', t', satisfying (33), $v(t', x', \xi) > 0$ when $\xi < \xi^- - 2b$ and $v(t', x', \xi) = O(\xi^2/2t')$ as $\xi \to -\infty$. Therefore there exists the number $A > 0$ such that $v(t', x', \xi) > A(\xi - \xi^-)^2$ when $\xi \leq \xi^- - 2b$. Using the estimate, let us write the following expression for the numerator in formula (40):

$$\int_{-\infty}^{\xi^- - 2b}\omega[\xi^- - \xi]\exp\left\{-\frac{v(t', x', \xi)}{2\mu}\right\}d\xi <$$
$$< \omega\int_{-\infty}^{\xi^- - 2b}(\xi^- - \xi)\exp\left\{-\frac{A}{2\mu}(\xi - \xi^-)^2\right\}d\xi = \frac{\omega\mu}{A}\exp\left\{-\frac{2Ab^2}{\mu}\right\}. \quad (42)$$

On the other hand, since $v(t', x', \xi^-(t', x')) = 0$, then there exists a $\delta > 0$ such that $v(t', x', \xi) < 4Ab^2$ for all ξ from interval $\xi(x', t') - \delta \leq \xi \leq \xi^-(x', t')$. Therefore the denominator in formula (40) can be estimated as follows:

$$\int_{-\infty}^{\xi^-(x',t')} \exp\left\{-\frac{v(t',x',\xi)}{2\mu}\right\} d\xi > \int_{\xi^-(x',t')-\delta}^{\xi^-(x',t')} \exp\left\{-\frac{2Ab^2}{\mu}\right\} d\xi = \delta \exp\left\{-\frac{2Ab^2}{\mu}\right\}. \quad (43)$$

The right side of inequality (40) does not exceed the ratio of the right side of (42) to (43), i.e., it does not exceed the value of $\omega\mu/A\delta$, where its estimate holds uniformly for all x', t' satisfying (33). Since a similar estimate can be readily obtained also for the second term in (39), then we can write:

$$u_\mu(x', t') > l + O(\mu). \quad (44)$$

Similar operations for (38) lead to the inequality

$$u_\mu(x', t') < L + O(\mu). \quad (45)$$

Estimates (44), (45) hold uniformly for x', t', satisfying (33). From the definition of the numbers ℓ and L (34) and the arbitrary choice of the quantity ε it follows that estimates (44), (45) prove formulas (31) and (32).

Let us determine the generalized solution of Cauchy's problem (1), (2) by means of the formula

$$u(x, t) = \lim_{\mu \to 0} u_\mu(x, t) \quad (46)$$

at all points x, t, at which this limit exists, i.e., $\xi^-(x, t) = \xi^+(x, t)$. The function u(x, t) is continuous at these points over the set of argument x, t.

At the points at which $\xi^-(x, t) \neq \xi^+(x, t)$, we will assume for definiteness that $u(x, t) = u(x - 0, t) = \frac{x - \xi^-(x, t)}{t}$.

Suppose $\xi^-(x, t) \neq \xi^+(x, t)$ at any point (x, t). Then from formulas (31) and (24) it follows that

$$u(x - 0, t) > u(x + 0, t), \quad (47)$$

since

$$u(x - 0, t) = \frac{x - \xi^-(x, t)}{t}, \quad u(x + 0, t) = \frac{x - \xi^+(x, t)}{t}. \quad (48)$$

Inequality (47) shows that the generalized solution u(x, t) given by formula (46) satisfies the stability condition which was introduced in Section I.

Suppose now $\xi^-(x, t) = \xi^+(x, t) = \xi(x, t)$. If $\xi = \xi(x, t)$ is a point of continuity of the initial function $u_0(\xi)$, then since $\lambda(t, x, \xi)$ takes on the minimum value of this point, then

$$u_0(\xi(x, t)) = \frac{x - \xi(x, t)}{t}, \qquad (49)$$

$$u_0'(\xi(x, t) - 0) \geqslant -\frac{1}{t}, \quad u_0'(\xi(x, t) + 0) \geqslant -\frac{1}{t} \qquad (50)$$

we assume that the point $\xi = \xi(x, t)$ can be a point of discontinuity of the derivative of the function $u_0(\xi)$. But if the point $\xi = \xi(x, t) = \xi^-(x, t) = \xi^+(x, t)$ is a point of discontinuity of the initial function $u_0(\xi)$, then

$$u_0(\xi(x, t) - 0) < u_0(\xi(x, t) + 0). \qquad (51)$$

Now suppose $\xi^-(x, t) \neq \xi^+(x, t)$. Then

$$u_0(\xi^-(x, t) - 0) = u(x - 0, t) \leqslant u_0(\xi^-(x, t) + 0), \qquad (52)$$

$$u_0(\xi^+(x, t) - 0) \leqslant u(x + 0, t) = u_0(\xi^+(x, t) + 0). \qquad (53)$$

Finally, let us note yet another property of solution $u(x, t)$. Suppose $x = x(t)$ is the discontinuity line $u(x, t)$. Then

$$\lim_{\substack{t' \to t+0 \\ t'' \to t+0}} \frac{x(t'') - x(t')}{t'' - t'} = \frac{1}{2}\left[u(x(t) - 0, t) + u(x(t) + 0, t)\right] \qquad (54)$$

This property shows that piecewise-smooth solutions $u(x, t)$ of Cauchy's problem (1), (2) satisfy Hugoniot's condition for equation (1).

Now let us proceed to clarifying the problem in which sense function $u(x, t) = \lim_{\mu \to 0} u_\mu(x, t)$ satisfy equation (1) and initial condition (2).

Let us consider the function

$$\Phi(x, t) = \lim_{\mu \to 0} \Phi_\mu(x, t) = \lim_{\mu \to 0} \frac{\ln 2\sqrt{\pi\mu t}}{1/2\mu} - \lim_{\mu \to 0} \frac{\ln \int_{-\infty}^{\infty} \exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\} d\xi}{1/2\mu}.$$

applying in this equality l'Hopital's rule, we get

$$\Phi(x, t) = \lim_{\mu \to 0} \frac{\int_{-\infty}^{\infty} \lambda(t, x, \xi) \exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\} d\xi}{\int_{-\infty}^{\infty} \exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\} d\xi}. \quad (55)$$

From formula (55) it follows that

$$\Phi(x, t) = \lambda_{\min}(x, t) = \lambda(t, x, \xi^-(x, t)) = \lambda(t, x, \xi^+(x, t)) = \\ = \inf_{\xi}\left[\Phi_0(\xi) + \frac{(x-\xi)^2}{2t}\right] = \inf_{\xi} \mathcal{G}(t, x, \xi), \quad (56) \\ \mathcal{G}(t, x, \xi) = \Phi_0(\xi) + \frac{(x-\xi)^2}{2t}.$$

The proof of (56) is analogous to the proof of inequalities (31). Hence it follows that $\Phi(x, t)$ is a function of the variable x, t that is continuous in the half-plane $t > 0$. Since as $t \to 0$, $\xi^-(x, t) \to x$, $\xi^+(x, t) \to x$, then from (56) it follows that $\Phi(x, t)$ is continuous also when $t = 0$, i.e., it is continuous when $t \geqslant 0$.

Let us consider the arbitrary point (x, t) of the half-plane $t > 0$, at which $\xi^-(x, t) = \xi^+(x, t)$. Obviously, there exists the neighborhood of the point (x, t) in which $\xi^-(x', t') = \xi^+(x', t')$ for all of its points (x', t'). By (6)

$$\frac{\partial \Phi_\mu}{\partial x} = u_\mu, \quad \frac{\partial \Phi_\mu}{\partial t} = \mu \frac{\partial u_\mu}{\partial x} - \frac{u_\mu^2}{2}. \quad (57)$$

Since the sequence u_μ converges uniformly in this neighborhood to u(x, t), and $\Phi_\mu(x, t) \rightrightarrows \Phi(x, t)$ according to the proof given above, then obviously $\Phi(x, t)$ is differentiable at the point (x, t); here

$$\frac{\partial \Phi(x, t)}{\partial x} = u(x, t). \quad (58)$$

Differentiating formula (16) relative to the variable x, we find

$$\mu \frac{\partial u_\mu}{\partial x} = \frac{1}{2}\left[u_\mu^2 - \overline{u_\mu^2}\right] + \frac{\mu}{t}, \tag{59}$$

where we denote the quantity

$$\overline{u_\mu^2} = \frac{\int_{-\infty}^{\infty}\left(\frac{x-\xi}{t}\right)^2 \exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\} d\xi}{\int_{-\infty}^{\infty}\exp\left\{-\frac{\lambda(t, x, \xi)}{2\mu}\right\} d\xi} \tag{60}$$

by $\overline{u_\mu^2}$. At the points (x, t) at which $\xi^-(x, t) = \xi^+(x, t)$, we have

$$\lim_{\mu\to 0} u_\mu^2 = \lim_{\mu\to 0} \overline{u_\mu^2} = \left[\frac{x-\xi(x, t)}{t}\right]^2. \tag{61}$$

Therefore in this neighborhood of the point (x, t), according to (57), the sequence $\partial\Phi_\mu/\partial t$ uniformly converges as $\mu \to 0$:

$$\lim_{\mu\to 0} \frac{\partial \Phi_\mu}{\partial t} = \lim_{\mu\to 0}\left(-\frac{u_\mu^2}{2}\right) = -\frac{u^2(x, t)}{2}. \tag{62}$$

Consequently, the function $\Phi(x, t)$ is differentiable in the point under consideration (x, t) relative to the variable t, where

$$\frac{\partial \Phi(x, t)}{\partial t} = -\frac{u^2(x, t)}{2}. \tag{63}$$

And thus, at all points (x, t) for which $\xi^-(x, t) = \xi^+(x, t)$, the function $\Phi(x, t)$ is continuously differentiable and satisfies the equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 = 0. \tag{64}$$

By formula (25) the functions $\xi^-(x, t)$ and $\xi^+(x, t)$ are monotonically increasing functions of the variable x and are semicontinuous relative to the variable x and are bounded for finite x, t. Hence it follows that for any segment of the straight line t = constant the set of points in which $\xi^-(x, t) \neq \xi^+(x, t)$ is not more than countable. Hence, it follows further, that in any domain G of variables (x, t) there exists not more than a countable number of lines outside of which $\Phi(x, t)$ is continuously differentiable and satisfies equations (64).

Since $\xi^-(x, t) \to x$, $\xi^+(x, t) \to x$ as $t \to 0$, then

$$\Phi(x, 0) = \Phi_0(x) = \int_0^x u_0(\xi)\, d\xi.$$

By virtue of the continuity of $\Phi(x, t)$, we conclude that

$$\int_0^x u(\xi, t)\, d\xi \to \int_0^x u_0(\xi)\, d\xi \quad \text{as } t \to 0. \tag{65}$$

Relations (65) and (64) show that the function u(x, t) given by formula (46) is the generalized solution of Cauchy's problem (1), (2).

Let us show by yet another method that the function u(x, t) given by formula (46) is a generalized solution of Cauchy's problem (1), (2). Integrating equation (3) over the domain G of the half-plane $t > 0$ bounded by the closed contour C, we obviously get

$$\oint_C u_\mu\, dx - \frac{u_\mu^2}{2}\, dt = -\oint_C \mu \frac{\partial u_\mu}{\partial x}\, dt. \tag{66}$$

By (59), $\mu \frac{\partial u_\mu}{\partial x} \to 0$ at the point which $\xi^-(x, t) = \xi^+(x, t)$. Therefore if we assume that at the contour C the measure of the set of points (x, t) at which $\xi^-(x, t) \neq \xi^+(x, t)$ is equal to zero, then as $\mu \to 0$ the integral $\oint_C \mu \frac{\partial u_\mu}{\partial x}$ dt appearing in the right side of (66) since to zero.

Under the same assumption, the integral in the left side of (66) as $\mu \to 0$ retains its meaning, since at the contour C almost everywhere $u(x, t) \to u(x, t)$. Therefore passing to the limit as $\mu \to 0$ in (66), we get

$$\oint_C u(x, t)\, dx - \frac{u^2(x, t)}{2}\, dt = 0. \tag{67}$$

And thus, for an arbitrary closed contour C at which the measure of the set of points of discontinuity u(x, t) is equal to zero, the function u(x, t) satisfies the integral law of conservation (67) for equation (1).

Since the function u(x, t) takes on initial values (2) in the sense of (65), then u(x, t) is a stable generalized solution of Cauchy's problem (1), (2) also in the sense of integral law of conservation (67).

Now we note that if at the points of the discontinuity of function u(x, t) a certain value is assigned, for example, $u(x, t) = u(x - 0, t)$, (68) then equality (67) is valid in general for any closed contour C lying in the half-plane $t \geqslant 0$.

Actually, the contour C can be considered as the limit of the contours C' at which the measure of the set of points of discontinuity of function u(x, t) is equal to zero.

Equality (67) is satisfied for each such contour C' according to the focal point. If at each discontinuity point (x, t) of the function u(x, t) the corresponding point (x', t') of contour C' tends toward the left, i.e., $x' < x$, $x' \to x$, then $u(x', t') \to u(x, t)$ by (68) and, passing to the limit in equality (67) we conclude that it has been satisfied for the arbitrary contour C.

Multiplying equation (3) by an arbitrary smooth finite function g(x, t) and integrating the result over the half-plane $t \geqslant 0$, we get

$$\iint\limits_{t>0} \left[\frac{\partial g}{\partial t} u_\mu + \frac{\partial g}{\partial x}\frac{u_\mu^2}{2}\right] dx\, dt + \int\limits_{-\infty}^{\infty} g(x, 0)\, u_0(x)\, dx = \iint\limits_{t>0} \mu \frac{\partial u_\mu}{\partial x}\frac{\partial g}{\partial x}\, dx\, dt. \quad (69)$$

Since almost everywhere $u_\mu \to u(x, t)$, $u_\mu^2 \to u^2(x, t)$ and by (64) for $\mu \frac{\partial u_\mu}{\partial x} \to 0$, then passing to the limit in (69) as $\mu \to 0$, we get

$$\iint\limits_{t>0} \left[\frac{\partial g}{\partial t} u(x, t) + \frac{\partial g}{\partial x}\frac{u^2(x, t)}{2}\right] dx\, dt + \int\limits_{-\infty}^{\infty} g(x, 0)\, u_0(x)\, dx = 0. \quad (70)$$

Equality (70) shows that the constructed function u(x, t) is a generalized solution of Cauchy's problem (1), (2) also in the sense of the last of the three definitions which were introduced in Section I. This definition of a generalized solution of quasilinear equations was first suggested by E. Hopf.

3. Cauchy's problem for the equation $u_t + \varphi_x = 0$ given the condition $\varphi''_{uu} > 0$. Suppose the function $\varphi(u, x, t)$ exhibits two first continuous derivatives relative to all their variables as $t \geqslant 0$, $-\infty < x < \infty$ and for all the constraints on u. We will assume that $\varphi''_{uu}(u, x, t) > 0$ in this domain of variables x, t, u(the case of $\varphi''_{uu}(u, x, t) < 0$ is considered quite analogously).

For the equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = 0 \qquad (1)$$

we pose the initial condition $u(0, x) = u_0(x)$, (2)

assuming that the function $u_0(x)$ is piecewise-continuous and has a piecewise-continuous first derivative for all bounded values of the variable x.

Cauchy's problem (1), (2) was considered and solved for the first time by O. A. Oleynik, and this was done even for broader class of initial functions — bounded and measurable functions $u_0(x)$. Our consideration of Cauchy's problem (1), (2) will be less detailed and somewhat different from that given by O. A. Oleynik.

For the characteristics system of equations (1)

$$\frac{dx}{dt} = \varphi'_u(u, x, t), \quad \frac{du}{dt} = -\varphi'_x(u, x, t) \qquad (3)$$

let us consider the Cauchy's problem with the initial conditions

$$x(0) = x_0, \quad u(0) = u_0. \qquad (4)$$

We will denote this solution of this Cauchy's problem by the latters

$$x(t) = X(t, x_0, u_0), \quad u(t) = U(t, x_0, u_0). \qquad (5)$$

These functions, by the definition, satisfy conditions (4), which we will now rewrite as $X(0, x_0, u_0) \equiv x_0$, $U(0, x_0, u_0) \equiv u_0$. (6)

We will assume that the function X and U remain bounded for any finite x_0, u_0 and $t > 0$. The conditions in which this obtains are relatively involved, and we will not present them here.

If $X(t, x_0, u_0)$ and $U(t, x_0, u_0)$ are bounded, then they have continuous first derivatives with respect to all their variables. This follows from the continuous differentiability of the right side of the characteristics in (3).

Finally, we further assume that $X(t, x_0, u_0) \to \pm\infty$ as $x_0 \to \pm\infty$. (7)

The second problem which we will consider for the characteristic system (3) is the boundary value problem with the following conditions.

It is required to find the solution $x(t) = \bar{X}(t, \tau, \xi, x_0)$, $u(t) = \bar{U}(t, \tau, \xi, x_0)$ of characteristic system (3) satisfying the following boundary conditions:

$$x(0) = \bar{X}(0, \tau, \xi, x_0) \equiv x_0, \quad x(\tau) = \bar{X}(\tau, \tau, \xi, x_0) \equiv \xi. \tag{8}$$

where x_0, ξ are arbitrary numbers, and $\tau > 0$.

As before we will assume that this problem has a unique bounded solution for arbitrary x_0, ξ, $\tau > 0$, without detailing the complicated sufficient conditions under which this obtains. However, if this is assumed, then the functions $\bar{X}$, $\bar{U}$ are continuously differentiable relative to all their arguments.

Let us enumerate the following obvious relations:

$$\bar{X}(0, \tau, \xi, x_0) \equiv x_0, \quad \bar{X}(\tau, \tau, \xi, x_0) \equiv \xi. \tag{9}$$
$$X(t, x_0, \bar{U}(0, \tau, \xi, x_0)) \equiv \bar{X}(t, \tau, \xi, x_0). \tag{10}$$
$$U(t, x_0, \bar{U}(0, \tau, \xi, x_0)) \equiv \bar{U}(t, \tau, \xi, x_0). \tag{11}$$

For the E. Hopf equation considered in previous section, $\varphi(u, x, t) = u^2/2$ and

$$X(t, x_0, u_0) = x_0 + u_0 t, \qquad U(t, x_0, u_0) = u_0. \tag{12}$$
$$\bar{X}(t, \tau, \xi, x_0) = x_0 + \frac{\xi - x_0}{\tau} t, \quad \bar{U}(t, \tau, \xi, x_0) = \frac{\xi - x_0}{\tau}. \tag{13}$$

From formulas (13) we can readily note that the solution $u(x, t)$ of Cauchy's problem (2.2.1), (2.2.2), given by formulas (2.2.48) can also be written as

$$\left.\begin{aligned} u(x-0, t) &= \bar{U}(t, t, x, \xi^-(x, t)), \\ u(x+0, t) &= \bar{U}(t, t, x, \xi^+(x, t)), \end{aligned}\right\} \tag{14}$$

where the quantities $\xi^-(x, t)$ and $\xi^+(x, t)$ are defined (cf subsection 2) as the exact lower and upper bounds of the set of ξ values for which the function

$$\lambda(t,\ x,\ \xi)=\Phi_0(\xi)+\frac{(x-\xi)^2}{2t}=\int_0^{\xi} u_0(\eta)\,d\eta+\frac{(x-\xi)^2}{2t}$$

takes on a minimum value for fixed x, t. Noting that for the E. Hopf equation

$$I(t,\ x,\ \xi)=\lambda(t,\ x,\ \xi)-\frac{x^2}{2t}=\int_0^{\xi}[u_0(\eta)-\overline{U}(0,\ t,\ x,\ \eta)]\,d\eta=$$
$$=\int_0^{\xi}\left[u_0(\eta)+\frac{\eta-x}{t}\right]d\eta, \quad (15)$$

we can assert that in this special case the $\xi^-(x, t)$ and $\xi^+(x, t)$ values can be also defined as the exact lower and exact upper bounds of the set of ξ values for which the function

$$I(t,\ x,\ \xi)=\int_0^{\xi}[u_0(\eta)-\overline{U}(0,\ t,\ x,\ \eta)]\,d\eta \qquad (16)$$

takes on its smallest value for fixed values of the variable x, $t > 0$.

As shown by O. A. Oleynik, formulas (14) give the solution of Cauchy's problem (1), (2) under the requirements that we imposed above on the function $\varphi(u, x, t)$, if we let $\xi^-(x, t)$ and $\xi^+(x, t)$ now stand for the exact upper and exact lower bounds of the set of ξ values for which $I(t, x, \xi)$ takes on its smallest value.

It follows from the requirements imposed on $\varphi(u, x, t)$ that

$$\overline{U}(0,\ t,\ x,\ \xi) \to \mp\infty \qquad \text{as } \xi \to \pm\infty, \qquad (17)$$

and since we assume the initial function $u_0(x)$ to be bounded, then

$$I(t,\ x,\ \xi)\to+\infty \qquad \text{as } |\xi| \to \infty . \qquad (18)$$

This means that the continuous function $I(t, x, \xi)$ for fixed values x, $t > 0$ takes on its smallest value $I_{min}(x, t)$ on some bounded set $m(x, t)$ of values of the variable ξ. We can denote exact lower and exact upper bounds of this set by, respectively, $\xi^-(x, t)$ and $\xi^+(x, t)$. The quantities $\xi^-(x, t)$ and $\xi^+(x, t)$ satisfy relations (2.2.24), (2.2.25), and (2.2.26). The proof is given analogously as in subsection 2 and will be omitted here. The generalized solution $u(x, t)$ of Cauchy's problem (1), (2), defined by formulas (14), satisfies inequality (2.2.47).

Let us introduce into consideration the function

$$\vartheta(t, x, \xi) = \int_0^\xi u_0(\eta)\,d\eta + \int_0^t [\bar{U}\varphi'_u(\bar{U}, \bar{X}, \tau) - \varphi(\bar{U}, \bar{X}, \tau)]\,d\tau, \quad (19)$$

where for brevity we omit the arguments at the functions $\bar{U} = \bar{U}(\tau, t, x, \xi)$, $\bar{X} = \bar{X}(\tau, t, x, \xi)$. Differentiate the integrand in the second term of formula (19) relative to the variable ξ. We get

$$[\bar{U}\varphi'_u(\bar{U}, \bar{X}, \tau) - \varphi(\bar{U}, \bar{X}, \tau)]'_\xi = \bar{U}'_\xi\varphi'_u(\bar{U}, \bar{X}, \tau) + \bar{U}[\varphi'_u(\bar{U}, \bar{X}, \tau)]'_\xi -$$
$$- \varphi'_u(\bar{U}, \bar{X}, \tau)\bar{U}'_\xi - \varphi'_x(\bar{U}, \bar{X}, \tau)\bar{X}'_\xi =$$
$$= \bar{U}[\varphi'_u(\bar{U}, \bar{X}, \tau)]'_\xi - \varphi'_x(\bar{U}, \bar{X}, \tau)\bar{X}'_\xi. \quad (20)$$

But $\bar{U}$ and $\bar{X}$ represent the solution of characteristic system (3), therefore

$$\varphi'_u(\bar{U}, \bar{X}, \tau) = \bar{X}'_\tau, \quad -\varphi'_x(\bar{U}, \bar{X}, \tau) = \bar{U}'_\tau. \quad (21)$$

Inserting formulas (21) into (20), we get

$$[\bar{U}\varphi'_u - \varphi]'_\xi = \bar{U}\bar{X}''_{\tau\xi} + \bar{U}'_\tau\bar{X}'_\xi = \frac{\partial}{\partial\tau}[\bar{U}(\tau, t, x, \xi)\bar{X}'_\xi(\tau, t, x, \xi)].$$

Inserting this expression into formula (19), we find

$$\vartheta(t, x, \xi) = \int_0^\xi u_0(\eta)\,d\eta + \int_0^t\int_0^\xi \frac{\partial}{\partial\tau}[\bar{U}(\tau, t, x, \eta)\bar{X}'_\eta(\tau, t, x, \eta)]\,d\tau\,d\eta +$$
$$+ \int_0^t [\bar{U}_0\varphi'_u(\bar{U}_0, \bar{X}_0, \tau) - \varphi(\bar{U}_0, \bar{X}_0, \tau)]\,d\tau, \quad (22)$$

where

$$\bar{U}_0 = \bar{U}(\tau, t, x, 0), \quad \bar{X}_0 = \bar{X}(\tau, t, x, 0).$$

Let us inspect the second term in formula (22):

$$\int_0^t\int_0^\xi \frac{\partial}{\partial\tau}[\bar{U}(\tau, t, x, \eta)\bar{X}'_\eta(\tau, t, x, \eta)]\,d\tau\,d\eta =$$
$$= \int_0^\xi \{\bar{U}(t, t, x, \eta)\bar{X}'_\eta(t, t, x, \eta) - \bar{U}(0, t, x, \eta)\bar{X}'_\eta(0, t, x, \eta)\}\,d\eta.$$

But by formulas (8) $\bar{X}'_\eta(0, t, x, \eta) = 1$, $\bar{X}'_\eta(t, t, x, \eta) = 0$. Therefore

$$\vartheta(t, x, \xi) = \int_0^\xi [u_0(\eta) - \bar{U}(0, t, x, \eta)]\,d\eta + F(x, t), \quad (23)$$

where

$$F(x, t) = \int_0^t [\bar{U}_\varphi \varphi'_z(\bar{U}_\varphi, \bar{X}, \tau) - \varphi(\bar{U}_\varphi, \bar{X}, \tau)] d\tau. \quad (24)$$

Formulas (23) and (19) give a new representation for $I(t, x, \xi)$, from which we can conclude that for fixed x, $t > 0$ the continuous function $\mathcal{J}(t, x, \xi)$ takes on its smallest value on the same set $m(x, t)$ of variable ξ as does the function $I(t, x, \xi)$. This means that the function

$$\Phi(x, t) = \mathcal{J}(t, x, \xi^-(x, t)) = \mathcal{J}(t, x, \xi^+(x, t)) = \inf_\xi \mathcal{J}(t, x, \xi) \quad (25)$$

is a unique and continuous function of the variables x, t.

From formulas (25) and (19) we conclude that the continuous function $\Phi(x, t)$ takes on, given $t = 0$, the following values:

$$\Phi(x, 0) = \int_0^x u_0(\xi) d\xi = \Phi_0(x). \quad (26)$$

Since the argument $\mathcal{J}(t, x, \xi)$ is a Lipschitz-continuous function of all its arguments in any bounded domain of variables $t > 0$, x, ξ, i.e.,

$$|\mathcal{J}(t+\Delta t, x+\Delta x, \xi+\Delta\xi) - \mathcal{J}(t, x, \xi)| \leqslant M\{|\Delta t| + |\Delta x| + |\Delta\xi|\}. \quad (27)$$

then its absolute minimum relative to the variable ξ — the function $\Phi(x, t)$ — exhibits the same property, i.e.,

$$|\Phi(x+\Delta x, t+\Delta t) - \Phi(x, t)| \leqslant \bar{M}\{|\Delta x| + |\Delta t|\}. \quad (28)$$

Lipschitz-continuous function $\Phi(x, t)$ has almost everywhere continuous derivatives relative to the variables x, t.

Let us calculate these derivatives formally at least. We have

$$\Phi(x, t) = \int_0^{\xi(x, t)} u_0(\eta) d\eta + \int_0^t [\bar{U}\varphi'_z(\bar{U}, \bar{X}, \tau) - \varphi(\bar{U}, \bar{X}, \tau)] d\tau, \quad (29)$$

where as before we denote $\xi(x, t) = \xi^-(x, t)$, $\bar{U} = \bar{U}(\tau, t, x, \xi(x, t))$.

By differentiating (29) with respect to variable x, we get

$$\frac{\partial\Phi}{\partial x}=u_0(\xi(x,t))\frac{\partial\xi(x,t)}{\partial x}+\int_0^t\frac{\partial}{\partial x}[\bar{U}\varphi'_u-\varphi]\,d\tau. \quad (30)$$

Just as in the preceding case,

$$\frac{\partial}{\partial x}[\bar{U}\varphi'_u(\bar{U},\bar{X},\tau)-\varphi(\bar{U},\bar{X},\tau)]=\frac{\partial}{\partial\tau}(\bar{U}\bar{X}'_x) \quad (31)$$

where

$$\bar{X}'_x=\frac{\partial}{\partial x}\bar{X}(\tau,t,x,\xi(x,t)).$$

Inserting (31) into (30), we find

$$\frac{\partial\Phi}{\partial x}=u_0(\xi(x,t))\frac{\partial\xi(x,t)}{\partial x}+\bar{U}(t,t,x,\xi(x,t))\bar{X}'_x(t,t,x,\xi(x,t))-$$
$$-\bar{U}(0,t,x,\xi(x,t))\bar{X}'_x(0,t,x,\xi(x,t)). \quad (32)$$

But in formulas (6) it follows that

$$\bar{X}'_x(0,t,x,\xi(x,t))=\frac{\partial\xi(x,t)}{\partial x},\quad \bar{X}'_x(t,t,x,\xi(x,t))=1, \quad (33)$$

therefore (32) becomes

$$\frac{\partial\Phi}{\partial x}=[u_0(\xi(x,t))-\bar{U}(0,t,x,\xi(x,t))]\frac{\partial\xi(x,t)}{\partial x}+\bar{U}(t,t,x,\xi(x,t))=$$
$$=[u_0(\xi(x,t))-\bar{U}(0,t,x,\xi(x,t))]\frac{\partial\xi(x,t)}{\partial x}+u(x,t). \quad (34)$$

Similar operations lead to the formula

$$\frac{\partial\Phi}{\partial t}=[u_0(\xi(x,t))-\bar{U}(0,t,x,\xi(x,t))]\frac{\partial\xi(x,t)}{\partial t}-\varphi(\bar{U},\bar{X},t)|_{\tau=t}=$$
$$=[u_0(\xi(x,t))-\bar{U}(0,t,x,\xi(x,t))]\frac{\partial\xi(x,t)}{\partial t}-\varphi(u(x,t),x,t). \quad (35)$$

Formula (34) was derived by us formally, and here we assume to the existence of the derivatives $\frac{\partial\xi(x,t)}{\partial x}$ and $\frac{\partial\xi(x,t)}{\partial t}$.

But these derivatives exist at the (x, t) points for which $\xi^-(x,t)=\xi^+(x,t)$, regardless of whether the initial function $u_0(\xi)$ is differentiable or not at the point $\xi=\xi^+(x,t)$. A vigorous validation of these formulas is not complicated, but is laborious. For example, validating formulas (34) and (35) can be done if we regard $u_0(x)$ as the limit of continuously differentiable functions. Now, however, we can easily note that if $\xi^-(x, t) = \xi^+(x, t)$, then the expressions

$$\{u_0(\xi(x, t)) - \bar{U}(0, t, x, \xi(x, t))\}\frac{\partial \xi(x, t)}{\partial x},$$
$$\{u_0(\xi(x, t)) - \bar{U}(0, t, x, \xi(x, t))\}\frac{\partial \xi(x, t)}{\partial t}$$

tend to zero*) and formulas (34) and (35) under this condition are converted into simpler expressions:

$$\frac{\partial \Phi}{\partial x} = u(x, t), \quad \frac{\partial \Phi}{\partial t} = -\varphi(u(x, t), x, t) \qquad (36)$$

But if $\xi^-(x, t) \neq \xi^+(x, t)$, then

$$\frac{\partial \Phi}{\partial x}(x-0, t) = u(x-0, t), \quad \frac{\partial \Phi}{\partial x}(x+0, t) = u(x+0, t) \qquad (37)$$

And thus, the derivatives $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial t}$ exist almost everywhere, and they were computed by formulas (36). From this it follows that the continuous function $\Phi(x, t)$ almost everywhere satisfies the equation

$$\frac{\partial \Phi}{\partial t} + \varphi\left(\frac{\partial \Phi}{\partial x}, x, t\right) = 0 \qquad (38)$$

and takes on the initial values (26).

And thus we conclude that the solution $u(x, t)$ given by formula (14), for any closed contour C, satisfies the identity

$$\oint_C u(x, t)\,dx - \varphi(u(x, t), x, t)\,dt = 0. \qquad (39)$$

Since, moreover, $u(x, t)$ takes on initial values to in the following sense:

$$\int_0^x u(x, t)\,dx \to \int_0^x u_0(x)\,dx \quad \text{as} \quad t \to 0, \qquad (40)$$

---

*) If $\xi^-(x, t) = \xi^+(x, t)$ and $u_0(\xi)$ is continuous at the point $\xi = \xi(x, t)$, then $u_0(\xi(x, t)) = \bar{U}(0, t, x, \xi(x, t))$; but if $\xi(x, t)$ is the discontinuity point $u_0(\xi)$, then $\frac{\partial \xi(x,t)}{\partial x} = \frac{\partial \xi(x,t)}{\partial t} = 0$.

and satisfies the stability condition $u(x - 0, t) \geqslant u(x + 0, t)$, therefore the discontinuous function $u(x, t)$ given by formula (14) is a stable generalized solution of Cauchy's problem (1), (2).

Above we obtained proof that formula (14) defines the generalized stable solution of Cauchy's problem (1), (2). But this proof does not indicate by what means we arrive at formula (14). Therefore we present here another method of constructing a discontinuous solution of Cauchy's problem (1), (2) which automatically leads to formula (14) for the generalized solution. This method was used under the same assumptions about the problem (1), (2) which were made above and even when some of them are rejected. However, for clarity and simplicity of exposition, we begin with an analysis of the case of a smooth initial function $u_0(x)$, and then consider the case of a piecewise-smooth and piecewise-continuous function $u_0(x)$.

And thus, we assume that the function $\varphi(u, x, t)$ satisfies our former requirements, and that the initial function $u_0(x)$ has a continuous first derivative and is bounded for any finite values of the variable x. As long as the equation

$$x = X(t, x_0, u_0(x_0)) \tag{41}$$

has a unique solution $x_0 = x_0(x, t)$ relative to x_0, as we have seen in Chapter One, the solution is continuous and is given by the formula

$$u(x, t) = U(t, x_0, u_0(x_0(x, t))). \tag{42}$$

But if equation (41) has more than one solution relative to x_0 at several points or domains of the variables x, t, then formula (42) defines some multivalue function of the variables x, t from whose branches the generalized solution must be constructed.

The existence of a generalized solution $u(x, t)$ of Cauchy's problem (1), (2) presupposes the existence of Lipschitz-continuous potential $\Phi(x, t)$, which almost everywhere satisfies the nonlinear equation

$$\frac{\partial \Phi}{\partial t} + \varphi\left(\frac{\partial \Phi}{\partial x}, x, t\right) = 0 \tag{43}$$

and the initial condition

$$\Phi(x, 0)=\Phi_0(x)=\int_0^x u_0(\xi)\,d\xi. \qquad (44)$$

Let us denote the geometrical integral of Cauchy's problem (43), (44) by $\Phi = \tilde{\Phi}(t, x_0)$, i.e., the equation of the integral surface of this problem expressed in parameters t, x_0, where x_0 is the point through which characteristic (41) passes. Generally, this surface is not uniquely projected onto the plane of variables x, t. According to subsection 2 of Section IX of Chapter One, the function $\tilde{\Phi}(t, x_0)$ is defined by means of integration of the "strip equation"

$$\frac{d\tilde{\Phi}(t, x_0)}{dt}=U\varphi'_u(U, X, t)-\varphi(U, X, t), \qquad (45)$$

where

$$U=U(t, x_0, u_0(x_0)), \quad X=X(t, x_0, u_0(x_0)).$$

Using initial condition (44), integrate equation (45):

$$\tilde{\Phi}(t, x_0)=\Phi_0(x_0)+\int_0^t [U\varphi'_u(U, X, t)-\varphi(U, X, t)]_{t=\tau}\,d\tau. \qquad (46)$$

For the case when equation (41) in uniquely solvable relative to parameter x_0, the potential $\Phi(x, t)$ is given by the formula

$$\Phi(X(t, x_0, u_0(x_0)), t)=\tilde{\Phi}(t, x_0) \qquad (47)$$

or by the explicit formula $\Phi(x, t) = \tilde{\Phi}(t, x_0(x, t))$. (48)

But if the function (41) is nonuniquely solved relative to x_0, formulas (47) and (48) define the multivalued function $\tilde{\Phi}(x, t)$, from whose branches a unique and continuous function $\Phi(x, t)$ must be constructed -- the potential of the generalized solution.

Let us fix an arbitrary $t > 0$ and consider the behavior of the curve

$$x = X(t, x_0, u_0(x_0)) \qquad (49)$$

in the plane of variables x, x_0 (Figure 4.16). The function (49) obviously is represented by some continuous curve, which however, for sufficiently large values of the argument $t > 0$, is not uniquely projected onto the straight line $x_0 = 0$. The curve (49) is always uniquely projected onto the straight line $x = 0$.

Therefore we can assume that corresponding to each point on this curve, for a fixed value of variable t, is a definite value of the parameter x_0. Consequently, we can assume that the continuous functions $U(t, x_0, u_0(x_0))$ and $\tilde{\Phi}(t, x_0)$ (functions of the variable x_0) are assigned at the continuous curve (49).

Our problem is to select from the branches of the multivalued function (48) a unique and continuous function -- the potential $\Phi(x, t)$. Let us show how this separation is made, and incidentally establish that it is unique.

Suppose that on some section $a < x_0 < b$ the curve (49) has at three points intersections with the straight lines $x = c$ when $x_1 < c < x_2$, i.e., is triply projected onto the segment $x_1 \leq x \leq x_2$ of the $x_0 = 0$ (Figure 4.17). Suppose $x_1 < x < x_2$ and suppose $\overset{(1)}{x}_0 < \overset{(2)}{x}_0 < \overset{(3)}{x}_0$ are values of the variable x_0 satisfying*) equality (49). Let $\overset{(1)}{u}(x, t), \overset{(2)}{u}(x, t), \overset{(3)}{u}(x, t), \overset{(1)}{\Phi}(x, t), \overset{(2)}{\Phi}(x, t), \overset{(3)}{\Phi}(x, t)$ denote the corresponding values of the functions $U(t, x_0, u_0(x_0))$ and $\tilde{\Phi}(t, x_0)$, i.e.,

$$\overset{(i)}{u}(x, t) = U(t, \overset{(i)}{x}_0, u_0(\overset{(i)}{x}_0)),$$
$$\overset{(i)}{\Phi}(x, t) = \tilde{\Phi}(t, \overset{(i)}{x}_0).$$

For each of these branches, in view of the "strip condition" (45) the following equalities are satisfied:

$$\frac{\partial \overset{(i)}{\Phi}(x, t)}{\partial x} = \overset{(i)}{u}(x, t). \qquad (50)$$

We assume earlier that the boundary value problem for the characteristic system is uniquely solvable. From this assumption it follows that

$$\varphi_u'(\overset{(1)}{u}(x, t), x, t) < \varphi_u'(\overset{(2)}{u}(x, t), x, t) < \varphi_u'(\overset{(3)}{u}(x, t), x, t), \qquad (51)$$

and since $\varphi''_{uu}(u, x, t) > 0$, then also

$$\overset{(1)}{u}(x, t) < \overset{(2)}{u}(x, t) < \overset{(3)}{u}(x, t). \qquad (52)$$

By formula (50), we can write

*) Curve (49) can have vertical segment. In this case an infinite set of points of the curve (49) can correspond to some x. As we have seen, this does not affect the course of our constructions.

$$\frac{\partial \overset{(1)}{\Phi}(x, t)}{\partial x} < \frac{\partial \overset{(2)}{\Phi}(x, t)}{\partial x} < \frac{\partial \overset{(3)}{\Phi}(x, t)}{\partial x}. \tag{53}$$

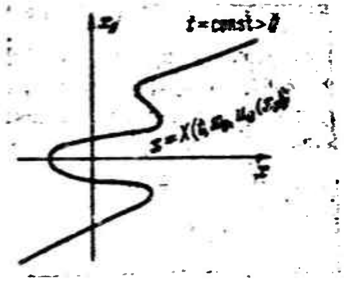

Figure 4.16

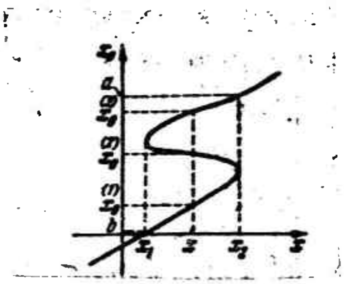

Figure 4.17

In Figure 4.18 we represent at the section $[x_1, x_2]$ the function (48), which according to our assumption is three-valued. The branch $\overset{(1)}{\Phi}(x, t)$ is defined at $x \leqslant x_2$, and the branch $\overset{(2)}{\Phi}(x, t)$ -- at $x_1 \leqslant x \leqslant x_2$; here, in view of the continuity of the function $\tilde{\Phi}(t, x_0)$ relative to the variable x_0, we have the equalities

$$\overset{(1)}{\Phi}(x_2, t) = \overset{(2)}{\Phi}(x_2, t),$$
$$\overset{(2)}{\Phi}(x_1, t) = \overset{(3)}{\Phi}(x_1, t).$$

Using, finally, inequality (53), we conclude that the graph of the function (48) is of the form shown in Figure 4.18.

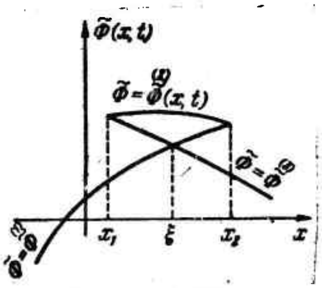

Figure 4.18

And thus, at the section $x_1 \leqslant x \leqslant x_2$ for which the function $\tilde{\Phi}(x, t)$ is three-valued, the separation of a unique and continuous functions -- the potential $\Phi(x, t)$ — is obviously carried out in a natural manner and is given by the formula

$$\Phi(x, t) = \min \tilde{\Phi}(x, t), \qquad (54)$$

where the minimum is taken over all values $\overset{(i)}{\Phi}(x, t)$ at the given point x, t. Figure 4.18 shows that this minimum is attained at the first and third branches of $\tilde{\Phi}(x, t)$, i.e.,

$$\Phi(x, t) = \overset{(1)}{\Phi}(x, t) \quad \text{where} \quad x \leq \xi, \ \Phi(x, t) = \overset{(3)}{\Phi}(x, t) \quad \text{where} \quad x > \xi. \qquad (55)$$

Correspondingly, the function u(x, t) is also thus defined:

$$u(x, t) = \overset{(1)}{u}(x, t) \quad \text{where} \quad x \leq \xi, \ u(x, t) = \overset{(3)}{u}(x, t) \quad \text{where} \quad x > \xi. \qquad (56)$$

Now if curve (49) has sections which are projected more than three times on the axis $x_0 = 0$, then by decomposing it into individual pieces at which $\tilde{\Phi}(x, t)$ is three-valued, we reduce the entire procedure of constructing the continuous potential $\Phi(x, t)$ to the case just considered.

Thus, the generalized solution u(x, t) of Cauchy's problem (1), (2) is given by the formula

$$u(x, t) = U(t, \xi(x, t), u_0(\xi(x, t))), \qquad (57)$$

where $\xi(x, t)$ is a value of the parameter x_0 for which

$$\Phi(x, t) = \tilde{\Phi}(t, \xi(x, t)) \qquad (58)$$

is the smallest value of the function $\tilde{\Phi}(t, x_0)$ for x_0 values bounded by condition (49), i.e., $x = X(t, x_0, u_0(x_0))$.

Now using formulas (46), (19), and (23), we can readily establish that formula (54) is equivalent to the requirement that function $I(t, x, \xi)$ have a minimum.

Thus, we again arrive at formula (14) for the generalized solution u(x, t).

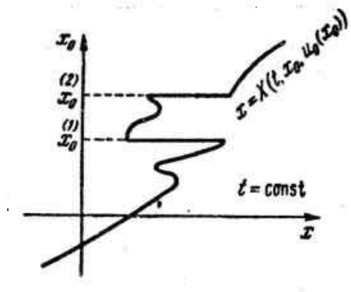

Figure 4.19

We have considered, however, only the case when the initial function $u_o(x)$ is smooth and, in particular, a continuous function of the variable x. We now see that nothing essential is changed even in the case of a piecewise-smooth and piecewise-continuous initial function $u_o(x)$.

If we consider the discontinuous initial function $u_o(x)$ as the limit of continuous functions, then this only leads to the situation that at the discontinuity points $u_o(x)$, when solving the characteristic system, we must given the function $u_o(x)$ all intermediate values between the left and the right limit values $u_o(x - 0)$, $u_o(x + 0)$. Then the function (49) as before will be represented by a continuous curve in the plane of variable x, x_o. The only difference is that at the discontinuity points this curve has horizontal segments (Figure 4.19). Thus, for example, in Figure 4.19 corresponding to the point $\overset{(1)}{x_o}$ is the case when $u_o(\overset{(1)}{x_o} - 0) > u_o(\overset{(1)}{x_o} + 0)$,
and corresponding to the point $\overset{(2)}{x_o}$ -- the case when $u_0(\overset{(2)}{x_0} - 0) < u_0(\overset{(2)}{x_0} + 0)$.

A bundle of characteristics $x = X(t, x_o, u_o(x_o))$ leaves the discontinuity point of the initial function $u_o(x)$, and this bundle of course cannot be described by only the parameter x_o. Therefore we introduce at the discontinuity point $u_o(x)$ yet another parameter α $(0 \leq \alpha \leq 1)$ and define the functions $X(t, x_o, u_o(x_o), \alpha)$, $U(t, x_o, u_o(x_o), \alpha)$, and $\widetilde{\Phi}(t, x_o, \alpha)$, thusly:

$$\left.\begin{aligned} X(t, x_0, u_0(x_0), \alpha) &= X(t, x_0, \alpha u_0(x_0 - 0) + (1 - \alpha) u_0(x_0 + 0)), \\ U(t, x_0, u_0(x_0), \alpha) &= U(t, x_0, \alpha u_0(x_0 - 0) + (1 - \alpha) u_0(x_0 + 0)), \end{aligned}\right\} \quad (59)$$

$$\Phi(t, x_0, \alpha) = \Phi_0(x_0) + \int_0^t [U\varphi'_x(U, X, \tau) - \varphi(U, X, \tau)]\, d\tau. \quad (60)$$

The functions (59) at $t = \tau$ are denoted by U and X in formula (60). Now if the parameter x_o runs through values from $-\infty$ to ∞, and α is made from 0 to 1, then the curve $x = X(t, x_o, u_o(x_o), \alpha)$ will be continuous in the plane of variables x, x_o; $\widetilde{\Phi}(t, x_o, \alpha)$ there is continuously along this continuous curve. Therefore, by repeating the former operations we conclude that also for the case of discontinuous initial functions $u_o(x)$ formulas (57) and (58) remain in effect for the generalized solution.

These formulas will now appear as follows:

$$u(x, t)=U(t, \xi(x, t), u_0(\xi(x, t)), \alpha(x, t)),$$

where $\xi(x, t)$ and $\alpha(x, t)$ are values of the parameters x_0 and α from which

$$\Phi(x, t)=\tilde{\Phi}(t, \xi(x, t), \alpha(x, t))$$

is the least value of the function $\tilde{\Phi}(t, x_0, \alpha)$ relative to the parameters x_0 and α and governed by the condition $x = X(t, x_0, u_0(x_0), \alpha)$.

For the inhomogeneous law of conservation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\varphi(u, x, t)=f(u, x, t) \qquad (61)$$

with initial condition (2), formulas (14) and (58) as before define the generalized solution in the case when $f(u, x, t) = f_1(x, t) + f_2(u, x, t)$ (62)

Only now $U(t, x_0, u_0)$, $X(t, x_0, u_0)$, $\bar{U}$, and $\bar{X}$ denote the solution of the characteristic system for equation (61):

$$\frac{dx}{dt}=\varphi_u'(u, x, t), \quad \frac{du}{dt}=f(u, x, t)-\varphi_x'(u, x, t), \qquad (63)$$

and $\tilde{\Phi}(t, x_0)$ represents the solution of the "strict condition" for equation (61):

$$\frac{d\tilde{\Phi}}{dt}=U\varphi_u'(U, X, t)-\varphi(U, X, t)+F(x, t)+f_2(t)\tilde{\Phi}, \qquad (64)$$

where

$$F_x'(x, t)=f_1(x, t).$$

To be convinced of this, let us transform equation (61) to become

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}[\bar{\varphi}(v, x, t)]=0 \qquad (65)$$

where

$$v=u\exp\{-\int_0^t f_2(\tau)\,d\tau\},$$

$$\bar{\varphi}(v, x, t)=\exp\{-\int_0^t f_2(\tau)d\tau\}\left[\varphi(v\cdot\exp\{\int_0^t f_2(\tau)d\tau\}, x, t)-F(x, t)\right].$$

Let us apply the method of constructing the generalized solution to the Cauchy's problem outlined above to equation (65) and, in particular, here formulas (14) and (58) are valid. Returning again to the variable $u = v \exp\int_0^t f_2(\tau)d\tau$, we get the result that the latter is defined by formulas (14). The solution satisfies the integral law of conservation

$$\oint_C u\,dx - \varphi(u, x, t)\,dt + \iint_{\mathscr{T}C} [f_1(x, t) + f_2(t)u]\,dx\,dt = 0,$$

deriving from equation (61) under condition (62).

4. Cauchy's problem for an inhomogeneous law of conservation. Cauchy's problem for an inhomogeneous law of conservation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = f(u, x, t) \quad (\varphi''_{uu}(u, x, t) > 0) \qquad (1)$$

with initial condition $u(0, x) = u_0(x)$ (2)
was first examined by A. N. Tikhonov and A. A. Samarskiy (1954), who assumed that the functions φ and f are doubly continuously differentiable, and that the initial function $u_0(x)$ is piecewise-continuous and piecewise-differentiable.

Their method can be called the method of integrating Hugoniot's condition.

The solution of the characteristic system

$$\frac{dx}{dt} = \varphi'_u(u, x, t), \quad \frac{du}{dt} = f(u, x, t) - \varphi'_x(u, x, t), \qquad (3)$$

satisfying the initial conditions $x(0) = x_0$, $u(0) = u_0$, (4)
as before is denoted by $x = X(t, x_0, u_0)$, $u = U(t, x_0, u_0)$. (5)
We will assume that the function X and U remain bounded in the domain of the variation of variables t, x_0, and u_0 under consideration.

It is sufficient to consider the case when the initial function $u_0(x)$ is assigned on the finite segment $|x| \leq a$ and to solve the problem (1), (2) in the domain of definition of the segment.

Let us select the value of a to be sufficiently small that the initial function $u_0(x)$ has on the segment $|x| \leq a$ a unique first-order discontinuity point which can, without restricting generality, be taken as the point $x = 0$. Let us consider two possible cases.

1) Suppose $u_0(-0) < u_0(+0)$. Since $\varphi''_{uu} > 0$, then it follows that

$$\varphi'_u(u_0(-0), 0, 0) < \varphi'_u(u_0(+0), 0, 0). \qquad (6)$$

Let us draw two characteristics $x = x^-(t)$ and $x = x^+(t)$ given by the equations

$$x^-(t) = X(t,\ 0,\ u_0(-0)),\quad x^+(t) = X(t,\ 0,\ u_0(+0)). \qquad (7)$$

through the point (0, 0). From condition (6) it follows that at least for sufficiently small values of the variable $t > 0$ the following condition will be satisfied:

$$x^-(t) < x^+(t), \qquad (8)$$

and if it is assumed, as earlier, that the boundary value problem (2.3.8) is uniquely solvable for characteristic system (3), then inequalities (8) will be satisfied for all $t > 0$.

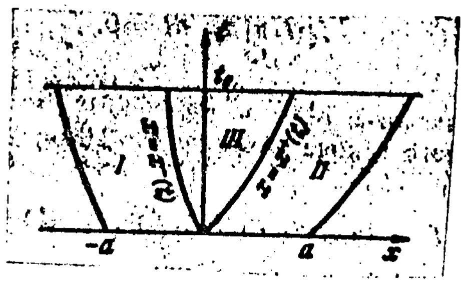

Figure 4.20

Figure 4.20 shows the domain of determinacy of the segment $|x| \leqslant a$ and the characteristic (7). According to Chapter One, a unique classical solution of equation (1) taking on initial values (2) exists in zones I and II (Figure 4.20) for sufficiently small $t_0 > 0$. This solution is given by the formula

$$u(X(t,\ x_0,\ u_0(x_0)),\ t) = U(t,\ x_0,\ u_0(x_0)) \qquad (9)$$

implicitly by means of parameter x_0 or, if we can solve the function

$$x = X(t,\ x_0,\ u_0(x_0)) \qquad (10)$$

relative to x_0:

$$x_0 = x_0(x,\ t), \qquad (11)$$

and also by the explicit formula $u(x,\ t) = U(t,\ x_0(x,\ t),\ u_0(x_0(x,\ t)))$. (12)

Formulas (9) and (12) define the solution to Cauchy's problem (1), (2) in zones I and II. It remains to determine the solution in zone III.

Let us draw the characteristics $x = X^\alpha$ given by the conditions

$$\left.\begin{aligned} &x = X^\alpha = X(t,\ 0,\ \alpha u_0(-0) + (1-\alpha)u_0(+0)),\quad 0 \leqslant \alpha \leqslant 1, \\ &X^0 = x^+(t),\quad X^1 = x^-(t). \end{aligned}\right\} \qquad (13)$$

through the point (0, 0). The equation $x = X^a$ (14)

is uniquely solvable in zone III relative to parameter α:

$$\alpha = \alpha(x, t). \qquad (15)$$

Therefore the solution u(x, t) is given in zone III by the formula

$$u(X^{\alpha}, t) = U(t, 0, \alpha u_0(-0) + (1-\alpha) u_0(+0)). \qquad (16)$$

or, if we know function (15), by the formula

$$u(x, t) = U(t, 0, \alpha(x, t) u_0(-0) + [1-\alpha(x, t)] u_0(+0)). \qquad (17)$$

The set of formulas (9) and (16) define the solution of Cauchy's problem (1), (2) in zones I, II, and III, which is continuous when $t > 0$, has discontinuities of first derivatives located on the lines $x = x^-(t)$, $x = x^+(t)$, and has a singularity at the point (0, 0). To clarify the nature of this singularity let us note that

$$\lim_{t \to 0} u(X^{\alpha}(t), t) = u_0^{\alpha} = \alpha u_0(-0) + (1-\alpha) u_0(+0). \qquad (18)$$

$$\lim_{t \to 0} \frac{dX^{\alpha}(t)}{dt} = \lim_{t \to 0} \varphi'_u(u(X^{\alpha}, t), X^{\alpha}, t) = \varphi'_u(u_0^{\alpha}, 0, 0). \qquad (19)$$

Hence it follows that in zone III solution (17) has a singularity of the type

$$u(x, t) = g\left(\frac{x}{t}\right) + O(t), \quad g' = \frac{1}{\varphi''_{uu}(u, 0, 0)}. \qquad (20)$$

A singularity of this kind is called a centered rarefaction wave in gas dynamics.

Thus, when inequality (6) is satisfied there exists a continuous solution which is given by formulas (9) and (16).

2) Now let us consider a second case, when $u_0(-0) > u_0(+0)$, i.e.,

$$\varphi'_u(u_0(-0), 0, 0) > \varphi'_u(u_0(+0), 0, 0). \qquad (21)$$

In this case inequality (8) is changed into its converse and zones I and II overlap each other (Figure 4.21). The intersection of zones I and II is denoted as zone III (Figure 4.21). In this case formula (9) defines the function u(x, t) twice in zone III: one function, which we denote by $u^-(x, t)$ is determined relative to the values $x_0 < 0$, and the other, $u^+(x, t)$ -- relative to the values $x_0 > 0$. The solution u(x, t) in this case is discontinuous. We will assume that the discontinuity line OD (Figure 4.22) is drawn through the point $\omega(0, 0)$ in zone III and that the equation of this discontinuity line will be assumed to be written in the form $x = x(t)$, $x(0) = 0$; to the left of the discontinuity line OD $u(x, t) = u^-(x, t)$; to the right of OD $u(x, t) = u^+(x, t)$.

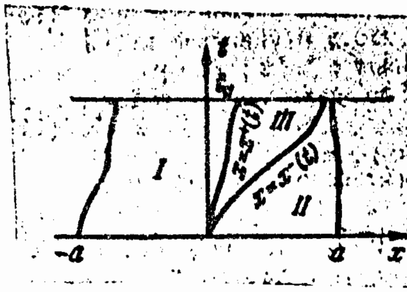

Figure 4.21

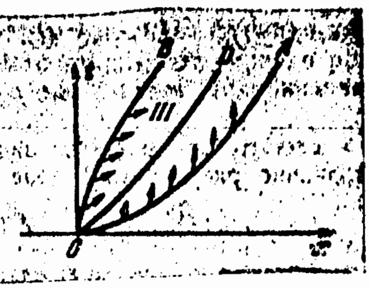

Figure 4.22

Hugoniot's conditions (Section I) must in this case be satisfied at the discontinuity line $x = x(t)$, and the conditions are written in the form

$$\frac{dx(t)}{dt} = D = \frac{\varphi(u^+(x, t), x, t) - \varphi(u^-(x, t), x, t)}{u^+(x, t) - u^-(x, t)} = A(x, t). \quad (22)$$

Under the existence theorems established in Chapter One, the functions $u^-(x, t)$ and $u^+(x, t)$, for sufficiently small t_0, have bounded first derivatives. Therefore the function $A(x, t)$ appearing in equation (22) has a bounded first derivative.

Let us consider any point $\mathcal{P}$ on the line OB. Suppose we let $D_{\mathcal{P}}$ denote the slope of the line OB to the axis $t = 0$ at the point $\mathcal{P}$, i.e.,

$$D_{\mathcal{P}} = \frac{dx^+(t)}{dt}\bigg|_{t=t_{\mathcal{P}}}.$$

Let a single characteristic $x = X(t, x_0, u_0(x_0))$ corresponding to the value $x_0 < 0$ be drawn through the point $\mathcal{P}$. Hence we conclude that

$$\varphi'_u(u^-(x_{\mathcal{P}}, t_{\mathcal{P}}), x_{\mathcal{P}}, t_{\mathcal{P}}) = X'_t(t_{\mathcal{P}}, x_0^-, u_0(x_0^-)) >$$
$$> \varphi'_u(u^+(x_{\mathcal{P}}, t_{\mathcal{P}}), x_{\mathcal{P}}, t_{\mathcal{P}}) = D_{\mathcal{P}} = X'_t(t_{\mathcal{P}}, 0, u_0(+0)). \quad (23)$$

Figure 4.23 is a graph of the function $\varphi = \varphi(u, x_{\mathcal{P}}, t_{\mathcal{P}})$, considering that $\varphi''_{uu} > 0$. We conclude from condition (23) that $u^-(x_{\mathcal{P}}, t_{\mathcal{P}}) > u^+(x_{\mathcal{P}}, t_{\mathcal{P}})$. Obviously the quantity $A(x_{\mathcal{P}}, t_{\mathcal{P}})$ is the slope of the chord joining the points $\{u^+(x_{\mathcal{P}}, t_{\mathcal{P}}), \varphi(u^+(x_{\mathcal{P}}, t_{\mathcal{P}}), x_{\mathcal{P}}, t_{\mathcal{P}})\}, \{u^-(x_{\mathcal{P}}, t_{\mathcal{P}}), \varphi(u^-(x_{\mathcal{P}}, t_{\mathcal{P}}), x_{\mathcal{P}}, t_{\mathcal{P}})\}$.
As is clear from Figure 4.23, a consequence of inequalities (23) is the inequality

$$A(x_{\mathcal{P}}, t_{\mathcal{P}}) > \varphi'_u(u^+(x_{\mathcal{P}}, t_{\mathcal{P}}), x_{\mathcal{P}}, t_{\mathcal{P}}) = D_{\mathcal{P}}. \quad (24)$$

Thus, on the line OB $A(x_{\mathcal{T}}, t_{\mathcal{T}}) > D_{\mathcal{T}}$, i.e., the velocity of the integral curves of differential equation (22) on the line OB is greater than the velocity of the line OB. Put briefly, the field of directions on the line OB has the shape shown in Figure 4.22 for equation (22). We similarly conclude that the field of directions at the line OA is also of the form shown in Figure 4.22. Since in zone III, as we have already stated above, $A(x, t)$ is continuously differentiable relative to its variables, therefore there exists a unique integral curve OD of ordinary differential equation (22) lying entirely within the zone III.

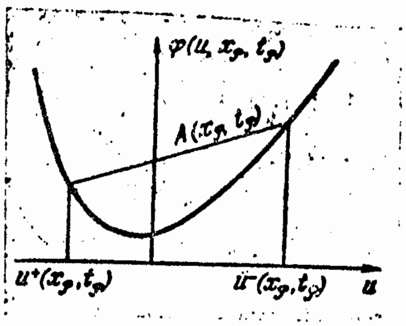

Figure 4.23

After the finding the line OD, let us determine the solution in the zone I, II, and III:

$$u(x, t) = \begin{cases} u^-(x, t) & \text{to the left of the line OD,} \\ u^+(x, t) & \text{to the right of the line OD.} \end{cases} \qquad (25)$$

Solution (25) is continuous everywhere except the line OD and has a bounded first derivative. On the line OD $u(x - 0, t) \quad u(x + 0, t)$, i.e., this solution satisfies the stability condition. Hugoniot's condition is satisfied at the discontinuity line OD as a consequence of equation (22); therefore formula (25) defines the stable generalized solution of Cauchy's problem (1), (2).

Thus, for the case when the initial function has an isolated first-order discontinuity, we construct a generalized solution of Cauchy's problem by the above described technique in some neighborhood of the discontinuity point. If the initial function has several discontinuity points, then by decomposing the initial segment into parts, we reduce the problem to the case under present study.

Finally, let us note yet another fact. The value t_o by which we delimit from above the domain of definition is bounded from above by the fact that the solutions $u^-(x, t)$ and $u^+(x, t)$ in zones I and II must have bounded first derivatives.

However, as we have seen in Chapter One, derivatives of a solution of quasilinear equations do not remain bounded and can grow in absolute value up to infinity. The essential thing is that most representative case in the behavior of solutions of quasilinear equations is that in which derivatives of the solution at any point become infinite, but the equation itself remains continuous.

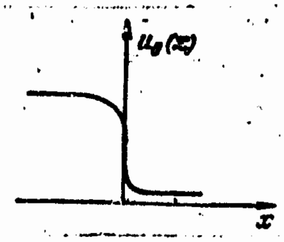

Figure 4.24

Thus, to have the possibility of successively, step by step, employing the above described method of explicit isolation of singularities, we must further consider the case when the initial function $u_o(x)$ has an unbounded derivative at the point $x = 0$, but itself remains continuous (Figure 4.24). Here it is sufficient to consider only the case when in the neighborhood of the point $x = 0$

$$\frac{\partial u_0(x)}{\partial x} < 0, \qquad (26)$$

since the derivative $p = \partial u/\partial x$ satisfies the equation

$$\frac{\partial p}{\partial t} + \varphi'_u(u, x, t)\frac{\partial p}{\partial x} = -\varphi''_{uu}p^2 + [f'_u - \varphi''_{ux}]\,p + f'_x \qquad (27)$$

and remains bounded from above: $p(x, t) < A(\varphi''_{uu} > 0)$ if the derivative of the function $u_o(x)$ is bounded.

And thus, we assume that when $|x| \leq$ a condition (26) and

$$\frac{\partial u_o(x)}{\partial x} \to -\infty \text{ as } \quad x \to 0. \qquad (28)$$

are satisfied. In this case we limit ourselves only to remarks, since the construction of the solution in fundamental features is similar to that given for the case (21). The main distinction of this case from the preceding is that the Cauchy's problem for equation (1) considered separately for the segments $[-a, 0]$ and $[0, a]$ has a solution with an unbounded derivative for any $t_0 > 0$. Therefore we must now take as the line OA (Figure 4.22) the envelope of a family of characteristics $x = X(t, x_0, u_0(x_0))$, $x_0 < 0$, i.e., characteristics departing to the left from point O; and the line OB must be taken as an envelope of characteristics of the family $x = X(t, x_0, u_0(x_0))$ $x_0 > 0$.

In this case lines OA and OB are tangent to each other at $t = 0$; the solutions $u^-(x, t)$ and $u^+(x, t)$ have unbounded derivatives, respectively, at the lines OA and OB; for the rest, the pattern is wholly similar with that considered above. Thus, the field of directions for differential equation (22) is of the form shown in Figure 4.22, and the uniqueness of the discontinuity line OD when derivatives of the function A are unbounded follows from the tangency of the line OA and OB at $t = 0$.

Thus, by using the method of explicitly integrating Hugoniot's condition, we can define the solution $u(x, t)$ of Cauchy's problem (1), (2) in any domain of interest to us in which the number of discontinuity lines remain bounded, which is most often the case in practical problems.

Let us further note that for the case when $f(u, x, t) \equiv 0$, after deriving the potential of the generalized solution $\Phi(x, t)$, we can write:

$$\left.\begin{aligned} u^-(x, t) &= \frac{\partial \Phi^-(x, t)}{\partial x}, & u^+(x, t) &= \frac{\partial \Phi^+(x, t)}{\partial x}, \\ \varphi(u^-(x, t), x, t) &= -\frac{\partial \Phi^-(x, t)}{\partial t}, & \varphi(u^+(x, t), x, t) &= -\frac{\partial \Phi^+(x, t)}{\partial t}, \end{aligned}\right\} \tag{29}$$

where $\Phi^-(x, t)$ and $\Phi^+(x, t)$ are uniquely defined in the domain AOB from the "strip equations." Therefore Hugoniot's condition takes on the form

$$\frac{\partial}{\partial t}[\Phi^+(x, t) - \Phi^-(x, t)]\,dt + \frac{\partial}{\partial x}[\Phi^+(x, t) - \Phi^-(x, t)]\,dx = 0, \tag{30}$$

from whence it follows that the discontinuity line OD $x = x(t)$ is a line where

$$\Phi^+(x, t) = \Phi^-(x, t). \tag{31}$$

Equality (31) shows that for the case of piecewise-smooth solutions u(x, t) the method of the explicit isolation of discontinuity lines is equivalent for homogeneous laws of conservation to the analytic method of finding the solution presented in Subsection 3.

Cauchy's problem (1), (2) can in some sense be reduced to the problem for the homogeneous law of conservation by using the method of successive approximations. Let us set $\overset{(0)}{u}(x, t) = u_0(x)$. Taking as known $\overset{(s-1)}{u}(x, t)$, let us define $\overset{(s)}{u}(x, t)$ as the solution of Cauchy's problem

$$\frac{\partial \overset{(s)}{u}}{\partial t} + \frac{\partial}{\partial x}\varphi(\overset{(s)}{u}, x, t) = f(\overset{(s-1)}{u}(x, t), x, t), \tag{32}$$

$$\overset{(s)}{u}(x, 0) = u_0(x). \tag{33}$$

Introducing the function $$\overset{(s)}{F}(x, t) = \int_0^x f(\overset{(s-1)}{u}(\xi, t), \xi, t)\, d\xi. \tag{34}$$

Write equation (32) as

$$\frac{\partial \overset{(s)}{u}}{\partial t} + \frac{\partial}{\partial x}[\varphi(\overset{(s)}{u}, x, t) - \overset{(s)}{F}(x, t)] = 0. \tag{35}$$

The theory of constructing discontinuous solutions developed in subsection 3 is applicable for equation (35), however we must allow for several details.

The characteristics system for equation (35) is written in the form

$$\frac{d\overset{(s)}{X}}{dt} = \varphi_u'(\overset{(s)}{U}, \overset{(s)}{X}, t), \quad \frac{d\overset{(s)}{U}}{dt} = -\varphi_x(\overset{(s)}{U}, \overset{(s)}{X}, t) + f(\overset{(s-1)}{u}(\overset{(s)}{X}, t), \overset{(s)}{X}, t), \tag{36}$$

here the function $f(\overset{(s-1)}{u}(x, t), x, t)$ is a discontinuous function of the variables x, t. We impose the continuity condition for the variables $\overset{(s)}{X}$ and $\overset{(s)}{U}$ at the discontinuity points f.

As above, the solution of problem (35) with initial condition (33) is given by the formula

$$\overset{(s)}{\Phi}(x, t) = \min \overset{(s)}{\Phi}(t, x_0). \tag{37}$$

where $\overset{(s)}{\Phi}(t, x_0)$ is defined by the quadrature

$$\overset{(s)}{\Phi}(t, x_0) = \Phi_0(x_0) + \int_0^t [\overset{(s)}{U}\varphi'_u(\overset{(s)}{U}, \overset{(s)}{X}, \tau) - \varphi(\overset{(s)}{U}, \overset{(s)}{X}, \tau) + \overset{(s)}{F}(\overset{(s)}{X}, \tau)]\, d\tau \quad (38)$$

(in formula (38) $\overset{(s)}{U} = \overset{(s)}{U}(\tau, x_0, u_0(x_0))$.

The sequence $\overset{(s)}{\Phi}(x, t)$ converges uniformly in any bounded domain of variable x, t. We will not present the proof here, instead we refer the reader to the work [28].

5. Uniqueness of the generalized solution under the condition $\varphi''_{uu} > 0$. We will now see that the generalized solution of the Cauchy's problem

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = f(u, x, t) \quad (\varphi''_{uu} > 0). \quad (1)$$

$$u(x, 0) = u_0(x). \quad (2)$$

satisfying the stability condition $u(x - 0, t) \geqslant u(x + 0, t)$, (3)
is unique.

We prove this theorem for the class of piecewise-continuous and piecewise-differentiable solutions.

Let us assume that there exist two bounded piecewise-continuous and piecewise-differentiable solutions of the problem (1), (2) $u(x, t)$ and $\bar{u}(x, t)$ where $t > 0$*), each of which satisfies equation (1) everywhere outside the discontinuity lines, and satisfies the stability condition (2) at the discontinuity lines.

Suppose $\Phi(x, t)$ and $\bar{\Phi}(x, t)$ are potentials corresponding to these two solutions. These functions are continuous and satisfy the equation

$$\frac{\partial \Phi}{\partial t} + \varphi\left(\frac{\partial \Phi}{\partial x}, x, t\right) = \int_0^x f\left(\frac{\partial \Phi}{\partial \xi}, \xi, t\right) d\xi. \quad (4)$$

everywhere except at the discontinuity lines of the solutions u and $\bar{u}$, and when $t = 0$ they satisfy the initial condition

$$\Phi(x, 0) = \Phi_0(x) = \int_0^x u_0(\xi)\, d\xi. \quad (5)$$

*) We are considering solution which can also have singularities of these types $g(\frac{x - x_0}{t}, t)$ and which are not continuous at $t = 0$.

The difference $v = \phi - \bar{\phi}$ satisfies everywhere, except for the discontinuity lines u and $\bar{u}$, the equation

$$\frac{\partial v}{\partial t}+A(x,t)\frac{\partial v}{\partial x}=\int_0^x B(\xi,t)\frac{\partial v(\xi,t)}{\partial \xi}d\xi=$$

$$=-\int_0^x \frac{\partial B(\xi,t)}{\partial \xi}v(\xi,t)d\xi+B(x,t)v(x,t)-B(0,t)v(0,t)+$$

$$+\sum_i v(x_i(t),t)[B(x_i(t)+0,t)-B(x_i(t)-0,t)]. \quad (6)$$

The summation in (6) is carried out over all discontinuity lines $x = x_j(t)$ of solutions $u(x, t)$ and $\bar{u}(x, t)$ such that $0 < x_i(t) < x$, and

$$A(x,t)=\begin{cases}\dfrac{\varphi(u(x,t),x,t)-\varphi(\bar{u}(x,t),x,t)}{u(x,t)-\bar{u}(x,t)} & \text{when } u \neq \bar{u},\\ \varphi'_u(u(x,t),x,t) & \text{when } u = \bar{u}.\end{cases} \quad (7)$$

$$B(x,t)=\begin{cases}\dfrac{f(u(x,t),x,t)-f(\bar{u}(x,t),x,t)}{u(x,t)-\bar{u}(x,t)} & \text{when } u \neq \bar{u},\\ f'_u(u(x,t),x,t) & \text{when } u = \bar{u}\end{cases} \quad (8)$$

and

$$v(x,0)\equiv 0. \quad (9)$$

Suppose

$$\left.\begin{array}{l}|u(x,t)|<M_0,\quad |\bar{u}(x,t)|<M_0,\\ |\varphi'_u(u,x,t)|\leqslant M_1 \quad \text{when} \quad |u|\leq M_0.\end{array}\right\} \quad (10)$$

Let us prove that for an arbitrary $a > 0$ in the trapezium

$$x+a > M_1 t, \quad x-a < -M_1 t, \quad 0 \leqslant t \leqslant T \quad (11)$$

the function $\qquad v(x,t)\equiv 0.$

Under these assumptions made above u and $\bar{u}$ and the requirements in subsection (4) imposed on functions φ and f, the function $B(x, t)$ has obviously a bounded variation relative to the variable x, and the number of discontinuity lines $x = x_i(t)$ of the functions u and $\bar{u}$ (they in fact are the discontinuity lines of the quantities $A(x, t)$ and $B(x, t)$) is finite in the trapezium (11) under consideration. Therefore the right side of equation (6) is estimated by the quantity $M_2V(t)$, where $\qquad V(t) = \max |v(\xi,\tau)|, \qquad (12)$

and the maximum in formula (12) is taken over the intersection of trapezium (11) with the strip $0 \leqslant \tau \leqslant t$.

And thus, in the trapezium (11) the function $v(x, t)$ satisfies the condition

$$\left|\frac{\partial v}{\partial t} + A(x, t)\frac{\partial v}{\partial x}\right| < M_2 V(t). \tag{13}$$

and at $t = 0$ — the condition (9).

Let us consider the ordinary differential equation $dx/dt = A(x, t)$, (14) in which the right side is discontinuous at the lines $x = x_1(t)$. We will call the continuous curve $x = x(t)$ which satisfies equation (14) at all continuity points of $A(x, t)$ the integral curve of equation (14).

Let us assume that the integral curve of equation (14) intersecting the axis $t = 0$ at the base of the trapezium $-a \leq x \leq a$ passes through each point of the trapezium (11), then by integrating inequality (13) along the integral curve passing through the point (x, t), we get -- using condition (9) --

$$|v(x, t)| < \int_0^t M_2 V(\tau)\, d\tau. \tag{15}$$

Hence it follows that

$$V(t) < M_2 \int_0^t V(\tau)\, d\tau \tag{16}$$

and, based on lemma 1 from subsection 5 of Section VI of Chapter One, it also follows that

$$v(x, t) \quad 0. \tag{17}$$

Since $u - \bar{u} = \frac{\partial v(x, t)}{\partial x}$, then from (17) follows the proof of the theorem we formulated.

And thus, the proof of the theorem reduces to proving the following fact:

The integral of curve of equation (14) intersecting the axis $t = 0$ passes through any point (x, t) of the trapezium (11).

Let us prove that if u and $\bar{u}$ satisfy the stability condition (3), then this actually is so. To do this, let us note that the integral curve of equation (14) defined at $t \leq \tau$ passes through the point (ξ, τ) if it is a continuity point of $A(x, t)$. If this integral curve at $t < \tau$ does not intersect the discontinuity line $x = x_1(t)$, then it intersects the axis $t = 0$ at segment $-a \leq x \leq a$ since $|A(x, t)| \leq M_1$.

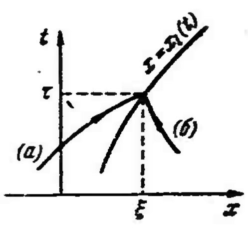

Figure 4.25

Therefore, if the integral curve of equation (14) defined at $t < \tau$ passes through any point (ξ, τ) lying on the discontinuity line $x = x_i(t)$, then through any point of the trapezium (11) passes the integral curve intersecting the axis $t = 0$.

Let us consider the case when the point (ξ, τ) lies on the discontinuity line of function $A(x, t)$ $x = x_i(t)$ (Figure 4.25). Under our assumptions, the left and right limit values of $A(\xi - 0, \tau)$ and $A(\xi + 0, \tau)$ exist at the point (ξ, τ). If

$$A(\xi - 0, \tau) > A(\xi + 0, \tau), \quad (18)$$

then there must necessarily pass through the point (ξ, τ) the integral curve of equation (14) defined at $t < \tau$.

Actually, if we denote $D_i = x_i'(t)$, then in this case one of the two inequalities is met: a) $A(\xi - 0, \tau) > D_i$, b) $A(\xi + 0, \tau) < D_i$. (19) Figure 4.25 gives the directions of the integral curves of equation (14) in the cases a) and b). As we can see from Figure 4.25, in each of these cases an integral curve defined at $t < \tau$ passes through the point (ξ, τ).

So the proof of the theorem has now been reduced to establishing inequality (18). Since (ξ, τ) is a discontinuity point of $A(x, t)$, then either $u(x, t)$ or $\bar{u}(x, t)$ is discontinuous at this point, or else they are discontinuous simultaneously. Therefore we will assume that

$$u(\xi - 0, \tau) > u(\xi + 0, \tau), \quad \bar{u}(\xi - 0, \tau) \geqslant \bar{u}(\xi + 0, \tau). \quad (20)$$

Let us give the graph of the function $\varphi(u, \xi, \tau)$ for fixed ξ and τ in Figure 4.26. Since $\varphi''_{uu} > 0$, this curve is convex downward.

According to formulas (7) the quantity $A(\xi - 0, \tau)$ is equal to the slope of the chord joining the points A^- and B^-, and $A(\xi + 0, \tau)$ is equal to the slope of the chord B^+A^+. From condition (20), it does follow that each endpoint of the interval $[\bar{u}(\xi - 0, \tau), u(\xi - 0, \tau)]$ lies to the right of the corresponding endpoints of the interval $[\bar{u}(\xi + 0, \tau), u(\xi + 0, \tau)]$. For the convex curves ($\varphi''_{uu} > 0$) it therefore follows that the slope of the chord A^-B^- is greater than that of the chord A^+B^+. This same relation in fact follows from Figure 4.26.

So condition (18) is satisfied at each discontinuity point from the function $A(x, t)$. This means that at least one integral curve of equation (14) intersecting the axis $t = 0$ passes through each point of the trapezium (11). This means also that condition (18) holds in this trapezium along with $u(x, t) \equiv \bar{u}(x, t)$.

Let us consider the problem of the continuous dependence of generalized solutions on input data.

Suppose $u(x, t)$ and $\bar{u}(x, t)$ are piecewise-smooth stable generalized solutions of quasilinear equations defined by the conditions

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = f(u, x, t), \quad u(x, 0) = u_0(x), \quad \varphi''_{uu} > 0, \quad (21)$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{\varphi}(\bar{u}, x, t)}{\partial x} = \bar{f}(\bar{u}, x, t), \quad \bar{u}(x, 0) = \bar{u}_0(x), \quad \bar{\varphi}''_{uu} > 0. \quad (22)$$

We will assume that the functions φ, $\bar{\varphi}$, f, $\bar{f}$, u_0, $\bar{u}_0$ satisfy the requirements which were imposed on them in subsection 3 and 4.

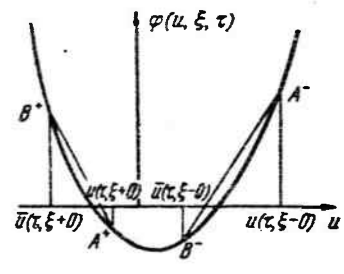

Figure 4.26

Corresponding to solutions u and $\bar{u}$ are the potentials Φ and $\bar{\Phi}$ satisfying the equations

$$\frac{\partial \Phi}{\partial t}+\varphi\left(\frac{\partial \Phi}{\partial x},\ x,\ t\right)=\int_{0}^{x} f\left(\frac{\partial \Phi}{\partial \xi},\ \xi,\ t\right) d\xi \qquad (23)$$

$$\frac{\partial \bar{\Phi}}{\partial t}+\bar{\varphi}\left(\frac{\partial \bar{\Phi}}{\partial x},\ x,\ t\right)=\int_{0}^{x} \bar{f}\left(\frac{\partial \bar{\Phi}}{\partial \xi},\ \xi,\ t\right) d\xi \qquad (24)$$

and the initial conditions

$$\Phi(x,\ 0)=\Phi_0(x)=\int_{0}^{x} u_0(\xi)\, d\xi, \quad \bar{\Phi}(x,\ 0)=\bar{\Phi}_0(x)=\int_{0}^{x} \bar{u}_0(\xi)\, d\xi \qquad (25)$$

The difference $v = \Phi - \bar{\Phi}$ satisfies the equation

$$\frac{\partial v}{\partial t}+A(x,\ t)\frac{\partial v}{\partial x}=$$

$$=\int_{0}^{x} B(\xi,\ t)\frac{\partial v}{\partial \xi}\, d\xi+[\bar{\varphi}(\bar{u}(x,\ t),\ x,\ t)-\varphi(\bar{u}(x,\ t),\ x,\ t)]+$$

$$+\int_{0}^{x} [f(\bar{u}(\xi,\ t),\ \xi,\ t)-\bar{f}(\bar{u}(\xi,\ t),\ \xi,\ t)]\, d\xi \qquad (26)$$

and initial condition

$$v(x,\ 0)=v_0(x)=\Phi_0(x)-\bar{\Phi}_0(x)=\int_{0}^{x} [u_0(\xi)-\bar{u}_0(\xi)]\, d\xi \qquad (27)$$

and the quantity A and B are defined by formulas (7) and (8).

Suppose the conditions $|u| \leq M_0$, $|\bar{u}| \leq M_0$ are satisfied in trapezium (11) and suppose that when $|u| \leq M_0$

$$|\varphi'_u(u,\ x,\ t)| \leqslant M_1, \quad |\varphi(u,\ x,\ t)-\bar{\varphi}(u,\ x,\ t)| \leqslant \Delta\varphi,$$

$$|f(u,\ x,\ t)-\bar{f}(u,\ x,\ t)| \leqslant \Delta f.$$

Then from equation (26) follows the estimate

$$\left|\frac{\partial v}{\partial t}+A(x,\ t)\frac{\partial v}{\partial x}\right| \leqslant \left|\int_{0}^{x} B(\xi,\ t)\frac{\partial v}{\partial \xi}\, d\xi\right|+\Delta\varphi+a\Delta f.$$

Similarly, we find the new estimate:

$$\left|\frac{\partial v}{\partial t}+A(x, t)\frac{\partial v}{\partial x}\right| \leqslant MV(t)+\Delta\varphi+a\Delta f, \qquad (28)$$

where the constant M depends, in particular, on the number of discontinuity lines of the solutions u and $\bar{u}$.

We assume that each of the solutions u and $\bar{u}$ satisfies the stability condition, i.e., $u(x-0, t) \geqslant u(x+0, t), \quad \bar{u}(x-0, t) \geqslant \bar{u}(x+0, t).$

Just as earlier, from this it follows that the inequality A(x, t) is satisfied at the discontinuity points of the function A(x 0, t) $\geqslant$ A(x + 0, t). Therefore, at least one integral curve of equation (14) intersecting the axis t = 0 passes through each point of the trapezium (11). By integrating inequality (28) along this curve and applying lemma (1) from subsection 5 of section VI of Chapter One, we get

$$|v(x, t)| \leqslant v_0 e^{Mt}+[\Delta\varphi+a\Delta f]\frac{e^{Mt}-1}{M}, \qquad (29)$$

where

$$v_0=\max_{|x|\leqslant a}|\Phi_0(x)-\bar{\Phi}_0(x)|$$

Using the fact that

$$v(x, t)=\int_0^x[u(\xi, t)-\bar{u}(\xi, t)]\,d\xi.$$

We conclude that inequality (29) establishes the continuous dependence of generalized solutions of quasilinear equations on the input data of Cauchy's problem in the potential metric.

Unfortunately, however, the constant M dependent on the number of discontinuity lines of solutions u and $\bar{u}$ appears in estimate (29). Here we note that this quantity can be estimated by variation of the functions u(x, t) and $\bar{u}$(x, t), and the latter are estimated from initial conditions.

For homogeneous laws of conservation when $f = \bar{f} \equiv 0$, estimate (29) becomes simplified: $|v(x, t)| \leq v_0 + \Delta\varphi t$. In particular, if we consider only the dependence of the solutions of Cauchy's problem of one quasilinea equation on the initial data, i.e., if we assume that $\Delta\varphi = 0$, this estimate shows that the solutions constructed above satisfy the principle of the continuous dependence of (1.5.4) and (1.5.5) from Chapter One.

One method of constructing discontinuous solutions, which we call the "method of potential smoothing" is related to the problem of the continuous dependence of generalized solutions on initial data. For simplicity, let us consider the homogeneous law of conservation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u, x, t)}{\partial x} = 0, \quad \varphi_{uu} > 0 \qquad (30)$$

for which the initial condition $u(x, 0) = u_0(x)$. (31)

is formulated. We will assume that the function φ satisfies the preceding requirements, and that $u_0(x)$ is assigned on the segment $|x| \leq a$, has a single first-order discontinuity point $x = 0$, and at all the remaining points of the segment has a bounded Lipschitz constant. Let us denote $u^- = u_0(-0)$ and $u^+ = u_0(+0)$. As we have seen above, if $u^- \leq u^+$, the solution of problems (30), (31) is continuous when $t \geq 0$ in some neighborhood $0 \leq t \leq T$ of the initial axis. In particular, it can be obtained as the limit of the classical solutions $u_\delta(x, t)$ as $\delta \to 0$, where $u_\delta(x, t)$ is the solution of equation (30) with the initial condition $u_\delta(x, 0) = u_\delta^0(x)$ (32)

here $u_\delta^0(x) \equiv u_0(x)$ as $|x| \geq \delta$ and $u^0(x)$ is monotonic as $|x| \leq \delta$.

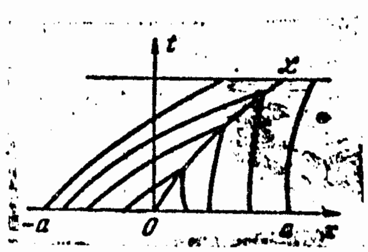

Figure 4.27

Therefore we have considered only the second case, when $u^- > u^+$. In this case the solution $u(x, t)$ for sufficiently small T has in the strip $0 \leq t \leq T$ a single discontinuity line $O\mathscr{L}$ issuing from the point (0, 0) of the initial axis (Figure 4.27). Figure 4.27 shows the discontinuity line

$\mathcal{OL}$ and the characteristics of the solution u(x, t). We will solve, instead of the problem (30), (31), the problem (30), (32), assuming that $u^0_\delta(x) \equiv u_0(x)$ when $|x| \geq \delta$, that the function $u^0_\delta(x)$ is monotonic when $|x| \leq \delta$, and that it has a Lipschitz constant on the segment $|x| \leq \delta$ not exceeding the quantity M/δ, and finally, that

$$\int_{-\delta}^{\delta} [u_0(x) - u_\delta^0(x)]\, dx = 0. \tag{33}$$

Let us introduce the potentials of the solutions u and u_δ by using the relations

$$\Phi(x, t) = \int_{(-a, 0)}^{(x, t)} u\, dx - \varphi(u, x, t)\, dx,$$

$$\Phi_\delta(x, t) = \int_{(-a, 0)}^{(x, t)} u_\delta\, dx - \varphi(u_\delta, x, t)\, dt.$$

Then from relation (33) it follows that $\Phi(x, 0) \equiv \Phi_\delta(x, 0)$ when $|x| \geq \delta$ (34)
With these limitations made on the function $u^0_\delta(x)$, there exists the function $u_\delta(x, t)$ in the broad sense of problem (30), (32), which is continuous and which can be constructed by the classical method of characteristics when $0 \leq t \leq t_1^\delta$, where $t_1^\delta \sim 1/\delta$. Let $x = X_1^{-\delta}$ and $x = X_1^\delta$ denote the equations of the characteristics of this problem departing from, respectively, from the points $x = -\delta$ and $x = \delta$ of the initial axis (Figure 4.28). Obviously, in view of condition (34) $u(x, t) \equiv u_\delta(x, t)$, $\Phi(x, t) \equiv \Phi_\delta(x, t)$

when $[x - X_1^{-\delta}(t)]\,[x - X_1^{\delta}(t)] > 0,\ 0 \leq t \leq t_1^\delta$, i. e., u(x, t), (x, t) coincides, respectively, with $u_\delta(x, t)$, $\Phi_\delta(x, t)$ outside the curvilinear trapezium formed by the straight lines $t = 0$, $t = t_1^\delta$ and by the segments of the characteristics $x = X_1^{-\delta}$ and $x = X_1^\delta$.

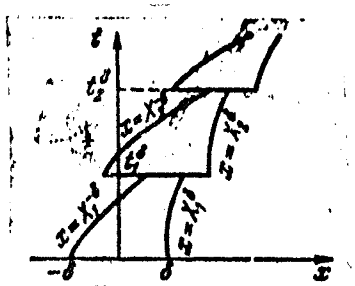

Figure 4.28

Suppose that at $t = t_1^{\delta}$ solution $u_{\delta}(x, t)$ has a singularity (the unboundedness of the Lipschitz constant or a first-order discontinuity) at the segment $[X_1^{-\delta}(t_1^{\delta}), X_1^{\delta}(t_1^{\delta})]$ of the straight line $t = t_1^{\delta}$. Obviously,

$$X_1^{\delta}(t_1^{\delta}) - X_1^{-\delta}(t_1^{\delta}) < 2\delta.$$

Let us select a segment of the straight line $t = t_1^{\delta}$ with length 2δ within which the segment $[X_1^{-\delta}(t_1^{\delta}), X_1^{\delta}(t_1^{\delta})]$ lies completely and again smooth the function $u_{\delta}(x, t_1^{\delta})$ on the segment, i.e., introduce the function $u_{\delta}^1(x, t_1^{\delta})$ which satisfies the same requirements as $u_{\delta}^0(x)$, namely: $u_{\delta}^1(x, t_1^{\delta}) \equiv u_{\delta}(x, t_1^{\delta})$ outside the smoothing segment, $\int_{a_1}^{b_1}[u_{\delta}(x, t_1^{\delta}) - u_{\delta}^1(x, t_1^{\delta})]dx = 0$, where $[a_1, b_1]$ is the smoothing segment. We require as before that the function $u_{\delta}^1(x, t_1^{\delta})$ have a Lipschitz constant bounded by the quantity M/δ and that it be monotonic on the smoothing segment $[a_1, b_1]$. We will seek the solution $u_{\delta}(x, t)$ when $t_1^{\delta} \leq t \leq t_2^{\delta}$, assuming that $u(x, t)$ satisfy equation (30) and the initial condition $u_{\delta}(x, t_1^{\delta}) = u_{\delta}^1(x, t_1^{\delta})$. To construct $u_{\delta}(x, t)$ in the strip $t_1^{\delta} \leq t \leq t_2^{\delta}$, let us again employ the classical method of characteristics. We again discover that outside the trapezium formed by the straight lines $t = t_1^{\delta}$ and $t = t_2^{\delta}$ and by the segments of characteristic $x = X_2^-(t)$ and $x = X_2(t)$ issuing from the endpoints of the smoothing segment, the solution $u(x, t)$ and $u_{\delta}(x, t)$ and the potentials $\Phi(x, t)$ and $\Phi_{\delta}(x, t)$ coincide.

Continuing this process we can successively by smoothing and solving Cauchy's problems by the classical method of characteristics arrive at the straight line $t = T$, i.e., delimit the domain of interest to us. Here we will have to solve Cauchy's problem with smooth initial conditions of the T/δ order.

As a result of this process we get the solution $u_{\delta}(x, t)$, which is continuous everywhere except for the smoothing segments on the lines $t = t_k^{\delta}$. Segments of the characteristics issuing from the endpoints of the smoothing segments form a "belt" within lie singularities and discontinuities of the function $u_{\delta}(x, t)$. When the value of δ is decreased, this "belt" will be drawn toward at the discontinuity line $O\mathcal{L}$.

Since the potentials $\Phi(x, t)$ and $\Phi_{\delta}(x, t)$ coincide outside the "belt", and the solution $u(x, t)$ and $u(x, t)$ are assumed to be bounded, then

$$|\Phi(x, t) - \Phi_{\delta}(x, t)| < 2M\delta$$

From this follows the convergence of the "potential smoothing method" as $\delta \to 0$ in the potential metric. It is however clear that outside the "belt" the solutions u(x, t) and u (x, t) coincide.

The method outlined permits an approximate construction of discontinuous solutions of quasilinear equations by approximating them with continuous solutions. However, of greater interest is the application of this method to the case of a system of quasilinear equations. Unfortunately, thus far no results have yet been achieved in this area.

6. Asymptotic behavior of generalized solution as $t \to \infty$. Suppose u(x, t) is a generalized solution of the homogeneous law of conservation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \quad \varphi''_{uu}(u) > 0, \tag{1}$$

whose coefficients do not depend on the independent variables x, t. Further assume that

$$u(x, 0) = u_0(x) = \begin{cases} u^- & \text{when} \quad x < 0, \\ u^+ & \text{when} \quad x > b > 0. \end{cases} \tag{2}$$

and suppose that $u_0(x)$ takes on arbitrary bounded values in the interval (0, b). We will assume the function $u_0(x)$ to be piecewise-continuous. Thus,

$$|u_0(x)| \leq M, \quad |u(x, t)| \leq M. \tag{3}$$

Let us denote

$$\left.\begin{array}{l} \xi = \max\limits_{|u| < M} |\varphi'_u(u)|, \quad A = \max\limits_{|u| \leq M} \varphi''_{uu}(u), \\ a = \min\limits_{|u| < M} \varphi''_{uu}(u), \quad 0 < a \leq A \end{array}\right\} \tag{4}$$

and

$$\int_0^x u_0(\xi)\, d\xi = \Phi_0(x). \tag{5}$$

We will bring the potential

$$\Phi(x, t) = \int_{(0,0)}^{(x,t)} u\, dx - \varphi(u)\, dt, \quad \Phi(x, 0) = \Phi_0(x), \tag{6}$$

in correspondence with the solution u(x, t). By conditions (2) and (3) we have:

$$\left.\begin{array}{ll} \Phi_0(x) = u^- x & \text{when} \quad x \leq 0, \\ \Phi_0(x) = \Phi_0(b) + (x - b)u^+ & \text{when} \quad x \geq b \\ |\Phi_0(x)| \leq Mx. & \end{array}\right\} \tag{7}$$

We are studying the behavior of the solution $u(x, t)$ as $t \to \infty$.

If the initial data (1) were .sign only when $x < 0$ ($u_0(x) \equiv u^-$), then obviously the solution of Cauchy's problem (1), (2) could be defined only where $x < \varphi'_u(u^-)t$ and would coincide there identically with u^-. Let us estimate the width of the domain to the left of the straight line $x = \varphi'_u(u^-)t$ in which the solution $u(x, t)$ of problem (1), (2) does not coincide with u^-, i.e., $u(x, t) \neq u^-$; here $u(x, t) \neq u^-$ by virtue of the characteristics departing from the segment $[0, b]$. Obviously, the lower characteristic $x = X(t, x_0, u_0(\tilde{x}_0))$ when $0 \leq \tilde{x}_0 \leq b$ cannot lie to the left of straight line $x = -\xi t$ in view of the boundedness of the initial function $u_0(x)$; however, this estimate of the domain of influence of the segment $[0, b]$ is to course. Suppose $u(x, t) \neq u^-$, $x - \varphi'_u(u^-)t < 0$ and that the characteristic issuing from the point $\tilde{x}_0$ of the segment $0 \leq x_0 \leq b$ arrive at the point (x, t) and suppose that $u(x, t) \neq u_0(\tilde{x}_0)$ (Figure 4.29). For simplicity we denote

$$u_0(\tilde{x}_0) = \tilde{u}, \quad \varphi'_u(u^-)t - x = h.$$

We have the equalities*)

$$h = \varphi'_u(u^-)t - x, \tag{8}$$

$$x - \tilde{x}_0 = \varphi'_u(\tilde{u})t \quad \text{or} \quad h = t[\varphi'_u(u^-) - \varphi'_u(\tilde{u})] - \tilde{x}_0. \tag{9}$$

Let us compute $\tilde{\Phi}(t, \tilde{x}_0)$ by formula (2.3.46):

$$\tilde{\Phi}(t, \tilde{x}_0) = \Phi_0(\tilde{x}_0) + t[\tilde{u}\varphi'_u(\tilde{u}) - \varphi(\tilde{u})]. \tag{10}$$

using formula (9) let us cancel out the quantity $tu\varphi'_u(u)$ from (10):

$$t\tilde{u}\varphi'_u(\tilde{u}) = \tilde{u}t\varphi'_u(u^-) - \tilde{u}\tilde{x}_0 - \tilde{u}h; \tag{11}$$

$$\tilde{\Phi}(t, \tilde{x}_0) = \Phi_0(\tilde{x}_0) - \tilde{u}\tilde{x}_0 - \tilde{u}h + t[\tilde{u}\varphi'_u(u^-) - \varphi(\tilde{u})]. \tag{12}$$

On the other hand, the characteristic $x = X(t, x_0^-, u^-)$ at the value $x_0^- < 0$ (Figure 4.29) arrive at the same point (x, t) $(x - \varphi'_u(u^-)t < 0)$. This characteristic translates to the point (x, t) the value $u = u^-$ and the value of the potential $\tilde{\Phi}(t, x_0^-)$.

---

*) For the case when $\tilde{x}_0$ is a discontinuity point of the initial function $u_0(x)$, we must take $\tilde{u}$ to stand for $u_0^\alpha(\tilde{x}_0)$ given the corresponding value of the parameter α.

Thus, $\quad x - x_0^- = \varphi'_u(u^-)t, \quad$ or $\quad x_0^- = -h. \qquad (13)$

and by formula (2.3.46)

$$\Phi(t, x_0^-) = \Phi_0(x_0^-) + t\,[u^-\varphi'_u(u^-) - \varphi(u^-)] = \\ = u^- x_0^- + t\,[u^-\varphi'_u(u^-) - \varphi(u^-)] = t\,[u^-\varphi'_u(u^-) - \varphi(u^-)] - hu^-. \qquad (14)$$

According to formulas (2.3.57) and (2.3.58), in order that $u(x, t) = u_0(\tilde{x}_0) = \tilde{u} \neq u^-$, it is necessary that the inequality

$$\tilde{\Phi}(t, \tilde{x}_0) \leq \tilde{\Phi}(t, x_0^-) \quad \text{or} \quad \tilde{\Phi}(t - x_0) - \tilde{\Phi}(t, x_0^-) \leq 0. \qquad (15)$$

is satisfied. Subtracting formula (14) from formula (12), we get

$$\Phi(t, \tilde{x}_0) - \Phi(t, x_0^-) = [\Phi_0(\tilde{x}_0) - \tilde{u}\tilde{x}_0] + h(u^- - \tilde{u}) + \\ + t\,[\varphi'_u(u^-)(\tilde{u} - u^-) + \varphi(u^-) - \varphi(\tilde{u})]. \qquad (16)$$

Now let us estimate the last term in formula (16). Using the fact that $\tilde{u} < u^-$ and $\varphi''_{uu}(u) > 0$, we get by the Taylor formula

$$\varphi(u^-) - \varphi(\tilde{u}) \geqslant \varphi'_u(u^-)(u^- - \tilde{u}) + \frac{a}{2}(u^- - \tilde{u})^2. \qquad (17)$$

Inserting this expression into (16), we obtain

$$\Phi(t, \tilde{x}_0) - \Phi(t, x_0^-) \geqslant [\Phi_0(\tilde{x}_0) - \tilde{u}\tilde{x}_0] + h(u^- - \tilde{u}) + \frac{ta}{2}(u^- - \tilde{u})^2. \qquad (18)$$

From formula (9), since $u^- > \tilde{u}$, we can readily obtain

$$\varphi'_u(u^-) - \varphi'_u(u) = \frac{h + \tilde{x}_0}{t} \quad \text{and} \quad (u^- - \tilde{u}) \geqslant \frac{h + \tilde{x}_0}{At} \qquad (19)$$

The last inequality lets us rewrite (18) as

$$\Phi(t, \tilde{x}_0) - \Phi(t, x_0^-) \geqslant [\Phi_0(\tilde{x}_0) - \tilde{u}\tilde{x}_0] + \frac{h(h + \tilde{x}_0)}{At} + \frac{a(h + \tilde{x}_0)^2}{2A^2t}. \qquad (20)$$

And thus, for inequality (15) to be satisfied it is necessary that

$$[\Phi_0(\tilde{x}_0) - \tilde{u}\tilde{x}_0] + \frac{h(h + \tilde{x}_0)}{At} + \frac{a(h + \tilde{x}_0)^2}{2A^2t} \leqslant 0 \qquad (21)$$

or

$$\frac{h(h + \tilde{x}_0)}{At} + \frac{a(h + \tilde{x}_0)^2}{2A^2t} \leqslant [\tilde{u}\tilde{x}_0 - \Phi_0(\tilde{x}_0)]. \qquad (22)$$

Inequality (22) delimits the domain of variables x, t lying in the half-plane $t \geqslant 0$ to the left of the straight line $x = \varphi'_u(u^-)t$, since

$$h = \varphi'_u(u^-)t - x.$$

We can assume that the point (x, t) belongs to the domain of influence of the segment [0, b] if at least one value $\tilde{x}_0$ $(0 \leq \tilde{x}_0 \leq b)$ exists for which at given x, t inequality (22) is valid. Since $|\tilde{u}| \leq M$, $|\Phi_0(\tilde{x}_0)| \leq Mx_0$, then by strengthening *) inequality (22), we write:

$$h + \frac{(h+\tilde{x}_0)a}{2A} < \frac{2M\tilde{x}_0 At}{h+\tilde{x}_0} < 2AMt \quad (23)$$

and

$$h < 2AMt. \quad (24)$$

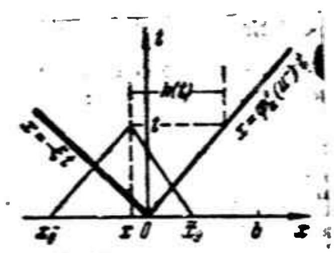

Figure 4.29

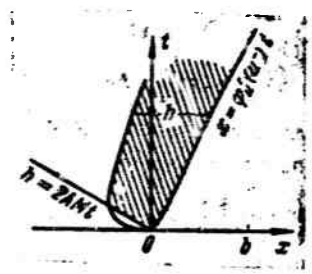

Figure 4.30

Inequality (24) shows that at t the value of h increases linearly with t. Conversely, we will clearly enlarge the domain of influence of the segment [0,b] if we set $\tilde{x}_0 = 0$ in (22) and in the right side take the maximum for all $\tilde{x}_0$. Then we write:

$$h^2\left(\frac{2A+a}{2A}\right) \leqslant 2MbAt \qquad \text{or} \qquad h^2 < \frac{4A^2Mbt}{2A+a} < 2AMbt. \quad (25)$$

We see that for large t values h increases as $t^{\frac{1}{2}}$. Thus, the domain of influence lies within the domain that is hatched in Figure 4.30:

$$h < \sqrt{2AMbt} = h(t). \quad (26)$$

We quite similarly estimate the domain of influence of the segment [0, b] to the right of the straight line $x = b + \varphi'_u(u^+)t$. Inequality (22) remains in force only if now we take h to stand for the quantity

$$h = x - b - \varphi'_u(u^+)t, \quad (27)$$

and take the right side with the opposite sign. Inequality (26) remains in general unchanged.

---

*) That is, we clearly expand the domain of influence of the segment [0,b].

In the case $u^- > u^+$ there exists a $t_1 > 0$ such that when $t \geqslant t_1$

$$u(x, t) = \begin{cases} u^- & \text{where} \quad x - c < Dt, \\ u^+ & \text{where} \quad x - c > Dt, \end{cases} \tag{28}$$

where

$$D = \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}. \tag{29}$$

Actually, if $u^- > u^+$ then the straight lines $x = \varphi'_u(u^-)t$ and $x = b + \varphi'_u(u^+)t$ lie as shown in Figure 4.31. Since the distance between these straight lines increases proportional to t, and the bounds of the domains of influence of the segment [0, b] are separated from the straight lines by a distance of the order of $(t)^{\frac{1}{2}}$, then we find such an instant $t = t_1$ for which the domain of influence of the segment [0, b] disappears (Figure 4.31). That is, the curves

$$h_1 = \varphi'_u(u^-)t - x = \sqrt{2AMbt},$$

$$h_2 = -b - \varphi'_u(u^+)t + x = \sqrt{2AMbt}$$

intersect each other. To determine the value of t_1 we have the equation

$$2\sqrt{2AMbt_1} + b = t_1[\varphi'_u(u^-) - \varphi'_u(u^+)]. \tag{30}$$

Suppose $t_1[\varphi'_u(u^-) - \varphi'_u(u^+)] > 2b$. Then from (30) we get

$$t_1 \leqslant \frac{4\sqrt{2AMbt_1}}{\varphi'_u(u^-) - \varphi'_u(u^+)} \quad \text{or} \quad t_1 \leqslant \frac{32AMb}{a^2[u^- - u^+]^2} = T. \tag{31}$$

Now we clearly conclude that when $t_1 = T$ formula (28) is valid. Actually, since the domain of values of the variable x, t is given by the conditions

$$b + \varphi'_u(u^+)t < x < \varphi'_u(u^-)t,$$

lies to the left of the straight line $x = \varphi'_u(u^-)t$ and to the right of the straight line $x = \varphi'_u(u^+)t + b$, in order that in this domain $u(x, t) \neq u^-$ and $u(x, t) \neq u^+$ is necessary that the point (x, t) belong to the domain of influence of the segment [0, b], which is impossible when $t \geqslant T$.

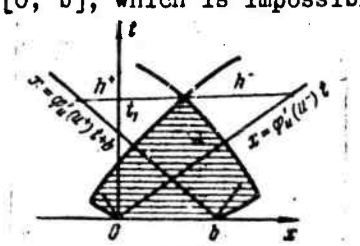

Figure 4.31

To determine the value of c in formula (28) let us consider a certain point x_A on the line $t = t_1$. Let us determine this point from the conditions

$$x_A - \varphi'_u(u^-)t_1 = x_0^-, \quad x_A - \varphi'_u(u^+)t_1 = x_0^+, \tag{32}$$
$$\bar{\Phi}(t_1, x_0^-) = \bar{\Phi}(t_1, x_0^+). \tag{33}$$

Condition (33) can be rewritten in the form

$$u^- x_0^- + t_1\left[u^-\varphi'_u(u^-) - \varphi(u^-)\right] = \\ = \Phi_0(b) + u^+(x_0^+ - b) + t_1\left[u^+\varphi'_u(u^+) - \varphi(u^+)\right]. \tag{34}$$

By inspecting equations (32) and (34) as equations defining the values of x_0^-, x_0^+, and x_A, let us find the quantities x_A and c:

$$x_A = \frac{\Phi_0(b) - u^+b}{u^- - u^+} + Dt_1 = c + Dt_1, \quad c = \frac{\Phi_0(b) - u^+b}{u^- - u^+}. \tag{35}$$

And thus, we have established that the solution $u(x, t)$ of Cauchy's problem (1), (2) given the condition $u^- > u^+$ coincides when $t \geqslant T$ with the solution $u^0(x, t)$ of the problem of the decay of discontinuity

$$\frac{\partial u^0}{\partial t} + \frac{\partial \varphi(u^0)}{\partial x} = 0, \quad u^0(x, 0) = \begin{cases} u^-, & x < c, \\ u^+, & x > c. \end{cases} \tag{36}$$

This result can be formulated in a different fashion. Suppose u and $\bar{u}$ are generalized solutions of equation (1), and suppose

$$u(x, 0) \equiv \bar{u}(x, 0) \quad \text{when} \quad |x| \geqslant b. \tag{37}$$

If, further,

$$u(x, 0) \equiv \bar{u}(x, 0) = \begin{cases} u^- & \text{when} \quad x \leqslant -b, \\ u^+ & \text{when} \quad x \geqslant b \end{cases} \tag{38}$$

and $u^- > u^+$, then there exists $t_1 > 0$ such that when $t \geqslant t_1$

$$u(x, t) \equiv \bar{u}(x + c, t), \tag{39}$$

where

$$c = \frac{1}{u^- - u^+} \int_{-b}^{b} [\bar{u}(x, 0) - u(x, 0)]\, dx. \tag{40}$$

The property of generalized solutions we have proven expresses the fact that in the case $u^- > u^+$ generalized solutions of Cauchy's problem (1) - (7) do not depend, "with an accuracy up to displacement," for sufficiently large times on initial values for any finite segment of the initial axis $t = 0$.

Note that this result can be strengthened in two directions:

1) for more general equations when $\varphi = \varphi(u, x, t)$ given certain assumptions on φ;

2) when requirement (38) is weakened:

$$|\bar{u}(x, 0) - u(x, 0)| \to 0 \quad \text{as} \quad x \to +\infty,$$

$$u(x, 0) \to \begin{cases} u^- & \text{as } x \to -\infty, \\ u^+ & \text{as } x \to +\infty; \end{cases}$$

here a specific order of approach to the limit is required.

Let us study the asymptotic behavior of solution $u(x, t)$ in the case $u^- < u^+$. In this case, as we can easily see from Figure 4.32, the domain of influence of the segment $[0, b]$ does not vanish as $t \to \infty$, but rather becomes unbounded. In this case we can no longer assert that for sufficiently large t

$$u(x, t) = u^0(x, t),$$

where $u^0(x, t)$ is the solution of the decomposition problem. However, in this case as well the solution $u(x, t)$ is close to the solution $u^0(x, t)$ for sufficiently large t. Specifically, we will prove that

$$|u(x, t) - u^0(x, t)| \rightrightarrows 0 \quad \text{as } t \to \infty. \tag{41}$$

Suppose u^0 is the solution of the decay problem

$$\left.\begin{aligned} &\frac{\partial u^0}{\partial t} + \frac{\partial \varphi(u^0)}{\partial x} = 0 \\ &u^0(x, 0) = \begin{cases} u^- \text{ when } x < 0, \\ u^+ \text{ when } x > 0, \end{cases} \quad u^- \leqslant u^+. \end{aligned}\right\} \tag{42}$$

Function u^0 is given by the formulas

$$u^0(x, t) = \begin{cases} u^- & \text{when} \quad x \leqslant \varphi'_u(u^-)t, \\ f\left(\frac{x}{t}\right) & \text{when} \quad \varphi'_u(u^-)t \leqslant x \leqslant \varphi'_u(u^+)t, \\ u^+ & \text{when} \quad x \geqslant \varphi'_u(u^+)t, \end{cases} \tag{43}$$

where

$$\varphi'_u(f(\xi)) = \xi, \quad \text{i.e.,} \quad f(\xi) = [\varphi'_u]^{-1}(\xi). \tag{44}$$

Thus, to the left of the ray OA $u^0(x, t) \equiv u^-$, and to the right of the ray OC $u^0(x, t) \equiv u^+$ (Figure 4.32); but in the zone AOC $u^0 = f(x/t)$. As we see, outside the domain of influence of the segment [0, b] it also holds for u(x, t), i.e., when

and when

$$\varphi'_u(u^-)t - x > h(t), \quad u(x, t) \equiv u^-, \tag{45}$$

$$x - \varphi'_u(u^+)t - b > h(t), \quad u(x, t) \equiv u^+. \tag{46}$$

Therefore, we need prove property (41) only in domain of influence of the segment [0, b] -- the zone EOBD. Suppose (x, t) is any point lying within the zone EOBD. Then, since not a single characteristic $x = X(t, x_o, u_o(x_o))$ when x_o [0, b] passes through this point, therefore

$$u(x, t) = u_0(\tilde{x}_0), \quad x - \varphi'_u(u_0(\tilde{x}_0))t = \tilde{x}_0 \tag{47}$$

where $$0 \leq \tilde{x}_o \leq b. \tag{48}$$

From (47) we get

$$\varphi'_u(u_0(\tilde{x}_0)) = \varphi'_u(u(x, t)) = \frac{x}{t} - \frac{\tilde{x}_0}{t}. \tag{49}$$

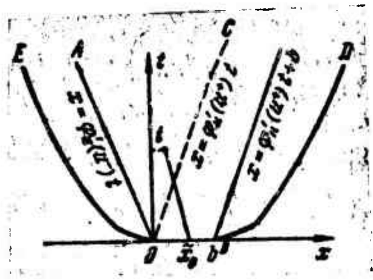

Figure 4.32

Comparing (49) with (44), we conclude that

$$u(x, t) = u_0(\tilde{x}_0) = f\left(\frac{x}{t} - \frac{\tilde{x}_0}{t}\right). \tag{50}$$

Thus, in the zone AOC

$$|u(x, t) - u^0(x, t)| = \left|f\left(\frac{x}{t} - \frac{\tilde{x}_0}{t}\right) - f\left(\frac{x}{t}\right)\right| \leqslant \max_{|\eta| < \xi} |f'(\eta)| \left|\frac{\tilde{x}_0}{t}\right|. \tag{51}$$

Since $f'_\eta = 1/\varphi''_{uu}$, then from (51) it follows that

$$|u(x, t) - u^0(x, t)| \leqslant \frac{\tilde{x}_0}{at} \leqslant \frac{b}{at}. \tag{52}$$

Thus, property (41) in the zone AOC is proven. Now suppose (x, t) is any point in the zone EOBD, for example, suppose that this point lies to the right of the straight line $x = \varphi'_u(u^+)t$, i.e., in the zone COBD (Figure 4.32). Then at this point $u^0 = u^+$, and u(x, t) as before is defined from formulas (47) and (48).

By the definition of the domain of influence of the segment [0, b], we have

$$x - \varphi'_u(u^+)t \leqslant b + h(t) = b + \sqrt{2AMbt}. \qquad (53)$$

Therefore,

$$|u(x, t) - u^0(x, t)| = \left| f\left(\frac{x - \tilde{x}_0}{t}\right) - u^+ \right| = \left| f\left(\frac{x - \tilde{x}_0}{t}\right) - f(\varphi'_u(u^+)) \right| \leqslant$$
$$\leqslant \frac{|x - \tilde{x}_0 - \varphi'_u(u^+)t|}{at} \leqslant \frac{|b - \tilde{x}_0 + h(t)|}{at} =$$
$$= \frac{b}{at} + \frac{h(t)}{at} = \frac{b}{at} + \frac{\sqrt{2AMb}}{a\sqrt{t}}. \qquad (54)$$

Since to the left of the ray OA inequality (54) is proven quite analogously, we have thus proven the property of generalized solution (41) and have established in so doing the order in which u(x, t) approaches $u^0(x, t)$, namely:

$$|u(x, t) - u^0(x, t)| < \frac{4b}{at} + \frac{8}{a\sqrt{t}} \sqrt{AMb}. \qquad (55)$$

Combining the result for the cases $u^- < u^+$ and $u^- > u^+$, we can formulate it as follows.

Suppose u(x, t) and $\bar{u}(x, t)$ are bounded generalized solutions of equation (1) whose initial values differ only by a finite segment of the initial axis, i.e.,

$$|u(x, 0) - \bar{u}(x, 0)| \equiv 0 \quad \text{where} \quad |x| \geqslant b. \qquad (56)$$

Then there exist the constants c, D, and t_1 such that where $t \geqslant t_1$

$$|u(x, t) - u(x + c, t)| \leqslant \frac{D}{\sqrt{t}}. \qquad (57)$$

Let us emphasize that in this formulation it is not assumed that

$$u(x, 0) \to u^-, u^+ \text{ as } x \to -\infty, +\infty.$$

The asymptotic properties of generalized solutions of a single quasilinear equation proven above are essentially associated with the nonlinearity of the equation. Actually, solutions of linear equations differing in initial values differ one from the other for all values t > 0, and this "discrepancy" generally speaking does not tend to zero as $t \to \infty$.

7. Method of viscosity. In subsection 1 of this section we already consider one of the applications of the viscosity method -- the generalized solution of the problem

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0, \quad u(x, 0)=u_0(x)$$

which was obtained as the limit, as $\mu \to 0$, of the solutions $u_\mu(x, t)$ of another Cauchy's problem:

$$\frac{\partial u_\mu}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u_\mu^2}{2}\right)=\mu\frac{\partial^2 u_\mu}{\partial x^2}, \quad u_\mu(x, 0)=u_0(x), \quad \mu>0.$$

Here we obtained for u_μ an explicit formula which enables us to make the passage to the limit as $\mu \to 0$ and to study the properties of the generalized solution $u(x, t)$. Naturally, it is difficult to anticipate obtaining analytic formulas for the solution for more involved equations. However, since we wish to familiarize the reader with applications of the viscosity method, we will consider several simple cases from which we can judge the potentialities of this method.

We will discuss the Cauchy's problem

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=0, \quad \varphi''_{uu}(u)>0, \tag{1}$$

$$u(x, 0)=u_0(x), \quad |u_0(x)|\leqslant M. \tag{2}$$

In addition to this problem, we will consider another:

$$\frac{\partial u_\mu}{\partial t}+\frac{\partial \varphi(u_\mu)}{\partial x}=\mu\frac{\partial^2 u_\mu}{\partial x^2} \quad (\mu>0). \tag{3}$$

$$u_\mu(x, 0)=u_0(x). \tag{4}$$

We will assume that the initial function $u_0(x)$ has a continuous first derivative, and that $|u_0'(x)| \leq K \quad (-\infty < x < \infty)$. (5)

The first problem, which in any case emerges at once is the problem of the existence of the solution u_μ and its properties.

The study of this problem will lead us far beyond the scope of our subject, which deals mainly with hyperbolic systems of quasilinear equations and with equations in gas dynamics. Therefore, referring the reader whose seeks a more detail exposition to special studies (cf, for example, [9, 39, 42]), we will assume that the following is known:

1) a bounded solution $u_\mu(x)$ exists for any $t > 0$, and does so uniquely.

2) For any initial function $u_0(x)$ satisfying conditions (2) and (5) derivatives of u_μ exist and remain bounded for all $t > 0$. In particular, when $t > 0$ the continuous derivatives appearing in equation (3) exist.

3) The solution u_μ and its derivative $p_\mu = \frac{\partial u_\mu}{\partial x}$ satisfy the maximum principle, which is formulated in the form of the inequalities

$$|u_\mu| \leqslant M, \tag{6}$$

$$p_\mu = \frac{\partial u_\mu}{\partial x} \leqslant K \tag{7}$$

which are valid for any x, $t \geqslant 0$.

Let us illustrate the principle of the maximum (6) by the following graphical arguments. Suppose the function $u_\mu(x, t_1)$ considered as a function of variable x at straight line $t = t_1$ has at the point $x = x_1$ a relative maximum (minimum). Then the relations

$$\frac{\partial u_\mu}{\partial x}(x_1, t_1) = 0, \quad \frac{\partial^2 u_\mu}{\partial x^2}(x_1, t_1) \leqslant 0.$$

are satisfied at this point. According to equation (3), we conclude:

$$\frac{\partial u_\mu(x_1, t_1)}{\partial t} = \left[\mu \frac{\partial^2 u_\mu}{\partial x^2} - \varphi'_u(u_\mu)\frac{\partial u_\mu}{\partial x}\right]_{\substack{x=x_1 \\ t=t_1}} \leqslant 0.$$

This inequality shows that each relative maximum with respect to the variable x of the function $u_\mu(x, t)$ does not increase with time t. However, inequality (6) does not yet follow, since the function u_μ cannot take on its maximum (minimum) value at any finite point.

A more detailed proof of the principle of the maximum considers that the initial function $u_0(x)$ is bounded.

As for the maximum principle for the derivative p_μ, we direct attention to the fact that inequality (7) restricts the derivative p_μ only from one side, namely from above. This is because the equation obtained by differentiating (3):

$$\frac{\partial p_\mu}{\partial t} + \varphi'_u(u_\mu)\frac{\partial p_\mu}{\partial x} = \mu \frac{\partial^2 p_\mu}{\partial x^2} - \varphi''_{uu}(u_\mu)p_\mu^2, \quad \varphi'_{uu} > 0,$$

enables us only to assert that the relative maximum $p_\mu(x, t)$ cannot increase with time t. As for the minima of p_μ, they can also decrease.

4. For all $\mu > 0$ and arbitrary x, $t \geqslant 0$ the inequality

$$\left| \mu \frac{\partial u_\mu}{\partial x} \right| \leqslant C. \tag{8}$$

is satisfied, where C is some positive quantity not dependent on μ.

Let us prove*) inequality (8). Denoting

$$z = \mu \frac{\partial u_\mu}{\partial x} - \varphi(u_\mu), \tag{9}$$

we write equation (3) in the form $\partial u_\mu / \partial t = \partial z / \partial x$. Differentiating equation (9) relative to variable t, we find

$$\frac{\partial z}{\partial t} = \mu \frac{\partial}{\partial x}\left(\frac{\partial u_\mu}{\partial t}\right) - \varphi'_u(u_\mu)\frac{\partial u_\mu}{\partial t}.$$

Here inserting $\frac{\partial u_\mu}{\partial t} = \frac{\partial z}{\partial x}$, we get $\quad \frac{\partial z}{\partial t} + \varphi'_u(u_\mu)\frac{\partial z}{\partial x} = \mu \frac{\partial^2 z}{\partial x^2}$.

Thus, the function z satisfies the nonlinear equation of thermal conductivity. Therefore, from the principle of the maximum we get

$$|z(x, t)| \leqslant \max_x |z(x, 0)|.$$

Hence follows inequality (8) and, in particular, the more exact inequality

$$\min_x \left[\mu \frac{du_0(x)}{dx} - \varphi(u_0(x))\right] + \varphi(u_\mu) \leqslant \mu \frac{\partial u_\mu}{\partial x} \leqslant \leqslant \max_x \left[\mu \frac{du_0(x)}{dx} - \varphi(u_0(x))\right] + \varphi(u_\mu).$$

When requirements 1 - 4 are satisfied, we can also prove several more properties of the solutions $u_\mu(x, t)$. Let us show, in particular, that the function $\mu \frac{\partial u_\mu}{\partial x}$ on the average for any finite domain of variables x, t tends to zero as $\mu \to 0$.

Multiplying equality (3) by u_μ, let us write the result in the form

$$\frac{\partial}{\partial t}\left(\frac{u_\mu^2}{2}\right) + \frac{\partial}{\partial x} F(u_\mu) = \mu \frac{\partial^2}{\partial x^2}\frac{u_\mu^2}{2} - \mu\left(\frac{\partial u_\mu}{\partial x}\right)^2, \tag{10}$$

where

$$F'(u_\mu) = u_\mu \varphi'(u_\mu).$$

Integrate equation (10) over the rectangle $0 \leq t \leq t_1$, $x_1 \leq x \leq x_2$:

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*) This proof was communicated to us by a student at Moscow State University, V. G. Shushkov, in 1964.

$$\int_0^{t_1}\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\frac{u_\mu^2}{2}+\frac{\partial}{\partial x}F(u_\mu)\right]dt\,dx=\int_0^{t_1}\int_{x_1}^{x_2}\left[\mu\frac{\partial^2}{\partial x^2}\frac{u_\mu^2}{2}-\mu\left(\frac{\partial u_\mu}{\partial x}\right)^2\right]dt\,dx.$$

Let us rewrite this relation in the form

$$\mu\int_0^{t_1}\int_{x_1}^{x_2}\left(\frac{\partial u_\mu}{\partial x}\right)^2 dt\,dx=\int_{x_1}^{x_2}\left[\frac{u_\mu^2(x,0)-u_\mu^2(x,t_1)}{2}\right]dx+$$
$$+\int_0^{t_1}\left\{\mu u_\mu(x_2,t)\frac{\partial u_\mu}{\partial x}(x_2,t)-F(u_\mu(x_2,t))-\mu u_\mu(x_1,t)\frac{\partial u_\mu(x_1,t)}{\partial x}+\right.$$
$$\left.+F(u_\mu(x_1,t))\right\}dt.$$

In view of inequalities (6) and (8), the right side is estimated from above:

$$\mu\int_0^{t_1}\int_{x_1}^{x_2}\left(\frac{\partial u_\mu}{\partial x}\right)^2 dt\,dx\leqslant\frac{1}{2}M^2(x_2-x_1)+2Ft_1+2MCt_1=$$
$$=\frac{M^2}{2}(x_2-x_1)+2t_1(F+MC), \qquad (11)$$

where the constant F is defined in such a way that

$$F\geqslant|F(u)|\quad\text{when}\quad|u|\leqslant M.$$

Since the right side of (11) does not depend on μ, hence it follows that for any finite domain $\mathcal{G}$ of the half-plane $t\geqslant 0$

$$\mu\iint_{\mathcal{G}}\left(\frac{\partial u_\mu}{\partial x}\right)^2 dx\,dt\leqslant D,$$

where the constant D depends only on the domain $\mathcal{G}$.

Let us use the Bunyakovsky inequality

$$\iint_{\mathcal{G}}|uv|\,dt\,dx\leqslant\sqrt{\iint_{\mathcal{G}}u^2\,dt\,dx\iint_{\mathcal{G}}v^2\,dt\,dx},$$

in which we set $u=\mu\frac{\partial u_\mu}{\partial x}$, $v\equiv 1$. Then we get

$$\mu\iint_{\mathcal{G}}\left|\frac{\partial u_\mu}{\partial x}\right|dt\,dx\leqslant\sqrt{S_{\mathcal{G}}\mu^2\iint_{\mathcal{G}}\left(\frac{\partial u_\mu}{\partial x}\right)^2 dt\,dx}\leqslant\sqrt{\mu}\sqrt{S_{\mathcal{G}}D}, \qquad (12)$$

where $S_{\mathcal{G}}$ is the area of the domain $\mathcal{G}$. Inequality (12) shows that $\iint_{\mathcal{G}}\mu\left|\frac{\partial u_\mu}{\partial x}\right|dt\,dx$ tends to zero as $\mu\to 0$ as $(\mu)^{\frac{1}{2}}$.

Now let us outline the proof of the convergence in the mean as $\mu\to 0$ of solutions $u_\mu(x,t)$ to the solution $u(x,t)$ of Cauchy's problem (1), (2). Let us introduce the potential $\Phi_\mu(x,t)$. According to equation (3) the contour integral

$$\Phi_\mu(x, t) = \int_{(0,0)}^{(x,t)} u_\mu dx + \left[\mu \frac{\partial u_\mu}{\partial x} - \varphi(u_\mu)\right] dt$$

does not depend on the path of integration and defines the continuous and differentiable function whose derivatives satisfy the equalities

$$\frac{\partial \Phi_\mu}{\partial x} = u_\mu, \quad \frac{\partial \Phi_\mu}{\partial t} = \mu \frac{\partial u_\mu}{\partial x} - \varphi(u_\mu).$$

Canceling u_μ from these equalities, we get

$$\frac{\partial \Phi_\mu}{\partial t} + \varphi\left(\frac{\partial \Phi_\mu}{\partial x}\right) = \mu \frac{\partial^2 \Phi_\mu}{\partial x^2}. \tag{13}$$

The potential Φ_μ takes on the following initial values:

$$\Phi_\mu(x, 0) = \Phi_0(x) = \int_0^x u_0(\eta)\, d\eta.$$

In view of inequalities (6) and (8), we conclude that

$$|\Phi_\mu(x, t)| < M|x| + (c + \varphi)t, \tag{14}$$

where $|\varphi(u)| \leq \varphi$ when $|u| \leq M$ and the family of functions $\Phi_\mu(x, t)$ is bounded for all $\mu > 0$ at any point (x, t). Finally, we write the obvious inequalities:

$$\left|\frac{\partial \Phi_\mu}{\partial x}\right| \leq M, \quad \left|\frac{\partial \Phi_\mu}{\partial t}\right| \leq c + \varphi.$$

Let us prove the uniform convergence in any finite section of the half-plane $t \geqslant 0$ of the potential $\Phi_\mu(x, t)$ as $\mu \to 0$. To do this, let us differentiate equation (13) relative to the variable μ. Denoting $\Phi'_\mu = \frac{\partial \Phi_\mu}{\partial \mu}$, we get

$$\frac{\partial \Phi'_\mu}{\partial t} + \varphi'(u_\mu)\frac{\partial \Phi'_\mu}{\partial x} = \mu \frac{\partial^2 \Phi'_\mu}{\partial x^2} + \frac{\partial^2 \Phi_\mu}{\partial x^2} = \mu \frac{\partial^2 \Phi'_\mu}{\partial x^2} + \frac{\partial u_\mu}{\partial x}. \tag{15}$$

and according to the initial condition $\Phi'_\mu(x, 0) \equiv 0$.

Taking the function u_μ as given, we see that the quantity Φ'_μ satisfies linear equation of thermal conductivity (15) and the zero initial condition. According to inequality (14), $\Phi_\mu(x, t)$ increases not more rapidly than the linear function as $x \to \pm\infty$, therefore Φ'_μ satisfies the principle of the maximum for equation (15):

$$\Phi'_{\mu}(x, t) \leq t \max_{(\xi, \tau)} \frac{\partial u_{\mu}(\xi, \tau)}{\partial \xi}.$$

Since according to (7) $\frac{\partial u_{\mu}}{\partial \xi} \leq K$, then $\Phi'_{\mu}(x, t) \leq Kt$. Therefore for the function $\psi_{\mu} = \Phi_{\mu} - 2Kt\mu$ the following inequality is satisfied: $\frac{\partial \psi_{\mu}}{\partial \mu} \leq -Kt$, such that the sequence of functions $\psi_{\mu}(x, t)$ decreases monotonically as $\mu \to 0$, and therefore owing to its uniform boundedness from below the sequence converges uniformly relative to x, t. Therefore there exists the limit $\Phi(x, t) = \lim_{\mu \to 0} \Phi_u(x, t)$.

Let us clarify certain very simple properties of the function $\Phi(x, t)$. Since $|u_{\mu}| \leq M$ and $\frac{\partial \Phi_{\mu}}{\partial x} = u_{\mu}$, then

$$|\Phi(x+\Delta x, t) - \Phi(x, t)| \leq M|\Delta x|.$$

Similarly

$$|\Phi(x, t+\Delta t) - \Phi(x, t)| \leq [c+\varphi]|\Delta t|.$$

Thus, the potential $\Phi(x, t)$ is a Lipschitz-continuous function of its variables, which takes on the initial values

$$\Phi(x, 0) = \Phi_0(x) = \int_0^x u_0(\eta)\, d\eta$$

and, like any Lipschitz-continuous function, has almost everywhere the derivatives $\frac{\partial \Phi}{\partial x}$, $\frac{\partial \Phi}{\partial t}$. Let us show that almost everywhere these derivatives satisfy the equation

$$\frac{\partial \Phi}{\partial t} + \varphi\left(\frac{\partial \Phi}{\partial x}\right) = 0, \qquad (16)$$

and that the sequence u_{μ} tends to $u(x, t)$ almost everywhere as $\mu \to 0$, where

$$u(x, t) = \frac{\partial \Phi(x, t)}{\partial x}.$$

To do this, let us note that owing to the uniform boundedness of u_{μ} (6) and the one-sided boundedness of the derivative p_{μ} (7), follows the boundedness of the variation of the functions u_{μ} on any finite segment of the straight lines t = constant.

Using this fact, we can show that almost everywhere

$$\frac{\partial \Phi_{\mu}}{\partial x} \to \frac{\partial \Phi}{\partial x}, \quad \frac{\partial \Phi_{\mu}}{\partial t} \to -\varphi\left(\frac{\partial \Phi}{\partial x}\right)$$

as $\mu \to 0$, such that the potential $\Phi(x, t)$ almost everywhere satisfies equation (16).

Thus, the potential $\Phi(x, t)$ is Lipschitz-continuous, satisfies equation (16) almost everywhere, and takes on the necessary values when $t = 0$. Therefore the function $u = \frac{\partial \Phi}{\partial x}$ is defined almost everywhere, bounded and measurable, satisfies stability condition as a consequence of (7):

$$\frac{\Phi(x+\Delta x, t) - 2\Phi(x, t) + \Phi(x-\Delta x, t)}{\Delta x^2} < K,$$

and is thus a generalized solution of Cauchy's problem (1), (2).

Let us note further that using the viscosity method we can consider the Cauchy's problem also for the case of an arbitrary, bounded, measurable initial function $u_0(x)$ (cf [45]).

Let us indicate yet another of the applications of the viscosity method. According to Gel'fand, we obtain stability conditions of a generalized solution of a single quasilinear equation with nonconvex function $\varphi(u)$ (cf also Section I).

Suppose the piecewise-continuous solution $u(x, t)$ is the limit of solutions $u_\mu(x, t)$ of an equation with viscosity

$$\frac{\partial u_\mu}{\partial t} + \frac{\partial \varphi(u_\mu)}{\partial x} = \mu \frac{\partial}{\partial x}\left(B(u_\mu)\frac{\partial u_\mu}{\partial x}\right), \quad B(u_\mu) > 0, \quad \mu > 0. \tag{17}$$

Suppose that at the point x_1, $t_1 > 0$ the solution $u(x, t)$ is discontinuous and $u(x_1 + 0, t_1) = u^+$, and suppose that the discontinuity is displaced in the x, t plane along the curve $x = x(t)$, and here $x'(t_1) = D$.

It is natural to assume that in a small neighborhood of the point x_1, t_1 the solution $u(x, t)$ is represented approximately in the form $u(x, t) = \bar{u}(x - Dt)$, where $\bar{u}(x) = u^-$ when $x < 0$ and $\bar{u}(x) = u^+$ when $x > 0$. We will assume that the solution $u_\mu(x, t)$ in a small neighborhood of the point x_1, t_1 is also a travelling wave, that is, $u_\mu(x, t) = \bar{u}_\mu\left(\frac{x - Dt}{\mu}\right)$; here $\bar{u}_\mu(x) \to u^-$ as $x \to -\infty$, $\bar{u}_\mu(x) \to u^+$ as $x \to \infty$ (in this case we assume that the parameter μ is sufficiently small).

And thus, we will seek the steady solution $u_\mu(x, t)$ of equation (17) in the form

$$u_\mu(x, t) = \bar{u}_\mu\left(\frac{x - Dt}{\mu}\right), \quad \bar{u}_\mu(\xi) \to \begin{cases} u^- & \text{as } \xi \to -\infty, \\ u^+ & \text{as } \xi \to +\infty. \end{cases} \tag{18}$$

Denoting $\xi = \frac{x - Dt}{\mu}$, let us derive an ordinary differential equation for $\bar{u}_\mu(\xi)$ from (17):

$$[\varphi_u'(\bar{u}_\mu(\xi)) - D]\frac{d\bar{u}_\mu(\xi)}{d\xi} = \frac{d}{d\xi}\left(B(\bar{u}_\mu)\frac{d\bar{u}_\mu(\xi)}{d\xi}\right); \tag{19}$$

here conditions (18) yield boundary conditions for equation (19):

$$\bar{u}_\mu(\xi) \to \begin{cases} u^- & \text{as } \xi \to -\infty, \\ u^+ & \text{as } \xi \to +\infty. \end{cases} \tag{20}$$

Integrating equation (19) from the point $\xi = -\infty$ to ξ and assuming that $\frac{d\bar{u}_\mu}{d\xi} \to 0$ as $\xi \to \pm\infty$, we get

$$B(\bar{u}_\mu)\frac{d\bar{u}_\mu}{d\xi} = \varphi(\bar{u}_\mu) - \varphi(u^-) - D(\bar{u}_\mu - u^-) = F(\bar{u}_\mu). \tag{21}$$

In order that $\frac{d\bar{u}_\mu}{d\xi} \to 0$ as $\bar{u}_\mu \to u^+$, it is necessary that $F(u^+) = 0$, i.e.,

$$D = \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}, \tag{22}$$

which, as we have seen in Section I, also follows from Hugoniot's conditions.

Thus, the existence of the integral curve of the problem (19), (20) necessitates that $F(u^-) = F(u^+) = 0$. This, however, is insufficient.

Another necessary condition for the existence of $\bar{u}_\mu$ is the requirement that there will be no alternation of sign at the function $F(\bar{u}_\mu)$ on the interval (u^-, u^+). Actually, if a point u^* exists in this interval such that to the left and to the right of u^* the function $F(u)$ has different signs, then obviously no interval curve $\bar{u}_\mu = \bar{u}_\mu(\xi)$ exists.

Multiplying equation (21) by the quantity $2(\bar{u}_\mu - u^-)$, we give it the following form:

$$B(\bar{u}_\mu)\frac{d}{d\xi}(\bar{u}_\mu - u^-)^2 = 2(\bar{u}_\mu - u^-)F(\bar{u}_\mu). \tag{23}$$

Now it is obvious that the existence of $\bar{u}_\mu(\xi)$ also necessitates that the right side of (23) be nonnegative, that is,

$$2(\bar{u}_\mu - u^-)F(\bar{u}_\mu) \geqslant 0.$$

Actually, since $B(\bar{u}_\mu) > 0$, from the existence of $\bar{u}_\mu(\xi)$ it follows that the quantity $(\bar{u}_\mu - u^-)^2$ does not decrease with increase in the variable ξ, since otherwise the function $F(u)$ would change sign on the interval (u^-, u^+).

We can similarly easily obtain the results that the quantity $(\bar{u} - u^+)^2$ does not increase with ξ, therefore $2(\bar{u}_\mu - u^+)F(\bar{u}_\mu) \leqslant 0$.

Dividing each of the last two inequalities by the positive quantities $2(\bar{u}_\mu - u^-)^2$ and $2(\bar{u}_\mu - u^+)^2$, we give it the following form:

$$\frac{\varphi(\bar{u}_\mu) - \varphi(u^-)}{\bar{u}_\mu - u^-} \geqslant D = \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-} \geqslant \frac{\varphi(\bar{u}_\mu) - \varphi(u^+)}{\bar{u}_\mu - u^+}; \qquad (24)$$

here, obviously, $\bar{u}_\mu$ is any number from the interval (u^-, u^+). And so, we conclude that the discontinuous solution $u(x, t)$ can be viewed as the limit of solutions $u_\mu(x, t)$ as $\mu \to 0$ only for the case when condition (24) is satisfied at the discontinuity line $x = x(t)$ as each of its points. We can easily note that the condition (24) obtained here coincides with the stability condition we obtained earlier in Section I from wholly different considerations.

In conclusion let us note that we have now proven the existence and uniqueness of a generalized solution of Cauchy's problem for the case of a non-convex function $\varphi(u)$. Here the stability of the generalized solution is understood as the satisfaction of conditions (24). We must note that these theorems have been proven in a much narrower generality than for the case of the convex function φ, namely for the case of piecewise-continuous and piecewise-smooth solutions.

III. System of Quasilinear Equations

1. Introductory remarks. Now we will consider the conservative system of quasilinear equations

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u, x, t)}{\partial x} = f_i(u, x, t) \qquad (i = 1, \ldots, n). \qquad (1)$$

which for brevity will be written in the form

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u, x, t)}{\partial x}=f(u, x, t), \tag{2}$$

where we take u, φ, and f as vector functions with n components. We will further assume that the vector-functions $\varphi(u, x, t)$ and $f(u, x, t)$ have two continuous derivatives relative to all their variables in the domain of variation of the variables u, x, and t we are considering.

The system of differential equations (2) in the hyperbolic case is reduced, as we have seen in Chapter One, to the form

$$l_\alpha^k(u, x, t)\left[\frac{\partial u_\alpha}{\partial t}+\xi_k(u, x, t)\frac{\partial u_\alpha}{\partial x}\right]=g_k(u, x, t), \tag{3}$$

where

$$\mathrm{Det}\left(\left(l_\alpha^k(u, x, t)\right)\right)\neq 0. \tag{4}$$

The vector $\ell^k(u, x, t) = \{\ell_\alpha(u, x, t)\}$ is the left eigenvector of the matrix $A(u, x, t) = \left(\left(\frac{\partial \varphi_i}{\partial u_j}\right)\right)$, and $\xi_k(u, x, t)$ is the corresponding eigenvalue, that is, $\ell^k(u, x, t)A(u, x, t) = \xi_k(u, x, t)\,\ell^k(u, x, t)$. (5)
Throughout this section we will deal only with systems of equations (2) and (3) that are hyperbolic in the small, i.e., we will assume that

$$\xi_1(u, x, t)<\xi_2(u, x, t)<\ldots<\xi_n(u, x, t). \tag{6}$$

The problem of constructing classical solutions of system (2) was dealt with in Chapter One, and here we will study generalized (discontinuous) solutions.

A generalized solution of system (1) satisfies integral laws of conservation

$$\oint_C u\,dx-\varphi(u, x, t)\,dt+\iint_{G_C} f(u, x, t)\,dx\,dt=0, \tag{7}$$

but in the domain where the first derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ exist this solution satisfies differential equations (2) or (3); we will also write the latter more concisely:

$$l^k\left(\frac{\partial u}{\partial t}+\xi_k\frac{\partial u}{\partial x}\right)=g_k. \tag{8}$$

We will formulate the Cauchy's problem with the initial conditions

$$u(x, 0) = u_0(x), \tag{9}$$

for the system of quasilinear equations (2), where $u_0(x)$ is, in general, a discontinuous bounded vector-function.

As we have seen in Section I, the reasonable formulation of Cauchy's problem leads to certain stability conditions that must be satisfied by generalized solutions of this problem.

In this section we can consider mainly generalized solutions that have piecewise-smooth discontinuity lines $x = x(t)$, outside of which they are classical solutions of system (2).

Stability conditions, in accordance with Section I, will be understood in the following sense.

Some number k, called the index of the discontinuity line, will be brought to correspondence to each discontinuity line $x = x(t)$. The inequalities

$$\xi_k(u(x(t)-0, t), x(t), t) > D > \xi_k(u(x(t)+0, t), x(t), t), \tag{10}$$

$$\xi_{k-1}(u(x(t)-0, t), x(t), t) < D < \xi_{k+1}(u(x(t)+0, t), x(t), t), \tag{11}$$

$$D = x'(t),$$

are satisfied at the line $x = x(t)$ for this number k.

In their simplest form, properties of generalized solutions of systems of quasilinear equations can be studied with the example of a homogeneous nonlinear system of equations

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \tag{12}$$

i.e., in the case when the vector φ does not depend on variables x, t, and $f \equiv 0$.

2. Self-modeling solutions of a system of quasilinear equations. Solutions of system of equations (3.1.12) that depend only on a single variable $y = (x - x_0)/(t - t_0)$ are called self-modeling. Without restricting generality, we will assume $t_0 = x_0 = 0$. We seek the solution $u(y)$ of system (3.1.12) defined for $t \geqslant 0$ and dependent only on the variable $y = x/t$. Performing the substitution in (3.1.12) based on the following formulas

$$\frac{\partial}{\partial t} = -\frac{x}{t^2}\frac{d}{dy} = -\frac{y}{t}\frac{d}{dy}, \quad \frac{\partial}{\partial x} = \frac{1}{t}\frac{d}{dy},$$

we get the result that the self-modeling solution $u = u(y)$ satisfies the system of ordinary differential equations

$$[A(u) - yE]\frac{du}{dy} = 0, \quad A(u) = \left(\left(\frac{\partial \varphi_i(u)}{\partial u_j}\right)\right), \tag{1}$$

where E is the unit matrix of n-th order.

It follows from system of equations (1) that if

$$y \neq \xi_k(u) \qquad (k = 1, 2, \ldots, n),$$

then the vector du/dy is identically zero: $\frac{du}{dy} \equiv 0$ and the solution $u(y) \equiv$ constant. So we will assume that on some segment of the variable y the following equality is identically satisfied: $$y = \xi_k(u), \tag{2}$$
where k is any of these number 1, 2, ..., n. Then system of equations (1) has a nontrivial solution relative to the vector of the derivatives du/dy:

$$du/dy = \lambda r^k(u), \tag{3}$$

where $r^k(u)$ as usual denotes the right eigenvector of the matrix A(u) (Chapter One), i.e., $$A(u)r^k(u) = \xi_k(u)r^k(u).$$

System of equations (3) does not yet enable us to determine u(y) by integration, since an unknown cofactor λ appears in these equations. To determine it, let us differentiate equation (2) relative to the variable y. We get the equation

$$\sum_{j=1}^{n} \frac{\partial \xi_k(u)}{\partial u_j}\frac{du_j}{dy} = 1,$$

which we rewrite in symbolic form $$du/dy \text{ grad } \xi_k(u) = 1. \tag{4}$$
introducing the notation $$\text{grad } \xi_k(u) = \left\{\frac{\partial \xi_k(u)}{\partial u_1}, \ldots, \frac{\partial \xi_k(u)}{\partial u_n}\right\}.$$

Finally, inserting in equation (4) expression (3) for du/dy, we derive an equation that enables us to determine λ:

$$\lambda r^k(u) \text{ grad } \xi_k(u) = \lambda \sum_{j=1}^{n} \frac{\partial \xi_k(u)}{\partial u_j} r_j^k = 1. \tag{5}$$

Let us consider the following two possibilities:

a) The following inequality $r^k(u) \text{ grad } \xi_k(u) \neq 0$, (6)
is satisfied for the interval of variation of the variable y we are considering. Systems of equations for which inequality (6) is satisfied for all values $k = 1, 2, \ldots, n$ and any $u = u_1, \ldots, u_n$ are called purely nonlinear by Lax [38].

If inequality (6) is satisfied, then from (5) λ is uniquely determined, and when we insert it into (3) we get the following system of ordinary differential equations:

$$\frac{du}{dy} = \frac{r^k(u)}{r^k(u) \text{ grad } \xi_k(u)} \qquad (7)$$

from which by integration we can define the solution u(y) if we know the value $u_o = u(y_o)$ such that relation (2) is satisfied, i.e.,

$$y_0 = \xi_k(u_0) = \xi_k(u(y_0))$$

Let us denote the solution of system of ordinary differential equations (7) passing through the point $u_o = \{u_1^o, \ldots, u_n^o\}$ as follows:

$$u(y) = U^k(y, u_0), \quad U^k(\xi_k(u_0), u_0) \equiv u_0. \qquad (8)$$

b) The second possibility obtains when at the given point $y = y_o$

$$r^k(u(y)) \text{ grad } \xi_k(u(y)) = 0. \qquad (9)$$

In this case equation (5) is not satisfied when $\lambda \neq 0$, and the solution u(y) is either constant in some neighborhood of the point y_o or else discontinuous at this point.

The case of a discontinuity of a self-modeling solution will be discussed below.

In general, in the following we will limit ourselves, if otherwise not specifically stipulated, to the case when conditions (6) are satisfied for all $k = 1, 2, \ldots, n$ and for any u.

Confining ourselves to this case, we will now study in greater detail the solution u(y) of system (7). We will call solution (8), following the analogy with equations in gas dynamics, a "centered rarefaction wave" or a centered travelling wave.

Let us consider the differential equation in partial derivatives in the space of variables $u_1, u_2, \ldots, u_n$ for a single (scalar) function $v = v(u) = v(u_1, \ldots, u_n)$:

$$r^k(u)\,\mathrm{grad}\,v(u) = \sum_{a=1}^{n} r_a^k(u)\frac{\partial v}{\partial u_a} = 0. \qquad (10)$$

Suppose $v = v(u)$ is any solution of equation (10). We will show that

$$v(U^k(y, u_0)) = \mathrm{const.} \qquad (11)$$

Actually,

$$dv(U^k(y, u_0)) = \sum_{a=1}^{n} \frac{\partial v}{\partial u_a}\frac{\partial U_a^k}{\partial y}\,dy =$$

$$= \sum_{a=1}^{n} \frac{\partial v}{\partial u_a}\lambda r_a^k(U)\,dy = \lambda\,dy\,r^k(u)\,\mathrm{grad}\,v = 0.$$

Thus, each solution $v(u)$ of equation (10) is constant at the rarefaction wave (8). This can also be formulated in the following manner.

Integral curve (8) of system of equations (7) lies on the hypersurface $v(u)$ = constant if $v(u)$ satisfies equation (10).

Equation (10) has $n - 1$ independent solutions, which we will denote by

$$v = v_1^k(u), \quad v = v_2^k(u), \ \ldots, \quad v = v_{n-1}^k(u);$$

here the system of n vectors $r^k(u)$, grad $v_1^k(u)$, ..., grad $v_{n-1}^k(u)$ is linearly independent.

From equation (11) it follows that in the traveling wave $u = u(y)$ given by formula (8), the $(n - 1)$-dimensional vector $v^k(u) = \{v_1^k(u), \ldots, v_{n-1}^k(u)\}$ is constant, i.e., $v^k(U^k(y,u_o)) = c^k$, where c^k is a constant $(n - 1)$-dimensional vector.

Thus, integral curve (8) of equations (7) can be given in another fashion:

$$v^k(u) = \text{constant}, \quad \xi_k(u) = y,$$

and the problem of integrating system (7) reduces to determining $n - 1$ independent solutions of equation (10).

We will call the vector $v^k(u) = \{v_1^k(u), \ldots, v_{n-1}^k(u)\}$ the $(n - 1)$-dimensional vector-Riemann invariant. We can easily show that in cases when ordinary Riemann invariants $r_k = r_k(u)$ exist, the vector-variant $v^k(u)$ is represented in the form

$$v^k(u) = \{r_1(u), r_2(u), \ldots, r_{k-1}(u), r_{k+1}(u), \ldots, r_n(u)\}.$$

Actually, according to Section III of Chapter One $dr_i(u) = \mu_i(u)\ell^i(u)du$, i.e., $\text{grad}\, r_i(u) = \mu_i(u)\ell^i(u)$, and therefore $v_i^k(u) = r_i(u)$ when $i \neq k$ is solution of equation (10), since the right $r^k(u)$ and left $\ell^k(u)$ eigenvectors of matrix $A(u)$ are biorthogonal: $\ell^i(u)r^k(u) = 0$ when $i \neq k$. In particular, when $n = 2$ Riemann invariants always exist and therefore when $n = 2$ $v^k(u) = r_i(u)$, $i \neq k$.

Solution (8) of system (7) is a doubly continuously differentiable curve located in the space of variables $u_1, u_2, \ldots, u_n, y$. If the quantity y is taken as a parameter, then this will be a curve passing through the point u_0 in the space of variables u.

By condition (6) the quantity $\xi_k(u(y))$ varies monotonically along this curve. Since $y = x/t$ and $t > 0$, then from equation (2) we conclude that part of this curve given by the condition $\xi_k(u(y)) > \xi_k(u_0)$ corresponds to the rays $y = x/t$ lying to the left of the point $y = y_0$, and the part of the curve on which $\xi_k(u(y)) < \xi_k(u_0)$ corresponds to the rays $y = x/t$ lying in the plane of variables x, t to the left of the ray $x/t = y_0$.

Now assigning the number k inequality (2) all values from 1 to n, we arrive at the following conclusion:

Through the point u_0 in the space of n variables $u_1, u_2, \ldots, u_n$ pass n smooth curves which are the solutions of equation (3) when $k = 1, 2, \ldots, n$, which are not tangent to each other anywhere. The direction of the monotonic increase of the variable $y = x/t$ is indicated along each of these curves.

Now let us consider the case of discontinuity of the self-modeling solution. Self-modeling solution $u(y) = u(x/t)$ can have discontinuities only along the lines $x/t = y = \text{constant}$. Suppose that the solution $u(y)$ is discontinuous at the point y. Then the following Hugoniot's conditions (Section I) must be satisfied at this line $x = yt$:

$$y[u(y-0) - u(y+0)] = \varphi(u(y-0)) - \varphi(u(y+0)), \quad D = y.$$

We will assume that one of the values $u(y + 0)$ or $u(y - 0)$ is fixed and we will denote it by u_0, and this system of equations will serve for the determination of another quantity, which we will denote by u. Let us rewrite this system in the form $D(u - u_0) = \varphi(u) - \varphi(u_0)$, $D = y$. (12)

Let us assume that equations (12) define in the space of variables u in smooth curves passing through the point $u = u_o$.

Let us consider one of these curves and introduce some parameter ξ, assuming that the solution of equations (12) is parametrically expressed by ξ:

$$u = u(\xi), \qquad D = D(\xi) \qquad \text{and} \qquad u(\xi_o) = u_o.$$

Differentiating system (12) relative to parameter ξ, we find

$$D\dot{u} + \dot{D}[u(\xi) - u_o] = A(u(\xi))\dot{u},$$

where $\dot{u}$ and $\dot{D}$ denote derivatives of the corresponding functions relative to parameter ξ. Here setting $\xi = \xi_o$, we get $\quad [A(u_o) - DE]\dot{u}(\xi_o) = 0, \qquad (13)$
since $u(\xi_o) = u_o$.

The quantity $\dot{u}(\xi_o)$ is distinct from zero only when

$$\mathrm{Det}((A(u_o) - D(\xi_o)E)) = 0.$$

Therefore we assume $\qquad D(\xi_o) = \xi_k(\xi_o) = \xi_k(u_o) \qquad$ and we will denote this branch of the curve determined from system (12) by $u = \bar{U}^k(\xi)$.

Now let us select parameter ξ in a wholly specific manner: $\xi = \xi_k(u) \quad (14)$
Equations (13) lead to the consequence $\qquad \dot{\bar{U}}^k(\xi_o) = \lambda r^k(u_o), \qquad (15)$
and from (14) we get, by differentiation relative to ξ,

$$\dot{\bar{U}}^k(\xi_o)\,\mathrm{grad}\,\xi_k(\bar{u}_o) = 1. \qquad (16)$$

A comparison of formulas (15) and (16) with formulas (3) and (4) leads us to the conclusion that

$$\dot{\bar{U}}^k(\xi_o) = \frac{r^k(u_o)}{r^k(u_o)\,\mathrm{grad}\,\xi_k(u_o)}, \qquad (17)$$

i.e., the derivative $\dot{\bar{U}}^k(\xi_o)$ has the same value as in the rarefaction wave (7). We will now show that the second derivatives $\ddot{\bar{U}}^k(\xi_o)$ coincide with the second derivatives computed at the rarefaction wave. Twice differentiating formulas (12) relative to $\xi = \xi_k(u)$ and then setting $\xi_k(u) = \xi_k(u_o)$, we get

$$[A(u_o) - \xi_k(u_o)E]\ddot{\bar{U}}^k(\xi_o) = 2\dot{D}_k(\xi_o)\dot{\bar{U}}^k(\xi_o) - [\nabla A(u_o)\dot{\bar{U}}^k(\xi_o)]\dot{\bar{U}}^k(\xi_o) \qquad (18)$$

where the symbol $[\nabla A\dot{\bar{U}}^k]\dot{\bar{U}}^k$ denotes the quantity

$$[\nabla A\dot{\bar{U}}^k]\dot{\bar{U}}^k = \sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial^2\varphi_i}{\partial u_j\,\partial u_l}\dot{U}^k_j\dot{U}^k_l. \qquad (19)$$

For comparison, differentiate equations (1) relative to the variable y; we obtain equations similar to (18) if in them we set $y = y_0$:

$$[A(u_0) - \xi_k(u_0)E]\frac{d^2u}{dy^2} = \frac{du}{dy} - \left[\nabla A(u_0)\frac{du}{dy}\right]\frac{du}{dy}. \qquad (20)$$

In the left side of equations (18) and (20) the matrix of coefficients is degenerate, since $\xi_k(u_0)$ is the eigenvalue of matrix $A(u_0)$. Therefore for system (20) to be solvable relative to d^2u/dy^2, the right side must be orthogonal to the left eigenvector $\ell^k(u_0)$ of the matrix $A(u_0)$, since the left side of the equality is plainly orthogonal to this factor.

Since by formula (17) the function $u(y)$ has fully determinate second derivatives, this condition is clearly satisfied. Therefore by writing the orthogonality condition of the right side of (20) to the vector $\ell^k(u_0)$, we get

$$\frac{d\xi(y_0)}{dy} l^k(u_0) = \frac{r^k(u_0)\, l^k(y_0)}{r^k(u_0)\,\mathrm{grad}\,\xi_k(u_0)} = \frac{l^k(u_0)[\nabla A(u_0)\, r^k(u_0)]\, r^k(u_0)}{[r^k(u_0)\,\mathrm{grad}\,\xi_k(u_0)]^2}, \qquad (21)$$

The orthogonality condition of the right side of system (18) to the vector $\ell^k(u_0)$ can be written, using (17), in the following form:

$$\dot{D}_k(\xi_0) = \frac{1}{2}\,\frac{l^k(u_0)[\nabla A(u_0)\, r^k(u_0)]\, r^k(u_0)}{(r^k(u_0)\,\mathrm{grad}\,\xi_k(u_0))\,(r^k(u_0)\, l^k(u_0))}. \qquad (22)$$

Referring to formula (21), from (22) we get $\dot{D}_k(\xi_0) = \frac{1}{2}$. (23)

Now from a comparison of system of equations (18) with system (20), we conclude that

$$\dot{U}^k(\xi_0) = \frac{dU^k}{dy}(y_0, u_0). \qquad (24)$$

Let us denote by $u = \bar{U}^k(\xi_k(u), u_0)$ the solution of equations (12). Then from equalities (17) and (24) follows that

$$U^k(\xi_k(u), u_0) = \bar{U}^k(\xi_k(u), u_0) + O[(\xi_k(u) - \xi_k(u_0))^2] \qquad (25)$$

and

$$D_k(\xi_k(u), u_0) = \xi_k(u_0) + \frac{1}{2}[\xi_k(u) - \xi_k(u_0)] + O[(\xi_k(u) - \xi_k(u_0))^2] =$$
$$= \frac{\xi_k(u) + \xi_k(u_0)}{2} + O[(\xi_k(u) - \xi_k(u_0))^2]. \qquad (26)$$

Formulas (25) reflect the fact that the state $\vec{U}^k$ associated with the state u_0 by means of the shock wave of the k-th index is distinct from the state U^k associated with u_0 by the rarefaction wave by a quantity of the third order of smallness relative to the difference $[U^k - u_0]$. This fact was first noted by Lax [37] for general systems of quasilinear equations, although in gas dynamics this property of weak shock waves is well known.

Thus, two smooth curves $u = U^k(\xi_k(u), u_0)$ and $u = \vec{U}^k(\xi_k(u), u_0)$ pass through the point $u = u_0$. The first curve represents the family of states which can be associated with the state u_0 by means of a rarefaction wave of the k-th type. The second curve is formed of states which can be associated with the state u_0 by way of a shock wave of index k. These two curves at the point u_0 are in second-order tangency.

If we assume the state u_0 to be the left state, then only one half of the curve $u = U^k(\xi_k(u), u_0)$ given by the condition $\xi_k(u) > \xi_k(u_0)$ is meaningful for the values $y > y_0$, as we stated above.

Stability conditions (3.1.10) and (3.1.11) require that the following inequalities be satisfied for the shock wave $u = \vec{U}^k(\xi_k(u), u_0)$:

$$\xi_k(\bar{U}^k(\xi_k(u), u_0)) > D_k(\xi_k(u), u_0) > \xi_k(\bar{U}^k(\xi_k(u), u_0)),$$
$$\xi_{k-1}(\bar{U}^k(\xi_k(u), u_0)) < D_k(\xi_k(u), u_0) < \xi_{k+1}(\bar{U}^k(\xi_k(u), u_0)),$$

i.e., we conclude that if $u = u_0$ is the left state, then corresponding to the right states is only half of the curve $u = \vec{U}^k(\xi_k(u), u_0)$ given by the inequality

$$\xi_k(u) < \xi_k(u_0).$$

Therefore the curve

$$u = \begin{cases} U^k(\xi_k(u), u_0) & \text{when} \quad y = \xi_k(u) \geqslant \xi_k(u_0), \\ \bar{U}^k(\xi_k(u), u_0) & \text{when} \quad \xi_k(u) \leqslant \xi_k(u_0) \end{cases} \qquad (27)$$

has, by virtue of equations (25), two continuous derivatives, passes through the point u_0, and represents a family of states which can be associated with the state u_0, considered as the left state relative to u given by formula (27) by means of a rarefaction wave or a shock wave.

Conversely, if the state u_0 is taken as the right state, then the analogous curve is given by the formula

$$u = \begin{cases} U^k(\xi_k(u), u_0) & \text{where} \quad \xi_k(u) < \xi_k(u_0) \\ \bar{U}^k(\xi_k(u), u_0) & \text{where} \quad \xi_k(u) > \xi_k(u_0) \end{cases}$$

Finally, let us consider the case when for any value $k = 1, 2, \ldots, n$

$$r^k(u) \operatorname{grad} \xi_k(u) \equiv 0. \qquad (29)$$

We will call the characteristic $\xi = \xi_k(u)$ satisfying condition (29) a contact characteristic.

Obviously, along the line $v^k(u) = \{v_1^k(u), v_2^k(u), \ldots, v_{n-1}^k(u)\} = c^k$ (30) the quantity $\xi_k(u)$ is constant by virtue of equation (10). Therefore the quantity $\xi_k(u)$ cannot be selected, as we did above, as a parameter defining a point on this curve.

Let us introduce some other parameter on the curve (30), for example, the length of the arc of this curve measured from any point on it. Then curve (30) is an integral curve of the system of equations $du/ds = r^k(u)$, since we assume that $\| r^k(u) \| = \sum_{i=1}^{n} (r_i^k)^2 = 1$. Multiplying this equation on the left side by the matrix $A(u) = \frac{\partial \varphi}{\partial u}$, we get

$$A(u)\frac{du}{ds} = \frac{\partial\varphi}{\partial u}\frac{du}{ds} = \frac{d\varphi}{ds} = A(u)\,r^k(u) = \xi_k(u)\,r^k(u) = \xi_k(u)\frac{du}{ds}.$$

But along the curve (30) the quantity $\xi_k(u)$ is constant and therefore these equations are integrated. Let us integrate these equations from $u = u_0$ up to the arbitrary point $u = u(s)$, then

$$\varphi(u(s)) - \varphi(u_0) = \xi_k(u)\,[u(s) - u_0] = \xi_k(u_0)\,[u(s) - u_0]. \qquad (31)$$

Thus, we see that any two points on the curve (30) satisfy Hugoniot's conditions where $D = \xi_k(u_0) = \xi_k(u)$. Discontinuities of this kind are called contact discontinuities in gas dynamics.

Finally, let us note that if condition (29) is met, then a centered rarefaction wave of the k-th type does not exist. This rarefaction wave changes into shock wave (30).

3. Problem of the decay of an arbitrary discontinuity. The Cauchy's problem

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \qquad (1)$$

$$u(x, 0) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0, \end{cases} \qquad (2)$$

is called the problem of the decay of an arbitrary discontinuity. We can easily note that this problem is invariant relative to the similarity transformation

$$t = kt', \quad x = kx', \quad k = \text{const} > 0. \qquad (3)$$

Therefore we presume the uniqueness of the solution of Cauchy's problem (1), (2), from this follows the self-modeling status of the solution. Actually, suppose

$$u = u(x, t) \qquad (4)$$

is a solution of the problem (1), (2). Carrying out the similarity transformation (3), we see that solution $u(x', t')$, by virtue of the assumption of its uniqueness, coincides with solution (4), i.e.,

$$u\left(\frac{x}{k}, \frac{t}{k}\right) = u(x, t). \qquad (5)$$

Equality (5) is satisfied for any values of the parameter k 0. Therefore by setting $k = 1/t > 0$, we get $u(x, t) = u(1, x/t) = u(1, y) = u_0(y)$. (6)

We can easily see that if we do not assume the uniqueness of the solution of Cauchy's problem (1), (2), we cannot assert that all solutions of this problem are self-modeling, that they depend only on the variable $y = x/t$. Therefore proof of the uniqueness of the self-modeling solution of problem (1), (2) does not allow us to assert that this Cauchy's problem has a unique solution. None the less the problem of the uniqueness of a self-modeling solution of problem (1), (2) is of definite independent significance, first of all because in several cases we succeed in directly proving that any stable generalized solution of this problem is self-modeling, and secondly because in the course of proving the uniqueness of the self-modeling solution fundamental difficulties which are encountered in studying the general Cauchy's problem (1), (2) are unearthed.

Let us give a geometrical interpretation to the problem of the decay of an arbitrary discontinuity.

We will consider the self-modeling solution $u(y)$ of the problem (1), (2), assuming that it exists. Then in the domains of smoothness of the vector function $u(y)$ it satisfies system of equations (3.2.1):

$$A(u)\frac{du}{dy} = y\frac{du}{dy}, \qquad (7)$$

and at the discontinuity points — Hugoniot's conditions

$$y[u(y+0)-u(y-0)]=\varphi(u(y+0))-\varphi(u(y-0)). \quad (8)$$

We will assume the solution u(y) to be bounded. Then we can assert that the vector u(y) is inconstant only in a bounded interval of values of variable y. Actually, suppose $|u(y)| \leq M, |\xi_k(u)| \leq M$; then if $y = y_0$ is a point of continuous variation of u(y), equations (7) are satisfied at this point and, as we have seen in Subsection 2, $|y_0| = |\xi_k(u(y_0))| \leq M$. (9)
But if $y = y_0$ is a discontinuity point of the function u(y), then from the stability conditions it follows that

$$|D| = |y_0| \leqslant \max_{k=1,\dots,n} \max_{\substack{y=y_0+0\\ y=y_0-0}} |\xi_k(u(y))| \leqslant M. \quad (10)$$

Thus, if $|\xi_k(u(y))| \leq M (k = 1, \dots, n)$, then outside the interval $[-M, M]$ the function u(y) is obviously constant. Condition (2) becomes as follows for the function $u(y) \to u^-$ as $y \to -\infty$, $u(y) \to u^+$ as $y \to \infty$, (11)
and due to boundedness of u(y)

$$u(y) = \begin{cases} u^- & \text{as} \quad y < -M, \\ u^+ & \text{as} \quad y > M. \end{cases} \quad (12)$$

Suppose that at the point $y \geqslant -M$ the solution $u(y) = u^-$. Let us consider the possible variation of the function u(y). If u(y) is varied, forming a rarefaction wave of the k-th type, then $y = \xi_k(u(y))$. The least among the quantities ξ_k is ξ_1. Therefore let us consider the section of the variable y on which $y = \xi_1(u(y))$. (13)

Since $u(y) = u^-$ when $y < -M$, then a solution of the type (13) can hold, beginning with y_1^- values where $y_1^- = \xi_1(u^-)$. (14)

Suppose that equality (13) is satisfied on the interval $[y_1^-, y_1^+]$, i.e., in the interval $[y_1^-, y_1^+]$ the solution u(y) forms a rarefaction wave corresponding to the first eigenvalue ξ_1.

In Figure 4.73 the value of the variable y is laid out on the abscissa axis, and the values of the quantities $\xi_1(u(y)), \xi_2(u(y)), \dots, \xi_n(u(y))$ are laid out on the ordinate axis, and the bisector $\xi = y$ is drawn. On the section $y_1^- \leqslant y \leqslant y_1^+$ $\xi_1(u(y)) = y$, all the remaining eigenvalues $\xi_2(u), \dots, \xi_n(u)$ are larger than y. The states $u = u(y)$ on the segment $[y_1^-, y_1^+]$ are associated with state u^-, just as with the left state, by a rarefaction wave corresponding to the eigenvalue $\xi = \xi_1(u)$. The plot of the function $\xi_1 = \xi_1(u(y))$ when $y_1^- \leq y \leq y_1^+$ lies on these straight lines $\xi = y$.

Suppose that on the interval $[y_2^-, y_2^+]$ $\xi_2(u(y)) = y$, i.e., on this interval the solution $u(y)$ forms a rarefaction wave corresponding to $\xi = \xi_2(u)$. Then it is quite obvious and, in particular, from Figure 4.33 it is clear that

$$y_2^- > y_1^+, \quad \xi_3(u(y)) > y, \quad \xi_1(u(y)) < y. \tag{15}$$

The states $u(y)$ considered on the interval $[y_2^-, y_2^+]$ are associated with the state $u(y_1^+)$, just as in the case of the left state, with a rarefaction wave of the second type $(y = \xi_2(u))$.

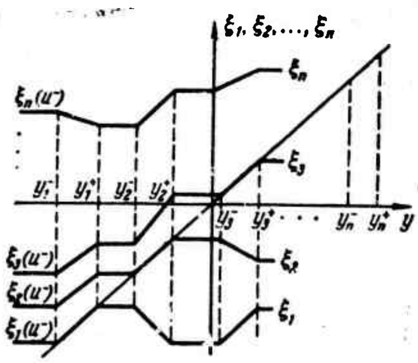

Figure 4.33

Now from Figure 4.33 it is altogether clear that each rarefaction wave corresponding to the k-th eigenvalue $(y = \xi_k(u(y))$ lies to the right of any rarefaction wave corresponding to the eigenvalue ξ_ℓ with a smaller ℓ number, and to the left of any rarefaction wave with a larger ℓ number.

Hence we conclude that the continuous solution $u(y)$ of problem (7), (12) contains not more than nine ordered rarefaction waves.

If we can select the quantities y_k^- and y_k^+ such that $u(y_1^-) = u^-$ and $u(y_n^+) = u^+$, the function $u(y)$ defined on the intervals $[y_k^-, y_k^+]$ in the form of a rarefaction wave, i.e.,

$$u(y) = U^k(y, U_+^{k-1}) \quad \text{where } y_k^- < y < y_k^+, \tag{16}$$

$$u(y) = U_+^k \quad \text{where } y_k^+ < y < y_{k+1}^-, \tag{17}$$

where $U_+^k = U^k(y_k^+, U_+^{k-1}), \quad U_+^0 = u^-.$

gives a self-modeling solution of the discontinuity decay problem.

Let us consider another possibility when the solution u(y) is discontinuous at the point $y = y_k$. Then by the stability conditions, there exists a k number such that

$$\xi_k(u(y_k - 0)) > y_k > \xi_k(u(y_k + 0)), \tag{18}$$

$$\xi_{k+1}(u(y_k + 0)) > y_k > \xi_{k-1}(u(y_k - 0)). \tag{19}$$

Suppose the index of the discontinuity line at the point y_1 is k = 1. Figure 4.34 again gives the function $\xi_k(u(y))$ when k = 1, 2, ..., n. In this case at the point y_1

$$\xi_1(y_1 - 0) = \xi_1(u^-) > y_1 > \xi_1(u(y_1 + 0)), \tag{20}$$

$$\xi_k(u(y_1^- + 0)) > y_1 \quad \text{when} \quad k > 1. \tag{21}$$

Hence it follows that once that if a shock wave with index k = 1 is present at the point $y = y_1$, then a rarefaction wave corresponding to the value $\xi = \xi_1(u) = y$ is absent in the solution $u = u(y)$.

In general, from a comparison of Figures 4.33 and 4.34 we conclude that a stable generalized self-modeling solution u(y) (satisfying stability conditions (18) and (19)) contains not more than n travelling waves (rarefaction or shock waves), which are ordered by their indexes, since the presence of a shock wave with index k precludes the possibility of a rarefaction wave with index k, and vice versa.

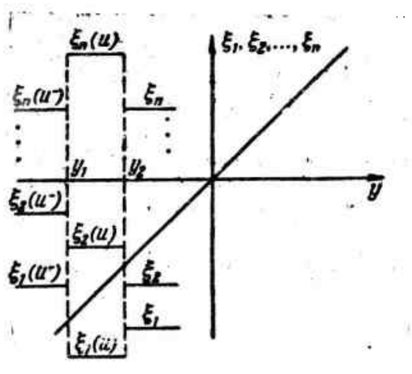

Figure 4.34

Thus, when solving the decay problem (1), (2) in the class of self-modeling solutions, it appears possible to construct a solution in the form of n travelling waves whose amplitudes must be chosen so as to satisfy conditions (11).

As we have seen in the preceding section, if we know the left state, then at a wave of the k-th type the family of states to which we can convert by means of this wave (shock wave or rarefaction wave of the k-th type) is described with a single parameter. Thus, the problem is, after selecting n such parameters, to satisfy conditions (11).

Based on the given u^- value, which is the left value for the solution $u = u(y)$, we define the solution in a wave (shock or rarefaction) with index 1 as the function of the single parameter $\xi_1 = \xi_1(u)$:

$$u^1 = F^1(\xi_1, u^-), \tag{22}$$

where F^1 is given, according to (3.2.27), by the formula

$$F^1(\xi_1, u^-) = \begin{cases} U^1(\xi_1, u^-) \text{ where } \xi_1 \geqslant \xi_1(u^-), \\ \bar{U}^1(\xi_1, u^-) \text{ where } \xi_1 \leqslant \xi_1(u^-); \end{cases} \tag{23}$$

here

$$\frac{\partial F^1}{\partial \xi_1}(\xi_1(u^-), u^-) = \frac{r^1(u^-)}{r^1(u^-)\,\mathrm{grad}\,\xi_1(u^-)}. \tag{24}$$

The state (22) is the left state for the wave with index 2. Therefore we introduce

$$u^2 = F^2(\xi_2, u^1) = F^2(\xi_2, F^1(\xi_1, u^-)), \tag{25}$$

where F^2 is determined by formula (3.2.27) analogously to (23). Continuing our reasoning, we find

$$\begin{aligned} u^n = F^n(\xi_n, u^{n-1}) = \\ = F^n(\xi_n, F^{n-1}(\xi_{n-1}, F^{n-2}(\ldots(F^1(\xi_1, u^-))\ldots))) = \\ = \Phi(\xi_1, \xi_2, \ldots, \xi_n, u^-). \end{aligned} \tag{26}$$

The solution of the decay problem in the class of self-modeling solutions reduces to determining the quantities $\xi_1, \xi_2, \ldots, \xi_n$ from a system of n equations

$$\Phi(\xi_1, \xi_2, \ldots; \xi_n, u^-) = u^+. \tag{27}$$

We will assume that the states $u^1, u^2, \ldots, u^n = u^+$ lie in a sufficiently small neighborhood of the point u^-. Since the Jacobian

$$\frac{\partial\Phi(\xi_1, \xi_2, \ldots, \xi_n, u^-)}{\partial(\xi_1, \xi_2, \ldots, \xi_n)}\Bigg|_{\xi_1=\xi_2=\ldots=\xi_n=\xi(u^-)} = \mathrm{Det}((r_i^k(u^-)))\prod_{i=1}^{n}\frac{1}{r^i(u^-)\,\mathrm{grad}\,\xi_i(u^-)} \qquad (28)$$

is distinct from zero owing to the assumption that system (1) is hyperbolic and that conditions (3.2.6) have been satisfied, there exists some neighborhood $|u - u^-| < \delta$ of the point u^- such that for $u^1, \ldots, u^n$ lying in this neighborhood, the Jacobian

$$\frac{\partial\Phi(\xi_1, \ldots, \xi_n, u^-)}{\partial(\xi_1, \ldots, \xi_n)}$$

is distinct from zero. Equations (27), moreover, are compatible when $u^+ = u^-$; $\xi_k = \xi_k(u^-)$. Therefore by the theorem of implicit functions there exists in this neighborhood the unique solution $\xi_1, \xi_2, \ldots, \xi_n$ of system of equations (27), to which corresponds the self-modeling solution $u = u(y)$.

Now let us note that the existence and uniqueness of the self-modeling solution $u(y)$ has just been proven not only on the assumption of the sufficient proximity of the points u^- and u^+, but also on the assumption of the proximity of the quantities $u^-, u^1, u^2, \ldots, u^n = u^+$.

Thus, here we have not received an answer to our question of the uniqueness of solution $u(y)$ as a function of the quantities u^- and u^+, though we can assume that for sufficiently large u^- and u^+ the solutions $u^1, u^2, \ldots, u^{n-1}$ will remain in a fairly small neighborhood of these points.

4. Example of the nonuniqueness of the self-modeling solution of the decay problem. We will now show that if we do not assume the sufficient proximity of vectors u^-, u^+, then without restrictions on this system of quasilinear equations we cannot rely on the uniqueness of the automodeling solution of this problem.

Let us first note that the concept of a discontinuous self-modeling solution of a decay problem was introduced by us only for conservative systems of the form (3.3.1); however the continuous solution $u(y)$, if it exists, is also determined for systems not written in the form of laws of conservation.

Therefore we now consider a system of three quasilinear equations that is hyperbolic in the small:

$$l^k(u)\left[\frac{\partial u}{\partial t}+\xi_k(u)\frac{\partial u}{\partial x}\right]=0, \quad u=\{u_1, u_2, u_3\} \quad (k=1, 2, 3), \qquad (1)$$

leaving for the moment the question of the possibility of formulating this system in the form of laws of conservation, and we will show that the decay problem for this system can have several continuous self-modeling solutions $u = u(y)$.

Let us note that this situation also holds for conservative systems of quasilinear equations [21]; but we are considering a nonconservative system, since the example is simpler in this case.

Let us set

$$l^1=\{\cos u_2, 0, \sin u_2\}, \quad l^2=\{0, 1, 0\}, \quad l^3=\{-\sin u_2, 0, \cos u_2\}, \qquad (2)$$

and suppose

$$\xi_1(u)<\xi_2(u)\equiv u_2<\xi_3(u). \qquad (3)$$

The vectors ℓ^1, ℓ^2, and ℓ^3 are mutually orthogonal, therefore we will assume that

$$r^k(u) = \ell^k(u). \qquad (4)$$

Conditions (3.2.6) take on the following form for system of equations (1):

$$\left.\begin{array}{ll} k=1: & \frac{\partial \xi_1}{\partial u_1}\cos u_2+\frac{\partial \xi_1}{\partial u_3}\sin u_2\neq 0, \\ k=2: & \frac{\partial \xi_2}{\partial u_2}=1>0, \\ k=3: & -\frac{\partial \xi_3}{\partial u_1}\sin u_2+\frac{\partial \xi_3}{\partial u_3}\cos u_2\neq 0. \end{array}\right\} \qquad (5)$$

We can easily note that we can always select the functions ξ_1 and ξ_3 such that conditions (5) and (3) are satisfied, for example:

$$\xi_1=u_2-\alpha(u_1\cos u_2+u_3\sin u_2), \quad \alpha>0, \quad \alpha'\neq 0, \qquad (6)$$

$$\xi_3=u_2+\beta(-u_1\sin u_2+u_3\cos u_2), \quad \beta>0, \quad \beta'\neq 0. \qquad (7)$$

Curves $u = U^k(y, u_0)$ describing states that can be associated by rarefaction waves are the following straight lines for system (1):

$$U^1(y, u_0) = u_0 + l^1(u_0)\,s, \quad \frac{dy}{ds} = l^1(u_0)\,\mathrm{grad}\,\xi_1(U^1), \tag{8}$$

$$U^2(y, u_0) = u_0 + l^2(u_0)(y - u_2^0), \tag{9}$$

$$U^3(y, u_0) = u_0 + l^3(u_0)\,s, \quad \frac{dy}{ds} = l^3(u_0)\,\mathrm{grad}\,\xi_3(U^3). \tag{10}$$

The straight line $u = U^1$ lies in the plane $u_2 = u_2^0$ and has the direction of the vector $\ell^1(u_0)$; similarly, the line $u = U^3$ also lies in the plane $u_2 = u_2^0$ and has the direction of the vector $\ell^3(u_0)$; the line $u = U^2$ is the line $u_1 = u_1^0$, $u_2 = u_2^0$ (Figure 4.35). The arrows at the lines $u = U^k$ denote the direction in which the quantity $\xi_k(u)$ increases.

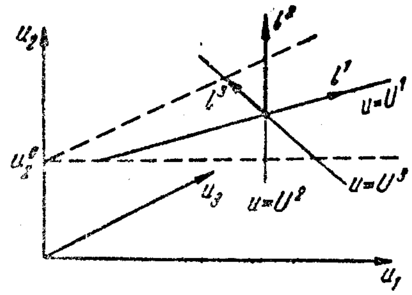

Figure 4.35

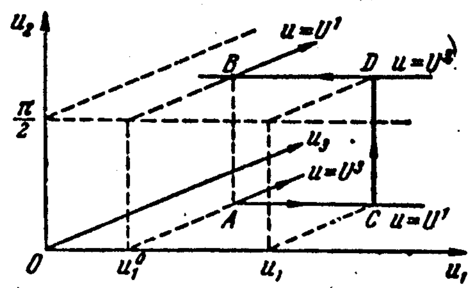

Figure 4.36

Let us consider any two planes $u_2 = c$, $u_2 = c + \pi/2$, for example, $u_2 = 0$ and $u_2 = \pi/2$ (Figure 4.36). Since $\ell^1(u_1, 0, u_3) = \ell^3(u_1, \pi/2, u_3)$, then the lines $u = U^1$ and $u = U^3$ lying in these two planes project onto each other. The arrows at these lines $u = U^k$ as before denote the direction of increase of the quantity $\xi_k(u)$. Now let us consider for system (1) the decay problem for the case when

$$u^- = \{u_1^0, 0, u_3^0\}, \quad u^+ = \left\{u_1^0, \frac{\pi}{2}, u_3^0\right\}. \tag{11}$$

Let us construct one of the solutions of this problem

$$u(y) = \begin{cases} u^- & \text{when} \quad y \leq 0, \\ U^2(y, u^-) & \text{when} \quad 0 \leq y \leq \pi/2, \\ u^+ & \text{when} \quad y \geq \pi/2, \end{cases} \tag{12}$$

which obviously is continuous relative to the variable y. Let us show that for this same problem there exists an infinite set of other self-modeling solutions $u(y)$. Suppose, for example, that the points C and D lie on the same straight line parallel to the u_2 axis: $C(u_1, 0, u_3^0)$, $D(u_1, \pi/2, u_3^0)$ (Figure 4.36). In view of our assumptions

$$\xi_1(u_1^0, 0, u_3^0) > \xi_1(u_1, 0, u_3^0), \quad \xi_3\left(u_1, \frac{\pi}{2}, u_3^0\right) > \xi_3\left(u_1^0, \frac{\pi}{2}, u_3^0\right). \qquad (13)$$

Therefore we can also write another self-modeling and continuous solution of this decay problem:

$$u(y) = \begin{cases} u^- & \ast\text{при} \quad y \leqslant \xi_1(u^-) < 0, \\ U^1(y, u^-) & \ast\text{при} \quad \xi_1(u_1^0, 0, u_3^0) \leqslant y \leqslant \xi_1(u_1, 0, u_3^0), \\ U^1(\xi_1(u_1, 0, u_3^0), u^-) & \ast\text{при} \quad \xi_1(u_1, 0, u_3^0) \leqslant y \leqslant 0, \\ U^2(y, (u_1, 0, u_3^0)) & \ast\text{при} \quad 0 \leqslant y \leqslant \frac{\pi}{2}, \\ U^2\left(\frac{\pi}{2}, (u_1, 0, u_3^0)\right) & \ast\text{при} \quad \frac{\pi}{2} \leqslant y \leqslant \xi_3\left(u_1, \frac{\pi}{2}, u_3^0\right), \\ U^3\left(y, \left(u_1, \frac{\pi}{2}, u_3^0\right)\right) & \ast\text{при} \quad \xi_3\left(u_1, \frac{\pi}{2}, u_3^0\right) \leqslant y \leqslant \xi_3\left(u_1^0, \frac{\pi}{2}, u_3^0\right), \\ u^+ = U^3\left(\xi_3\left(u_1^0, \frac{\pi}{2}, u_3^0\right), \left(u_1, \frac{\pi}{2}, u_3^0\right)\right) & \ast\text{при} \quad y \geqslant \xi_3\left(u_1^0, \frac{\pi}{2}, u_3^0\right), \end{cases} \qquad (14)$$

[* = when]

which obviously is distinct from solution (12). By arbitrarily varying the quantity $u_1 > u_1^0$, we get an infinite set of self-modeling solutions.

This example shows that the decay problem can have an infinite set of self-modeling solutions for a hyperbolic system of three quasilinear equations.

Finally, we further note that it is possible that the nonuniqueness of the solution of the decay problem of an arbitrary discontinuity for a system of the type (1) is a consequence of the more general property of hyperbolic systems of three and more quasilinear equations, which consists of the following.

For the system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \qquad (15)$$

of three or more quasilinear equations, the growth of solution u of Cauchy's problem with the initial data $u(x, 0) = u_o(x)$ is determined, in contrast to systems of two quasilinear equations, not only by the quantity $U_o = \max_x \|u_o(x)\|$, but also by the derivatives $\|\frac{\partial u_o}{\partial x}\|$. Therefore the solution of system (15) when $n \geq 3$ generally speaking becomes unbounded for some $t > 0$. This shows that, generally speaking, the problem of the decay of an arbitrary discontinuity is meaningless for such systems, since in this case the solution of the problem can not be considered as a limit solution with the initial data smoothed.

5. Decay problem for a system of two quasilinear equations. In the case $n = 2$ the system of quasilinear equations is reduced to Riemann invariants (Chapter One, Section III) and is written in the following form:

$$\frac{\partial r_k}{\partial t} + \xi_k(r)\frac{\partial r_k}{\partial x} = 0 \quad (k = 1, 2). \tag{1}$$

We will assume that condition (3.2.6) is satisfied, which can be written as

$$\partial \xi_k / \partial r_k \neq 0, \tag{2}$$

for system (1). The curve $u = U^k(y, u_o)$ in the plane of variables u_1, u_2 changes into the straight line r_j = constant ($j \neq k$) in the plane of variables r_1, r_2 (Figure 4.37). Thus the rarefaction wave for system (1) corresponds to a segment of the line r_j = constant. According to the foregoing, through each point $r = r_o$ there pass two curves $u = \bar{U}^k(y, u_o)$ representing families of states that can be associated with the state u_o by a shock transition. As we have seen above, these curves have at point u_o a second-order tangency with the lines $r_j = r_j^o$; therefore at least in some neighborhood of the point $r = r_o$ the equations of these curves can be written as

$$\left.\begin{aligned} r_2 &= R_2(r_1, r_1^0, r_2^0) = R_2(r_1, r_0), \\ r_1 &= R_1(r_2, r_1^0, r_2^0) = R_1(r_2, r_0), \end{aligned}\right\} \tag{3}$$

i.e., these curves are uniquely projected, respectively, onto the lines r_2 = constant and r_1 = constant. In Figure 4.37 the arrows indicate the direction of increase of the variable $y = x/t$ in the rarefaction waves r_j = constant and in the shock waves $r_j = R_j$; here we require that

$$\frac{\partial \xi_k(r)}{\partial r_k} > 0, \tag{4}$$

which can always be assumed to be satisfied in view of (2).

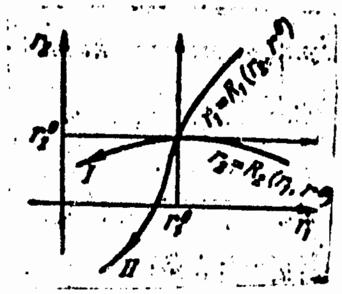

Figure 4.37

The question of the unique solvability of the decay problem in a class of self-modeling solutions depends essentially on the behavior of the curves $r_2 = R_2$ and $r_1 = R_1$ overall, that is, for sufficiently large values of $|r - r_0|$. However, it is difficult to study the behavior of these curves overall, since they are defined from substantially nonlinear equations. Therefore we will now indicate certain sufficient conditions under which the problem of the decay of an arbitrary discontinuity for a system of two quasilinear equations has a unique self-modeling solution.

With the assumptions made above, we will additionally assume that Hugoniot's conditions are solved in the form (3) for any r_1, r_2, r_1^0, r_2^0, i.e., the corresponding curves are uniquely projected onto the axes $r_2 =$ constant and $r_1 =$ constant for any r and r_0. Suppose, further, that

$$\left|\frac{\partial R_2(r_1, r_1^0, r_2^0)}{\partial r_1}\right| < 1, \quad \left|\frac{\partial R_1(r_2, r_1^0, r_2^0)}{\partial r_2}\right| < 1. \qquad (5)$$

We know the value $D = y$ appearing in the Hugoniot's conditions at each point of curve I $r_2 = R_2(r_1, r_0)$. Let us denote by D the expression

$$D = D_1(r_1, r_1^0, r_2^0) = D_1(r_1, r_0), \qquad (6)$$

along the curve I

and along curve II $\quad D = D_2(r_2, r_1^0, r_2^0) = D_2(r_2, r_0). \qquad (7)$

As we have shown in Subsection 2, at the point $r = r^0$

$$\left.\begin{aligned} D_1(r_1^0, r_0) &= \xi_1(r_0), \\ D_2(r_2^0, r_0) &= \xi_2(r_0) \end{aligned}\right\} \qquad (8)$$

and

$$\left.\frac{\partial D_1(r_1, r_2)}{\partial r_1}\right|_{r_1=r_1^0}=\frac{1}{2}\frac{\partial \xi_1(r_1)}{\partial r_1}>0, \quad \left.\frac{\partial D_2(r_1, r_2)}{\partial r_2}\right|_{r_2=r_2^0}=\frac{1}{2}\frac{\partial \xi_2(r_2)}{\partial r_2}>0, \quad (9)$$

$$\left.\frac{\partial R_2(r_1, r_2)}{\partial r_1}\right|_{r_1=r_1^0}=0, \quad \left.\frac{\partial R_1(r_1, r_2)}{\partial r_2}\right|_{r_2=r_2^0}=0. \quad (10)$$

We assume that for any r_1^0, r_2^0 and $r_1 < r_1^0$, $r_2 > r_2^0$, in addition to (5) following conditions are satisfied:

$$\left.\begin{array}{l} \xi_1(r_1, R_2(r_1, r_0))<D_1(r_1, r_0)<\xi_1(r_0) \\ \xi_2(r_1, R_2(r_1, r_0))>D_1(r_1, r_0), \quad r_1<r_1^0 \end{array}\right\} \quad (11)$$

$$\left.\begin{array}{l} \xi_2(R_1(r_2, r_0), r_2)>D_2(r_2, r_0)>\xi_2(r_0) \\ \xi_1(R_1(r_2, r_0), r_2)<D_2(r_2, r_0), \quad r_2>r_2^0 \end{array}\right\} \quad (12)$$

Conditions (11) and (12) denote, obviously, that the solutions $r_2 = R_2$, $D = D_1$ and $r_1 = R_1$, $D = D_2$ of Hugoniot's conditions satisfy the stability conditions.

If inequalities (11) and (12) are satisfied for any r_0 and $r_1 < r_1^0$, $r_2 > r_2^0$, then when $r_1 > r_1^0$ and $r_2 < r_2^0$ the signs in these equalities are reversed. Actually, suppose $r_1 > r_1^0$. The curve $r_2 = R_2$ passes through the point $(r_1, R_2(r_1, r_0))$, which according to Hugoniot's conditions, satisfies the equations

$$R_2(r_1^0, r_1, R_2(r_1, r_0))=r_2^0, \quad D_1(r_1^0, r_1, R_2(r_1, r_0))=D_1(r_1, r_0). \quad (13)$$

Equations (13) express the obvious fact that Hugoniot's conditions are unchanged if the left and the right values of the solution are interchanged. From formulas (13) we can now readily obtain the validity of our assertion.

Conditions (5), (11), and (12) are fairly complicated. Therefore we will verify that they have been satisfied with a simple example. Let us consider the system of two quasilinear equations describing the motion of an isothermal gas in Lagrangian variables (Chapter Two, Section II, Subsection 9):

$$\frac{\partial V}{\partial t}-\frac{\partial u}{\partial x}=0, \quad \frac{\partial u}{\partial t}+\frac{\partial p(V)}{\partial x}=0, \quad p'(V)<0, \quad p''(V)>0. \quad (14)$$

which can be written in Riemann invariants:

$$\frac{\partial r_1}{\partial t}-c(V)\frac{\partial r_1}{\partial x}=0, \quad \frac{\partial r_2}{\partial t}+c(V)\frac{\partial r_2}{\partial x}=0, \tag{15}$$

where

$$r_1=u-\int_{V_0}^{V} c(\eta)\,d\eta, \quad r_2=u+\int_{V_0}^{V} c(\eta)\,d\eta, \quad c^2(V)=-\frac{\partial p}{\partial V}. \tag{16}$$

Equations (16) enable us to express the quantity V in terms of the difference $r_2 - r_1$. Compute the derivatives $\frac{\partial \xi_k}{\partial r_k}$ to verify that condition (2) has been satisfied:

$$\frac{\partial \xi_1}{\partial r_1}=\frac{\partial \xi_2}{\partial r_2}=-\frac{c'(V)}{2c(V)}=\frac{p''(V)}{c^2(V)}>0. \tag{17}$$

Proceeding to verify conditions (5), (11), and (12) let us write the Hugoniot's conditions for system (14):

$$D(V-V_0)=u_0-u, \quad D(u_0-u)=p(V_0)-p(V), \tag{18}$$

whence

$$(u-u_0)^2=[p(V)-p(V_0)](V_0-V), \tag{19}$$

$$D^2=\frac{p(V)-p(V_0)}{V_0-V}. \tag{20}$$

From formula (20) we obtain the result that D can have positive and negative values. Suppose $D_1(r_1, r_0) < 0$, and $D_2(r_2, r_0) > 0$. Then since $p'(V) < 0$, $p''(V) > 0$, therefore if $V_0 > V$, then

$$-c(V)<D_1(r_1, r_0)<-c(V_0), \tag{21}$$

$$c(V_0)<D_2(r_2, r_0)<c(V). \tag{22}$$

To verify that inequalities (5) have been satisfied, we will differentiate Hugoniot's conditions (18), assuming that V, u, and D depend on r_1 and that $D = D_1(r_1, r_0) < 0$. Denoting the products by r_1 with a prime, we get from (18)

$$DV'+D'(V-V_0)=-u', \quad Du'+D'(u-u_0)=-c^2(V)V' \tag{23}$$

and

$$\frac{dr_1}{dr_1} = u' + c(V)V' = 1. \tag{24}$$

From equations (23) and (24) let us determine the values of V', u', and D'. Since

$$\frac{\partial R_2(r_1, r_0)}{\partial r_1} = u' - c(V)V' = \frac{[D_1(r_1, r_0) + c(V)]^2}{[D_1(r_1, r_0) - c(V)]^2} \tag{25}$$

and $D_1(r_1, r_0) < 0$, then

$$0 < \frac{\partial R_2(r_1, r_0)}{\partial r_1} < 1. \tag{26}$$

Similar computations lead to the results

$$\frac{\partial R_1(r_2, r_0)}{\partial r_2} = \frac{[D_2(r_2, r_0) - c(V)]^2}{[D_2(r_2, r_0) + c(V)]^2}. \tag{27}$$

and we have thus seen that requirements (5) have been satisfied for the system of two equations (14).

Since along the curve $r_2 = R_2(r_1, r_0)$

$$V' = \frac{dV}{dr_1} = \frac{-2D_1(r_1, r_0)}{[D_1(r_1, r_0) - c(V)]^2} > 0. \tag{28}$$

and along the curve $r_1 = R_1(r_2, r_0)$ $V' < 0$, inequalities (21) and (22) lead to the satisfaction of conditions (11), (12).

Returning to the general case, let us show that if a system of two quasi-linear equations satisfies our conditions, then the problem of the decay of a discontinuity for the system has not more than one self-modeling solution*).

Let us draw a segment of the curve $r_2 = R_2(r_1, r^-)$ through the point r^- for $r_1 \leq r_1^-$ and let us draw the ray $r_2 = r_2^-$ for $r_1 \geqslant r_1^-$ (Figure 4.38), and through the point r^+ let us draw a segment of the curve $r_1 = R_1(r_2, r^+)$ when $r_2 \geqslant r_2^+$ and the ray $r_1 = r_1^+$ when $r_2 \leq r_2^+$. Curve I represents a family of states

*) If in conditions (5) it is required that $\left|\frac{\partial R_i}{\partial r_j}\right| < q < 1$, then we can also assert that a self-modeling solution of the decay problem exists.

that can be associated with the state r^- if the latter is taken as the left; curve II represents states that can be associated with the state r^+, which is taken as the right. It follows from condition (5) that the intersection of curves I and II can occur only at a single point. This then denotes the uniqueness of a self-modeling solution of the decay problem.

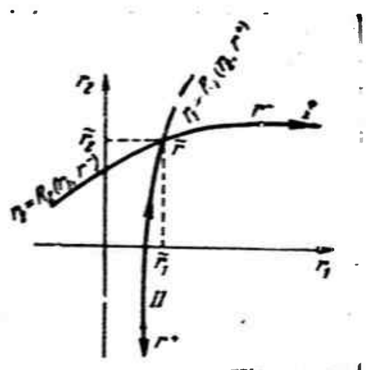

Figure 4.38

We can easily note that if $r_1^- \leqslant r_1^+$, $r_2^- \leqslant r_2^+$, the self-modeling solution consists of two rarefaction waves (Figure 4.39, a); if $r_1^- \leqslant r_1^+$, but $r_2^- > r_2^+$, then the solution consists either of a rarefaction wave and a shock wave (4.3.9, b), or two shock waves (4.3.9, c), and in the case $r_1^- > r_1^+$, $r_2^- > r_2^+$ also necessarily contain at least one shock wave. If we succeed in defining the functions R_2, D_1, R_1, and D_2 then in all cases, with the exception of the case when a self-modeling solution has two shock waves, the solution $r(y)$ is explicitly defined. For the case of two shock waves (Figure 4.3.8), construction of self-modeling solution $r(y)$ reduces to solving a system of two equations

$$R_2(\tilde{r}_1, r^-)=\tilde{r}_2, \quad R_1(\tilde{r}_2, r^+)=\tilde{r}_1, \qquad (29)$$

where $\tilde{r}_1$ and $\tilde{r}_2$ represent the state of the self-modeling solution between the shock waves. The system of equations (29) can be solved by the method of successive approximations:

$$\overset{(i+1)}{r_1} = R_1(\overset{(i)}{r_2}, r^+); \quad \overset{(i+1)}{r_2} = R_2(\overset{(i)}{r_1}, r^-); \tag{30}$$

here we can set $\overset{(0)}{r_1} = r_1^-$, $\overset{(0)}{r_2} = r_2^+$.

The condition for the convergence of successive approximations

$$\left|\frac{\partial R_2(r_1, r^-)}{\partial r_1} \cdot \frac{\partial R_1(r_2, r^+)}{\partial r_2}\right| < 1, \tag{31}$$

is obviously satisfied if inequalities (5) are satisfied. Now let us note that condition (31) is more general than (5) and is sufficient for the uniqueness of the self-modeling solution. Additionally, this condition is invariant relative to the change of dependent variables, in contrast to requirements (5). We adopted condition (5) for sake of simplicity, actually conditions (31) are sufficient.

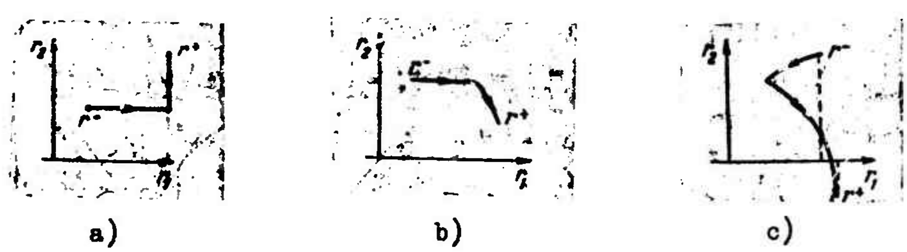

Figure 4.39

Suppose a system of two quasilinear equations satisfies requirements (2), (11), (12), and (31). Then the solution of the decay problem for such a system is self-modeling. Actually, suppose that the following Cauchy's problem is formulated for a system of quasilinear equations (1):

$$r(x, 0) = \begin{cases} r^-, & x < 0, \\ r^+, & x > 0. \end{cases} \tag{32}$$

Let us denote the bounded stable generalized solution of this problem by r(x, t) and prove that it is self-modeling, that is, that it depends only on the variable $y = x/t$.

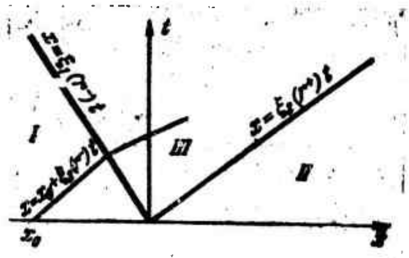

Figure 4.40

Suppose the solution $r = r(x, t)$ of Cauchy's problem (1), (32) is continuous when $t > 0$. This is possible only in the case when $\xi_1(r^-) < \xi_2(r^+)$. Let us draw two characteristics $x = \xi_1(r^-)t$ and $x = \xi_2(r^-)t$ through the origin of coordinates (0, 0) (Figure 4.40). Obviously, in zones I and II the solution is constant. We assume that in some neighborhood of the line $x = \xi_1(r^-)t$ in zone III solution $r(x, t)$ is variable. But in this neighborhood $r_2(x, t) = r_2^-$, since $\xi_2(r^-) > \xi_1(r^-)$. Therefore the function $r_1(x, t)$ satisfies in the neighborhood of the characteristic $x = \xi_1(r^-)t$ the following equation:

$$\frac{\partial r_1}{\partial t} + \xi_1(r_1, r_2^-)\frac{\partial r_1}{\partial x} = 0. \tag{33}$$

From equation (33) we conclude that near the line $x = \xi_1(r^-)t$ in zone III the function $r_1(x, t)$ is constant along the straight lines $x = \xi_1(r_1, r_2^-)t + \alpha$. Obviously, these straight lines can intersect only when $t \leqslant 0$ (Figure 4.41). Quite analogously we conclude that in zone III in the vicinity of the characteristic $x = \xi_2(r^+)t$, $r_1(x, t) = r_1^+$, but $r_2(x, t)$ is constant along the straight lines which can intersect each other only when $t \leqslant 0$. Thus, $r_1(x, t)$ is constant along the straight line $x = \xi_1(r_1, r_2^-)t + \alpha(r_1), \alpha \geqslant 0$, but $r_2(x, t)$ is constant along the lines $x = \xi_2(r_1^+, r_2)t + \beta(r_2), \beta \leqslant 0$. Let us show that $\alpha = \beta = 0$. If this is not so, then these two families of straight lines intersect each other when $t > 0$. At the point of intersection we will have

$$r_2(x, t) = r_2^-, \quad r_1(x, t) = r_1^+.$$

Hence we obtain the result that in the domain OABC (Figure 42) $r(x, t)$ = constant, which is possible only in the case when $r^- = r^+$. In this case the solution is in general constant, and, therefore, self-modeling. But if $r^- \neq r^+$, then $\alpha = \beta = 0$. If $\alpha = \beta = 0$, the continuous solution $r(x, t)$ is self-modeling.

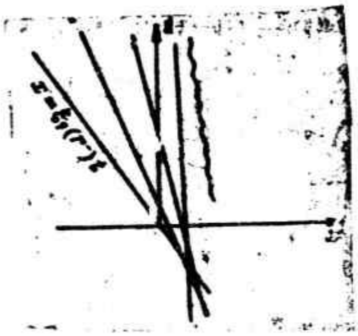

Figure 4.41

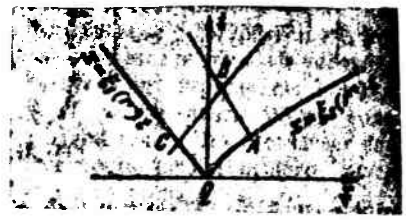

Figure 4.42

Now let us consider the case of discontinuous solutions $r(x, t)$ of the discontinuity decay problem. First let us note that it follows from the stability conditions for the generalized solution (3.1.10) and (3.1.11) that a stable generalized solution that is bounded and piecewise-continuous when $t > 0$ cannot have more than two discontinuity lines propagating from one point in the domain $t > 0$.

Therefore it is sufficient to inspect cases when the solution has one and two discontinuity lines departing from the point (0, 0) of discontinuity of the initial values. Let us consider the case of a single discontinuity line. Suppose the solution $r(x, t)$ has a single discontinuity line OA with index 2 (Figure 4.43). Obviously, to the right of OA $r(x, t) = r^+$, and to the left of the characteristic $x = \xi_1(r^-)t$ $r(x, t) = r^-$. Similarly, we conclude that in the zone BOA $r_2(x, t) = r_2^-$. Actually, from the stability conditions (3.1.10) and (3.1.11) it follows that the characteristics $\frac{dx_2}{dt} = \xi_2(r(x, t))$ intersect simultaneously the lines OB and OA and, therefore, $r_2(x - 0, t)|_{OA} = r_2^- = r_2(x, t)|_{OB}$. Therefore $r_2(x - 0, t) = r_2^-$, $r(x + 0, t) = r^+$. (34) are assigned at the line OA.

According to our assumptions made about the system of two quasilinear equations, the velocity D of the discontinuity line OA and the value $r_1(x - 0, t)$ are uniquely determined from these data; here these quantities will be constant:

$$D = D_2(r_2^-, r^+), \quad r_1(x-0, t) = R_1(r_2^-, r^+) \tag{35}$$

Thus, the discontinuity line OA is a straight line. Next, as we did above let us establish that in the zone BOA $r(x, t) = r(y)$, i.e., the solution of the decay problem with a single discontinuity line is self-modeling.

Finally, let us consider the case when the solution $r = r(x, t)$ has two discontinuity lines OA and OB (Figure 4.44) from which stability conditions (3.1.10) and (3.1.11) are satisfied. Omitting several details, we note that the integral curves of the equation $dx/dt = \xi_1(r(x, t))$ (36) in the zone BOA intersect simultaneously the lines OA and OB. In Figure 4.44 the integral curve of equation (36) is the curve CE, here $t_C > t_E$. Similarly, the integral curve CD of the equation $dx/dt = \xi_2(r(x, t))$ (37) intersect simultaneously the lines OB and OA, here $t_D > t_C$. Thus, we can write

$$r_2(D) = r_2(C), \quad r_2(C) = R_2(r_1(C), r^-) \tag{38}$$

and

$$r_1(C) = r_1(E), \quad r_1(E) = R_1(r_2(E), r^+), \tag{39}$$

where by $r(D)$ and $r(E)$ we denote values of the solution in zone III at the corresponding points. Inserting (39) into (38), we find

$$r_2(D) = R_2(R_1(r_2(E), r^+), r^-). \tag{40}$$

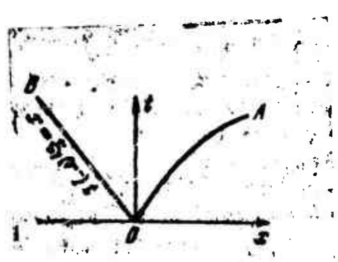

Figure 4.43

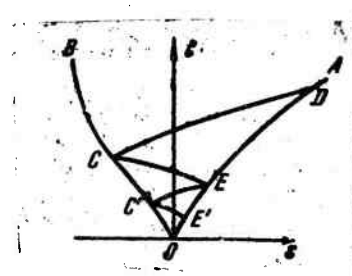

Figure 4.44

Similarly, we could obtain

$$r_2(D) = R_2(R_1(R_2(R_1(r_2(E'), r^+), r^-), r^+), r^-). \tag{41}$$

Extending this process and comparing it with the process of successive approximations (30), we conclude that $r_2(D) = \tilde{r}_2$. In view of the arbitrariness of point D, we conclude that the quantity $r_2(x - 0, t)$ is constant at the line OA and equal to $\tilde{r}_2$; similarly, $r_1(x + 0, t)$ at the line OB is equal to $\check{r}_1$. Thus, the solution $r(x, t)$ is constant in the zone BOA, and the lines OA and OB are straight lines. This then signifies the self-modeling status of the solution $r(x, t) = r(y)$ containing two discontinuity lines.

The proof that the solution of the decay problem is self-modeling together with the uniqueness of the self-modeling solution of the decay problem allow us to assert that the solution of Cauchy's problem on the decay of arbitrary discontinuity is unique and self-modeling.

6. The Goursat problem for a system of two quasilinear equations. Now for a system of two quasilinear equations

$$\frac{\partial r_k}{\partial t} + \xi_k(r, x, t)\frac{\partial r_k}{\partial x} = f_k(r, x, t) \qquad (k = 1, 2) \qquad (1)$$

we will consider several more general problems than the discontinuity decay problem. Suppose ξ_k, $f_k \in C_2$ and

$$\xi_1(r, x, t) < \xi_2(r, x, t), \qquad \frac{\partial \xi_k(r, x, t)}{\partial r_k} > 0 \qquad (k = 1, 2). \qquad (2)$$

Suppose the curve $O\mathscr{L}_2$, whose equation we will write in the form $x = x_2(t)$, is a characteristic of system (1). Differentiable functions $r_1^0(x, t)$ and $r_2^0(x, t)$ are known at the curve $O\mathscr{L}_2$. The curve $O\mathscr{L}_2$ has a continuous tangent and is the integral curve of the equation $dx/dt = \xi_2(r^0(x, t), x, t)$, and the function $r_2^0(x, t)$ satisfies at $O\mathscr{L}_2$ the compactibility condition

$$\frac{d}{dt} r_2^0(x_2(t), t) = f_2(r^0(x_2(t), t), x, t).$$

We will seek in some domain $\mathscr{L}_2 O \bar{\mathscr{L}}_2$ (Figure 4.45) the solution $r(x, t)$ of system (1) satisfying the conditions:

First. The solution $r(x, t)$ takes on the given values $r^0(x, t)$ at the line $O\mathscr{L}_2$:

$$r(x, t)|_{O\mathscr{L}_2} = r^0(x, t)|_{O\mathscr{L}_2} \qquad (3)$$

Second. The function $r_2(x, t)$ has a singularity at the point (0, 0) of the type of the self-modeling rarefaction wave. Analytically, this condition reaches thusly:

$$\lim_{t \to 0} r_2(t\xi_2(r_1^0(0, 0), \beta, 0, 0), t) = \beta, \qquad (4)$$

and parameter β takes on values from a certain interval

$$\beta_1 < \beta < \beta_2 = r_2^0(0, 0)$$

From condition (4) it follows that the function $r_2(x, t)$ must have at the point (0, 0) a singularity of the form

$$r_2(x, t) = g_2\left(\frac{x}{t}, t\right). \quad (5)$$

We will seek a solution of the problem (1), (3), (4) for such that the function $r_2(x, t)$ can be represented in the form (5); here $g_2(y, t)$ has continuous first derivatives in y and in t. We will seek the solution of the problem formulated by the method of successive approximations differing from that employing in Chapter One due to the singularity of the solution $r_2(x, t)$.

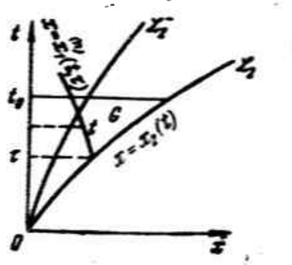

Figure 4.45

Here we must note that the boundedness of also successive approximations can be insured if we consider only a sufficiently small strip $0 \leq t \leq T$ of the variable t.

We assume that the approximation $\overset{(n-1)}{r}(x, t)$ satisfying conditions (3), (4) is known in the zone $\mathscr{L}_2^- O \mathscr{L}_2$.

Let us define $\overset{(n)}{r_1}(x, t)$ as the solution of the single quasilinear equation

$$\frac{\partial \overset{(n)}{r_1}}{\partial t} + \xi_1(\overset{(n)}{r_1}, \overset{(n-1)}{r_2}(x, t), x, t)\frac{\partial \overset{(n)}{r_1}}{\partial x} = f_1(\overset{(n)}{r_1}, \overset{(n-1)}{r_2}(x, t), x, t), \quad (6)$$

satisfying condition (3), i.e.,

$$\overset{(n)}{r_1}(x_2(t), t) = r_1^0(x_2(t), t). \quad (7)$$

This Cauchy's problem is a normal Cauchy's problem, since the line $O\mathcal{L}_2$ is a characteristic of the second family.

The function $\overset{(n)}{r_2}(x, t)$ is also defined as the solution of the single quasi-linear equation

$$\frac{\partial \overset{(n)}{r_2}}{\partial t} + \xi_2\left(\overset{(n-1)}{r_1}(x, t), \overset{(n)}{r_2}, x, t\right)\frac{\partial \overset{(n)}{r_2}}{\partial x} = f_2\left(\overset{(n-1)}{r_1}(x, t), \overset{(n)}{r_2}, x, t\right) \quad (8)$$

satisfying condition (4):

$$\lim_{t\to 0} \overset{(n)}{r_2}\left(\xi_2\left(\overset{(n)}{r_1}(0, 0), \beta, 0, 0\right) t, t\right) = \beta.$$

We will seek the solution of the two problems in the domain G formed by the intersection of the domain $\mathcal{L}_2^{-} O \mathcal{L}_2$ with the strip $0 \leq t \leq t_0$. We determine the value of t_0 below from the condition that all successive approximations $\overset{(n)}{r_1}$ and $\overset{(n)}{r_2}(y, t) = \overset{(\beta)}{r_2}(yt, t)$ have at G bounded first derivatives of $\overset{(n)}{r_1}$ in x and in t, and bounded first derivatives of $\overset{(n)}{g_2}(y, t)$ in y and in t.

Cauchy's problem (6), (7) is solved by the ordinary method of characteristics. As for the problem (8), (4) here we have a problem with a singularity at the point (0, 0). We denote by $x = \overset{(n)}{x_2}(t, \beta)$ the equation of a characteristic along which as $t \to 0$ the function $\overset{(n)}{r_2}(x, t)$ takes on the value of β. Obviously,

$$\overset{(n)}{x_2}(0, \beta) = 0 \text{ and } \lim_{t\to 0} \overset{(n)}{r_2}\left(\overset{(n)}{x_2}(t, \beta), t\right) = \beta.$$

For brevity, let us denote

$$\overset{(n)}{r_2}\left(\overset{(n)}{x_2}(t, \beta), t\right) \Rightarrow \overset{(n)}{r_2}(t, \beta).$$

The quantities $\overset{(n)}{r_2}(t, \beta)$ and $\overset{(n)}{x_2}(t, \beta)$ satisfy the characteristic system of equation (8):

$$\left.\begin{aligned} \frac{d\overset{(n)}{x_2}}{dt} &= \xi_2\left(\overset{(n-1)}{r_1}(\overset{(n)}{x_2}, t), \overset{(n)}{r_2}, \overset{(n)}{x_2}, t\right), \\ \frac{d\overset{(n)}{r_2}}{dt} &= f_2\left(\overset{(n-1)}{r_1}(\overset{(n)}{x_2}, t), \overset{(n)}{r_2}, \overset{(n)}{x_2}, t\right) \end{aligned}\right\} \quad (9)$$

and satisfy the initial conditions

$$\overset{(n)}{x_2}(0, \beta) = 0, \quad \overset{(n)}{r_2}(0, \beta) = \beta. \quad (10)$$

We will assume that t_0 is sufficiently small that all successive approximations $\overset{(n)}{r_1}$ and $\overset{(n)}{r_2}$ remain bounded in C:

$$|\overset{(s)}{r_i}(x, t)| < R, \quad i = 1, 2, \quad s = 1, 2, \ldots$$

Let us show that for sufficiently small t_0 derivatives of the functions $\overset{(n)}{r_1}$ and $\overset{(n)}{\xi_2}(y, t)$ remain bounded for all n = 1, 2, ...

Let us denote by E_1^0 the quantity that modulowise exceeds the first derivatives of $r_1^0(x, t)$:

$$\left|\frac{\partial r_1^0}{\partial x}\right|, \left|\frac{\partial r_1^0}{\partial t}\right| < E_1^0,$$

similarly

$$\left|\frac{\partial \overset{(s-1)}{r_1}(x, t)}{\partial x}\right| < \overset{(s-1)}{E_1}, \quad \left|\frac{\partial \overset{(s-1)}{g_2}(y, t)}{\partial y}\right| < \overset{(s-1)}{E_2},$$

and let us make the estimate $\left|\frac{\partial \overset{(n)}{r_1}(x, t)}{\partial x}\right|$. To do this, let us differentiate equation (6) relative to the variable x. We obtain the equation

$$\frac{\partial \overset{(s)}{p_1}}{\partial t} + \xi_2 \frac{\partial \overset{(s)}{p_1}}{\partial x} = -\frac{\partial \xi_1}{\partial r_1}(\overset{(s)}{p_1})^2 + \overset{(s)}{p_1}\left[\frac{\partial f_1}{\partial r_1} - \frac{\partial \xi_1}{\partial x} - \frac{1}{t}\frac{\partial \xi_1}{\partial r_2}\frac{\partial \overset{(s-1)}{g_2}}{\partial y}\right] +$$
$$+\left[\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial r_2}\frac{\partial \overset{(s-1)}{g_2}}{\partial y}\frac{1}{t}\right], \quad \overset{(s)}{p_1} = \frac{\partial \overset{(s)}{r_1}(x, t)}{\partial x}. \quad (11)$$

Let us compute the initial value of the derivative $\overset{(n)}{p_1}$ at the line $O\mathcal{L}_2$. To do this, let us determine $\overset{(n)}{p_1}$ from two conditions: Equation (6)

$$\overset{(s)}{q_1} + \xi_1 \overset{(s)}{p_1} = f_1,$$

and the differential consequence (7)

$$\overset{(s)}{p_1}\frac{dx_2}{dt} + \overset{(s)}{q_1} = \frac{\partial r_1^0}{\partial x}\frac{dx_2}{dt} + \frac{\partial r_1^0}{\partial t}.$$

Since $dx_2/dt = \xi_2$, from these two conditions we find

$$\overset{(s)}{p_1}\Big|_{Ox_1} = \frac{\frac{\partial r_1^0}{\partial x}\xi_2 + \frac{\partial r_1^0}{\partial t} - f_1}{\xi_2 - \xi_1}\Bigg|_{Ox_1}.$$

Since $\xi_2 - \xi_1 > \varepsilon > 0$, it follows that there exists the number C such that

$$|\overset{(s)}{p_1}|\Big|_{Ox_1} \leqslant C$$

The increase in the quantity $\overset{(n)}{p_1}$ is estimated by using the solution of the ordinary differential equation

$$\frac{dp_1}{dt} = Mp_1^2 + M\left[2 + \frac{\overset{(n-1)}{E}_2}{t}\right]p_1 + M\left[1 + \frac{\overset{(n-1)}{E}_2}{t}\right] <$$

$$< M(p_1 + 1)^2 + \frac{M\overset{(n-1)}{E}_2}{t}(p_1 + 1), \quad (12)$$

which takes on the value C when $t = \tau$.

By τ we denote the value of the variable t at the characteristic of equation (11) where it intersects with the line $O\mathcal{L}_2$ (Figure 4.45); the value of M is selected so that where $|r_i| \leq 2$ and $(x, t) \in G$ the following inequalities are satisfied:

$$|g_i| < M, \quad |f_i| < M, \quad \left|\frac{\partial g_i}{\partial r_j}\right| < M, \quad \left|\frac{\partial g_i}{\partial x}\right| < M, \quad \left|\frac{\partial f_i}{\partial r_j}\right| < M,$$

$$\left|\frac{\partial f_i}{\partial x}\right| < M \quad (i, j = 1, 2).$$

The solution of the equation

$$\frac{dp_1}{dt} = M(p_1 + 1)^2 + \frac{M\overset{(n-1)}{E}_2}{t}(p_1 + 1) \quad (13)$$

exceeds the solution of equation (12). Rewriting (13) in the form

$$\frac{d}{dt}(p_1 + 1)\left(\frac{\tau}{t}\right)^{M\overset{(n-1)}{E}_2} = M(p_1 + 1)^2\left(\frac{\tau}{t}\right)^{M\overset{(n-1)}{E}_2}.$$

Let us integrate it with reference to the condition $p_1(\tau) = C$:

$$p_1 + 1 = \frac{(C+1)\left(\frac{t}{\tau}\right)^{M\overset{(n-1)}{E}_2}}{1 - \frac{(C+1)\tau M}{M\overset{(n-1)}{E}_2 + 1}\left[\left(\frac{t}{\tau}\right)^{M\overset{(n-1)}{E}_2 + 1} - 1\right]}. \quad (14)$$

And thus, the solution $\overset{(n)}{p}_1(x, t)$ of equation (11) is estimated from above:

$$|\overset{(n)}{p}_2(x, t)| < \frac{(C+1)\left(\frac{t}{\tau}\right)^{M\overset{(n-1)}{E}_2}}{1-\frac{(C+1)\tau M}{M\overset{(n-1)}{E}_2+1}\left[\left(\frac{t}{\tau}\right)^{M\overset{(n-1)}{E}_2+1}-1\right]}. \qquad (15)$$

The ratio t/τ appearing in (15) is the ratio of the variable t at the characteristic of equation (11) to the value of this same variable at the point of intersection of the characteristic with the line $\mathcal{L}_2$ (Figure 4.45). If in the domain G the ratio t/τ is bounded, then for sufficiently small t_0 the boundedness of the derivatives of $\overset{(n)}{p}_1$ follows from estimate (15).

It is easier to estimate that value of the derivative $\partial \overset{(n)}{g}_2/\partial y$ by starting from the characteristic system (9). Differentiating equations (9) and initial conditions (10) relative to parameter β, we get

$$\left.\begin{aligned} \frac{d}{dt}\overset{(n)}{x'_2} &= \frac{\partial \xi_2}{\partial r_2}\overset{(n)}{r'_2} + \left[\frac{\partial \xi_2}{\partial r_1}\frac{\partial \overset{(n-1)}{r_1}}{\partial x} + \frac{\partial \xi_2}{\partial x}\right]\overset{(n)}{x'_2},\\ \frac{d}{dt}\overset{(n)}{r'_2} &= \frac{\partial f_2}{\partial r_2}\overset{(n)}{r'_2} + \left[\frac{\partial f_2}{\partial r_1}\frac{\partial \overset{(n-1)}{r_1}}{\partial x} + \frac{\partial f_2}{\partial x}\right]\overset{(n)}{x'_2},\\ \overset{(n)}{x'_2}(0, \beta) &= 0, \qquad \overset{(n)}{r'_2}(0, \beta) = 1, \end{aligned}\right\} \qquad (16)$$

where we introduce the notation

$$\overset{(n)}{x'_2} = \frac{\partial}{\partial \beta}\overset{(n)}{x_2}(t, \beta), \qquad \overset{(n)}{r'_2} = \frac{\partial}{\partial \beta}\overset{(n)}{r_2}(t, \beta).$$

From equation (16) it follows that

$$\left.\begin{aligned} \overset{(n)}{x'_2}(t, \beta) &= \int_0^t \frac{\partial \xi_2}{\partial r_2}\overset{(n)}{r'_2}(\tau, \beta)\exp\left\{\int_\tau^t\left[\frac{\partial \xi_2}{\partial r_1}\frac{\partial \overset{(n-1)}{r_1}}{\partial x} + \frac{\partial \xi_2}{\partial x}\right]d\xi\right\}d\tau,\\ \overset{(n)}{r'_2}(t, \beta) &= \exp\left\{\int_0^t \frac{\partial f_2}{\partial r_2}d\tau\right\} + \\ &\quad + \int_0^t\left[\frac{\partial f_2}{\partial r_1}\frac{\partial \overset{(n-1)}{r_1}}{\partial x} + \frac{\partial f_2}{\partial x}\right]\overset{(n)}{x'_2}(\tau, \beta)\exp\left\{\int_\tau^t \frac{\partial f_2}{\partial r_2}d\xi\right\}d\tau. \end{aligned}\right\} \qquad 17)$$

Ordinary estimates of formulas (17) lead to the inequalities

$$\delta \min_{0<\tau<t} \overset{(s)}{r'_2}(\tau,\beta)\, e^{-[\overset{(s-1)}{E_1}+2]Mt} < \overset{(s)}{x'_2}(t,\beta) < Mt \max_{0<\tau<t} \overset{(s)}{r'_2}(\tau,\beta)\, e^{[\overset{(s-1)}{E_1}+2]Mt}. \tag{18}$$

Here δ denotes the quantity

$$\delta = \min_{|r_1|<R,\ (x,\ t)\in G} \frac{\partial g_2}{\partial r_2}(r,\ x,\ t),$$

which, according to (2), is larger than zero.

Inserting estimates (18) into second formula (17), we get

$$e^{-Mt} - [\overset{(s-1)}{E_1}+1]\frac{M^2t^2}{2} \max_{0<\tau<t} \overset{(s)}{r'_2}(\tau,\beta) \exp[(\overset{(s-1)}{E_1}+2)Mt] < \overset{(s)}{r'_2}(t,\beta) <$$

$$< e^{Mt} + [\overset{(s-1)}{E_1}+1]\frac{M^2t^2}{2} \max_{0<\tau<t} \overset{(s)}{r'_2}(\tau,\beta) \exp[(\overset{(s-1)}{E_1}+2)Mt]. \tag{19}$$

Suppose t_0 is sufficiently small that

$$\frac{M^2t_0^2}{2}[\overset{(s-1)}{E_1}+1]\exp[(\overset{(s-1)}{E_1}+2)Mt_0] < \frac{1}{2}. \tag{20}$$

Then from (19) follow simpler estimates:

$$\frac{2}{3}e^{-Mt} < \overset{(s)}{r'_2}(t,\beta) < 2e^{Mt}. \tag{21}$$

Inserting estimates (21) into (18), we get

$$\frac{2}{3}t\delta e^{-[\overset{(s-1)}{E_1}+2]Mt} \leqslant \overset{(s)}{x'_2}(t,\beta) \leqslant 2Mte^{[\overset{(s-1)}{E_1}+2]Mt}. \tag{22}$$

Since

$$\frac{\partial \overset{(s)}{g_2}}{\partial y} = t\frac{\partial \overset{(s)}{r_2}}{\partial x} \quad \text{and} \quad \frac{\partial \overset{(s)}{r_2}}{\partial x} = \frac{\overset{(s)}{r'_2}(t,\beta)}{\overset{(s)}{x'_2}(t,\beta)},$$

then from (21) and (22) we have

$$\left|\frac{\partial \overset{(s)}{g_2}(y,t)}{\partial y}\right| < \frac{2e^{Mt}t}{{}^2/_3 t\delta e^{-[\overset{(s-1)}{E_1}+2]Mt}} = \frac{3}{\delta}e^{[\overset{(s-1)}{E_1}+3]Mt}. \tag{23}$$

The derivatives $\partial \overset{(n)}{p}_1/\partial t$ and $\partial \overset{(n)}{g}_2/\partial t$ can be similarly estimated. Assuming that in the domain G the ratio t/τ is bounded: $t/\tau < k$, from (15) we get

$$|\overset{(n)}{p}_1(x, t)| < \frac{(C+1)K^{M\overset{(n-1)}{E}_1}}{1-\frac{(C+1)\tau M}{M\overset{(n-1)}{E}_1+1}[K^{M\overset{(n-1)}{E}_1}-1]}. \tag{24}$$

Suppose that in domain G the inequality

$$[K^{M\overset{(n-1)}{E}_1+1}-1]\frac{(C+1)t_0 M}{M\overset{(n-1)}{E}_1+1} < \frac{1}{2}, \tag{25}$$

is satisfied, then from (24) we have

$$|\overset{(n)}{p}_1(x, t)| < 2(C+1)K^{M\overset{(n-1)}{E}_1}. \tag{26}$$

If, further, $[\overset{(n-1)}{E}_1 + 3]Mt_0 < \alpha$, (27)

then from (23) we get

$$\left|\frac{\partial \overset{(n)}{g}_2}{\partial y}\right| < \frac{3}{\delta}e^{\alpha} = E_2 \tag{28}$$

and, according to (26), $|\overset{(n)}{p}_1(x, t)| < 2(C+1)K^{(\frac{3Me^{\alpha}}{\delta})} = E_1.$ (29)

Thus, if in the domain G the following inequalities are simulataneously satisfied:

$$\frac{t}{\tau} < K, \tag{30}$$

$$M^2\frac{t_0^2}{2}[E_1+1]e^{(E_1+2)Mt_0} < \frac{1}{2}, \tag{31}$$

$$[K^{ME_1+1}-1]\frac{(C+1)t_0M}{ME_1+1} < \frac{1}{2}, \tag{32}$$

$$(E_1+3)Mt_0 < \alpha, \tag{33}$$

then the inequalities (20), (25), (27), and, therefore,

$$\overset{(n-1)}{E}_1 < E_1, \quad \overset{(n-1)}{E}_2 < E_2.$$

are simultaneously satisfied. So assuming that conditions (30) - (33) are satisfied in G, we have

$$\left|\overset{(n)}{F}_1(x, t)\right| = \left|\frac{\partial \overset{(n)}{r}_1(x, t)}{\partial x}\right| < E_1, \quad \left|\frac{\partial \overset{(n)}{g}_2(y, t)}{\partial y}\right| < E_2 \qquad (34)$$

$$(n = 1, 2, 3 \ldots),$$

and all successive approximations of $\overset{(n)}{r}_1$ and $\overset{(n)}{g}_2(y, t)$ have bounded first derivatives in G.

Let us consider the problem of whether inequalities (30) - (33) can be satisfied in domain G. Let us assign arbitrary end values $\alpha > 0$ and $K > 1$. Next we calculate the quantities E_1 and E_2 by formulas (29) and (28). Inequalities (31) - (33) can obviously be satisfied if we choose parameter t_0 sufficiently small. As for inequality (30), it bounds from beneath the domain of β values. Thus, in view of the arbitrary choice of $K > 1$ we can assume inequality (30) to be satisfied in domain G. Thus there exists a domain G of the type indicated in Figure 4.45 in which all successive approximations are $\overset{(n)}{r}_1(x, t)$ and $\overset{(n)}{g}_2(y, t)$ have bounded first derivatives.

Note that in the case of Lipschitz-continuous initial data of Goursat's problem, the uniform Lipschitz-continuity of successive approximations is similarly proven.

Let us prove the convergence of successive approximations. From equations (6) we have the consequence

$$\left|\frac{\partial}{\partial t}(\overset{(n)}{r}_1 - \overset{(n-1)}{r}_1) + \xi_1(\overset{(n)}{r}_1, \overset{(n-1)}{r}_2, x, t)\frac{\partial}{\partial x}(\overset{(n)}{r}_1 - \overset{(n-1)}{r}_1)\right| <$$
$$< \left[\max\left|\frac{\partial f_1}{\partial r_1}\right| + \max\left|\frac{\partial \xi_1}{\partial r_1}\right| E_1\right] |\overset{(n)}{r}_1 - \overset{(n-1)}{r}_1| +$$
$$+ \left[\max\left|\frac{\partial f_1}{\partial r_2}\right| + E_1\left|\frac{\partial \xi_1}{\partial r_2}\right|\right] |\overset{(n-1)}{r}_2 - \overset{(n-2)}{r}_2| \leqslant$$
$$\leqslant M(E_1 + 1)\left(|\overset{(n)}{r}_1 - \overset{(n-1)}{r}_1| + \max_G |\overset{(n-1)}{r}_2 - \overset{(n-2)}{r}_2|\right). \qquad (35)$$

Referring to condition (7), it follows that there exists a $B > 0$ such that

$$\left|\overset{(n)}{r}_1(x,t)-\overset{(n-1)}{r}_1(x,t)\right|<B(t-\tau)\max_G\left|\overset{(n-1)}{r}_2-\overset{(n-2)}{r}_2\right|<$$
$$<Bt\max_G\left|\overset{(n-1)}{r}_2-\overset{(n-2)}{r}_2\right|. \quad (36)$$

We can more readily estimate the quantity $\left|\overset{(n)}{r}_2-\overset{(n-1)}{r}_2\right|$ by starting from the characteristic system (9). From equations (9) and initial conditions (10) it follows that

$$\left.\begin{aligned}\left|\overset{(n)}{x}_2(t,\beta)-\overset{(n-1)}{x}_2(t,\beta)\right|&\leqslant Bt\max_G\left|\overset{(n-1)}{r}_1-\overset{(n-2)}{r}_1\right|,\\ \left|\overset{(n)}{r}_2(t,\beta)-\overset{(n-1)}{r}_2(t,\beta)\right|&\leqslant Bt\max_G\left|\overset{(n-1)}{r}_1-\overset{(n-2)}{r}_1\right|.\end{aligned}\right\} \quad (37)$$

According to our notation

$$\left|\overset{(n)}{r}_2(x,t)-\overset{(n-1)}{r}_2(x,t)\right|=\left|\overset{(n)}{r}_2(\overset{(n)}{x}_2(t,\beta),t)-\overset{(n-1)}{r}_2(\overset{(n)}{x}_2(t,\beta),t)\right|\leqslant$$
$$\leqslant\left|\overset{(n)}{r}_2(\overset{(n)}{x}_2(t,\beta),t)-\overset{(n-1)}{r}_2(\overset{(n-1)}{x}_2(t,\beta),t)\right|+$$
$$+\left|\overset{(n-1)}{r}_2(\overset{(n-1)}{x}_2(t,\beta),t)-\overset{(n-1)}{r}_2(\overset{(n)}{x}_2(t,\beta),t)\right|<\left|\overset{(n)}{r}_2(t,\beta)-\overset{(n-1)}{r}_2(t,\beta)\right|+$$
$$+\frac{E_2}{t}\left|\overset{(n)}{x}_2(t,\beta)-\overset{(n-1)}{x}_2(t,\beta)\right|.$$

Here inserting estimates (37), we get

$$\left|\overset{(n)}{r}_2(x,t)-\overset{(n-1)}{r}_2(x,t)\right|<B\max_G\left|\overset{(n-1)}{r}_1-\overset{(n-2)}{r}_1\right|\{t+E_2\}. \quad (38)$$

Estimates (36) and (38) prove the uniform convergence of successive approximations in domain G for a sufficiently small t_0.

Thus, we have proven the existence of a solution of the formulated Goursat's problem for a system of two quasilinear equations. It is interesting to note that in the case $n \geqslant 3$ not only is the question of the assistance of the solution unclear, but even the very formulation of Goursat's problem.

7. Construction of discontinuous solutions of a system of two quasilinear equations. Now we take up several cases of construction of discontinuous solutions of a system of two quasilinear equations, which we will write in invariants·

$$\frac{\partial r_k}{\partial t} + \xi_k(r, x, t)\frac{\partial r_k}{\partial x} = f_k(r, x, t) \quad (k = 1, 2), \tag{1}$$

and also in the form of the laws of conservation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u, x, t)}{\partial x} = g_i(u, x, t) \quad (i = 1, 2). \tag{2}$$

We will assume that system of equation (2) satisfies requirements (3.5.4) - (3.5.12), with the only difference that now the functions ξ_k, R_1, R_2, D_1, and D_2 appearing in these conditions, except for the arguments indicated in subsection 5, also depend on x and t.

We will seek the generalized solution $r = r(x, t)$ of system of equations (1) taking on the initial values $r(x, 0) = r_0(x)$. (3)
We will assume that $r_0(x)$ has a first-order discontinuity at point $x = 0$; with the exception of this point, the function $r_0(x)$ at the segment $|x| \leq a$ is assumed to be continuously differentiable.

For the case when $r_1^0(-0) \leq r_1^0(+0)$, $r_2^0(-0) \leq r_2^0(+0)$, the solution $r(x, t)$ will contain only centered rarefaction waves and will not have discontinuity lines (shock waves). The solution can be constructed in this case by using the solution of two Goursat's problems considered in the previous subsection. Therefore here we will deal with the case of discontinuous solutions and require of the initial function $r_0(x)$ that $r_1^0(-0) > r_1^0(+0)$. (4)
Just as for the case of the problem of the decay of arbitrary discontinuity considered in subsection 5, problem (1), (3) given condition (4) is decomposed into three mutually exclusive cases:

a) $$r_1^0(+0) < r_1^0(-0),\ R_2(r_1^0(+0),\ r_1^0(-0),\ r_2^0(-0),\ 0,\ 0) < r_2^0(+0); \tag{5}$$

b) $$r_2^0(+0) < r_2^0(-0),\ R_1(r_2^0(-0),\ r_1^0(+0),\ r_2^0(+0),\ 0,\ 0) > r_1^0(-0); \tag{6}$$

c) $$\left.\begin{aligned} R_2(r_1^0(+0),\ r_1^0(-0),\ r_2^0(-0),\ 0,\ 0) > r_2^0(+0), \\ R_1(r_2^0(-0),\ r_1^0(+0),\ r_2^0(+0),\ 0,\ 0) < r_1^0(-0). \end{aligned}\right\} \tag{7}$$

When (5) is satisfied, solution $r(x, t)$ of problem (1), (3) has the discontinuity line $O\mathcal{L}_{D_1}$ with index 1, issuing from the point (0, 0) and the rarefaction wave $\mathcal{L}_2^- O \mathcal{L}_2^+$ (Figure 4.46, a); when (6) is satisfied, rarefaction wave $\mathcal{L}_1^- O\mathcal{L}_1^+$ and discontinuity line $O\mathcal{L}_{D_2}$ with index 2 issue from point (0, 0) (Figure 4.46, b). Finally, when inequalities (7) are satisfied, two discontinuity lines $O\mathcal{L}_{D_1}$ with index 1 and $O\mathcal{L}_{D_2}$ with index 2 issue from point (0, 0) (Figure 4.46, c).

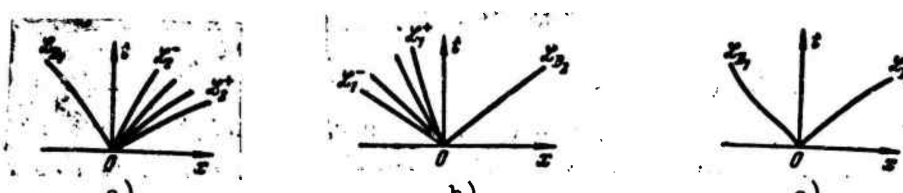

Figure 4.46

The construction of solution $r(x, t)$ is different in each of these cases, however cases a) and b) differ from each other only by the indexes of the discontinuity lines and the rarefaction waves. Therefore it is sufficient that we consider the question of whether the solution has been constructed either when inequality (5) or inequality (7) has been met.

Let us outline the method of constructing the solution $r(x, t)$ in each of the two cases.

The common ground for these two cases is the solution of Cauchy's problem for equation (1) with the initial conditions

$$r(x, 0) = r_0(x), \quad -a \leq x < 0,$$

assigned only to the left of the point $x = 0$, and with initial conditions

$$r(x, 0) = r_0(x), \quad 0 < x \leq a,$$

assigned only to the right of the point $x = 0$.

The solution of each of these two Cauchy's problems is continuously differentiable and can be defined in the domain of definition of each of these problems by the method of characteristics outlined in Chapter One. The solution of the first problem is defined here in domain I bounded on the right by a characteristic of the first family $O\mathcal{L}_1^-$; the solution of the second problem is in domain II, bounded from the left by a characteristic of the second family, which we denote by $O\mathcal{L}_2^+$ (Figure 4.47). Note that in some cases domains I and II can overlap each other, i.e., the line $O\mathcal{L}_2^+$ can lie to the left of the line $O\mathcal{L}_1^-$, however for our further consideration this is not of key importance. Solutions of the two problems in the domains I and II will be further denoted by $r_0(x, t)$. According to the results in Chapter One, $r_0(x, t)$ has bounded first derivatives in the variables x and t; we will assume that these derivatives are bounded modulowise by the number $C > 0$.

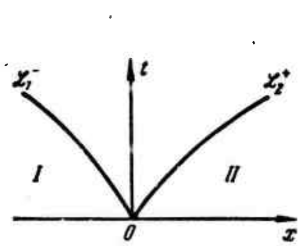

Figure 4.47

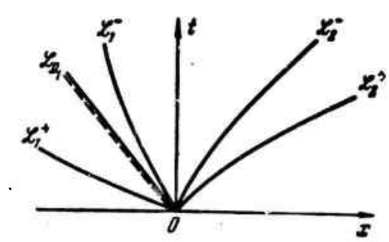

Figure 4.48

In the case when conditions (5) are satisfied, the construction of the solution begins with the solution of Goursat's problem for system of equations (1) with the conditions assigned at the characteristic $O\mathcal{L}_2^+$:

$$r_1(x, t)|_{O\mathcal{L}_2^+} = r_1^0(x, t),$$

and at the point (0, 0):

$$\lim_{t\to 0} r_2(t\xi_2(r_1^0(+0), \beta, 0, 0), t) = \beta,$$

where

$$R_2(r_1^0(+0), r_1^0(-0), r_2^0(-0), 0, 0) \leqslant \beta \leqslant r_2^0(+0).$$

The solution of this problem is constructed by the method outlined in subsection 6. Suppose that when $0 \leqslant t \leqslant T$ the solution of this problem is constructed in the zone $\mathcal{L}_2^- O\mathcal{L}_2^+$ (Figure 4.48). According to subsection 6, the solution r(x, t) of this Goursat's problem is a smooth function and, in particular, is differentiable along the characteristic $O\mathcal{L}_2^-$. The solution r(x, t) in the zone $\mathcal{L}_2^- O\mathcal{L}_2^+$ will also be denoted by $r_0(x, t)$.

Further constructions are aimed at defining the functions $\check{r}_1(x, t)$ and $\check{r}_2(x, t)$ satisfying the following conditions:

1) $\check{r}_1$ and $\check{r}_2$ are defined in the zone $\mathcal{L}_1^+ O\mathcal{L}_2^-$ containing the zone $\mathcal{L}_1^- O\mathcal{L}_2^-$ and satisfy system of equations (1) in the zone $\mathcal{L}_1^+ O\mathcal{L}_2^-$.

2) Integral curve $O\mathscr{L}_{D_1}$ of the equation

$$\frac{dx}{dt} = D_1(\tilde{r}_1(x, t), r_0(x, t), x, t), \tag{8}$$

passing through the point (0, 0): $x(0) = 0,$ (9)

lies entirely within the zone $\mathscr{L}_1^+ O \mathscr{L}_1^-$ when $0 \leqslant t \leqslant T$.

3) The condition $r_1(x, t)\big|_{O\mathscr{L}_2^-} = r_1^0(x, t),$ (10)

is satisfied at the line $O\mathscr{L}_2^-$ and the condition

$$\tilde{r}_2(x, t)\big|_{O\mathscr{L}_{D_1}} = R_2(\tilde{r}_1(x, t), r_0(x, t), x, t) \tag{11}$$

is satisfied at the line $O\mathscr{L}_{D_1}$.

We can easily note that if such functions $\tilde{r}_1$ and $\tilde{r}_2$ and the line $O\mathscr{L}_{D_1}$ are found, the generalized solution is given by the formulas

$$r(x, t) = \begin{cases} r(x, t) & \text{in the zone } \mathscr{L}_{D_1} O \mathscr{L}_2^-, \\ r_0(x, t) & \text{outside the zone } \mathscr{L}_{D_1} O \mathscr{L}_2^-. \end{cases} \tag{12}$$

Actually, formula (12) defines a function that is discontinuous at the line $O\mathscr{L}_{D_1}$, and which everywhere, except for the discontinuity line, satisfies system of equations (1). Solution $r(x, t)$ satisfies Hugoniot's conditions and stability conditions at the discontinuity line $O\mathscr{L}_{D_1}$.

Thus, the problem has been reduced to constructing the functions $\tilde{r}_1$ and $\tilde{r}_2$ satisfying requirements (1), (2), and (3) formulated above. We now indicate the method of successive approximations by means of which these functions can be constructed.

Let us define the function $\overset{(0)}{\tilde{r}_2}(x, t)$ to the left of the line $O\mathscr{L}_2^-$. We will assume that $\overset{(0)}{\tilde{r}_2}(x, t)$ does not depend on x and that at the line $O\mathscr{L}_2^-$ it takes on the same values as does $r_2^0(x, t)$. Thus, $\overset{(0)}{\tilde{r}_2}(x, t)$ is defined to the left of $O\mathscr{L}_2^-$ and is given by the formula

where $x = x(t)$ is the equation of the line $O\mathscr{L}_2^-$.

Next, let us determine $\overset{(1)}{\tilde{r}_1}(x, t)$ as the solution of the equation

$$\frac{\partial \overset{(1)}{r}_1}{\partial t} + \xi_1(\overset{(1)}{r}_1, \overset{(0)}{r}_2(x, t), x, t)\frac{\partial \overset{(1)}{r}_1}{\partial x} = f_1(\overset{(1)}{r}_1, \overset{(0)}{r}_2(x, t), x, t), \qquad (13)$$

satisfying the condition $\overset{(1)}{r}_1\big|_{O\mathcal{L}_2^-} = r_1^0(x, t)$ at the line $O\mathcal{L}_2^-$. The solution $\overset{(1)}{r}_1$ is uniquely defined in the zone $\overset{(1)}{\mathcal{L}}{}_1^+ O\mathcal{L}_2^-$ (Figure 4.49), where $O\overset{(1)}{\mathcal{L}}{}_1^+$ is the characteristic of equation (13) passing through the point (0, 0). According to our conditions, $r_1^0(+0) < r_1^0(-0)$ therefore for sufficiently small T the inequality $\overset{(1)}{r}_1(x, t) < r_1^0(x, t)$ will be satisfied in the zone $\overset{(1)}{\mathcal{L}}{}_1^+ O\mathcal{L}_1^-$.

From condition (3.5.11) it follows that

$$\begin{aligned}\xi_1(\overset{(1)}{r}_1(0, 0), \overset{(0)}{r}_2(0, 0), 0, 0) &= \\ &= \xi_1(r_1^0(+0), R_2(r_1^0(+0), r_0(-0), 0, 0), 0, 0) < \\ &< \xi_1(r_1^0(-0), r_2^0(-0), 0, 0). \qquad (14)\end{aligned}$$

In inequality (14) the slope of the line $O\overset{(1)}{\mathcal{L}}{}_1^+$ at the point (0, 0) appears in the left side, and in the right side — the slope of the characteristic $O\mathcal{L}_1^-$ at the point (0, 0). Therefore when this inequality is satisfied it follows that for sufficiently small T the curve $O\overset{(1)}{\mathcal{L}}{}_1^+$ lies to the left of $O\mathcal{L}_1^-$, as is shown in Figure 4.49.

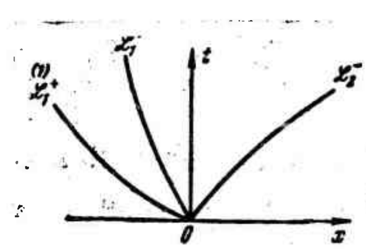

Figure 4.49

Now let us consider in the zone $\overset{(1)}{\mathcal{L}}{}_1^+ O\mathcal{L}_1^-$ the ordinary differential equation

$$\frac{dx}{dt} = D_1(\overset{(1)}{r}_1(x, t), r_0(x, t), x, t), \qquad (15)$$

for which we impose initial condition $x(0) = 0$. Noting that for sufficiently small T, $\overset{(1)}{\tilde{r}}_1(x, t)$ is a differentiable function of its variables, we conclude that the right side of equation (15) is also a differentiable function of its variables. At the point (0, 0), from conditions (3.5.11) it follows that

$$\xi_1(\overset{(1)}{\tilde{r}}_1(0, 0), \overset{(2)}{\tilde{r}}_-(0, 0), 0, 0) < D_1(\overset{(1)}{\tilde{r}}_1(0, 0), r_0(0, 0), 0, 0) < \xi_1(r_0(0, 0), 0, 0).$$

Hence, from the continuity of these functions we conclude that for sufficiently small T the inequality

$$D_1(\overset{(1)}{\tilde{r}}_1(x, t), r_0(x, t), x, t) < \xi_1(r_0(x, t), x, t) \tag{16}$$

is satisfied at the line $O\overset{(1)}{\mathscr{L}}{}_1^-$, and the inequality

$$\xi_1(\overset{(1)}{\tilde{r}}_1(x, t), \overset{(0)}{\tilde{r}}_2(x, t), x, t) < D_1(\overset{(1)}{\tilde{r}}_1(x, t), r_0(x, t), x, t). \tag{17}$$

-- at the line $O\overset{(1)}{\mathscr{L}}{}_1^+$.

Inequalities (16) and (17) signify that at the lines $O\mathscr{L}_1^-$ and $O\overset{(1)}{\mathscr{L}}{}_1^+$ field of directions for differential equation (15) is of the form sho in Figure 4.50. We conclude that the integral curve of equation (15) passing through the point (0, 0) exists, and that it is unique. We denote this curve by $O\overset{(1)}{\mathscr{L}}_{D_1}$. Obviously, the curve $O\overset{(1)}{\mathscr{L}}_{D_1}$ is a smooth curve, and also differentiable.

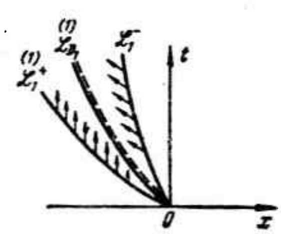

Figure 4.50

After determining the curve $O\overset{(1)}{\mathscr{L}}_{D_1}$, let us define the function $\overset{(1)}{\tilde{r}}_2(x, t)$ in the zone $\overset{(1)}{\mathscr{L}}{}_1^+ O\mathscr{L}_2^-$ as the solution of equation

$$\frac{\partial \overset{(1)}{\tilde{r}}_2}{\partial t} + \xi_2(\overset{(1)}{\tilde{r}}_1(x, t), \overset{(1)}{\tilde{r}}_2, x, t)\frac{\partial \overset{(1)}{\tilde{r}}_2}{\partial x} = f_2(\overset{(1)}{\tilde{r}}_1(x, t), \overset{(1)}{\tilde{r}}_2, x, t) \tag{18}$$

(in which $\overset{(1)}{\tilde{r}}_1(x, t)$ is a known function) satisfying the initial condition imposed on the curve $O\overset{(1)}{\mathcal{L}}_{D_1}$:

$$\overset{(1)}{r}_2(x, t)\Big|_{O\overset{(1)}{\mathcal{L}}_{D_1}} = R_2(\overset{(1)}{\tilde{r}}_1(x, t), r_0(x, t), x, t).$$

This problem obviously has the solution $\overset{(1)}{r}_2(x, t)$ uniquely defined over the entire zone $\overset{(1)}{\mathcal{L}}{}^{+}_{1}\,O\mathcal{L}^{-}_{2}$ and differentiable in this zone for sufficiently small T; here the values of the solution in the zone $\overset{(1)}{\mathcal{L}}{}^{+}_{1}\,O\mathcal{L}^{-}_{2}$ continuously adjoin the values of the function $r^0_2(x, t)$ at the line $O\mathcal{L}^{-}_{2}$.

To the left of the line $O\overset{(1)}{\mathcal{L}}{}^{+}_{1}$ we assume that the function $\overset{(1)}{\tilde{r}}_2(x, t)$ is independent of the variable x and that it takes on the same values as it does at the line $O\overset{(1)}{\mathcal{L}}{}^{+}_{1}$. Then the process of successive approximations is repeated.

Suppose we know the Lipschitz-continuous function $\overset{(n-1)}{\tilde{r}}_2(x, t)$ to the left of $O\mathcal{L}^{-}_{2}$. Let us define $\overset{(n)}{\tilde{r}}_1(x, t)$ in the zone $\overset{(n)}{\mathcal{L}}{}^{+}_{1}\,O\mathcal{L}^{-}_{2}$ as the solution of the problem

$$\frac{\partial \overset{(n)}{\tilde{r}}_1}{\partial t} + \xi_1(\overset{(n)}{\tilde{r}}_1, \overset{(n-1)}{r}_2(x, t), x, t)\frac{\partial \overset{(n)}{\tilde{r}}_1}{\partial x} = f_1(\overset{(n)}{\tilde{r}}_1, \overset{(n-1)}{\tilde{r}}_2(x, t), x, t). \qquad (19)$$

$$\overset{(n)}{\tilde{r}}_1(x, t)\Big|_{O\mathcal{L}^{-}_{2}} = r^0_1(x, t). \qquad (20)$$

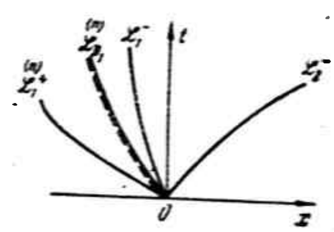

Figure 4.51

Analogously, the line $O\overset{(n)}{\mathcal{L}}{}^{+}_{1}$ is located to the left of $O\mathcal{L}_1$, and the unique differentiable integral curve $O\overset{(n)}{\mathcal{L}}_{D_1}$ of the equation

$$\frac{dx}{dt} = D_1(\overset{(n)}{\tilde{r}}_1(x, t), r_0(x, t), x, t), \quad x(0) = 0. \qquad (21)$$

exists in the zone $\overset{(n)}{\mathcal{L}}{}^{+}_{(n)1} O \mathcal{L}^{-}_{1}$ (Figure 4.51). Then the function $\overset{(n)}{r}_2(x, t)$ is defined in the zone $\mathcal{L}^{+}_{1} O \mathcal{L}^{-}_{2}$ as the solution of the problem

$$\frac{\partial \overset{(n)}{r_2}}{\partial t} + \xi_2(\overset{(n)}{r_1}(x, t), \overset{(n)}{r_2}, x, t)\frac{\partial \overset{(n)}{r_2}}{\partial x} = f_2(\overset{(n)}{r_1}(x, t), \overset{(n)}{r_2}, x, t), \quad (22)$$

$$\overset{(n)}{r_2}\Big|_{\overset{(n)}{O x D_1}} = R_2(\overset{(n)}{r_1}(x, t), r_2(x, t), x, t) \quad (23)$$

and it is assumed to be independent of x to the left of $O\overset{(n)}{\mathcal{L}}{}^{+}_{1}$ and continuous at the line $O\overset{(n)}{\mathcal{L}}{}^{+}_{1}$.

To prove the convergence of successive approximations, let us estimate, just as in subsection 6, the derivatives $\overset{(n)}{p_1} = \frac{\partial \overset{(n)}{r_1}}{\partial x}$, $\overset{(n)}{p_2} = \frac{\partial \overset{(n)}{r_2}}{\partial x}$.

Suppose

$$|\overset{(n)}{r_i}(x, t)| \leqslant R \quad (i = 1, 2), \quad n = 1, 2, \ldots,$$
$$|\overset{(n-1)}{p_i}(x, t)| \leqslant \overset{(n-1)}{E_i}, \quad |\overset{(n)}{p_1}|\Big|_{\partial x_2^-} \leqslant C,$$

and the quantity M has the same significance as in subsection 6.

From equation (19) we derive an equation analogous to (3.6.11):

$$\frac{\partial \overset{(n)}{p_1}}{\partial t} + \xi_1 \frac{\partial \overset{(n)}{p_1}}{\partial x} = -\frac{\partial \xi_1}{\partial r_1}(\overset{(n)}{p_1})^2 +$$
$$+\left[\frac{\partial f_1}{\partial r_1} - \frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_1}{\partial r_2}\frac{\partial \overset{(n-1)}{r_2}}{\partial x}\right]\overset{(n)}{p_1} + \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial r_2}\frac{\partial \overset{(n-1)}{r_2}}{\partial x}\right].$$

Similarly as with (12), we have an equation for the majorant of p_1:

$$\frac{dp_1}{dt} = Mp_1^2 + M\left[\overset{(n-1)}{E_2} + 2\right]p_1 + M\left[E_2 + 1\right] < M(E_2 + 1)(p_1 + 1)^2.$$

Using similar operations as in subsection 6, from this it readily follows that

$$|\overset{(n)}{p_1}(x, t)| < \frac{C + 1}{1 - (C + 1)M(\overset{(n-1)}{E_2} + 1)(t - \tau)}. \quad (24)$$

If

$$T < \frac{1}{2(C + 1)M(\overset{(n-1)}{E_2} + 1)},$$

then when $0 \leq \tau \leq t < T$

$$|\overset{(n)}{p}_1(x, t)| < 2(C+1). \qquad (25)$$

Let us also similarly write out the equation for $\overset{(n)}{p}_2$; the initial value $\overset{(n)}{p}_2(x,t)$ is calculated at the line $O\overset{(n)}{\mathcal{L}}_{D_1}$ from condition (23):

$$\overset{(n)}{p}_2(x, t) = \frac{\frac{\partial R_2}{\partial t} + D_1 \frac{\partial R_2}{\partial x} - f_1}{\xi_2(\overset{(n)}{\bar r}_1, \overset{(n)}{\bar r}_2, x, t) - D_1(\overset{(n)}{\bar r}_1, \overset{(n-1)}{\bar r}_2, x, t)}. \qquad (26)$$

Here we denote by $\frac{\partial R_2}{\partial t}$ and $\frac{\partial R_2}{\partial x}$, concisely, the corresponding derivatives of the right half of (23). According to conditions (3.5.12)

$$\xi_2(\overset{(n)}{\bar r}_1, \overset{(n)}{\bar r}_2, x, t) - D_1(\overset{(n)}{\bar r}_1, \overset{(n)}{\bar r}_2, x, t) > \varepsilon > 0,$$

therefore for sufficiently small T the denominator in formula (26) will not tend to zero, and in view of (25) we can assume that the constants D and E exist such that at the line $O\mathcal{L}_{D_1}$ $|\overset{(n)}{p}_2(x, t)| < DC + E = D_c$. Just as in the preceding, we get the result that

$$T < \frac{1}{2(D_2+1)M(\overset{(n)}{E}_1+1)} \qquad (27)$$

$$|\overset{(n)}{p}(x, t)| < 2D_r \qquad (28)$$

Now selecting

$$T = \min\left\{\frac{1}{2(C+1)M(2D_2+1)};\ \frac{1}{2(D_2+1)M(2C+3)}\right\}, \qquad (29)$$

we obtain the result that when $0 \leq t \leq T$, estimates (25) and (28) are simultaneously satisfied for any n = 1, 2, ..., i.e., all successive approximations have bounded first derivatives.

Let us proceed to proving the convergence of successive approximations of $\overset{(n)}{\bar r}(x, t)$, assuming that the value of T is given by formula (29). Let us selects the common part of the domains $\overset{(n)}{\mathcal{L}}{}^{-}_{1} O\mathcal{L}^{-}_{2}$ when n = 1, 2, 3, ... and let us bound it by the condition $0 \leq t \leq T' \leq T$, where T' is sufficiently small that the common part of domains $\overset{(n)}{\mathcal{L}}{}^{-}_{1} O\mathcal{L}^{-}_{2}$ always contains within it all lines $O\overset{(n)}{\mathcal{L}}_{D_1}$ where n = 1, 2, ... Since the inclination of the line $O\overset{(n)+}{\mathcal{L}}_1$ at the point (0, 0) does not depend on the number n, and the quantities $\overset{(n)}{\bar r}_1$ and $\overset{(n)}{\bar r}_2$ have uniformly bounded derivatives, then this can always be done.

Let us denote by G this common part of the domains $\overset{(n)+}{\mathcal{L}}_1$, $\mathcal{L}_2^-$ and introduce the following notation:

$$\Delta \overset{(n)}{\bar{\bar r}}_i(t) = \max_{G_t} \left| \overset{(n)}{\bar{\bar r}}_i(\xi, \tau) - \overset{(n-1)}{\bar{\bar r}}_i(\xi, \tau) \right|,$$

where G_τ is the intersection of the domain G with the strip $0 \leq \tau \leq t$,

$$\Delta \overset{(n)}{x}_{D_1}(t) = \max_{0 < \tau < t} \left| \overset{(n)}{x}_{D_1}(\tau) - \overset{(n-1)}{x}_{D_1}(\tau) \right|.$$

Here we denote by $x = \overset{(n)}{x}_{D_1}(t)$ the equation of the line $O\overset{(n)}{\mathcal{L}}_{D_1}$. From (19) and (20) we can readily obtain estimates (we produced similar estimates in subsection 6 of the section)

$$\Delta \overset{(n)}{\bar{\bar r}}_1(t) < \bar M t \Delta \overset{(n-1)}{\bar{\bar r}}_2(t),$$

from equations (21)

$$\Delta \overset{(n)}{x}_{D_1}(t) < \bar M t \Delta \overset{(n)}{\bar{\bar r}}_1(t) < \bar M^2 t^2 \Delta \overset{(n-1)}{\bar{\bar r}}_2(t),$$

and, finally, can obtain the estimates

$$\Delta \overset{(n)}{\bar{\bar r}}_2(t) < \bar M \Delta \overset{(n)}{\bar{\bar r}}_1(t) + \bar M t \Delta \overset{(n)}{\bar{\bar r}}_1(t) + \bar M \Delta \overset{(n)}{x}_{D_1}(t) < \bar{\bar M} t \Delta \overset{(n-1)}{\bar{\bar r}}_2(t),$$

from equations (22) and (23). Here $\widetilde M$ and $\bar{\bar M}$ are certain bounded quantities.

The uniform convergence of the sequences $\left\{\overset{(n)}{\tilde r}_1\right\}$, $\left\{\overset{(n)}{\tilde r}_2\right\}$, and $\left\{\overset{(n)}{x}_{D_1}\right\}$ readily follows from these formulas.

In view of their uniform convergence, the quantities

$$\bar r_1(x, t) = \lim_{n \to \infty} \overset{(n)}{\bar r}_1(x, t), \quad \bar r_2(x, t) = \lim_{n \to \infty} \overset{(n)}{\bar r}_2(x, t)$$

satisfy in G system of equations (?); the line $O\mathcal{L}_{D_1}$ ($x_{D_1}(t) = \lim_{n \to \infty} \overset{(n)}{x}_{D_1}(t)$) is the integral curve of equation (?), and the function $\bar r_2(x, t)$ satisfies the condition (??) at the line $O\mathcal{L}_{D_1}$.

Let us now proceed to examining the second case.

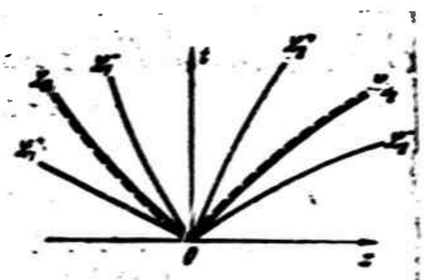

Figure 4.52

When inequalities (7) are satisfied, as we have already noted, the solution contains two discontinuity lines: $O\mathcal{L}_{D_1}$ with index 1 and $O\mathcal{L}_{D_2}$ with index 2, issuing from the point (0, 0) (Figure 4.52). On analogy with the preceding, let us introduce functions $\tilde{r}_1(x, t)$ and $\tilde{r}_2(x, t)$, which are required to comply with the following:

1) The functions $\tilde{r}_1$ and $\tilde{r}_2$ are defined in the zone $\mathcal{L}_1^+ O\mathcal{L}_2^-$ containing the zone $\mathcal{L}_{D_1} O\mathcal{L}_{D_2}$; they have in this zone bounded first derivatives satisfying system of equations (1).

2) The integral curve $O\mathcal{L}_{D_1}$ of the equation

$$\frac{dx}{dt} = D_1(\tilde{r}_1(x, t), r_2(x, t), x, t), \tag{30}$$

passing through the point (0, 0), when $0 \leqslant t \leqslant T$ lies in the zone $\mathcal{L}_1^+ O\mathcal{L}_2^-$, and the integral curve $O\mathcal{L}_{D_2}$ of equation

$$\frac{dx}{dt} = D_2(\tilde{r}_2(x, t), r_2(x, t), x, t), \tag{31}$$

issuing from the point (0, 0) lies in the zone $\mathcal{L}_2^+ O\mathcal{L}_2^-$ when $0 \leqslant t \leqslant T$.

3) The conditions

$$\tilde{r}_2(x, t)|_{O\mathcal{L}_{D_1}} = R_2(\tilde{r}_1(x, t), r_2(x, t), x, t), \tag{32}$$

$$\tilde{r}_1(x, t)|_{O\mathcal{L}_{D_2}} = R_1(\tilde{r}_2(x, t), r_2(x, t), x, t). \tag{33}$$

are satisfied at the lines $O\mathcal{L}_{D_1}$ and $O\mathcal{L}_{D_2}$. If the functions $\tilde{r}_1$ and $\tilde{r}_2$ and the lines $O\mathcal{L}_{D_1}$ and $O\mathcal{L}_{D_2}$ satisfying these requirements are constructed, then the solution of Cauchy's problem (1), (3) is given by the formula

$$r(x,t)=\begin{cases}\tilde{r}(x,t) & \text{in the zone } \mathcal{L}_{D_1}O\mathcal{L}_{D_2},\\ r_0(x,t) & \text{outside the zone } \mathcal{L}_{D_1}O\mathcal{L}_{D_2}.\end{cases}\qquad (34)$$

We outline the method of successive approximations by which the functions $\tilde{r}_1$ and $\tilde{r}_2$ can be constructed; here we omit several several details that are common to the preceding construction.

Let us introduce the notation

$$\overset{(0)}{r}_1(x,t)=\overset{(0)}{r}_1=R_1(r_2^0(-0),\ r_0(+0),\ 0,\ 0),$$
$$\overset{(0)}{r}_2(x,t)=\overset{(0)}{r}_2=R_2(r_1^0(+0),\ r_0(-0),\ 0,\ 0)$$

and let us define the line $O\overset{(0)}{\mathcal{L}}_{D_1}$ as the integral curve of the problem

$$\frac{dx}{dt}=D_1(\overset{(0)}{r}_1,\ r_0(x,t),\ x,\ t),\qquad x(0)=0,\qquad (35)$$

and the line $O\overset{(0)}{\mathcal{L}}_{D_2}$ — as the integral curve of the problem

$$\frac{dx}{dt}=D_2(\overset{(0)}{r}_2,\ r_0(x,t),\ x,\ t),\qquad x(0)=0.\qquad (36)$$

If T is sufficiently small, then when $0\leq t\leq T$ the curve $O\overset{(0)}{\mathcal{L}}_{D_1}$ to the left of $O\mathcal{L}_1^-$; correspondingly, $O\overset{(0)}{\mathcal{L}}_{D_2}$ to the right of $O\mathcal{L}_2^+$.

Next, let us define the subsequent approximation $\overset{(1)}{\tilde{r}}_1(x,t)$ as the solution of the Cauchy's problem

$$\frac{\partial \overset{(1)}{r}_1}{\partial t}+\xi_1(\overset{(1)}{r}_1,\ \overset{(0)}{r}_2(x,t),\ x,\ t)\frac{\partial \overset{(1)}{r}_1}{\partial x}=f_1(\overset{(1)}{r}_1,\ \overset{(0)}{r}_2(x,t),\ x,\ t),\qquad (37)$$
$$\overset{(1)}{r}_1(x,t)\big|_{O\overset{(0)}{\mathcal{L}}_{D_1}}=R_1(\overset{(0)}{r}_2(x,t),\ r_0(x,t),\ x,\ t),\qquad (38)$$

and $\overset{(1)}{\tilde{r}}_2(x,t)$ — as the solution of the Cauchy's problem

$$\frac{\partial \overset{(1)}{r}_2}{\partial t}+\xi_2(\overset{(0)}{r}_1(x,t),\ \overset{(1)}{r}_2,\ x,\ t)\frac{\partial \overset{(1)}{r}_2}{\partial x}=f_2(\overset{(0)}{r}_1(x,t),\ \overset{(1)}{r}_2,\ x,\ t),\qquad (39)$$
$$\overset{(1)}{r}_2(x,t)\big|_{O\overset{(0)}{\mathcal{L}}_{D_1}}=R_2(\overset{(0)}{r}_1(x,t),\ r_0(x,t),\ x,\ t).\qquad (40)$$

The solution of the problem (37), (38) can be defined in the zone $\overset{(1)}{\mathcal{L}}{}_1^+ O \overset{(1)}{\mathcal{L}}{}_2^-$; let us also define the solution of Cauchy's problem (39), (40) in this same zone, and exterior to this zone let us assume that $\overset{(1)}{\tilde{r}}_1$ and $\overset{(1)}{\tilde{r}}_2$ continuously adjoin the values at the lines $O\overset{(1)}{\mathcal{L}}{}_1^+$ and $O\overset{(1)}{\mathcal{L}}{}_2^-$ and do not depend on the coordinates of x.

Then, the process of successive approximations becomes routine. Suppose we know the approximation $\overset{(n-1)}{r}(x, t)$ with a bounded coordinate. First let us define the lines $O\overset{(n-1)}{\mathcal{L}}_{D_1}$ and $O\overset{(n-1)}{\mathcal{L}}_{D_2}$ as integral curves of the problems

$$\frac{dx}{dt} = D_1(\overset{(n-1)}{\tilde{r}_1}(x, t), r_0(x, t), x, t), \quad x(0) = 0, \tag{41}$$

$$\frac{dx}{dt} = D_2(\overset{(n-1)}{\tilde{r}_2}(x, t), r_0(x, t), x, t), \quad x(0) = 0. \tag{42}$$

These integral curves, for sufficiently small T, lie within the zones $\overset{(n-1)}{\mathcal{L}}{}_1^+ O \mathcal{L}_1^-$ and $\mathcal{L}_2^+ O \overset{(n-1)}{\mathcal{L}}{}_2^-$, respectively. We can be convinced of this by investigating the field of directions of differential equations (41) and (42) at the lines $O\overset{(n-1)}{\mathcal{L}}{}_1^+$, $O\mathcal{L}_1^-$ and $O\mathcal{L}_2^+$, $O\overset{(n-1)}{\mathcal{L}}{}_2^-$, quite analogously to the preceding.

Then, let us define $\overset{(n)}{\tilde{r}}_1(x, t)$ by means of the solution of Cauchy's problem

$$\left.\begin{aligned} &\frac{\partial \overset{(n)}{r_1}}{\partial t} + \xi_1(\overset{(n)}{\tilde{r}_1}, \overset{(n-1)}{\tilde{r}_2}(x, t), x, t)\frac{\partial \overset{(n)}{r_1}}{\partial x} = f_1(\overset{(n)}{\tilde{r}_1}, \overset{(n-1)}{\tilde{r}_2}(x, t), x, t), \\ &\overset{(n)}{r_1}(x, t)\Big|_{O\mathcal{L}_{D_1}}^{(n-1)} = R_1(\overset{(n-1)}{\tilde{r}_2}(x, t), r_0(x, t), x, t) \end{aligned}\right\} \tag{43}$$

in the zone $\overset{(n)}{\mathcal{L}}{}_1^+ O \overset{(n)}{\mathcal{L}}{}_2^-$; let us find $\overset{(n)}{\tilde{r}}_2(x, t)$ in this same zone from the conditions

$$\left.\begin{aligned} &\frac{\partial \overset{(n)}{r_2}}{\partial t} + \xi_2(\overset{(n-1)}{\tilde{r}_1}(x, t), \overset{(n)}{\tilde{r}_2}, x, t)\frac{\partial \overset{(n)}{r_2}}{\partial x} = f_2(\overset{(n-1)}{\tilde{r}_1}(x, t), \overset{(n)}{\tilde{r}_2}, x, t), \\ &\overset{(n)}{r_2}(x, t)\Big|_{O\mathcal{L}_{D_2}}^{(n-1)} = R_2(\overset{(n-1)}{\tilde{r}_1}(x, t), r_0(x, t), x, t) \end{aligned}\right\} \tag{44}$$

and let us take $\overset{(n)}{\tilde{r}}_1$ and $\overset{(n)}{\tilde{r}}_2$ as independent of x exterior to the zone $\overset{(n)}{\mathcal{L}}{}_1^+ O \overset{(n)}{\mathcal{L}}{}_2^-$ (Figure 4.53). We will not present here the cumbersome operations involved with the estimates of the first derivatives of the successive approximations $\overset{(n)}{r}_1$ and $\overset{(n)}{\tilde{r}}_2$, since they mainly repeat those given earlier, but we will present only the results. If system (1) satisfies the above enumerated requirements, and if solution $\overset{(n)}{r}_1$ and $\overset{(n)}{r}_2$ remain bounded for any n = 1, 2, ..., then there exists a $T > 0$ such that when $0 \leqslant t \leqslant T$ all successive approximations $\overset{(\nu)}{r}(x, t)$ have bounded first derivatives.

We will denote by the zone $\overset{(n)}{\mathscr{L}}{}^+_1 O \overset{(n)}{\mathscr{L}}{}^-_2$ the domain of the variables x, t in which the functions $\overset{(n)}{r}_1(x, t)$ and $\overset{(n)}{r}_2(x, t)$ are simultaneously determined (Figure 4.53).

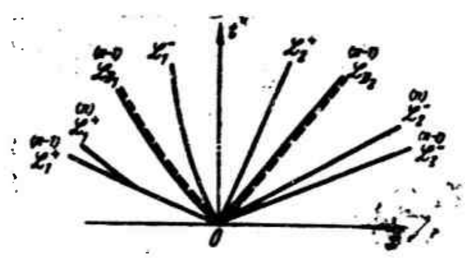

Figure 4.53

Exterior to the zone $\overset{(n)}{\mathscr{L}}{}^+_1 O \overset{(n)}{\mathscr{L}}{}^-_2$ we predetermine $\overset{(n)}{r}(x, t)$ in order that in the process of successive approximations there is no reduction in the domains $\overset{(n)}{\mathscr{L}}{}^+_1 O \overset{(n)}{\mathscr{L}}{}^-_2$ because the functions $\overset{(n-1)}{\tilde{r}}_1$ and $\overset{(n-1)}{\tilde{r}}_2$ are unknown in the domain of determinacy of the solutions of Cauchy's problem (43) and (44).

Finally, let us note the elements of the proof that the method of successive approximations converges in order to explain the requirements that insure convergence.

In the general portion of the zones $\overset{(n)}{\mathscr{L}}{}^+_1 O \overset{(n)}{\mathscr{L}}{}^-_2$ for all n = 1, 2, ... (such exists and for sufficiently small T contains the lines $O\overset{(n)}{\mathscr{L}}_{D_1}$ and $O\overset{(n)}{\mathscr{L}}_{D_2}$ for all n = 1, 2, ...), on analogy with the foregoing it is not difficult to obtain the estimates

$$\Delta\overset{(n)}{\tilde{r}}_i(t) \leqslant \max\left|\frac{\partial R_i(\tilde{r}_j, r_0, x, t)}{\partial \tilde{r}_j}\right| \Delta\overset{(n-1)}{\tilde{r}}_j(t) + \bar{M}t\,\Delta\overset{(n-1)}{\tilde{r}}_j(t) + \bar{M}\,\Delta\overset{(n-1)}{x}_{D_j}(t)$$

$$(i = 1,\ 2;\ j \neq i). \qquad (45)$$

$$\Delta\overset{(n)}{x}_{D_i}(t) \leqslant \bar{M}t\,\Delta\overset{(n)}{\tilde{r}}_i(t). \qquad (46)$$

In formula (45) the expression for $\frac{\partial R_i}{\partial \tilde{r}_j}$ is computed for some value of the first argument that is intermediate for the successive approximations. From formulas (45) and (46) follows that

$$\overset{(n)}{\Delta\tilde{r}_i}(t) \leqslant \left\{ \max \left| \frac{\partial R_i}{\partial \tilde{r}_j} \right| + \bar{M}t \right\} \Delta \overset{(n-1)}{\tilde{r}_j}(t) \qquad (i = 1,\ 2;\ j \neq i). \quad (47)$$

Hence it follows that, further,

$$\overset{(n)}{\Delta\tilde{r}_i}(t) \leqslant \left\{ \max \left| \frac{\partial R_i}{\partial \tilde{r}_j} \right| + \bar{M}t \right\} \left\{ \max \left| \frac{\partial R_j}{\partial r_i} \right| + \bar{M}t \right\} \Delta \overset{(n-2)}{\tilde{r}_i}(t) \quad (48)$$
$$(i = 1,\ 2;\ j \neq i).$$

Since

$$\left| \frac{\partial R_1(\tilde{r}_2, r_0, x, t)}{\partial \tilde{r}_2} \right| \cdot \left| \frac{\partial R_2(\tilde{r}_1, \bar{r}_0, x, t)}{\partial \tilde{r}_1} \right| < 1$$

for arbitrary $\tilde{r}_1$, $\tilde{r}_2$, r_0, and $\bar{r}_0$, from this it follows that there exists a small $T > 0$ for which inequality (48) can be strengthened such that

$$\overset{(n)}{\Delta\tilde{r}_i}(t) \leqslant \beta \Delta \overset{(n-2)}{\tilde{r}_i}(t), \quad 0 < \beta < 1 \qquad (i = 1,\ 2).$$

Hence follows the uniform convergence when $0 \leqslant t \leqslant T$ of successive approximations are $\overset{(n)}{\tilde{r}}(x, t)$ and the discontinuity lines $O\overset{(n)}{\mathcal{L}_{D_1}}$ and $O\overset{(n)}{\mathcal{L}_{D_2}}$.

Thus, we can show that there exist the limit functions $\tilde{r}_i(x, t) = \lim_{n \to \infty} \overset{(n)}{\tilde{r}_i}(x, t)$ $(i = 1, 2)$ which satisfy the above formulated conditions. This concludes the proof of the existence of the solution of Cauchy's problem (1), (3) for the case when conditions (*) are satisfied*).

8. Remarks on the uniqueness of the discontinuous solution of a system of two equations. We will briefly discuss here the uniqueness of the discontinuous solution of $r(x, t)$, whose construction was presented above.

From the known approximation $\overset{(n-1)}{\tilde{r}}(x, t)$, we construct in the preceding section the following approximation $\overset{(n)}{\tilde{r}}(x, t)$. We briefly outline the procedure of constructing the approximation $\overset{(n)}{\tilde{r}}(x, t)$ using the equality

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*) Note for more general systems, however, only for the initial functions differing little from the constant function was the existence of generalized solution recently proven by D. Glimm [66].

$$\overset{(n)}{\tilde r}(x, t) = T\overset{(n-1)}{\tilde r}(x), \qquad (1)$$

where by T we denote the nonlinear operator that converts approximation $\overset{(n-1)}{\tilde r}(x, t)$ to the approximation $\overset{(n)}{\tilde r}(x, t)$. We showed above that for each of the configurations considered the following equality obtains:

$$\|\overset{(n)}{\tilde r}(x, t) - \overset{(n-1)}{\tilde r}(x, t)\| \leqslant \beta \|\overset{(n-1)}{\tilde r}(x, t) - \overset{(n-2)}{\tilde r}(x, t)\|, \quad 0 < \beta < 1, \qquad (2)$$

where by the norm $\|r(x, t)\|$ we refer to the maximum of the modulus $r_i(x, t)$ in the G region. Inequality (2) can also be written as

$$\|T\overset{(n-1)}{\tilde r} - T\overset{(n-2)}{\tilde r}\| \leqslant \beta \|\overset{(n-1)}{\tilde r} - \overset{(n-2)}{\tilde r}\|, \quad 0 < \beta < 1. \qquad (3)$$

Now let us assume that there exists two distinct solutions r(x, t) and $\bar r(x,t)$ of the above-considered Cauchy's problem. Two possibilities are feasible here:

1) solution r and $\bar r$ correspond to two distinct configurations; and

2) solution r and $\bar r$ correspond to the same configuration of the discontinuity lines and the rarefaction waves.

However, the first possibility can be striken out at once, since it is equivalent to the nonuniqueness of the stable generalized solution of the problem of the decay of an arbitrary discontinuity, and, as we have seen in subsection 5, this is precluded by the requirements imposed on the system of two quasilinear equations.

Therefore it remains for us to consider the case when the solutions r(x, t) and $\tilde r(x, t)$ correspond to the second configuration. This means that there exist two solutions $\tilde r(x, t)$ and $\tilde{\tilde r}(x, t)$ satisfying the requirements that were formulated in subsection 7 and satisfying the same operator equation:

$$\tilde r(x, t) = T\tilde r(x, t), \quad \tilde{\tilde r}(x, t) = T\tilde{\tilde r}(x, t). \qquad (4)$$

The method of estimates which were presented above for successive approximations is applicable to solutions $\tilde{r}$ and $\tilde{\tilde{r}}$ of equation (4); therefore from (4) follows the estimate

$$\|\tilde{r} - \tilde{\tilde{r}}\| = \|T\tilde{r} - T\tilde{\tilde{r}}\| < \theta \|\tilde{r} - \tilde{\tilde{r}}\|, \quad 0 < \theta < 1, \qquad (5)$$

which is impossible. Hence we conclude that $\tilde{r} = \tilde{\tilde{r}}$. Thus we have proven the uniqueness of the stable generalized solution of Cauchy's problem (3.7.1), (3.7.3).

Our second remark deals with the region in which a discontinuous solution of a system of two quasilinear equations can be constructed. As long as the number of singularities of this solution (discontinuity lines and rarefaction waves) remains finite, we can adopt this method of constructing the solution by decomposing the region into areas in which the singularities are isolated. However, solution singularities arise even from smooth initial data, and their number can be multiplied, possibly, even unboundedly.

This fact makes difficult to the construction of discontinuous solutions of a system of two quasilinear equations in the large, i.e., for any $t > 0$.

Let us note, however, that in most cases of practical interest the number of singularity remains bounded.

9. Viscosity method for a system of quasilinear equations. Phenomena of the viscosity method. In Chapter Two we saw that shock waves in a gas or liquid can be considered as the limits of flows of a viscous and thermally conductive fluid and we became acquainted with the application of some linear viscosity (Neumann-Richtmyer viscosity). In subsections 2 and 7 of Section II of this chapter it was shown that a stable generalized solution of one quasilinear equation is the limit of solutions of an equation containing "viscosity" as the viscosity coefficient tends to zero.

The viscosity method has not yet been adequately studied for systems of quasilinear equations. With the example of a homogeneous system of quasilinear equations

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \quad u = \{u_1, \ldots, u_n\}, \qquad (1)$$

that is hyperbolic in the narrow sense:

$$\xi_1(u) < \xi_2(u) < \ldots < \xi_n(u),$$

we will show that the choice of any given viscosity is substantial and requires great care.

We will write the system of equations containing viscosity corresponding to (1) in the form

$$\frac{\partial u_\mu}{\partial t} + \frac{\partial \varphi(u_\mu)}{\partial x} = \mu \frac{\partial}{\partial x} B(u_\mu) \frac{\partial u_\mu}{\partial x}, \qquad (2)$$

where $B(u_\mu)$ is a $n \times n$ square matrix. Thus we will confine ourselves to the class of "divergent"*) viscosities.

First let us make several very general remarks. Obviously, the matrix B must be chosen so that the following requirements are met:

a) the formulation of Cauchy's problem is correct for system (2);

b) solutions u_μ are smooth when $t > 0$ for any piecewise-continuous and piecewise-smooth initial data;

c) the convergence (for any norm, for example, in the average) of solutions u_μ as $\mu \to 0$ to stable generalized solutions u(x, t) of system (1) holds.

At present it is impossible to indicate which are the sufficiency conditions, which one met will cause conditions a) - c) to obtain, in view of the fact that systems (2) have not been studied closely enough. Therefore we will attempt to delimit the class of matrices B by relying on certain very simple analogies.

Let us consider the case of linear systems (1) and (2) with constant coefficients, when $\varphi = Au$ and A, B are coefficient matrices.

First, let us note that when any solution $u_\mu(x, t)$ is representable in the form $u_\mu(x, t) = u(x/\mu, t/\mu)$,
where u(x, t) is the solution of the system

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = B \frac{\partial^2 u}{\partial x^2}. \qquad (3)$$

*) The divergent form of viscosity in the form $\frac{\partial}{\partial x} B \frac{\partial u}{\partial x}$ ensures that Hugoniot's conditions are satisfied at the front of a blurred shock wave (compare Chapter Two, Section V).

This system has particular solutions of the form

$$u = u_0 \exp\{i(\lambda t + \nu x)\}.$$

where u_0 is the eigenvector (right), and $(-\frac{i\lambda}{\nu^2})$ is the eigenvalue of the matrix $B + iA/\nu$. If β_j are the eigenvalues of matrix B, then as $|\nu| \to \infty$, $\lambda = i\beta_j \nu^2 + O(\nu)$. According to requirement a), we wish to satisfy the condition

$$u(x, t) \to 0 \quad \text{as} \quad u(x, 0) \to 0,$$

therefore we must require that $\operatorname{Re} \beta_j > 0 \quad (j = 1, 2, \ldots, n)$. (5)

hold. Actually, when $\operatorname{Re} \beta_{j_0} < 0$, we can specify the sequence of initial functions, say, $u_\nu(x, 0) = u_0 e^{i\nu x}/\nu$, which as $\nu \to \infty$ tends to zero, but for some solution $u_\nu(x, t)$, according to (4), it tends (in absolute value) to ∞ for any $t > 0$.

Condition (5) is the well-known condition for the correctness of Cauchy's problem for system of equations (3) according to Hadamard.

Recently attention was directed to the fact that (cf [65]) conditions (5) are in some sense insufficient for the correctness of the problem under consideration.

Let us consider the behavior of the numbers λ for small ν. If α_j are eigenvalues of matrix A and r_j and ℓ_j are the corresponding eigenvectors (as usual, we will assume that α_j are real and distinct), then as we can readily see,

$$\lambda = -\alpha_j \nu + i\nu^2 (l_j B r_j) + O(\nu^3).$$

If for some $j = j_0$ $(\ell_{j_0} B r_{j_0}) < 0$, then by (4) a system with viscosity will have particular solutions of the form

$$u_\mu = e^{\frac{t|(l_{j_0} B r_{j_0})|}{\mu}} v_\mu$$

where v_μ is a bounded (as $\mu \to 0$) function, which as $\mu \to 0$ will tend, for any $t > 0$, to ∞. According to requirement c), we must exclude this possibility and require that along with condition (5) the following condition is met:

$$(l_j B r_j) \geqslant 0 \quad (j = 1, 2, \ldots, n). \qquad (6)$$

Note that requirement (6) implies that the diagonal elements of the viscosity matrix in the system arising from (3) after it has been reduced to Riemann invariants must be positive.

On analogy with a linear case, conditions (5), (6) must also be satisfied for the nonlinear system as well.

Let us consider to have a simple example.

We will seek the particular solutions $u_\mu\left(\frac{x-Dt}{\mu}\right)$ of system (2) dependent only on a single variable $y = (x - Dt)/\mu$, i.e.,

$$u_\mu(x, t) = u_\mu(y), \qquad y = \frac{x - Dt}{\mu}.$$

The function $u_\mu(y)$ satisfies the system of equations

$$\frac{d}{dy} B(u_\mu) \frac{du_\mu}{dy} = \frac{d\varphi(u_\mu)}{dy} - D\frac{du_\mu}{dy} = [A(u_\mu) - DE]\frac{du_\mu}{dy},$$

which admits of integration

$$B(u_\mu)\frac{du_\mu}{dy} = \varphi(u_\mu) - Du_\mu - C = F(u_\mu), \qquad (7)$$

where C is an arbitrary, constant vector.

We require of this solution u_μ that it tends to constant values as $y \to \pm\infty$, i.e.,

$$u_\mu(y) \to \begin{cases} u^- & \text{as } y \to -\infty, \\ u^+ & \text{as } y \to +\infty. \end{cases}$$

For this solution to exist, it is necessary that the points $u = u^-$, $u = u^+$ be stationary points of system (7), i.e., $F(u^-) = F(u^+) = 0$.

These conditions can be rewritten as $\varphi(u^-) - \varphi(u^+) = D(u^- - u^+)$. Hence we conclude that the states u^-, u^+ must be related by Hugoniot's conditions.

First, however, it is still insufficient for the existence of the integral curve $u_\mu(y)$ of system (7) passing through the points u^- when $y = -\infty$ and u^+ when $y = +\infty$.

Let us explain several additional necessary conditions; to do this we rewrite system (7) as $du_\mu/dy = B^{-1}(u_\mu)F(u_\mu)$,
where B^{-1} is a matrix inverse to B. Expand the right side of this system in a series in the neighborhood of a stationary point, for example, $u = u^-$. Then

$$\frac{d}{dy}(u_\mu - u^-) = B^{-1}(u^-)\left(\frac{\partial F(u_\mu)}{\partial u}\right)\Big|_{u=u^-}(u_\mu - u^-) + O(|u_\mu - u^-|^2) =$$
$$= B^{-1}(u^-)[A(u^-) - DE](u_\mu - u^-) + O(|u_\mu - u^-|^2).$$

Multiplying the system scalarly by the vector $(u_\mu - u^-)$, we get

$$\frac{d}{dy}\frac{(u_\mu - u^-)^2}{2} =$$
$$= (u_\mu - u^-)B^{-1}(u^-)[A(u^-) - DE](u_\mu - u^-) + O(|u_\mu - u^-|^3).$$

When the variable y increases, the quantity $\frac{(u_\mu - u^-)^2}{2}$ does not decrease in the neighborhood $u = u^-$; therefore if the integral curve $u\ (y)$ of system (7) exists, the matrix $B^{-1}(u^-)[A(u^-) - DE]$ cannot be negatively determinate. We quite similarly establish that the matrix $B^{-1}(u^+)[A(u^+) - DE]$ cannot be positively determinate.

Suppose there exists the desired solution $u_\mu(y)$ of system (7). Then the limit

$$\lim_{\mu \to 0} u_\mu(x, t) = u(x, t) = \begin{cases} u^- & \text{when } x - Dt < 0, \\ u^+ & \text{when } x - Dt > 0. \end{cases}$$

is a discontinuous function which satisfies Hugoniot's conditions at the discontinuity line $x = Dt$ and, therefore, the integral laws of conservation of system (1).

However, this solution can be an unstable solution of system (1), since the conditions for the existence of solution $u\ (y)$, which were discussed above, and the conditions for the stability of the discontinuity u^-, u^+ are distinct.

Let us confirm this with a simple example, set up in the work [5]. Suppose

$$n = 2; \quad \varphi(u) = \{-u_2;\ p(u_1)\}; \quad p'(u_1) < 0, \quad p''(u_1) > 0.$$

Then system (1) is a system of equations of isothermal flow of a normal gas (cf Chapter Two, Section II).

Suppose $u^+ = \{1, 0\}$, $D > 0$. Let us show that there exists a solution $u\ (y)$ when $u_1^- > 1$. Such a solution, as $\mu \to 0$, changes a rarefaction wave into a shock wave, and as we have seen in Chapter Two, this solution is an unstable solution for equations in gas dynamics.

Let us select the coefficient, positively determinate matrix

$$B = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \quad a, b > 0, \quad c \neq 0.$$

Then

$$B^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ c/ab & \frac{1}{b} \end{pmatrix}.$$

Eigenvalues of the matrix $B^{-1}(A(u) - DE)$, as we can easily check, are roots of the quadratic equation

$$\lambda^2 + \left[\frac{(a+b)D - c}{ab}\right]\lambda + \frac{D^2 + p'(u_1)}{ab} = 0. \qquad (8)$$

According to Hugoniot's conditions, $D^2 = \frac{p(1) - p(u_1^-)}{u_1^- - 1}$,

and according to our assumptions $u_1^- > 1$, $D > 0$. Since $p'(u_1) < 0$ and $p''(u_1) < 0$, then there follow the inequalities:

$$\sqrt{-p'(1)} > D > \sqrt{-p'(u_1^-)}, \qquad (9)$$

or

$$p'(1) + D^2 < 0, \quad p'(u_1^-) + D^2 > 0.$$

Since a, b > 0, it follows that the roots of quadratic equation (8) when $u_1 = u_+ = 1$ have different signs. Thus, eigenvalues of the matrix $B^{-1}(A(u^+) - DE)$ are of different signs. This implies that the point $u = u^+$ is a saddle point.

Conversely, at the point $u = u^-$ the roots of quadratic equation (8) are of the same sign (if they are real). Therefore for the integral curve connecting the points u^-, u^+ to exist, we must require that the roots of equation (8) be real and positive, i.e., that the matrix $B^{-1}(u^-)[A(u^-) - DE]$ be positively defined. This will hold provided that the following inequalities are satisfied:

$$\frac{c - (a+b)D}{ab} > 0, \quad \frac{[c - (a+b)D]^2}{4ab} - D^2 - p'(u_1^-) > 0. \quad (10)$$

We can easily see that these two inequalities can be satisfied by fixing the arbitrary $c > 0$ and choosing the quantity $D > 0$ to be sufficiently small.

Suppose the quantities a, b, c, $D > 0$ are such that these inequalities are satisfied. Then the point $u^- = \{u_1^-, u_2^-\}$ is a node ($u_1^- > 1$, $u_2^- < 0$), and the point $u^+ = \{1, 0\}$ is a saddle point.

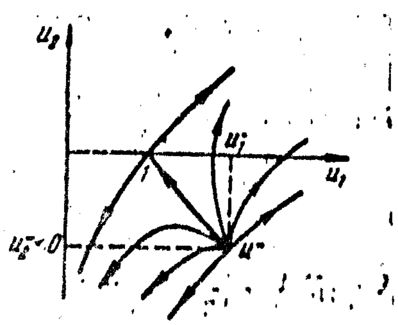

Figure 4.54

Thus, the pattern of integral curves of system (7) in the plane of variable u_1, u_2 is of the form shown in Figure 4.54. The arrows of the figure indicate the direction of increase of variable y. Thus, given the restrictions made, there exists an integral curve $u = u_\mu(y)$ passing through the points u^-, u^+.

By passing in the solution u_μ the parameter μ to zero, at the limit we get a shock wave of rarefaction, which is unstable.

Thus far we have not taken conditions (6) into account. In our case, as we can easily verify,

$$\ell_1 B r_1 = a\sqrt{-p'(u_1)} - c + b\sqrt{-p'(u_1)},$$
$$\ell_2 B r_2 = a\sqrt{-p'(u_1)} + c + b\sqrt{-p'(u_1)}.$$

If conditions (10) are satisfied, under which we obtained the unstable shock wave, then by (9) $(\ell_1 B r_1)_{u_1 = u_1^-} < (a + b)D - c < 0$,
i.e., (6) has been violated.

An even simpler and more striking example*) is represented by system (2) in the case

$$\varphi(u) = \left\{\frac{u_1^2}{2}, \frac{u_2^2}{2}\right\}, \quad B = \begin{vmatrix} \sqrt{2} - \left(\frac{1}{2} + \sqrt{2}\right)\alpha & 1 - \alpha \\ 1 - \alpha & \frac{1}{\sqrt{2}} \end{vmatrix}.$$

When

$$0 < \alpha < 1 - \frac{1}{2\sqrt{2}}$$

matrix B is positively defined, and moreover it is always symmetric and, obviously, satisfies condition (6). System (1) when $\varphi = \left\{\frac{u_1^2}{2}, \frac{u_2^2}{2}\right\}$ decomposes into two E. Hopf equations:

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\frac{u_1^2}{2} = 0, \quad \frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\frac{u_2^2}{2} = 0,$$

and system (2) in this case has the solution

$$u_2 = -\sqrt{2}u_1, \quad \frac{u_1 + u_1^0}{u_1 - u_1^0} = \exp\left\{\frac{2u_1^0 x}{\mu}\right\}, \quad u_1^0 > 0.$$

Hence it follows that as $\mu \to 0$, $u_1 = u_1(y) = u_1(x/\mu)$ tends to the unstable solution

$$u_1(x) = \begin{cases} -u_1^0, & x \leq 0, \\ u_1^0, & x > 0, \end{cases} \quad u_1^0 > 0,$$

of the Hopf equation ($u_2(y)$ tends to a stable solution).

---

*) The examplewas suggested by N. N. Kuznetsov.

Note that the integral curve in this case passes through the point $u_1 = u_2 = 0$ at which the eigenvalues of system (1) $\xi_1 = u_1$, $\xi_2 = u_2$ coincide (i.e., the system is not hyperbolic in the narrow sense). Precisely because of this, condition (6) is insufficient in the given case.

Whether similar examples with a symmetric viscosity matrix are possible for systems that are hyperbolic in the narrow sense is not known.

Let us consider the viscosity matrix $B = b(u)E$, which obviously satisfies condition (6).

Let us show that in this case when $n = 2$, system (7) cannot have solutions which as $\mu \to 0$ correspond to unstable discontinuities.

Actually, if the integral curve $u_\mu(y)$ passing through the points $u^-(y = -\infty)$ and $u^+(y = \infty)$ exists, then it is impossible for the following inequalities to be satisfied:

$$\xi_k(u^-) < D < \xi_k(u^+) \qquad (11)$$

for any $k = 1, 2$. Actually, suppose that these inequalities do hold, for example, for $k = 1$. Then, since $\xi_2(u) > \xi_1(u)$, then $\xi_2(u^+) > D$. Therefore the matrix $B^{-1}(u^+)[A(u^+) - DE] = \frac{E}{b(u^+)}[A(u^+) - DE] = \frac{1}{b(u^+)}[A(u^+) - DE]$ has the eigenvalues $\frac{\xi_k(u^+) - D}{b(u^+)}$ which are both positive according to (11). As we have seen above, in this case no integral curve of system (4) connecting the points u^- and u^+ exists, therefore it is impossible to satisfy (11). The impossibility of (11) when $k = 2$ is similarly proven.

And thus, in the case $B = b(u)$ (11) cannot be satisfied for any integral curve.

Note that from this it does not yet follow that whole solutions $u_\mu(y)$ as $\mu \to 0$ will tend to stable solutions of system (1), since stability conditions were not formulated in the form of inequalities (3.1.10) and (3.1.11) for arbitrary systems. None the less, for systems for which these inequalities guarantee uniqueness, viscosity matrices of this type yield only stable solutions.

Some advantages of a unit viscosity matrix can be established also in more general cases (cf [62]). General viscosity matrices, but for systems of a more special type, have been treated in the work [34]. The advantages of a unit viscosity matrix does not of course imply that it is to be preferred. In practice more complex viscosity matrices have to be used. Thus, for example, the "viscosity matrix" is not a unit matrix for a viscous and thermally conductive fluid.

Section IV. Applications of the General Theory of Systems of Hyperbolic Quasilinear Equations

In this section we will point out a number of problems in physics, chemistry, and mathematics that are related to the theory of systems of hyperbolic quasilinear equations.

The best known application of systems of hyperbolic quasilinear equations is the study of one-dimensional flows of compressible gases and fluids devoid of viscosity and thermal conductivity. Chapter Two discusses this problem in detail along with its relationship to the theory of systems of quasilinear equations.

Other well-known examples of problems related to systems of quasilinear equations include the motion of an incompressible fluid in shallow channels ("shallow water" theory), supersonic steady flow of gas or liquid in the two-dimensional case, problems in the nonlinear theory of elasticity, filtration theory, and certain others, We now dwell in brief on several of these.

1. "Shallow water" theory. Suppose that a heavy (in the gravity field) incompressible fluid flows in the channel that has the shape shown in Figure 4.55. We will assume that the liquid is devoid of internal friction, friction against walls and bottom of the channel, and that the level of the fluid above the bottom of the channel h is a small quantity compared with the dimensions of the bottom irregularities, characteristic flow dimensions, and so on. We will assume that the flow of the liquid characterized by one three-dimensional variable x and depend on time t. Thus we will assume that the velocity of the liquid u has a nonzero component u_x, which we will denote by u, and we can neglect the remaining components; in addition, we will assume that the level h depends also only on x and t.

With these assumptions, we will derive equations describing the flow of the liquid.

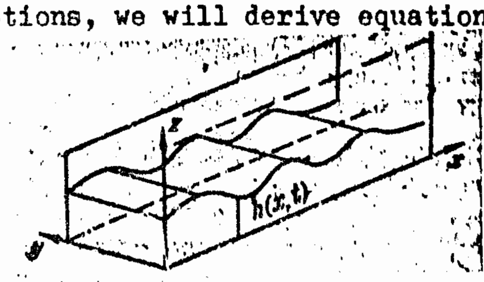

Figure 4.55

Suppose $h(x, t)$ is the level of the liquid measured from the channel bottom at the point x, ρ is the density of the liquid, ℓ is the channel width, and $u(x, t)$ is the velocity of the liquid directed along the x axis.

The amount of liquid present at instant t between two cross sections of the channel divided by the planes $x = x_1$, $x = x_2$ is obviously the quantity

$$\int_{x_1}^{x_2} \rho l h(x, t)\,dx = \rho l \int_{x_1}^{x_2} h(x, t)\,dx.$$

The change in the amount of liquid in this part of the channel between the instants $t = t_1$ and $t = t_2$ is the quantity

$$\rho l \int_{x_1}^{x_2} [h(x, t_2) - h(x, t_1)]\,dx, \tag{1}$$

which, obviously, must be equal to the amount of liquid flowing during the time from $t = t_1$ to $t = t_2$ to the planes $x = x_1$ and $x = x_2$, that is, the quantity

$$-\rho l \int_{t_1}^{t_2} [h(x_2, t)\,u(x_2, t) - h(x_1, t)\,u(x_1, t)]\,dt. \tag{2}$$

Equating (1) to the quantity (2), we get the equation

$$\int_{x_1}^{x_2} [h(x, t_2) - h(x, t_1)]\,dx + \int_{t_1}^{t_2} [hu|_{x=x_2} - hu|_{x=x_1}]\,dt = 0, \tag{3}$$

which obviously is the integral law of conservation of the mass of liquid.

As usual, the following integral law of conservation stems from (3):

$$\oint_C h\,dx - hu\,dt = 0; \tag{4}$$

it is valid for any closed contour C of the plane of variables x, t, and the differential equation $\partial h/\partial t + (\partial/\partial x)\,hu = 0$ (5)
in the case of smooth flows.

The change of total momentum of the liquid in that section of the channel during the time from $t = t_1$ to $t = t_2$ is equal to the quantity

$$\rho l \int_{x_1}^{x_2} [hu|_{t=t_2} - hu|_{t=t_1}]\,dx, \tag{6}$$

The momentum of the liquid varies in this part of the channel due to two effects: the transport of momentum by the flow through the planes $x = x_1$ and $x = x_2$ in the amount

$$-\rho l \int_{t_1}^{t_2} \left[u^2 h\big|_{x=x_2} - u^2 h\big|_{x=x_1}\right] dt \qquad (7)$$

and also the momentum of the pressure forces and the planes $x = x_1$ and $x = x_2$.

Calculate the total pressure $p(x, t)$ acting in the channel cross section. Assuming that at the free surface of the liquid $z = h$ the pressure is zero, we will have by the barometric formula $p = \rho g(h - z)$ (g is gravitational acceleration) and

$$p(x, t) = l \int_0^h p\, dz = l\rho g \int_0^h (h - z)\, dz = \frac{1}{2} l\rho g h^2. \qquad (8)$$

Therefore the momentum of the pressure forces in the sections $x = x_1$ and $x = x_2$ during the time from $t = t_1$ to $t = t_2$ is given by the quantity

$$-\frac{l\rho g}{2} \int_{t_1}^{t_2} [h^2(x_2, t) - h^2(x_1, t)]\, dt. \qquad (9)$$

Now equating (6) to the sum of (7) and (9), we get the integral law of conservation of momentum of the liquid

$$\int_{x_1}^{x_2} \left[hu\big|_{t=t_2} - hu\big|_{t=t_1}\right] dx + \int_{t_1}^{t_2} \left\{\left[hu^2 + g\frac{h^2}{2}\right]\Big|_{x=x_2} - \left[hu^2 + g\frac{h^2}{2}\right]\Big|_{x=x_1}\right\} dt = 0. \qquad (10)$$

Equation (10) can be written as the integral law of conservation of momentum

$$\oint_C hu\, dx - \left(hu^2 + g\frac{h^2}{2}\right) dt = 0, \qquad (11)$$

from which follows this differential equation for smooth flows:

$$\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{g}{2}h^2\right) = 0. \qquad (12)$$

Combining equations (5) and (12), we get a system of two quasilinear equations for $h(x, t)$, $u(x, t)$:

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0, \quad \frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left[hu^2 + \frac{g}{2}h^2\right] = 0. \qquad (13)$$

Now we can easily note that the system of eqration (13) coincides with the system of equations in gas dynamics of an isentropic flow of an ideal gas with adiabat index $\gamma = 2$. In fact, if the quantity h is denoted by ρ and if we assume that $p = (g/2)\rho^2$, then system (13) turns into the corresponding system for the case under consideration (cf Chapter Two).

From this comparison we can in particular conclude that the system (13) is a system of hyperbolic quasilinear equations and that its solutions are generally speaking discontinuous. Corresponding to the discontinuity of the solution of system (13) is the sudden increase in the level h(x, t), the so-called "water leap." The usual Hugoniot's conditions and stability conditions must be satisfied at the discontinuity front.

2. Plane steady state flow of a compressible gas. Another well-known example of a system of hyperbolic quasilinear equations is the system of equations describing a plane steady state supersonic flow of a compressible gas. If u, v are the components of the velocity vector q, then this system is of the form

$$\left.\begin{aligned} u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}+v\frac{\partial\rho}{\partial y}+\rho\frac{\partial v}{\partial y}&=0,\\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}&=0,\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}&=0,\\ u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}&=0.\end{aligned}\right\} \qquad (1)$$

This system describes only smooth flows. The conservative form of equations (1) require when considering discontinuous flows is presented below.

The characteristic equation (of the fourth degree) for system (1) is of the form

$$\begin{vmatrix} v-\xi u & -\xi\rho & \rho & 0\\ -\xi\frac{c^2}{\rho} & v-\xi u & 0 & \frac{-\xi p'_S}{\rho}\\ \frac{c^2}{\rho} & 0 & v-\xi u & \frac{p'_S}{\rho}\\ 0 & 0 & 0 & v-\xi u\end{vmatrix}=0$$

or

$$(v-\xi u)^2\left[\xi^2(u^2-c^2)-2uv\xi+(v^2-c^2)\right]=0. \qquad (2)$$

where $\xi = dy/dx$ is the characteristic direction of system (1).

The first cofactor gives a double root $\xi = v/u$. The corresponding characteristic is obviously a streamline. Thus it is doubly degenerate. The second cofactor in the left side of (2) has real roots only when $q^2 = u^2 + v^2 \geqslant c^2$. In the acoustic case ($q = c$), both of these roots coincide and system (1), as we can easily check, is not hyperbolic. In contrast, in the supersonic case ($q > c$) it is of the hyperbolic type. To see this, it suffices to verify that corresponding to each eigenvalue $\xi = v/u$ there are two linearly independent eigenvectors, that is, two independent equations in the characteristic form of system (1). Uncomplicated manipulations lead to the following two equations that contain only differentiation along the streamlines:

$$\frac{dS}{dq} = 0, \qquad (3)$$

$$\frac{1}{\rho}\frac{dp}{dq} + \frac{d}{dq}\left(\frac{u^2+v^2}{2}\right) = 0 \qquad (4)$$

(here $d/dq = u\,\partial/\partial x + v\,\partial/\partial y$). Relation (3) implies the constancy of entropy at the streamline and, obviously, is independent of equation (4).

Introducing the function $H(\rho, S)$ given by the equation

$$H(\rho, S) = \int \frac{c^2(\rho, S)\,d\rho}{\rho},$$

from equations (3) and (4) we get $\quad d/dq\,(H(\rho, S) + q^2/2) = 0. \qquad (5)$

Thus, the streamlines correspond to two Riemann invariants: entropy S and $B = H + \frac{1}{2}q^2$. The equality B = constant following from (5) is called Bernoulli's integral.

The expression of the remaining (so-called acoustic) characteristic values of system (1) is as follows:

$$\xi^{\pm} = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - c^2} = \frac{v^2 - c^2}{uv \mp c\sqrt{q^2 - c^2}}.$$

The characteristics $dy/dx = \xi^{\pm}$ form with the streamlines the angles α and $-\alpha$, respectively, where $\sin\alpha = c/q = 1/M$, $M = q/c$ (the angle α is called Mach's angle, and the function M is the Mach number).

In conclusion, we present the divergent form of equations (1):

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,$$

$$\frac{\partial}{\partial x}(p + \rho u^2) + \frac{\partial}{\partial y}(\rho u v) = 0,$$

$$\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(p + \rho v^2) = 0,$$

$$\frac{\partial}{\partial x}\left[\rho u\left(e + \frac{p}{\rho} + \frac{q^2}{2}\right)\right] + \frac{\partial}{\partial y}\left[\rho v\left(e + \frac{p}{\rho} + \frac{q^2}{2}\right)\right] = 0.$$

3. Chemical sorption and chromatography problems [64]. Suppose that through a tube containing a sorptively active substance (sorbent) flows a liquid or gaseous mixture of compounds that are to be separated. Avoiding effects associated with the influence of tubing walls, we will assume the problem to be one-dimensional. Suppose t is time, x is coordinate along the axis of the sorption column, u_i is the concentration of i-th component in the mixture, a_i is the concentration of the i-th component in the sorbent, and V is the velocity of the mixture on the column, assumed constant.

Neglecting diffusion flows of substances both in the mixtures as well as in the sorbent, we write the equations of conservation of mass for each component:

$$\int_{x_1}^{x_2} \left\{[u_i + a_i]\big|_{t=t_2} - [u_i + a_i]\big|_{t=t_1}\right\} dx + \int_{t_1}^{t_2} V\,[u_i(x_2, t) - u_i(x_1, t)]\, dx = 0 \quad (1)$$

$$(i = 1, 2, \ldots, n),$$

which for smooth u_i, a_i reduced to the differential equations

$$V\frac{\partial u_i}{\partial x} + \frac{\partial}{\partial t}(u_i + a_i) = 0. \quad (2)$$

We will make the assumption that the sorption is instantaneous, that is, we will assume that at each point of the tube and at each time instant equilibrium obtains between the sorbed substance and the free mixture. Mathematically, this is expressed by the fact that the concentration of the sorbed substance is determined by the composition of the mixture, that is, the following functions hold:

$$a_i = f_i(u_1, \ldots, u_n) = f_i(u) \quad (i = 1, \ldots, n). \qquad (3)$$

Equation (3) usually called equations of the sorption isotherm. Given this condition, system of equations (2) is rewritten as

$$\frac{\partial}{\partial t}(u + f(u)) + V\frac{\partial u}{\partial x} = 0, \qquad (4)$$

where u and f are vectors with n components.

Characteristic values $\xi = \xi(u)$ of system (4) are determined from the equation

$$\mathrm{Det}\left(\left((V - \xi)\delta_{ij} - \xi\frac{\partial f_i}{\partial u_j}\right)\right) = 0. \qquad (5)$$

Let us denote by $\lambda = \lambda(u)$ the eigenvalue of the matrix $((\partial f_i/\partial u_j))$. Then, obviously, from equation (5) we get

$$\frac{V - \xi}{\xi} = \lambda, \quad \xi = \frac{V}{1 + \lambda}. \qquad (6)$$

We will assume (actually this follows from general principles of sorption) that all eigenvalues $\lambda_k(u)$ of the matrix $((\partial f_i/\partial u_j))$ are positive, i.e., the matrix $\partial f/\partial u$ is positively defined. Then from (6) it follows that the eigenvalues $\xi = \xi_k(u)$ of system (4) satisfies the inequalities

$$\xi_k(u) = \frac{V}{1 + \lambda_k(u)} < V \quad (V > 0). \qquad (7)$$

i.e., the velocity $\xi_k(u)$ of the characteristics $dx/dt = \xi_k(u)$ (8) of system (4) is smaller than the flow velocity V.

Let us consider in greater detail the case of a special sorption isotherm (3) when

$$a_i = \frac{a_i^{\infty} k_i u_i}{1 + \sum_{s=1}^{n} k_s u_s} \quad (i = 1, 2, \ldots, n). \qquad (9)$$

usually called the case of Langmuir sorption. Here a_i^{∞} is the saturation adsorption and k_i is the sorbability factor.

Introducing the notation $v_i = k_i u_i$, $\Gamma_i = a_i^{\infty} k_i$ (Henry's coefficients), let us rewrite the equations of Langmuir's isotherm (9) as

$$\varphi_i(v) = k_i a_i = \Gamma_i \frac{v_i}{1+\sum_{s=1}^{n} v_s} = \frac{\Gamma_i v_i}{p}, \qquad p = 1 + \sum_{s=1}^{n} v_s \quad (10)$$

and system (4) as

$$\frac{\partial}{\partial t}(v + \varphi(v)) + V\frac{\partial v}{\partial x} = 0. \quad (11)$$

We will assume that all Henry's coefficients Γ_i are distinct (if a series of Γ_i coincide, then the problem can be reduced to the case when Γ_i are distinct), and we will number them in increasing order:

$$\Gamma_1 < \Gamma_2 < \cdots < \Gamma_n. \quad (12)$$

For isotherm (10), the eigenvalues λ are determined from the equation

$$\mathrm{Det}(((\Gamma_i p - \lambda)\delta_{ij} - \Gamma_i v_i)) = 0. \quad (13)$$

Let us consider the case when $v_i \neq 0$ for all $i = 1, 2, \ldots, n$. Then equation (13) is transformed to become

$$\mathrm{Det}\left(\left(\frac{\Gamma_i p - \lambda}{\Gamma_i v_i}\delta_{ij} - 1\right)\right) = \left[\prod_{k=1}^{n} \frac{\Gamma_k p - \lambda}{\Gamma_k v_k}\right]\left(1 - \sum_{k=1}^{n} \frac{\Gamma_k v_k}{\Gamma_k p - \lambda}\right) = 0. \quad (14)$$

Equation (14) can also be written in the form $F(\lambda, v) = 1$, (15)

where

$$F(\lambda, v) = \sum_{k=1}^{n} \frac{\Gamma_k v_k}{\Gamma_k p - \lambda}. \quad (16)$$

Since

$$F'_\lambda(\lambda, v) = \sum_{k=1}^{n} \Gamma_k v_k (\Gamma_k p - \lambda)^{-2} > 0 \qquad (v_k > 0). \quad (17)$$

then the function $F(\lambda, v)$ is a monotonically increasing function of variable λ, which has zones at the points $\lambda_k^* = \Gamma_k p > 0$ (Figure 4.56). Since the roots $\lambda = \lambda_k(v)$ are the abscissae of the points of intersection of the graph of function $F(\lambda, v)$ with the straight line $F = 1$ (Figure 4.56) and

$$0 < F(0, v) = \sum_{k=1}^{n} \frac{\Gamma_k v_k}{\Gamma_k p} = 1 - \frac{1}{p} < 1,$$

We conclude that equation (15) has n real, distinct, positive roots $\lambda = \lambda_k(v)$. These roots satisfy the inequalities

$$0 < \lambda_1(v) < \lambda_1^\bullet, \quad \lambda_{l-1}^\bullet < \lambda_l(v) < \lambda_l^\bullet \qquad (18)$$

and by formula (6) the characteristic values $\xi = \xi_k(v)$ of system (11) are also real, distinct, and positive; here $V > \xi_1(v) > \xi_2(v) > \ldots > \xi_i(v) > 0$, (19) i.e., given the condition $\Gamma_i v_i \neq 0$ system (11) is hyperbolic in the narrow sense.

Calculating the left eigenvectors $\ell^k = \{\ell_i^k\}$ of system (11), we find (with an accuracy up to the multiplier)

$$l_l^k = \frac{1}{\Gamma_l p - \lambda_k(v)}, \qquad (20)$$

and the right $r^k = \{r_i^k\}$:

$$r_l^k = \frac{\Gamma_l v_l}{\Gamma_l p - \lambda_k(v)}. \qquad (21)$$

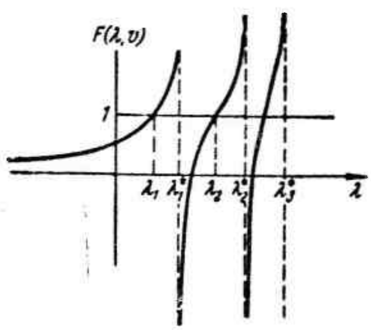

Figure 4.56

Note also certain singularities of system of equations (11).

For system (11) there exist n Riemann invariants, so that the system can be reduced to the form

$$\frac{\partial R_l}{\partial t} + \xi_k(R) \frac{\partial R_l}{\partial x} = 0 \qquad (l = 1, 2, \ldots, n), \qquad (22)$$

where

$$R_l(v) = \frac{\lambda_l(v)}{p}.$$

Discontinuities of the solution $v(x, t)$ of system (11), as usual, satisfy Hugoniot's conditions, which in this case become

$$D[v + \varphi(v)] = V[v]. \quad D = \frac{dx}{dt}. \qquad (23)$$

and the stability condition $\xi_k(v(x - 0, t) > D > \xi_k(v(x + 0, t))$. (24)

It is interesting to note that for system (11) the rarefaction waves (R_i = constant when $i \neq k$) of the k-th type coincide with Hugoniot's adiabat of the k-th type, i.e., a straight line $R_i(v)$ = constant ($i \neq k$) gives a solution of equations (23).

Due to the difference in the velocities of the characteristics $\xi_k(v)$ is explained by the means which the components of the mixture in the sorbent are separated. The method of separating the components, based on differences in Henry's coefficients, is called chromatography.

4. Applications in differential geometry. Problems in geometry are related to nonlinear differential equations. Therefore geometers first began systematic study of nonlinear differential equations and their solutions. It is not mere chance that the outstanding geometer of the past century Riemann obtained fundamental results in gas dynamics, which have in many respects remained unsurpassed even at the present time.

We will here point to the relationship between the theory of quasilinear equations with one of the fields of differential geometry -- the theory of surfaces.

Suppose that at some smooth surface in three-dimensional space the following metric is implemented: $ds^2 = dx^2 + B^2(x, y)dy^2$, (1)
where the lines y = constant are geodesic lines on the surface, and the lines x = constant represent a boundary of trajectories orthogonal to them. This system of coordinates (x, y) plotted at the surface is called a semigeodesic system.

The gaussian curvature $K(x, y)$ of the surface is defined only by the metric (1), using the formula $G''(x, y) + K(x, y)B(x, y) = 0$. (2)

If only metric (1) is given (the first quadratic form), then the question of whether a surface exists in three-dimensional Euclidean space that realizes this metric reduces to finding the coefficients $L(x, y)$, $M(x, y)$, and $N(x, y)$ of the second quadratic form. These coefficients must satisfy the main equations of the theory of surfaces — the Peterson-Codacci equations[*]):

$$M_x - L_y = -\frac{B_x}{B}M, \quad N_x - M_y = \frac{B_x}{B}(N + B^2 L) - \frac{B_y}{B}M. \qquad (3)$$

On the other hand, the gaussian curvature of the surface can be calculable also externally, through the coefficients of the second quadratic form. The Gauss formula

$$B^2 K = LN - M^2 \qquad (4)$$

is used for this purpose. Now if we cancel out of equation (3), using (4) one of the quantities L, M, and N, and we get a system of two quasilinear equations with two independent variables x, y with respect to the two unknown functions.

Investigating this system of equations, we can readily establish that it is a elliptical in the case $K > 0$, parabolic when $K = 0$, and hyperbolic in the case $K < 0$.

Thus, for the case of negative gaussian curvature $K < 0$, the Peterson-Codacci equations reduce to a system of two hyperbolic quasilinear equations. The integral curves of the equation

$$L\,dx^2 + 2M\,dx\,dy + N\,dy^2 = 0 \qquad (5)$$

will be characteristics in this case, lines which are called asymptotic lines of the surface, in geometry.

As we have seen in Chapter One, Riemann invariants can be introduced for any system of two hyperbolic quasilinear equations. Uncomplicated calculations lead to the following expressions for the invariants:

$$r = B\frac{-M - B\sqrt{-K}}{N}, \quad s = B\frac{-M + B\sqrt{-K}}{N}, \qquad (6)$$

after which equations (3) are reduced to the form

$$\left.\begin{aligned} &\frac{\partial r}{\partial x} + \frac{s}{B}\frac{\partial r}{\partial y} = -s(1 + r^2)\frac{B_x}{B} + \frac{(r - s)}{2}\left[\frac{\partial Q}{\partial x} + \frac{r}{B}\frac{\partial Q}{\partial y}\right], \\ &\frac{\partial s}{\partial x} + \frac{r}{B}\frac{\partial s}{\partial y} = -r(1 + s^2)\frac{B_x}{B} + \frac{(s - r)}{2}\left[\frac{\partial Q}{\partial x} + \frac{s}{B}\frac{\partial Q}{\partial y}\right], \\ &Q = \ln\sqrt{-K(x, y)}, \quad B = B(x, y). \end{aligned}\right\} \qquad (7)$$

---

[*]) cf Blyashke, Differential Geometry.

If we assume $r = \operatorname{tg} \varphi_1$, $s = \operatorname{tg} \varphi_2$, then φ_1, φ_2 are angles formed at the surface by the direction of the asymptotic lines,(characteristics) with the direction of the geodesic lines y = constant. Here the system (7) is also written in the following form:

$$\left.\begin{array}{l}\dfrac{d\varphi_1}{ds_2} = \cos\varphi_2 \dfrac{\partial \varphi_1}{\partial x} + \dfrac{\sin\varphi_2}{B} \dfrac{\partial \varphi_1}{\partial y} = -\sin\varphi_2 \dfrac{B_x}{B} + \dfrac{1}{2}\sin\omega \dfrac{dQ}{ds_1}, \\ \dfrac{d\varphi_2}{ds_1} = \cos\varphi_1 \dfrac{\partial \varphi_2}{\partial x} + \dfrac{\sin\varphi_1}{B} \dfrac{\partial \varphi_2}{\partial y} = -\sin\varphi_1 \dfrac{B_x}{B} - \dfrac{1}{2}\sin\omega \dfrac{dQ}{ds_2}, \\ \qquad\qquad \omega = \varphi_1 - \varphi_2 \end{array}\right\} \quad (8)$$

Now we construct on the basis of system of equation (7) or (8), we can readily note that system (7) is a weakly-nonlinear system of quasilinear equations, since

$$\xi_1 = \frac{s}{B(x, y)}, \quad \xi_2 = \frac{r}{B(x, y)}.$$

The regular surface of negative gaussian curvature $K(x, y) < 0$ exhibits different directions of the asymptote lines (characteristics), so that we can assume that at the regularity points $r \neq s$, i.e., $\xi_1 \neq \xi_2$, i.e., systems (7) and (8) are hyperbolic in the narrow sense at the regularity points.

As we have seen in Chapter One, weakly nonlinear systems have a remarkable property: solutions of such systems remain continuous and smooth as long as the solution itself is bounded. A similar property also holds for solutions of system (7) or (8).

At the beginning of this century, D. Hilbert formulated a hypothesis which states that no complete regular surface with negative gaussian curvature $K(x, y) < \varepsilon < 0$ embedded in a three-dimensional Euclidian space exists. He also gave a proof of this statement for the case of constant gaussian curvature $K = -1$. Since system (8) is weakly-nonlinear, the cause of the nonexistence of the surface $K(x, y) \leqslant -1$ is not the formation of discontinuity of the solution (as we would have thought), and also not the unboundedness of the solution (as can be seen from the expression (8), the solution remains bounded at the endpoint x, y). Therefore the cuase of the nonexistence of solutions of system (8) in the large is the degeneracy of the surface,i.e., the case $\varphi_1 = \varphi_2$, $\omega = 0$. In all known cases, actually, at the edge of the surface we obtain the result that $\omega = 0$ and the surface cannot be extended smoothly beyond the bound (edge).

Note that in recent years proof of the hypothesis formulated given certain restrictions on the derivatives of $K(x, y)$ has been obtained.

5. Equations of magnetic hydrodynamics. In this section we will obtain differential*) equations describing the one-dimensional motion of an electroconductive gas in a magnetic field.

If we let E and H denote, respectively, the intensity of an electrical and a magnetic field, then the force f acting on the side of the electromagnetic field for unit volume of gas can, as we know**) be written as

$$f = \rho_e E + \frac{1}{c}[j \times B], \qquad (1)$$

where ρ_e is the density of the electrical charge, j is the density of the electrical current, and c is the speed of light. The gaussian system of units for electromagnetic quantities has been adopted in formula (1).

When an electrical current j passes through a fixed substance, the following amount of energy (joule heat) is released per unit time per unit volume:

$$Q = Ej.$$

For the case when the conductor moves at velocity u, $Q = E'j'$, where E' and j' can be written in the following form, based on formulas of electrodynamics with an accuracy up to terms of the order u^2/c^2:

$$E' = E + \frac{1}{c}[u \times B], \quad j' = j - \rho_e u, \qquad (2)$$

where B is the induction of the magnetic field.

Electromagnetic fields satisfy the Maxwellian system of equations

$$\left.\begin{array}{ll} \operatorname{rot} H = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial D}{\partial t}, & \operatorname{rot} E = -\frac{1}{c}\frac{\partial B}{\partial t}, \\ \operatorname{div} D = 4\pi\rho_e, & \operatorname{div} B = 0. \end{array}\right\} \qquad (3)$$

---

*) In studying discontinuous solutions, we must derive the integral laws of conservation. For simplicity we will confine ourselves here to use smooth solutions of equations of magnetic hydrodynamics.

**) I. Ye. Tamm, Osnovy teorii elektrishestva (Fundamentals of the Theory of Electricity).

Here $D = \varepsilon E$, $B = \mu H$ (ε and μ are the electrical and magnetic permeabilities of the gases).

We will assume that the medium we are considering satisfies the condition of quasineutrality. This implies that the combined electrical charge of any volume element is zero ($\rho_e = 0$). Then $j' = j$. Usually a medium satisfying the condition of quasineutrality is called a plasma.

Finally, for a well-insulated plasma we can assume with sufficient accuracy that $B = H$, $D = E$.

Let us use Ohm's law to determine the density of current j:

$$j = \sigma E' = \sigma\left(E + \frac{1}{c}[u \times H]\right). \quad (4)$$

We obtain equations describing plasma motion. They consist of two groups: Maxwell equations in a moving medium and hydrodynamic equations. The latter must allow for the action of electromagnetic force (1) and the release per unit volume of Joule heat $Q = j^2/\sigma$.

Obviously, the continuity equation remains unchanged:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho u = 0, \quad (5)$$

and the force f enters into the equations of motion:

$$\frac{\partial u}{\partial t} + (u\nabla)u + \frac{1}{\rho}\operatorname{grad} p = \frac{f}{\rho}, \quad (6)$$

and Joule heat appears in the equation of the energy balance written for entropy S per unit mass of gas:

$$\frac{\partial S}{\partial t} + (u\nabla)S = \frac{Q}{\rho T} = \frac{j^2}{\rho T \sigma}. \quad (7)$$

Thus, a complete system of equations describing the motion of plasma in an electrical field is the system (3)-(7). Here the electromagnetic equations (3), (4) are related with the "hydrodynamic" equations (5)-(7) only by the right sides: the function j dependent on velocity u appears in (3), and the functions j and f dependent on E and H appear in (5)-(7).

In most case of practical interest, the system (3)-(7) can be somewhat simplified. The point is that even for a case of a fairly cold plasma, we can ordinarily neglect in system (3) the displacement current $\frac{i}{4\pi}\frac{\partial D}{\partial t}$ compared with the conductivity current j. Here obviously, we must discard the equation div D = 0. Then from the system of equations (3)-(7) we cancel out E, and equations (3) become

$$\left.\begin{aligned} &\operatorname{div} H = 0, \\ &\frac{\partial H}{\partial t} + \operatorname{rot}\left[\frac{c^2}{4\pi\sigma}\operatorname{rot} H - (u \times H)\right] = 0. \end{aligned}\right\} \qquad (8)$$

Consider the further simplification of the system, assuming that the electroconductivity of the plasma σ is infinitely great, that is, the gas under consideration is an ideal conductor. With this assumption, Ohm's law (4) is replaced by the condition of the finiteness of current j, i.e., with the equation

$$E = -(1/c)[u \times H],$$

and equations (8) become

$$\operatorname{div} H = 0,$$

$$\frac{\partial H}{\partial t} + \operatorname{rot}[u \times H] = 0. \qquad (9)$$

The system of equations (5)-(7) is very considerably simplified in this case. Since

$$j = \frac{c}{4\pi}\operatorname{rot} H, \quad f = \frac{1}{c}[j \times H] = \frac{1}{4\pi}[\operatorname{rot} H \times H]$$

and

$$\operatorname{rot} H \times H = -\operatorname{grad}\frac{H^2}{2} + \frac{\partial}{\partial x}(H_x H) + \frac{\partial}{\partial y}(H_y H) + \frac{\partial}{\partial z}(H_z H),$$

then equations(6) are transformed to become

$$\frac{\partial u}{\partial t} + (u\nabla)u + \frac{1}{\rho}\operatorname{grad}\left(p + \frac{H^2}{8\pi}\right) = \frac{1}{4\pi\rho}\left[\frac{\partial}{\partial x}(H_x H) + \frac{\partial}{\partial y}(H_y H) + \frac{\partial}{\partial z}(H_z H)\right]. \qquad (10)$$

and equation (7) changes into adiabaticity condition

$$\frac{\partial S}{\partial t} + (u\nabla)S = 0. \qquad (11)$$

Let us consider one-dimensional motion, that is, let us assume that all the quantities depend only on x and t. Then from equations (9) it follows that at $H_x = H_0$ = constant.

Let us write our system of equations (5), (9), (10), and (11):

$$
\left.
\begin{aligned}
\frac{\partial H_y}{\partial t} + \frac{\partial}{\partial x}(u_x H_y - u_y H_x) &= 0\\
\frac{\partial H_z}{\partial t} + \frac{\partial}{\partial x}(u_x H_z - u_z H_x) &= 0\\
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{H_y^2 + H_z^2}{8\pi}\right) &= 0\\
\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} - \frac{1}{4\pi\rho}\frac{\partial}{\partial x}(H_x H_y) &= 0\\
\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} - \frac{1}{4\pi\rho}\frac{\partial}{\partial x}(H_x H_z) &= 0\\
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} &= 0\\
\frac{\partial S}{\partial t} + u_x \frac{\partial S}{\partial x} &= 0
\end{aligned}
\right| \qquad (12)
$$

System (12) associates seven variables: ρ, S, u_x, u_z, H_y, H_z with seven equations. Writing the characteristic equation for system (12), we get

$$
(u_x - \xi)[(u_x - \xi)^2 - a^2]\left\{(u_x - \xi)^2\left[(u_x - \xi)^2 - \frac{H^2}{4\pi\rho}\right] - c^2[(u_x - \xi)^2 - a^2]\right\} = 0 \qquad (13)
$$

where

$$
c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S, \quad a^2 = \frac{H_x^2}{4\pi\rho}, \quad b^2 = \frac{H_y^2 + H_z^2}{4\pi\rho}.
$$

The first cofactor gives the usual entropy characteristic (streamline) $\xi_0 = u_x$. The second cofactor gives the so-called Alfven characteristics $\xi_{\pm 1} = u_x \pm a$. Finally, the last cofactor in ([illegible]) gives an additional [illegible] for real roots:

$$
\xi_{\pm 2} = u_x \pm a_-; \quad \xi_{\pm 3} = u_x \pm a_+,
$$

where

$$
a_- = \frac{1}{\sqrt{2}}\sqrt{(c^2 + a^2 + b^2) - \sqrt{(c^2 + a^2 + b^2)^2 - 4c^2a^2}},
$$

$$
a_+ = \frac{1}{\sqrt{2}}\sqrt{(c^2 + a^2 + b^2) + \sqrt{(c^2 + a^2 + b^2)^2 - 4c^2a^2}}.
$$

The characteristics are called, respectively, slow and fast magnetoacoustic characteristics.

In the case when the longitudinal component of the magnetic field is absent ($H_c = 0$), we have $a = a_- = 0$, such that system (12) has a fivefold degenerate characteristic $\xi = u_x$.

The three independent equations for this characteristic are the fourth, fifth, and seventh equations of system (12). To obtain an additional two equations, we compute $\frac{\partial}{\partial t}\left(\frac{H_y}{\rho}\right)$, $\frac{\partial}{\partial t}\left(\frac{H_z}{\rho}\right)$.

From system (12) we get

$$\frac{\partial}{\partial t}\left(\frac{H_y}{\rho}\right) = -\frac{1}{\rho}\frac{\partial u_x H_y}{\partial x} - \frac{H_y}{\rho^2}\frac{\partial\rho}{\partial t} = -\frac{1}{\rho}\frac{\partial u_x H_y}{\partial x} + \frac{H_y}{\rho^2}\frac{\partial\rho u_x}{\partial x} =$$

$$= \frac{-1}{\rho}\left(\rho u_x \frac{\partial\left(\frac{H_y}{\rho}\right)}{\partial x} + \frac{H_y}{\rho}\frac{\partial\rho u_x}{\partial x}\right) + \frac{H_y}{\rho^2}\frac{\partial\rho u_x}{\partial x} = -u_x\frac{\partial\left(\frac{H_y}{\rho}\right)}{\partial x}.$$

Therefore the desired relations are as follows:

$$\frac{\partial}{\partial t}\left(\frac{H_y}{\rho}\right) + u_x\frac{\partial}{\partial x}\left(\frac{H_y}{\rho}\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{H_z}{\rho}\right) + u_x\frac{\partial}{\partial x}\left(\frac{H_z}{\rho}\right) = 0.$$

From these it follows that along the streamline H/ρ = constant. (14)

We can state that equality (14) expresses the "frozenness" of the magnetic field.

Using integral (14), we reduce system (12) in the case $H_c = 0$ to three equations:

$$\left.\begin{array}{c}\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_x}{\partial x} = 0, \quad \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}p_э = 0, \\ \frac{\partial S}{\partial t} + u_x\frac{\partial S}{\partial x} = 0.\end{array}\right\}$$

where $p_э$ denotes the so-called effective magnetohydrodynamic pressure:

$$p_э = p(\rho, S) + A\rho^2. \quad [э = \text{effective}]$$

The quantity A, just as entropy S, can vary in the transition from one streamline to another, but is constant along any of them.

In the case under consideration $H_c = 0$, the fast magnetoacoustic waves propagate relative to the gas with the velocity $a_+ = \sqrt{c^2 + b^2} > c$.

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