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UNSTABLE LINEAR DIFFERENTIAL GAMES

by
M. S. Nikol'skiy

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13. ABSTRACT

The direct method developed by I. S. Pontryagin (On linear differential games. II. Doklady AN SSSR, Vol 175, No 4, 1967) for solving linear differential games is generalized to the non-stationary case.

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NOTICE

§1. Suppose the motion of n-dimensional vector z, in Euclidean space R^n, is described by a linear vector differential equation

$$\dot{z} = A(t)\ z - u + v, \tag{1}$$

where A(t) is a quadratic matrix of order n, continually dependent on t ($- \infty < < t < + \infty$); the control parameters u and v belong to the convex compacts P(t), Q(t) respectively, which are embedded in R^n and change continually over ($- \infty < < t < + \infty$). Parameter u is controlled by the pursuer; parameter v -- by the evader. Suppose a convex closed terminal set M is fixed in R^n. Pursuit begins from point $z_0 \bar{\in} M$ at moment t_0 and is considered completed when z(t) (see (1)) first contacts M.

The goal of the pursuer is to bring point z(t) to M as rapidly as possible. It is assumed that the pursuer knows z(t) and v(t) at each moment in time t, i.e. pursuit with discrimination of the evader is studied. The evader acts arbitrarily, using measurable control v(t), which follows the requirement $v(t) \in Q(t)$.

We will say that game (1) can be completed from position (z_0, t_0) in a finite time if there is a number $t(z_0, t_0)$ such that with any measurable change v(t) the pursuer, using his information, can construct a measurable change u(t) ($u(t) \in P(t)$), such that point z(t) strikes M not later than momemnt $t_0 + t(z_0, t_0)$.

One of the most important problems arising in the theory of pursuit is the problem of separation of those points (z_0, t_0) from which the game can be completed in a finite time. Strong results have been produced in this direction for stable games (see [1-7] and others). The most complete results have been produced by L. S. Pontryagin in [4]. They were produced by a direct method with a wider area of application than that of the first direct method developed in [3].

The present article is dedicated to a generalization of the second direct method of L. S. Pontryagin (see [4]) to the unstable case (see (1)).

§2. In this paragraph, we will introduce certain concepts which will be useful for the future.

A. Let U(τ) be a convex compact belonging to R^n, continually dependent on τ in sector [p, q], where $p \leq q$. Let us study all possible measurable vector functions u(·) in [p, q], satisfying the condition $u(\tau) \in U(\tau)$. Let us study the set of vector integrals $\int_p^q u(\tau)\,d\tau$ and represent it by $\int_p^q U(\tau)\,d\tau$. It is not difficult to see that this set is a convex limited set. Using [8], it is not difficult to prove that $\int_p^q U(\tau)\,d\tau$ is a closed set. Thus, we have the operation of integration of a closed set dependent on a parameter.

B. The geometric difference of two convex sets $\mathfrak{A}_1$, $\mathfrak{A}_2$, belonging to R^n refers (see [3]) to the set $\mathfrak{A}_3$, which consists of all vectors a translating $\mathfrak{A}_2$ into , i.e. $a + \mathfrak{A}_2 \subset \mathfrak{A}_1$. This operation is represented as: $\mathfrak{A}_3 = \mathfrak{A}_1 \overset{*}{-} \mathfrak{A}_2$. It is not difficult to show that closure of $\mathfrak{A}_1$ indicates closure of $\mathfrak{A}_3$.

C. Suppose $\mathfrak{A}_1, \mathfrak{A}_2$ are arbitrary sets from R^n. The algebraic sum of these sets refers to the set $\mathfrak{A}_3$ of all vectors a_3 of the form $a_3 = a_1 + a_2$, where $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, and will be written as $\mathfrak{A}_3 = \mathfrak{A}_1 + \mathfrak{A}_2$.

D. In [4], L. S. Pontryagin introduced the concept of the alternative integral from U(τ), V(τ), belonging to R^n and changing continually over the sector [p, q] ($p \leq q$) of convex compacts with the initial closed convex set B. This integral is represented by the symbol $\int_{B,p}^{q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau]$.

We require the altered interval

$$\int_{E,p}^{q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau], \qquad (2)$$

fixing not the initial integration set, but rather the final integration set. In constructing integral (2), we will base ourselves on rational subdivisions ω of sector [p, q] by means of points $p = \tau_0 < \tau_1 < \dots < \tau_k = q$, where $\tau_1, \dots, \tau_{k-1}$ are rational numbers. This rational subdivision ω is compared with the convex set

$$\Sigma_\omega = \left(\left(\left(\left(B + \int_{\tau_{k-1}}^{\tau_k} U(\tau)\,d\tau\right) \overset{*}{-} \int_{\tau_{k-1}}^{\tau_k} V(\tau)\,d\tau\right) + \right.\right.\right. \tag{3}$$

$$+ \int_{\tau_{k-2}}^{\tau_{k-1}} U(\tau)\,d\tau\Bigr) \overset{*}{-} \int_{\tau_{k-2}}^{\tau_{k-1}} V(\tau)\,d\tau\Bigr) + \ldots,$$

which we will call the integral sum.

By integral (2), we refer to the intersection of sets Σ_ω with respect to all rational divisions ω:

$$\int_p^q [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau] = \bigcap_\omega \Sigma_\omega. \tag{4}$$

We note that the existence of integral (2) as a non-empty set requires non-emptiness of all Σ_ω. If $\bigcap_\omega \Sigma_\omega$ is not empty, the altered integral (2) is a closed convex set.

Suppose integral (2) is not empty and rational point $t_1 \in (p, q)$ is selected on sector $[p, q]$. Let us study the rational divisions ω' of the form $p = \tau_0 < < \tau_1 = t_1 < \tau_2 < \ldots < \tau_k = q$.

Obviously,

$$\bigcap_\omega \Sigma_\omega \subset \bigcap_{\omega'} \Sigma_{\omega'}. \tag{5}$$

Let us study the rational subdivision ω'' of sector $[t_1, q]$, generated by subdivision ω'. It follows from formula (3) that

$$\Sigma_{\omega'} = \left(\Sigma_{\omega''} + \int_p^{t_1} U(\tau)\,d\tau\right) \overset{*}{-} \int_p^{t_1} V(\tau)\,d\tau. \tag{6}$$

Rational subdivisions ω'' are always even numbers. They can be renumbered: ω_1'', ω_2'', ... Let us represent by $\mu_i (i = 1, 2, \ldots)$ the rational subdivision produced by combining the points of the subdivisions ω_1'', ..., ω_i''. The integral sum (3) corresponding to rational subdivision μ_i will be represented by Σ_{μ_i}.

Work [4] gives the following formulas:

$$(A+U) \overset{*}{-} V = A \overset{*}{-} (U+V), \quad (A+U) \overset{*}{-} V \supset (A \overset{*}{-} V) + U,$$

where A, U, V are convex sets in R^n. Using definition Σ_{μ_i} and these formulas, it is not difficult to show that $\Sigma_{\mu_1} \supset \Sigma_{\mu_2} \supset \ldots$ and that

$$\int_{t_1}^{p,q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau] = \bigcap_{i=1}^{\infty} \Sigma_{\mu_i}. \tag{7}$$

Let us show that

$$\bigcap_{\omega'} \Sigma_{\omega'} \subset \left(\bigcap_{i=1}^{\infty} \Sigma_{\mu_i} + \int_p^{t_1} U(\tau)\,d\tau\right) \overset{*}{-} \int_p^{t_1} V(\tau)\,d\tau. \tag{8}$$

Let us study the rational subdivision of sector [p, q] ω_i', generated by point t_1 (t_1 is a rational number) and subdivisions μ_i. Obviously $\bigcap_{\omega'} \Sigma_{\omega'} \subset$ $\subset \bigcup_{i=1}^{\infty} \Sigma_{\omega_i'}$.

To prove inclusion (8), it is sufficient to prove inclusion

$$\bigcap_{i=1}^{\infty} \Sigma_{\omega_i'} \subset \left(\bigcap_{i=1}^{\infty} \Sigma_{\mu_i} + \int_p^{t_1} U(\tau)\,d\tau\right) \overset{*}{-} \int_p^{t_1} V(\tau)\,d\tau. \tag{9}$$

Let us study point ξ_0, satisfying the condition $\xi_0 \in \bigcap_{i=1}^{\infty} \Sigma_{\omega_i'}$. Due to equation (6), this inclusion indicates the relationship

$$\xi_0 + \int_p^{t_1} V(\tau)\,d\tau \subset \Sigma_{\mu_i} + \int_p^{t_1} U(\tau)\,d\tau,$$

which is correct with any i = 1, 2, ... It follows from this that for any given measurable vector function $v(\tau)$, $p \leq \tau \leq t_1$ ($v(\tau) \in V(\tau)$), a measurable vector function $u_i(\tau)$, $p \leq \tau \leq t_1$ ($u_i(\tau) \in U(\tau)$), can be found such that

$$\xi_0 + \int_p^{t_1} v(\tau)\,d\tau - \int_p^{t_1} u_i(\tau)\,d\tau = \eta_i \in \Sigma_{\mu_i}. \tag{10}$$

On the strength of the assumed continuity of sets $U(\tau)$, $V(\tau)$ in sector $[p, q]$, the estimate $|\eta_i| \leq$ const is correct for η_i. Therefore, it can be considered that a certain subsequence of vectors η_i, which we will represent by η_{i_k}, converges to a certain vector η^*. The embeddedness of closed sets Σ_{μ_i} indicates $\eta^* \in \bigcap_{i=1}^{\infty} \Sigma_{\mu_i}$.

We can consider that subsequence $u_{i_k}(\cdot)$ converges weakly in sector $[p, t_1]$ to a certain function $u_0(\cdot)$ $(u_0(\tau) \in U(\tau))$. Then from equation (10) it follows easily that

$$\xi_0 + \int_p^{t_1} v(\tau)\,d\tau - \int_p^{t_1} u_0(\tau)\,d\tau = \eta^* \in \bigcap_{i=1}^{\infty} \Sigma_{\mu_i}.$$

From this we get

$$\xi_0 \in \left(\bigcap_{i=1}^{\infty} \Sigma_{\mu_i} + \int_p^{t_1} U(\tau)\,d\tau\right) \overset{*}{-} \int_p^{t_1} V(\tau)\,d\tau,$$

i.e. inclusion (9) is proven and, consequently, inclusion (8) is proven. Using inclusion (5) and equations (4), (7), we produce an important formula:

$$\int_p^{B,q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau] \subset \left(\int_{t_1}^{B,q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau] + \int_p^{t_1} U(\tau)\,d\tau\right) \overset{*}{-} \int_p^{t_1} V(\tau)\,d\tau, \tag{11}$$

where t_1 is either any rational number in $[p, q]$, or a number corresponding with one of the ends of $[p, q]$.

The following will be useful in producing further properties of the altered integral (2).

E. Let $\mathfrak{A}_1(s)$, $\mathfrak{A}_2(s)$ be convex compacts from R^n, dependent on parameter s. Suppose in a certain area of point s_0, set $\mathfrak{A}_1(s) \overset{*}{-} \mathfrak{A}_2(s)$ is not empty, while at point $\mathfrak{A}_1(s)$, it is upward semicontinuous relative to inclusions (see [8]); $\mathfrak{A}_2(s)$ is continuous. We then have the following

Lemma. The set $\mathfrak{A}_1(s) \underline{*} \mathfrak{A}_2(s)$ is upper semicontinuous relative to inclusions at point s_0.

The proof of the lemma is simple, and we will not present it.

F. Let us represent integral (2) as a function of the lower limit of p through B(p). Let us assume in formula (11) $-p + t_1 = \varepsilon$. Then inclusion (11) can be rewritten as:

$$B(p) \subset \left(B(p+\varepsilon) + \int_p^{p+\varepsilon} U(\tau)\,d\tau \right) \underline{*} \int_p^{p+\varepsilon} V(\tau)\,d\tau, \tag{12}$$

where $p + \varepsilon$ is either any rational number from sector [p, q] or one of the ends of sector [p, q].

Let us prove that inclusion (12) is correct with any $p + \varepsilon$ belonging to sector [p, q].

Let us study the sequence of numbers $p_i \leq p$ such that $p_i + \varepsilon \in [p, q]$, $p_i + \varepsilon$ is a rational number and $p_i + \varepsilon \to p + \varepsilon$, where $p + \varepsilon$ is an arbitrary fixed number from interval (p, q). From inclusion (12)

$$B(p) + \int_p^{p_i+\varepsilon} V(\tau)\,d\tau \subset B(p_i + \varepsilon) + \int_p^{p_i+\varepsilon} U(\tau)\,d\tau. \tag{13}$$

Let us study a certain rational subdivision ω of sector $[p + \varepsilon, q]$, generated by points $p + \varepsilon = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k = q$, and rational subdivisions ω_i (i = 1, 2, ...) of sector $[p_i + \varepsilon, q]$, generated by points $p_i + \varepsilon = \tau_0^i < \tau_1 < \tau_2 < \ldots < \tau_k = q$. Thus, subdivisions ω and ω_i differ only in their left point.

Let us represent by Σ_ω and Σ_{ω_i} (i = 1, 2, ...) the integral sums (3), corresponding to divisions ω, ω_i with finite set B. It follows from inclusion (13) that

$$B(p) + \int_p^{p_i+\varepsilon} V(\tau)\,d\tau \subset \Sigma_{\omega_i} + \int_p^{p_i+\varepsilon} U(\tau)\,d\tau, \qquad i = 1, 2, \ldots$$

Let us take arbitrary vector $b \in B(p)$. It follows from the inclusion produced

$$b + \int_{p}^{p_i+r} V(\tau)\,d\tau \subset \Sigma_{\omega_i} + \int_{p}^{p_i+\varepsilon} U(\tau)\,d\tau. \tag{14}$$

Using the definition of the operation $\underset{-}{*}$, formula (3), the even limited nature of sets U(τ), V(τ) in sector [p, q] and inclusion (14), we can prove the inclusion

$$b + \int_{p}^{p_i+r} V(\tau)\,d\tau \subset \Sigma_{\omega_i}(B_1) + \int_{p}^{p_i+\varepsilon} U(\tau)\,d\tau, \qquad i = 1, 2, \ldots, \tag{15}$$

where B_1 is a convex compact belonging to B, while $\Sigma_{\omega_i}(B_1)$ is an integral sum constructed with respect to subdivision ω_i but with finite set B_1. This statement is trivial when B is limited and interesting when B is unlimited.

Using the lemma of point "E" and inclusion (15), it is not difficult to produce the relationship

$$b + \int_{p}^{p+r} V(\tau)\,d\tau \subset \Sigma_{\omega}(B_1) + \int_{p}^{p+\varepsilon} U(\tau)\,d\tau \subset \Sigma_{\omega} + \int_{p}^{p+\varepsilon} U(\tau)\,d\tau, \tag{16}$$

where $\Sigma_\omega(B_1)$ is an integral sum corresponding to rational subdivision ω, with finite set B_1.

Earlier in point "D" we showed that the altered integral (2) can be produced as the intersection of integral sums Σ_{μ_i} (i = 1, 2, ...), forming a sequence of sets embedded in each other $\Sigma_{\mu_1} \supset \Sigma_{\mu_2} \supset \ldots$ Taking such a sequence of subdivision μ_i(i = 1, 2, ...) as the ω in inclusion (16), we produce

$$b + \int_{p}^{p+\varepsilon} V(\tau)\,d\tau \subset \Sigma_{\mu_i} + \int_{p}^{p+\varepsilon} U(\tau)\,d\tau.$$

Using the limited nature of closed set $\int_{p}^{p+\varepsilon} U(\tau)\,d\tau$, , the embeddedness of the closed sets Σ_{μ_i} and the equation $B(p+\varepsilon) = \bigcap_{i=1}^{\infty} \Sigma_{\mu_i}$, , it is not difficult to prove that

$$b + \int_{p}^{p+\varepsilon} V(\tau)\,d\tau \subset B(p+\varepsilon) + \int_{p}^{p+\varepsilon} U(\tau)\,d\tau.$$

Since b is an arbitrary element from B(p), inclusion (12) is proven for arbitrary point p + ε $\in$ (p, q).

§3. Everything is now prepared for investigation of game (1) using the altered integral. Let us represent by $C(t, \tau)$ $(t \geq \tau)$ the matrixant of the homogeneous system $\dot{y} = A(t)y$ (for a definition of a matrixant and its properties, see [9]). We recall only that if measurable controls $u(\cdot)$, $v(\cdot)$ ($u(\tau) \in P(\tau)$, $v(\tau) \in Q(\tau)$) are fixed in the sector $[t_0, t]$, then according to the Cauchy formula

$$z(t) = C(t, t_0) z_0 + \int_{t_0}^{t} C(t, \tau)(-u(\tau) + v(\tau))\, d\tau.$$

Let us study the altered integral

$$W(t, t_0) = \int_{t_0}^{M, t} [C(t, \tau) P(\tau)\, d\tau \overset{*}{-} C(t, \tau) Q(\tau)\, d\tau] \tag{17}$$

where $t \geq t_0$. We will assume that the set $W(t, t_0)$ is not empty with all t, t_0 $(t \geq t_0)$. Let us study also vector $C(t, t_0)z_0 (t \geq t_0)$. Two cases are possible: 1) with no t does vector $C(t, t_0)z_0$ belong to $W(t, t_0)$; 2) there is at least one $\bar{t}$ with which the inclusion

$$C(\bar{t}, t_0)z_0 \in W(\bar{t}, t_0). \tag{18}$$

is true.

In the first case, we can say nothing concerning the possibility of completion of pursuit from point (z_0, t_0).

Let us study the second case.

Lemma. There is a minimum $\bar{t}$ for which inclusion (18) is fulfilled.

Proof. There are two possibilities: a) there is a finite number of moments $\bar{t}$ at which inclusion (18) is fulfilled; b) there is an infinite number of moments $\bar{t}$, at which inclusion (18) is fulfilled. In case "a" everything is clear. In case "b" we can take the decreasing sequence of numbers $\bar{t}_i$, which converges to the lower bound of all numbers $\bar{t}$ satisfying condition (18). Let us assume that the limit of $\bar{t}_i$ is equal to $t_0 + T(z_0, t_0)$.

For brevity we will write T in place of $T(z_0, t_0)$. Let us take a certain rational subdivision of sector $[t_0, t_0 + T]$. It is generated by the points $\tau_0 = t_0 < \tau_1 < \ldots < \tau_k = t_0 + T$. Let us study also the rational subdivisions ω_i of sectors $[t_0, t_i]$, produced from subdivision ω as follows: $\tau_0 = t_0 <$

$< \tau_1 < \ldots < \tau_{k-1} < \tau_k^i = \bar{t}_i$ $(i = 1, \ldots)$. Thus, they differ from subdivision ω only in the rightmost point.

Let us represent by Σ_ω the integral sum (3) corresponding to subdivision ω with finite set M and

$$U(\tau) = C(t_0 + T, \tau) P(\tau), \quad V(\tau) = C(t_0 + T, \tau) Q(\tau), \qquad t_0 \leqslant \tau \leqslant t_0 + T.$$

We represent by Σ_{ω_i} the integrals sum (3), corresponding to the division ω_i $(i = 1, 2, \ldots)$ with finite set M and

$$U_i(\tau) = C(t_0 + \bar{t}_i, \tau) P(\tau), \quad V_i(\tau) = C(t_0 + \bar{t}_i, \tau) Q(\tau), \quad t_0 \leqslant \tau \leqslant t_0 + t_i.$$

It follows from the definition of $\bar{t}_i$ that $C(\bar{t}_i, t_0) z_0 \in \Sigma_{\omega_i}$.

Let us study the curve $C(t, t_0) z_0$ as a function of parameter t in sector $[t_0, \bar{t}_1]$. Obviously, there is a sphere D with its center at the coordinate origin so large that this curve will be within it where $t_0 \leq t \leq \bar{t}_1$. For the following, it is sufficient to study the set $\Sigma_{\omega_i} \cap D$, $\Sigma_\omega \cap D$.

Using formula (3) for Σ_{ω_i} and Σ_ω, the even limitation of sets $P(\tau)$, $Q(\tau)$ in $[t_0, \bar{t}_1]$ and the definition of the operation $\underset{-}{*}$, it is not difficult to prove that Σ_{ω_i} and Σ_ω correspond in sphere D with the integral sums $\Sigma_{\omega_i}(M_1)$ and $\Sigma_\omega(M_1)$ respectively, constructed on the basis of the subdivisions ω_i and ω and the same sets $U_i(\tau)$, $V_i(\tau)$, $U(\tau)$, $V(\tau)$, as Σ_{ω_i}, Σ_ω, but with finite set M_1, where M_1 is a convex compact, independent of the number i and belonging to M. This statement is trivial with limited M and interesting for unlimited M. It follows from the above that $C(\bar{t}_i, t_0) z_0 \in \Sigma_{\omega_i}(M_1)$. We note that $\Sigma_{\omega_i}(M_1) \subset \Sigma_{\omega_i}$, $\Sigma_\omega(M_1) \subset \Sigma_\omega$.

Let us now study set $\Sigma_{\omega_i}(M_1)$ as a function of $\bar{t}_i$. Using the upper semi-continuity of operation $\underset{-}{*}$ relative to inclusions (see paragraph 2, "E"), it is easy to prove that the fixed ε can be used to find a number $N(\varepsilon)$, such that $i > N(\varepsilon)$

$$C(\bar{t}_i, t_0) z_0 \in \Sigma_\omega(M_1) + S_\varepsilon \subset \Sigma_\omega + S_\varepsilon, \tag{19}$$

where S_ε is a sphere of radius ε with its center at the coordinate origin. From inclusion (19) it is not difficult to see that $C(t_0 + T, t_0)z_0 \in \Sigma_\omega + S_\varepsilon$. Since was an arbitrary rational subdivision of sector $[t_0, t_0 + T]$, while ε is an arbitrary positive number, it follows from this that

$$C(t_0 + T, t_0) z_0 \in W(t_0 + T, t_0), \tag{20}$$

which was to be proven.

Theorem. Pursuit can be completed from point z_0, t_0 in time $T(z_0, t_0)$ if the pursuer knows control v(s) of the evader at each moment t on sector $t \leq s \leq t + \varepsilon (\varepsilon > 0$ is arbitrary).

The basis of the reality of this hypothesis of information in the hands of the pursuer is presented in [4].

Proof. According to our assumption, the pursuer knows the control of the evader in sector $[t_0, t_0 + \varepsilon]$; suppose this control is $v(\cdot)$ $(v(\tau) \in Q(\tau)$ where $t_0 \leq \tau \leq t_0 + \varepsilon)$.

Without limiting generality, we can consider $\varepsilon \leq T(z_0, t_0)$. Subsequently to simplify our inscription, let us write T in place of $T(z_0, t_0)$. Using inclusion (12) for the altered integral $W(t, t_0)$ (see (17)), we produce

$$W(t_0 + T, t_0) \subset (W(t_0 + T, t_0 + \varepsilon) + \\ + \int_{t_0}^{t_0+\varepsilon} C(t_0 + T, \tau) P(\tau) d\tau \overset{*}{-} \int_{t_0}^{t_0+\varepsilon} C(t_0 + T, \tau) Q(\tau) d\tau,$$

from which, using the definition of the operation $\overset{*}{-}$ (see paragraph 2, "B"), we easily produce the inclusion

$$W(t_0 + T, t_0) \subset W(t_0 + T, t_0 + \varepsilon) + \int_{t_0}^{t_0+\varepsilon} C(t_0 + T, \tau) P(\tau) d\tau - \\ - \int_{t_0}^{t_0+\varepsilon} C(t_0 + T, \tau) v(\tau) d\tau.$$

from which and from formula (20) it follows that a measurable control $u(\cdot)$ $(t_0 \leq \tau \leq t_0 + \varepsilon,\ u(\tau) \in P(\tau))$ is found such that

$$C(t_0+T, t_0)z_0 - \int_{t_0}^{t_0+\varepsilon} C(t_0+T, \tau)u(\tau)d\tau + \\ + \int_{t_0}^{t_0+\varepsilon} C(t_0+T, \tau)v(\tau)d\tau \in W(t_0+T, t_0+\varepsilon). \tag{21}$$

As we know (see [9]), matrixant $C(t, t_0)$ has the property $C(t, t_0 = C(t, t_1)C(t_1, t_0)$, where $t_0 \le t_1 \le t$.

Using this property, we produce from formula (21)

$$C(t_0+T, t_0+\varepsilon)(C(t_0+\varepsilon, t_0)z_0 - \int_{t_0}^{t_0+\varepsilon} C(t_0+\varepsilon, \tau)u(\tau)d\tau + \\ + \int_{t_0}^{t_0+\varepsilon} C(t_0+\varepsilon, \tau)v(\tau)d\tau = C(t_0+T, t_0+\varepsilon)z(\varepsilon) \in W(t_0+T, t_0+\varepsilon).$$

From which it follows that for point $(z(\varepsilon), t_0 + \varepsilon)$, the inequality $T(z(\varepsilon), t_0 + \varepsilon) \le T - \varepsilon$ is correct. Thus, we have decreased time $T(z_0, t_0)$ by at least ε in time ε. Performing similar steps further, the pursuer will complete pursuit in a time $\le T(z_0, t_0)$ (we note that $W(t, t) = M$).

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