PROJECT ON TECHNIQUES OF OBJECTIVE FACTOR ANALYSIS

An Investigation of the Goodness of Fit of the Maximum Likelihood Estimation procedure in Factor Analysis

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Report TR 2

Sponsored by
Personnel & Training Research Programs
Office of Naval Research
Arlington, Virginia 22217
Contract No. N00014-67-A-0305-0012

June 1971

Report No. 451

AN INVESTIGATION OF THE GOODNESS OF FIT OF THE MAXIMUM LIKELIHOOD ESTIMATION PROCEDURE IN FACTOR ANALYSIS

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JUNE 1971

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This work was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Psychology, 1971.

ACKNOWLEDGMENT

The author wishes to thank all of those individuals and organizations who contributed to one or more phases of the research presented in this thesis. The members of the thesis committee who gave many helpful suggestions when the thesis was in the planning stage and who carefully read and criticized the earlier drafts of the thesis deserve special thanks. The committee members were: Professor Lloyd G. Humphreys, Chairman, and Professors Ledyard R. Tucker, Maurice M. Tatsuoka, Kern W. Dickman and William F. Stout.

The author is also indebted to the Office of Naval Research, contract N00014-67-A-0305-0012, which paid for the computer time. The work is part of a project "Development of Objective Factor Analytic Procedures" with Dr. Humphreys as the principal investigator.

The author also wishes to thank the entire Statistical Consulting (SOUPAC) group of the Department of Computer Science of the University of Illinois for their programming aid. Special thanks go to Joseph Kolman, Paul Chouinard, and Bill Walter.

A special thanks is also due to Pam Rosborough who deciphered the author's handwriting and typed all of the drafts of the thesis. Thanks also to the Offset Department of the Department of Computer Science for printing and collating the thesis.

And finally, a very special thank you to the author's wife, Dale Montanelli for her patience and encouragement.

TABLE OF CONTENTS

I. INTRODUCTION

A. Background

Factor analysis is a frequently used model building technique, especially in sciences where a large number of variables need to be studied. Unfortunately, little work has been done on ways of testing the goodness of fit of the model to the data. Several techniques for testing this goodness of fit have been evaluated in this investigation. In addition, these techniques were used to evaluate the maximum likelihood estimation procedure as a factor analytic method.

For many years, factor analysis has been used as a research tool for finding the major or meaningful influences on a set of variables. Many different methods for finding these influences (or factors) have been proposed and used (Harman, 1967), but most are strictly non-inferential in nature. That is, they treat the observed correlations as though they are population values, and the resulting factor loadings are calculated directly from the sample correlations. Thus, the statistical problems of the sampling of individuals are ignored, and results are usually considered as though they are population values. On the other hand, the method of maximum-likelihood estimation provides a statistical procedure for the estimation of the parameters of the factor analytic model, based on the assumption that the original variables follow a multivariate normal distribution.

The basic assumption of factor analysis is that

$$x = \Lambda f + \mu + e$$

where x is a random vector of p variables, f is a random vector of $m < p$ common factor scores, e is a random vector of p uniquenesses, Λ is a $p \times m$

matrix of factor loadings, and μ is a p vector of means. It is further assumed that $E(f) = 0$, $E(e) = 0$, $E(ff') = I$, $E(ee') = D^2$, and $E(fe') = 0$. This model is usually written

$$P = AA' + D^2$$

where P is the correlation matrix. Here, x is assumed to be multivariate normal and the rows of A are proportional to the rows of Λ as Lawley (1940) has shown that the results are independent of the scale of measurement.

B. Algebraic Methods of Factor Analysis

The principal axis method of factor analysis, developed by Hotelling (1933), is today probably the most commonly used factor analytic technique. Its basic objective is to determine factors which account for the maximum amount of variance of the observed variables. The first factor is that linear combination of the original variables which accounts for maximum variance. Each subsequent factor accounts for a maximum amount of the remaining variance, while remaining uncorrelated with all previous factors. Thus, the factors are derived by an algebraic rule from the sample correlations, and can be strongly influenced by sampling variability. The factors are determined by the matrix equation $A = V \Lambda^{1/2}$ where V contains the normalized eigenvectors of the sample correlation matrix as columns, and Λ is a diagonal matrix containing the eigenvalues, i.e., $R = V \Lambda V'$.

Minres (minimum residual) factor analysis (Harman and Jones, 1966) is another algebraic approach to the problem of obtaining factors. Its aim is the best possible reproduction of the observed correlations, where best is defined in terms of a least squares fit. Thus the factor matrix A is determined such that the sum of squares of the off diagonal elements of $R - AA'$ is minimized.

C. Maximum Likelihood Factor Analysis

In contrast to the algebraic techniques mentioned previously, the method of maximum-likelihood estimation requires the writing down of the density function of the observations, given the population parameters. Then the sample values are considered as fixed, and the parameters are considered as the variables. The resulting function, called the likelihood function, is then maximized with respect to the parameters. The values of the parameters which maximize the likelihood function are termed the maximum-likelihood estimates of the population parameters (Lehmann, 1959).

The possibility of the use of the maximum-likelihood method for the estimation of factor loadings has existed for at least thirty years (Lawley, 1940). The method has always required an iterative procedure with a large number of calculations to be performed on each iteration. Thus it seemed natural that the development of computers would encourage the use of the method, but Lawley and Maxwell (1963) reported that in some cases convergence of the likelihood function to its maximum was a very slow process or might not even be attained, unless good initial estimates for the factor loadings were used.

Jöreskog (1967 a and b) has developed a new computational method which has the advantage that the iterative procedure always converges. In other papers, Jöreskog (1969, 1970) has extended the maximum-likelihood estimation method to cover a wide variety of models, including factor analytic ones. In conjunction with these efforts, Jöreskog, Gruvaeus, and van Thillo (1970) have developed a general computer program to calculate maximum-likelihood estimates. The maximum-likelihood estimation procedure also provides a likelihood-ratio test of the number of factors, and this test has been made available in their general computer program.

In applying the method of maximum likelihood to the general factor analytic model ($P = AA' + D^2$), one writes the log of the likelihood function, omitting a function solely of the observations (Lawley and Maxwell, 1963)

$$\log_e L = -\frac{n}{2} [\log_e |P| + tr(RP^{-1})]$$

where R is a sample correlation matrix based on a sample of size n + 1. This expression is then maximized with respect to the elements of the matrices A and D, to obtain a maximum-likelihood solution.

Since the method of maximum-likelihood estimation is an established statistical procedure, it is desirable to see how well it does in practice. Browne (1968) has already shown that maximum-likelihood estimates are preferable to many other types when dealing with sample correlation matrices drawn from populations which exactly satisfy the factor analytic model, but it remains to be seen how well it will work with data more like real data.

D. <u>The Tucker, Koopman, and Linn Study</u>

Tucker, Koopman, and Linn (1969) generated 54 population correlation matrices in order to study factor analytic methods. One of their simulated correlation matrices was defined by

$$R = B_1 P_1 B_1 + B_2 P_2 B_2 + B_3 P_3 B_3$$

where B_1, B_2, and B_3 were diagonal matrices with real positive diagonal elements b_{1j}, b_{2j}, and b_{3j} ($j = 1, 2, \ldots, p$, the number of variables), respectively. Since b_{1j}^2 was the proportion of variance of variable j due to the major factors, b_{2j}^2 the proportion due to the minor factors, and b_{3j}^2 the proportion due to the unique factors they had:

$$b_{1j}^2 + b_{2j}^2 + b_{3j}^2 = 1.$$

Thus, $b_{1j}^2 + b_{2j}^2$ equaled the communality (common variance) of variable j. Three different relationships between these coefficients defined the three types of population matrices used by Tucker et al. Correlation matrices with $b_{2j}^2 = 0$ and $b_{3j}^2 = (1 - b_{1j}^2)$ exactly fitted the mathematical factor analytic model (common factors + uniquenesses). With $b_{3j}^2 = 0$ and $b_{2j}^2 = (1 - b_{1j}^2)$, the correlation matrices constituted the simulation model, as P_2 contained the accumulated effect of 180 minor factors. It was hoped that these simulation matrices would approximate real data population-correlation matrices which could be thought of as arising from a few major and many minor influences. They also employed a third model which contained influences of both minor factors and uniquenesses. For this middle model, they had $b_{2j}^2 = b_{3j}^2 = \frac{1 - b_{1j}^2}{2}$.

All correlation matrices contained 20 variables. Each P_s matrix (s = 1,2,3) was constructed from the relationship $P_s = A_s^* A_s^{*\prime}$ where A_s^* was obtained by adjusting the rows of A_s to be of unit length.

The A_1 matrices were generated by random processes and contained either three or seven columns, representing the number of major factors in each correlation matrix. The A_2 matrices were generated by another random process so that the effect of P_2 was as though there were 180 minor factors in it. P_3 was an identity matrix, as the factor analytic model assumes a unique factor for each variable, and that these unique factors are uncorrelated. Tucker et al also used three levels of entries in the B_1 matrices; hi (.6, .7, .8), wide (.2, .3, .4, .5, .6, .7, .8), and low (.2, .3, .4). Thus their design was three (models) x two (number of major factors) x three (levels of B_1 coefficients), and they generated three correlation matrices for each of the eighteen cells. Tucker, Koopman, and Linn were interested in comparing several factor analytic techniques, but the data they generated are useful for studying any procedures related to factor analysis.

In the Tucker, Koopman, and Linn (1969) study, the authors used a random process to generate conceptual input factor loadings $\bar{A}_1$ for the major factor domain. They combined these with random normal deviates, applied a skewing function, and multiplied by the matrices B_1, in order to get to actual input factor loadings A_1. ($A_1 = B_1 A_1^*$ where $A_1^* A_1^{*\prime} = P_1$). The authors used joint rotations of actual input factors with output factors, and also rotations of output factors only, to assess the degree to which actual input factors were found on output. Thus there were two methods of comparison used, and each resulted in a separate index (coefficient of congruence) for each actual input factor.

Although the raw data of the Tucker, Koopman, and Linn study consisted of population-correlation matrices and not samples, some of their results can serve as standards for some of the results of the current study. In general, the reproduction of the actual input factors in the output factors was very good for the formal model, and poorer for the simulation model. The reproduction was good with a high level of b_{1j}^2 and poorer for a low level. Thirdly, results were better for three factors than for seven. Finally, the combination of simulation model, low b_{1j}^2, and seven factors produced extremely poor results. These results led Tucker et al to conclude that the quality of factor analytic results depended heavily on the design and conduct of the study.

E. Goodness of Fit

The set of factor analytic methods can be divided into two parts: exploratory methods which are used in early investigations in an area, with the purpose of reducing a large number of variables to a smaller number of factors when the investigator has no a priori hypotheses as to the composition of the factors; and confirmatory methods which are used by investigators with specifiable hypotheses about the factors. The present study considers confirmatory factor analysis only, and a major interest is in the discovery or development of a measure which would reflect the degree of fit of the final solution to the specified hypothesis. It is possible to test this hypothesis via the likelihood-ratio technique, and although the distribution of the likelihood-ratio statistic has not been tabled, it is distributed approximately as a χ^2 in large samples (Lawley and Maxwell, 1963). Unfortunately, this test sets up the hypothesis as a null hypothesis, and as the sample size increases, it is more likely to be rejected, as no hypothesis is exactly true. Thus, this test is of little use to many researchers who are interested in how well their data agree with their model, fully realizing that their model cannot be exactly true in the population. Therefore what is needed is a measure to assess the goodness of fit of the model to the data. Thus, the problem in this study is different from the one considered by Tucker, Koopman, and Linn as they were interested in factor matching, while here, the aim is to have one index to measure the total goodness of fit.

Tucker (personal communication) has suggested a measure

$$\rho_{1m} = 1 - \frac{\left(\frac{\chi^2}{df} - 1\right)_m}{\left(\frac{\chi^2}{df} - 1\right)_0}$$

where χ^2/ d.f. is the chi-square approximate test criterion for the likelihood-ratio test statistic divided by its degrees of freedom, taken after zero factors and after m factors have been extracted. This measure is analogous to a percent of variance accounted for by the model, as the expected value of a χ^2 random variable divided by its degrees of freedom is one. More recently, Tucker and Lewis (1970) have developed a second reliability coefficient,

$$\rho_{2m} = \frac{M_0 - M_m}{M_0 - \frac{1}{n'_m}}$$

where $n'_m = N - 1 - \frac{1}{6}(2p + 5) - \frac{2}{3}m$, p = number of variables, $M_m = F_m/df_m$, F_m = minimum value of $F_m(A, D) = \log_e |P| + tr(RP^{-1}) - \log_e |R| - P$ (Jöreskog, 1967b) for m factors, and df_m = degrees of freedom for m factors. It was hoped that this coefficient would be independent of the sample size and would provide an estimate of the goodness of fit of the factor analytic model in the population. Tucker and Lewis calculated ρ_2 for the number of major factors for some of the population-correlation matrices of Tucker, Koopman, and Linn. These values (Table 1) can serve as targets for the current study. These two measures (ρ_{1m} and ρ_{2m}) are similar (as can be seen by substituting $\chi^2_m = n'_m F_m$ in ρ_{1m}), but not identical. It is hoped that one or both of them are good indicators of goodness of fit for maximum-likelihood factor analysis.

Table 1

Values of ρ_2 Obtained by Tucker and Lewis from Eight of the Tucker, Koopman, and Linn Population Correlation Matrices $(N = \infty,\ p = 20)$.

3 Factors in Major Domain, Reliabilities for 3 Common Factor Models

	Formal Model	Simulation Model
high b_{1j}^2	1.00	.83
low b_{1j}^2	1.00	.55

7 Factors in Major Domain, Reliabilities for 7 Common Factor Models

	Formal Model	Simulation Model
high b_{1j}^2	1.00	.71
low b_{1j}^2	1.00	.48

Another possible measure of goodness of fit is the sum of squares of differences between the correlations implied by the model and those reproduced by the actual output factors. Browne (1968) suggested this measure of goodness of fit:

$$c_1 = \sum_{i=1}^{p} \sum_{j=1}^{i} [\Phi\Phi' - AA']_{ij}^2$$

where A is the sample factor matrix, Φ is the population factor matrix, and p is the number of variables. Of course, another possibility is to exclude the diagonal elements. This would emphasize reproduction of the correlations, while ignoring the communalities:

$$c_2 = \sum_{i=2}^{p} \sum_{j=1}^{i-1} [\Phi\Phi' - AA']_{ij}^2$$

Both measures were scaled by the total sum of squares in order to produce coefficients, r_1 and r_2, with upper limits of 1.00. In most cases they should vary between zero and one.

$$r_1 = 1 - \frac{c_1}{\sum_{i=1}^{p} \sum_{j=1}^{i} [\Phi\Phi']_{ij}^2}$$

$$r_2 = 1 - \frac{c_2}{\sum_{i=2}^{p} \sum_{j=1}^{i-1} [\Phi\Phi']_{ij}^2}$$

These measures (c_1, c_2, r_1, r_2) are all invariant under orthogonal rotation of the sample factor matrix A, and of the hypothesis factor matrix Φ. All six measures (including ρ_1 and ρ_2) were obtained for all 96 sample correlation matrices.

II. METHOD

A. Data Used

Due to limitations on computer time, it was necessary to use only some of the population correlation matrices from the Tucker, Koopman, and Linn study. In order to preserve the effects due to the independent variables used in generating those matrices, it was decided to randomly select one matrix from each of eight cells in their design. The eight cells were created by using two levels of each of the three independent variables used by Tucker et al; i.e. model (formal vs simulation), level of B_1 (high vs low), and number of factors in the major domain (3 vs 7). The eight matrices used are identified in Table 2. The level of battery (1, 2, or 3) was used by Tucker, Koopman, and Linn to designate a particular correlation matrix, as they had three such matrices in each cell in their design. In the current study, one battery was randomly selected from each of the eight cells of interest. In order to include the parameter of sample size, it was decided to draw samples of size 100, 400, and 1600 from each population-correlation matrix. To achieve some stability of results, four sample correlation matrices were drawn from each population-correlation matrix, at each level of sample size, yielding 96 sample correlation matrices.

Table 2

Matrix	Level of b_{1j}	Model	Number of Factors	Battery
1	high	formal	3	2
2	high	formal	7	2
3	high	simulation	3	3
4	high	simulation	7	3
5	low	formal	3	3
6	low	formal	7	1
7	low	simulation	3	1
8	low	simulation	7	1

B. Generation of Sample Correlation Matrices

The intuitive way to generate sample correlation matrices is to generate samples of random variables from a multivariate normal distribution with a specified correlation matrix (Kaiser and Dickman, 1962) and to calculate the sample correlation matrices directly from this raw data. However, this method requires a large quantity of random numbers and a large amount of computer time, especially when large sample sizes are required. To avoid this problem, a more economical procedure, described by Odell and Feiveson (1966) and used by Browne (1968), was used in this study.

In order to compute a sample correlation matrix R when given the population correlation matrix P, one uses

$$R = (\text{Diag}\,[A])^{-1/2}\, A\, (\text{Diag}\,[A])^{-1/2} \quad \text{and}$$

$$A = (\Omega T)(\Omega T)' \quad \text{where}$$

$P = \Omega\Omega'$ and the elements of T(lower triangular) are chosen as independently distributed variables:

t_{ij} is distributed as N(0,1) $(i > j)$

t_{ii} is distributed as Chi with (N-i) degrees of freedom

$t_{ij} = 0 \quad i < j$

For convenience of calculation, Ω was chosen to be lower triangular and was obtained by the square root method for triangular factoring (Dwyer, 1945).

Thus, this method requires only the generation of $\frac{p(p-1)}{2}$ random normal deviates and p (p = 20, the number of variables) random Chi variables for each sample correlation matrix, regardless of the sample size. Also a large amount of computational time is saved in the calculation of the correlation matrix.

In order to generate the random normal deviates for the T matrix, it was first necessary to generate random integers on the computer. These integers were converted to real numbers uniformly distributed between zero and one, and then these were normalized. Unfortunately, there is no way to pick truly random numbers on the computer, so the random integers needed were produced by a simple arithmetic process. These random integers are often called pseudo random integers, because they are produced by a deterministic process. Richardson (1969) reviewed several methods of generating pseudo random integers and chose the multiplicative congruential method as the best for the IBM 360, on the basis of randomness (passing statistical tests), length of period (number of integers generated before the sequence repeats itself), and generation time needed. This method is based on the relation $X_{i+1} = aX_i \pmod{m}$ which means that aX_i is divided by m and the $i+1^{st}$ random integer X_{i+1} is set equal to the remainder.*

Muller (1959) compared several methods of generating pseudo random normal deviates from pseudo random numbers on the interval (0, 1). The direct approach (Box and Muller, 1958) was picked as best because of the resulting reliability in the tails of the distribution and the relatively greater accuracy when compared with other methods. The transformations are:

$$X_1 = (-2 \log_e U_1)^{1/2} \cos 2\pi U_2$$

$$X_2 = (-2 \log_e U_1)^{1/2} \sin 2\pi U_2$$

---

* The modulus m was set to 2^{24} in order to provide the maximum possible period. The constant a was chosen by Richardson from 1500 different multipliers, as the one which produced the integers with the best statistical properties. Integers on the IBM 360 occupy 32 binary digits (bits), but real numbers use only 24 bits (the remaining 8 are used for the exponent). Thus, the pseudo random integers were converted to a uniform distribution by merely inserting the appropriate exponent in the first eight bits, so that the real numbers would lie between zero and one.

where U_1 and U_2 are pseudo random numbers from the interval (0, 1), and X_1 and X_2 are independent variables from the normal distribution with mean zero and unit variance [N(0, 1)].

In order to determine the pseudo random Chi variables for the diagonals of the T matrix, the following approximation was used (Abramowitz and Stegun, 1966)

$$\chi_p = \nu\left[1 - \frac{2}{9\nu} + (X_p - h_\nu)\sqrt{\frac{2}{9\nu}}\,\right]^3 \qquad (\nu > 30)$$

where ν = degrees of freedom and X_p is a pseudo random normal deviate. The value for h_ν is gotten from the relation $h_\nu = \frac{60}{\nu} h_{60}$ where h_{60} is tabled against values of X_p from -3.5 to +3.5 by Abramowitz and Stegun. A cubic equation was used to interpolate between the tabled values of h_{60}.

$$\hat{h}_{60} = -.000924X_p - .000159X_p^2 + .000308X_p^3 + .000189$$

The correlation between h_{60} and $\hat{h}_{60}$ (for the 15 tabled values) was 1.0000.

C. Factor Analyses

Each correlation matrix was factored using Joreskog's (1967a) maximum-likelihood factor analysis program. The maximum number of iterations was set to 100 and the probability of chance occurence was set to 1.0 so that all solutions were obtained. Solutions were obtained for the number of factors in the major domain. Additionally, the likelihood ratio tests of the number of common factors were obtained from zero up to the number of factors in the major domain, so that ρ_1 and ρ_2 could be calculated for each possible number of factors. The coefficients c_1, c_2, r_1, and r_2 were calculated for each factor matrix.

D. Analysis of Variance

In order to determine the effects of the four independent variables (model, level of b_{1j}^2, number of factors, and number of observations) on the measures of goodness of fit, six separate fixed-factor analyses of variance were performed, each being 2 x 2 x 2 x 3.

III. RESULTS

A. Reliability Coefficients ρ_1 and ρ_2

The means of ρ_1 and ρ_2, across the four samples of the same size for each population-correlation matrix, are presented in Appendix A. There were only small differences between the two coefficients. For very large sample sizes, the formulas yield quite similar results, as is illustrated by the samples of size 1600. The correlation between ρ_1 and ρ_2 for the number of factors in the major domain, across all 96 sample factorings was .998. All results are discussed mainly in terms of ρ_2, as it is the later, published version.

ρ_2 was an excellent measure of goodness of fit for the factor matrices obtained from samples from the population-correlation matrices (hereafter called sample factor matrices) of the formal model. In the three factor matrices, with high or low b_{1j}^2 (Table 3) ρ_2 was very close to 1.00, for all sample sizes. With seven factors and high b_{1j}^2 (Table 4) the results were as good. However, with seven factors but low b_{1j}^2, ρ_2 went above 1.00 after four factors with only 100 observations. The average value of ρ_2 after seven factors were obtained was 1.4285, and the individual values were 1.0956, 1.0428, 1.4710, and 2.1046. This value, 1.4285, was much larger than the population value of 1.00 obtained by Tucker and Lewis (1970). While this result was probably due to the small sample size of 100, it reflected an undesireable property for a reliability coefficient. However, with 400 and 1600 observations, results were much better. Thus, the method of maximum likelihood resulted in good solutions as measured by ρ_2 when the populations exactly fitted the factor analytic model. The only exception was with variables of low communality, in which case more observations were necessary to obtain a good fit.

Table 3

Means of ρ_2 for Matrices with 3 Factors in the Major Domain, After 3 Factors Have Been Obtained

	Sample size	Formal Model	Simulation Model
High b_{ij}^2	100	1.0011	.8535
	400	1.0018	.8253
	1600	1.0004	.8319
Low b_{ij}^2	100	.9929	.6157
	400	1.0127	.5690
	1600	1.0023	.5623

Table 4

Means of ρ_2 for Matrices with 7 Factors in the Major Domain, After 7 Factors Have Been Obtained

	Sample size	Formal Model	Simulation Model
High b_{ij}^2	100	1.0139	.7348
	400	1.0014	.6829
	1600	.9997	.7118
Low b_{ij}^2	100	1.4285	.6022
	400	1.0482	.5312
	1600	.9984	.4982

Table 5

Means of ρ_2 for Matrices with 3 Factors in the Major Domain, After 4 Factors Have Been Obtained

	Sample size	Formal Model	Simulation Model
High b_{ij}^2	100	1.0116	.8778
	400	1.0054	.8422
	1600	1.0011	.8442
Low b_{ij}^2	100	1.0416	.6743
	400	1.0257	.6029
	1600	1.0050	.6108

Results from the simulation model matrices were not nearly as good. In no case did ρ_2 reach the value of 1.00. The largest values were obtained with high b^2_{1j} and three factors (Table 3), but the highest was .8535. There was a trend for ρ_2 to decrease with increased sample size for simulation model matrices with low b^2_{1j} (Tables 3 and 4). ρ_2 also became smaller as the number of factors in the major domain increased and as b^2_{1j} decreased.

The calculation of ρ_2 was extended to four factors in the three factor matrices, in order to see how it behaved. It was thought that there might be some leveling off, after three factors. This did occur for the formal model (Table 5), after ρ_2 had already reached 1.000. There was some tendency for the values of ρ_2 to level off for the simulation model, high b^2_{1j}, as the increase from two factors to three factors was much greater than that from three factors to four factors (Appendix A). In the simulation model with low b^2_{1j}, there were no signs of a leveling off of ρ_2 after three factors.

B. Other Goodness of Fit Measures

The results for c_1 and c_2 (Table 6) were very similar, as were the results for r_1 and r_2. To get an idea of the degree of similarity, the coefficients were correlated across all 96 matrices. Since c_1 correlated .968 with c_2, and r_1 correlated .994 with r_2, results will be discussed in terms of c_1 and r_1 only. The coefficient c_1 behaved exactly as expected. For all eight matrices, c_1 got smaller as the sample size increased. In all cases, increasing the number of factors, while holding model, level of b^2_{1j}, and sample size fixed, caused an increase in c_1. In all cases, moving from the formal model to the simulation model while holding the other three independent variables fixed caused an increase in c_1. Finally, in all

Table 6

Means of the Coefficients c_1, c_2, r_1, and r_2, After the Number of Factors in the Major Domain Have Been Extracted

Matrix	Sample size	c_1	c_2	r_1	r_2
1	100	.9587	.9050	.9782	.9737
	400	.2472	.2296	.9944	.9933
	1600	.0464	.0426	.9990	.9988
2	100	1.6355	1.4322	.9413	.9201
	400	.4625	.4336	.9834	.9758
	1600	.1035	.0943	.9963	.9947
3	100	1.4689	1.3214	.9705	.9671
	400	.5290	.4547	.9894	.9887
	1600	.3523	.2865	.9929	.9929
4	100	2.5336	2.1420	.8915	.8403
	400	.9252	.6490	.9604	.9516
	1600	.6677	.4593	.9714	.9657
5	100	1.5298	1.2331	.8541	.8520
	400	.3131	.2732	.9701	.9672
	1600	.0749	.0623	.9929	.9925
6	100	3.7076	1.8881	.1256	.2423
	400	.7221	.3854	.8297	.8453
	1600	.2576	.1078	.9392	.9567
7	100	5.0938	4.1732	.5351	.5258
	400	2.9050	2.0161	.7349	.7709
	1600	2.3474	1.7914	.7858	.7964
8	100	6.7834	4.1999	-.5998	-.6854
	400	5.1870	2.0895	-.2233	-.2398
	1600	4.7623	2.7143	-.1231	-.0893

Table 7

Standard Deviations of the Coefficients c_1, c_2, r_1, and r_2, After the Number of Factors in the Major Domain Have Been Extracted

Matrix	Sample size	c_1	c_2	r_1	r_2
1	100	.2668	.2555	.0061	.0074
	400	.0696	.0632	.0016	.0018
	1600	.0066	.0071	.0001	.0002
2	100	.4390	.5500	.0177	.0307
	400	.1100	.1057	.0039	.0059
	1600	.0230	.0240	.0008	.0013
3	100	.9734	.9287	.0196	.0231
	400	.0416	.0459	.0008	.0011
	1600	.0534	.0457	.0011	.0011
4	100	.4661	.2975	.0200	.0222
	400	.1814	.1938	.0078	.0148
	1600	.1704	.1382	.0073	.0103
5	100	.2350	.0933	.0224	.0112
	400	.1363	.1325	.0130	.0159
	1600	.0130	.0061	.0012	.0007
6	100	.2419	.1978	.0571	.0794
	400	.1819	.0546	.0429	.0219
	1600	.1348	.0111	.0319	.0045
7	100	2.3294	2.2254	.2126	.2529
	400	.1106	.0656	.0101	.0075
	1600	.4228	.1850	.0386	.0210
8	100	.5732	.5566	.1503	.2234
	400	.6449	.6975	.1521	.2799
	1600	.3716	.1726	.0876	.0692

cases, c_1 increased when the b_{1j}^2 went from high to low. (Note, there was one reversal of this last finding with c_2, which decreased as the level of b_{1j}^2 went from high to low from matrix 2 to matrix 6.)

Also, r_1 increased with increased sample size, increased values of b_{1j}^2, fewer factors, and from the simulation model to the formal model. For high b_{1j}^2, formal model, the values of r_1 were good for three factors, all sample sizes, while seven factors required a sample size of 400 for a satisfactory result. With the simulation model, low b_{1j}^2, and three factors, r_1 reached only .7964 with 1600 observations. In matrix 8 (simulation model, low b_{1j}^2, seven factors), the values of r_1 were actually negative. This was partly due to the low total sum of squares in the model correlation matrix, but this result indicated, much better than did ρ_2, the inaccuracy of these solutions.

The standard deviations of the coefficients c_1, c_2, r_1, and r_2 were calculated (Table 7). There was a tendency for the standard deviations to be smaller with better fit, but there were more reversals than with the means. Also, since r_1 and r_2 had upper limits of 1.0, their standard deviations were forced to decrease as the means increased because the upper limit was being approached.

C. Use of the Likelihood-Ratio Test

Jöreskog (1967b) used the likelihood-ratio technique to test the hypothesis that the number of factors m was a given number. The exact distribution of the likelihood-ratio test statistic is not known, but for large N its distribution is approximately a χ^2 distribution with degrees of freedom $\frac{1}{2}[(p - m)^2 - (p + m)]$. If the hypothesis of m factors was rejected (due to a statistically significant value of the test statistic), Jöreskog refactored the matrix for m+1 factors. It was thought that by

Table 8

Range of Probability Levels for χ^2 Statistics

3 Factor Matrices

Matrix	Sample size	2 Factors Min.	2 Factors Max.	3 Factors Min.	3 Factors Max.	7 Factors Min.	7 Factors Max.
1	100	.0000	.0000	.1173	.8533	.9890	.9976
	400	.0	.0	.2697	.9211	.9820	1.0000
	1600	.0	.0	.2028	.8257	.9776	.9968
3	100	.0000	.0000	.0000	.0000	.0000	.0194
	400	.0	.0	.0	.0	.0000	.0000
	1600	.0	.0	.0	.0	.0	.0
5	100	.0066	.6504	.2060	.8027	.9376	.9960
	400	.0000	.0000	.1201	.9712	.8897	.9982
	1600	.0	.0	.5070	.9186	.9616	.9923
7	100	.0000	.0000	.0000	.0000	.0007	.0054
	400	.0	.0	.0	.0	.0000	.0000
	1600	.0	.0	.0	.0	.0	.0

7 Factor Matrices

		4 Factors Min.	4 Factors Max.	6 Factors Min.	6 Factors Max.	7 Factors Min.	7 Factors Max.
2	100	.0000	.0000	.0000	.0683	.5381	.8465
	400	.0	.0	.0000	.0000	.2903	.7948
	1600	.0	.0	.0	.0	.2043	.7330
4	100	.0000	.0000	.0000	.0000	.0000	.0000
	400	.0	.0	.0	.0	.0	.0
	1600	.0	.0	.0	.0	.0	.0
6	100	.0723	.9560	.5019	.9995	.6065	.9998
	400	.0000	.0000	.0840	.6060	.7133	.9684
	1600	.0000	.0000	.0005	.0112	.2279	.5371
8	100	.0000	.0000	.0000	.0000	.0000	.0000
	400	.0	.0	.0	.0000	.0000	.0000
	1600	.0	.0	.0	.0	.0	.0

Note: In the above table, the entry .0000 means that the number was a zero, when rounded to 4 decimal places. The entry .0 was an exact zero, to the accuracy of the computations (about 7 decimal places).

looking at the probability levels (probabilities of the chance occurence of the observed χ^2 values) of the test statistic for various numbers of factors, one might be able to determine the correct number of factors.

In Table 8 are presented the ranges of these probability levels, for selected numbers of factors. For the formal model (matrices 1 and 3 for three factors, and matrices 2 and 6 for seven factors), in all cases one would accept (at any reasonable probability level from .001 to .100) the hypothesis of the number of factors in the major domain. The probability levels ranged from a low of .1173 for one high b_{1j}^2, three factor matrix to a high of .9998 for a low b_{1j}^2, seven factor matrix. However, in one case, with low b_{1j}^2, three factors, and a sample size of 100, the hypothesis of two factors was also accepted ($p = .6504$). With seven factors, and low b_{1j}^2 (matrix 6), an hypothesis of only four factors was supported with 100 observations and an hypothesis of six factors was supported with 400 observations.

For all simulation model matrices, however, the hypothesis that the number of factors was equal to the number of factors in the major domain, was rejected. Even the hypothesis of seven factors for a sample factor matrix with high b_{1j}^2 and only three factors in the major domain was rejected (although one matrix of sample size 100 did have a $p = .0194$ which would not have been rejected at the .01 level). Thus, this test is appropriate for testing the hypothesis that the factor analytic model holds exactly in the data, but it is of no use as a measure of goodness of fit for data that do not fit the model.

D. Analyses of Variance

Separate analysis of variance summary tables for the six measures ρ_1, ρ_2, c_1, c_2, r_1, and r_2 are presented in Appendix B. These analyses were performed in order to discover the relative sizes of the effects of the four independent variables, level of b_{1j}^2, model, number of factors in the major domain, and sample size. Since the assumption of normality of analysis of variance was possibly violated, especially with ρ_1, ρ_2, r_1, and r_2, border line significant F ratios should not be taken too seriously.

The results for ρ_1 and ρ_2 were again very similar, so results are discussed in terms of ρ_2. The main effect of model accounted for 61.19% of the total sum of squares for ρ_2. The average value of ρ_2 for the formal model was 1.042, while for the simulation model, it was only .668. The only other large contributor to the total sum of squares (except for within cell) was the interaction between model and level of b_{1j}^2, which accounted for 9.07% of the total sum of squares.

	formal	simulation
high b_{1j}^2	1.003	.773
low b_{1j}^2	1.080	.563

Four other small but statistically significant effects were also found. The average value of ρ_2 was .888 for sample factor matrices with high b_{1j}^2 and .822 for those with low b_{1j}^2. There was also a significant trend for ρ_2 to decrease with increased sample size. The averages were .905, .834, and .826, for sample sizes 100, 400, and 1600, respectively. The BXF and MXF interactions were also significant at the .01 level.

	3 Factors	7 Factors
high b_{1j}^2	.919	.857
low b_{1j}^2	.792	.851

	3 Factors	7 Factors
formal	1.002	1.082
simulation	.710	.627

Although the results were similar for c_1 and c_2, only 6.68% of the total sum of squares was attributable to error for c_1, whereas 13.02% was error for c_2. This supported the earlier decision to discuss results in terms of c_1 only, as it was less variable, within cells. The main effect of level of b_{1j}^2 accounted for 25.17% of the total sum of squares. The mean value of c_1 was .827 for high b_{1j}^2 and 2.807 for low b_{1j}^2. Model accounted for 24.62% of the total sum of squares, and the mean for the formal model was .838 while for the simulation model, it was 2.796. Sample size accounted for 17.37% of the total sum of squares, with c_1 dropping as sample size increased. The means were 2.964, 1.411, and 1.077 for the sample sizes 100, 400, and 1600, respectively. The level of b_{1j}^2 by model interaction accounted for 13.58% of the total sum of squares.

	formal	simulation
high b_{1j}^2	.576	1.079
low b_{1j}^2	1.101	4.513

The main effect of numbers of factors in the major domain (c_1 = 1.322 for three factors, c_1 = 2.312 for seven factors) and the three interactions shown below were also significant at the .01 level.

	3 Factors	7 Factors
high b_{1j}^2	.600	1.055
low b_{1j}^2	2.044	3.570

	100	400	1600
high b_{1j}^2	1.649	.541	.292
low b_{1j}^2	4.279	2.282	1.861

	3 Factors	7 Factors
formal	.528	1.148
simulation	2.116	3.477

Unfortunately, due to negative values for low b_{1j}^2, simulation model, and 7 factors (summed across sample size), every main effect and interaction which did not involve sample size was highly significant for r_1 and r_2. While the negative values ($r_2 = -.338$, $r_1 = -.315$) indicated how poor results were for that combination, the effect was apparently strong enough to influence most other effects. The main effect of sample size did account for 5.14% of the total sum of squares, with r_1 increasing as sample size increased. The means .587, .780, and .819 for 100, 400, and 1600 observations, respectively.

IV. DISCUSSION

A major goal of the present study was to find or develop a measure of goodness of fit for the factor analytic model. One such measure studied was ρ_2. 61.19% of the sum of squares of ρ_2 was accounted for by the main effect of model. For samples from population-correlation matrices constructed to exactly fit the factor analytic model, ρ_2 worked exceedingly well. Only in the case of seven factors and low b_{1j}^2 was it necessary to have a sample size greater than 100. ρ_2 also had the desirable property of approaching unity (or nearly so) as the sample size increased, for the matrices developed from the formal model. The samples from the simulation model behaved quite differently. Even in the best case (three factors, high b_{1j}^2), the average value of ρ_2 was .8535. Thus, in all cases, ρ_2 reflected the presence of the minor factors.

There was also a significant decrease in ρ_2 for sample sizes 400 and 1600, when compared with 100. This is not a good property for a proposed measure of goodness of fit, as intuitively one would expect the fit to a good model to improve with more observations. However, this decrease is due in part to ρ_2 coming down to 1.000, after going over that value for samples of 100. There was a significant decrease in c_1 with increased correlations implied by the model better with more observations. This was further illustrated by the fact that the ρ_2 values were approaching the population values obtained by Tucker and Lewis. This can be seen by subtracting the population values (Table 1) from the sample values (Tables 3 and 4).

Table 9

Differences Between Sample Values and Population Values of ρ_2

		Sample Size		
		100	400	1600
High b_{1j}^2, Formal Model,	3 Factors	.0011	.0018	.0004
	7 Factors	.0139	.0014	-.0003
High b_{1j}^2, Simulation Model,	3 Factors	.0235	-.0047	.0019
	7 Factors	.0248	-.0271	.0018
Low b_{1j}^2, Formal Model,	3 Factors	-.0071	.0127	.0023
	7 Factors	.4285	.0482	-.0016
Low b_{1j}^2, Simulation Model,	3 Factors	.0657	.0190	.0123
	7 Factors	.1222	.0512	.0182

Since these values (Table 9) are only accurate to two decimal places (as the Tucker and Lewis figures are to two places), all except the low b_{1j}^2, simulation model matrices were within rounding error of the population values for samples of size 1600. Thus, the decreases in ρ_2 with increasing sample sizes were toward the population values.

An important result was pointed out by the significant interactions between level of b_{1j}^2 and the model for ρ_2 and c_1. In both cases, the results in the simulation model, low b_{1j}^2 cell were much poorer than would have been predicted from the main effects alone. These results, an average ρ_2 of .563 and an average c_1 of 4.513 (over four times greater than the next largest cell), showed that one cannot expect to support one's hypothesis with variables that have low percentage of variance accounted for in the major factors. It was interesting to note that while the values of c_1 were about the same in the two cells high b_{1j}^2, simulation model and low b_{1j}^2, formal model, the former had $\rho_2 = .773$ and the latter had $\rho_2 = 1.080$. Thus the model correlations were reproduced as well for simulation model, high b_{1j}^2 as for formal model, low b_{1j}^2.

The fact that level of b_{1j}^2 (25.17%) accounted for as large a percentage of the total sum of squares of c_1 as did model (24.62%) was encouraging. Thus, if the simulation model is a better model of the world, it is still possible for an experimenter to improve his results by constructing measures with high proportions of variance accounted for by the major factors.

The χ^2 statistic was useful for sample factor matrices for the formal model only. Even then, it lead to the acceptance of too few factors in some cases, with low b_{1j}^2 and/or too few observations.

The measure r_1 did well for the formal model matrices, although more observations were necessary before it neared its maximum of 1.00. Also, with seven factors, low b_{1j}^2 and 1600 observations it only attaimed .9392. However, the simulation model matrices with high b_{1j}^2 also gave high values of r_1. Thus the maximum-likelihood estimation procedure was doing a good job of reproducing the model correlations, but this was not reflected in ρ_2. The results on matrices 7 and 8 (simulation model, low b_{1j}^2) confirmed the importance of controlling the relationship between the major and minor influences on one's results. The major factors should predominate over minor factors in any study. The results did indicate that it is easier to reproduce a small number of factors in a poorly designed study.

Thus r_1 was shown to be useful as a measure of goodness of fit. It does require the writing down of an hypothesized factor matrix Φ, so it can not normally be used in exploratory studies. It has the advantage that it could be used with any factor analytic procedure. On the other hand, ρ_2 could be used with any study, as long as the maximum-likelihood estimation procedure is used. However, it seems somewhat less useful than r_1 when the factor analytic model does not hold in the population.

An investigator should strive to develop variables which strongly represent his major factors. He should have a large sample size (a ratio of five observations per variable was not always sufficient to insure good results, even with high b_{1j}^2 and the formal model). If he uses maximum-likelihood factor analysis, he should usually stop factoring when the χ^2 becomes non-significant, if his sample size is large enough. However, with real data, there may be statistically significant minor factors which are not of interest. In this case the χ^2 cannot give an indication of when to stop factoring. However, the investigator can use ρ_2 as an estimate of how well the formal factor analytic model holds in the population from which his data was taken. Although the statistical properties of this estimate are not known, and it may be high for small samples, it does have the desirable property of having a value of 1.00 in populations which exactly fit the formal factor analytic model. Small values of ρ_2 probably indicate a poorly controlled study, and the investigator may be able to improve his results by using better controls over minor factors, by having variables with high percents of variance in the major domain, and by having a higher ratio of variables to major factors. Finally, in a confirmatory study he should write down an hypothesized factor matrix and use c_1 and r_1 to determine its goodness of fit.

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APPENDIX A

MEANS OF ρ_1 AND ρ_2

Table 10

Means of ρ_1 for 3 Factor Matrices

Matrix	Sample size	1 Factor	2 Factors	3 Factors	4 Factors
1	100	.4321	.7741	1.0011	1.0135
	400	.4385	.7471	1.0017	1.0054
	1600	.4323	.7575	1.0004	1.0011
3	100	.4776	.6626	.8570	.8818
	400	.4648	.6458	.8262	.8432
	1600	.4679	.6522	.8321	.8444
5	100	.6739	.8879	.9932	1.0397
	400	.6584	.8639	1.0126	1.0255
	1600	.6581	.8641	1.0023	1.0050
7	100	.4719	.5591	.6264	.6863
	400	.4413	.5205	.5714	.6058
	1600	.4577	.5177	.5629	.6112

Table 11

Means of ρ_2 for 3 Factor Matrices

Matrix	Sample size	1 Factor	2 Factors	3 Factors	4 Factors
1	100	.4273	.7703	1.0011	1.0116
	400	.4375	.7462	1.0018	1.0094
	1600	.4320	.7573	1.0004	1.0011
3	100	.4734	.6571	.8535	.8778
	400	.4639	.6446	.8253	.8422
	1600	.4676	.6512	.8319	.8442
5	100	.6702	.8852	.9929	1.0416
	400	.6578	.8634	1.0127	1.0257
	1600	.6579	.8655	1.0023	1.0050
7	100	.4670	.5509	.6157	.6743
	400	.4403	.5187	.5690	.6029
	1600	.4574	.5173	.5623	.6108

Table 12

Means of ρ_1 for 7 Factor Matrices

Matrix	Sample size	Number of Factors: 1	2	3	4	5	6	7
2	100	.2979	.4835	.5976	.7014	.8062	.9172	1.0130
	400	.2753	.4987	.5957	.6728	.7659	.8836	1.0013
	1600	.2871	.5119	.6131	.6761	.7678	.8913	.9997
4	100	.1502	.2519	.3407	.4693	.5648	.6678	.7504
	400	.1232	.2609	.3523	.4468	.5435	.6460	.6868
	1600	.1302	.2594	.3486	.4347	.5326	.6623	.7126
6	100	.4415	.6681	.8778	1.0156	1.1465	1.2631	1.3663
	400	.4225	.5745	.6827	.7968	.9099	.9862	1.0474
	1600	.4096	.5276	.6344	.7624	.8775	.9674	.9984
8	100	.2085	.3025	.3651	.4299	.4849	.5692	.6305
	400	.2724	.3428	.3738	.4137	.4375	.4709	.5373
	1600	.2488	.3392	.3636	.3983	.4184	.4457	.4997

Table 13

Means of ρ_2 for 7 Factor Matrices

Matrix	Sample size	1	2	3	4	5	6	7
		Number of Factors						
2	100	.2918	.4744	.5869	.6906	.7975	.9127	1.0139
	400	.2740	.4969	.5935	.6705	.7634	.8823	1.0014
	1600	.2868	.5106	.6126	.6756	.7692	.8910	.9997
4	100	.1430	.2391	.3236	.4509	.5457	.6502	.7348
	400	.1217	.2582	.3488	.4428	.5394	.6422	.6829
	1600	.1298	.2588	.3478	.4338	.5316	.6615	.7118
6	100	.4314	.6564	.8725	1.0196	1.1656	1.3016	1.4285
	400	.4198	.5546	.6806	.7950	.9089	.9860	1.0482
	1600	.4093	.5237	.6339	.7620	.8772	.9673	.9984
8	100	.2004	.2880	.3451	.4058	.4573	.5412	.6022
	400	.2710	.3403	.3703	.4093	.4322	.4650	.5312
	1600	.2485	.3387	.3628	.3972	.4172	.4442	.4982

APPENDIX B

ANOVA SUMMARY TABLES

Table 14

Analysis of Variance Summary Table for ρ_1

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	.1160	2.38	.1160	14.39 *
Model (M)	1	3.1454	64.59	3.1454	390.19 *
Number of Factors (F)	1	.0003	.01	.0003	.04
Sample Size (N)	2	.1181	2.43	.0590	7.33 *
BXM	1	.4482	9.20	.4482	55.70 *
BXF	1	.0738	1.52	.0738	9.16 *
BXN	2	.0603	1.24	.0302	3.74
MXF	1	.1272	2.61	.1272	15.78 *
MXN	2	.0034	.07	.0017	.21
FXN	2	.0668	1.37	.0334	4.14
BXMXF	1	.0024	.05	.0024	.30
BXMSN	2	.0168	.34	.0084	1.04
BXFXN	2	.0488	1.00	.0244	3.03
MXFXN	2	.0291	.60	.0146	1.80
BXMXFXN	2	.0331	.68	.0166	2.05
Error (within cell)	72	.5804	11.92	.0081	
Total	95	4.8701	100.00	4.1096	

* $p < .01$

Table 15

Analysis of Variance Summary Table for ρ_2

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	.1058	1.93	.1058	9.65 *
Model (M)	1	3.3484	61.19	3.3484	305.48 *
Number of Factors (F)	1	.0001	.00	.0001	.01
Sample Size (N)	2	.1227	2.24	.0613	5.60 *
BXM	1	.4965	9.07	.4965	45.30 *
BXF	1	.0867	1.58	.0867	7.91 *
BXN	2	.0726	1.33	.0363	3.31
MXF	1	.1586	2.90	.1586	14.47 *
MXN	2	.0147	.27	.0073	.67
FXN	2	.0783	1.43	.0391	3.57
BXMXF	1	.0060	.11	.0060	.55
BXMXN	2	.0305	.56	.0152	1.39
BXFXN	2	.0639	1.17	.0319	2.91
MXFXN	2	.0492	.90	.0246	2.24
BXMXFMN	2	.0486	.89	.0243	2.22
Error (within cell)	72	.7892	14.42	.0110	
Total	95	5.4718	100.00	4.4533	

* $p < .01$

Table 16

Analysis of Variance Summary Table for c_1

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	94.0392	25.17	94.0392	271.15 *
Model (M)	1	92.0158	24.62	92.0158	265.31 *
Number of Factors (F)	1	23.5284	6.30	23.5284	67.84 *
Sample Size (N)	2	64.9046	17.37	32.4523	93.57 *
BXM	1	50.7547	13.58	50.7547	146.34 *
BXF	1	6.8922	1.84	6.8922	19.87 *
BXN	2	5.1895	1.39	2.5947	7.48 *
MXF	1	3.2917	.88	3.2917	9.49 *
MXN	2	.0409	.01	.0204	.06
FXN	2	2.0653	.55	1.0326	2.98
BXMXF	1	1.2974	.35	1.2974	3.74
BXMXN	2	.2981	.08	.1490	.43
BXFXN	2	.0055	.00	.0027	.01
MXFXN	2	1.9241	.51	.9620	2.77
BXMXFXN	2	2.4663	.66	1.2331	3.56
Error (within cell)	72	24.9709	6.68	.3468	
Total	95	373.6831	100.00	310.6118	

* $p < .01$

Table 17

Analysis of Variance Summary Table for c_2

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	30.3036	18.06	30.3036	99.86 *
Model (M)	1	43.7942	26.11	43.7942	144.32 *
Number of Factors (F)	1	3.8503	2.30	3.8503	12.69 *
Sample Size (N)	2	39.4955	23.54	19.7477	65.08 *
BXM	1	23.4395	13.97	23.4395	77.24 *
BXF	1	.1248	.07	.1248	.41
BXN	2	1.0886	.65	.5443	1.79
MXF	1	.4348	.26	.4348	1.43
MXN	2	.7133	.43	.3566	1.18
FXN	2	.1752	.10	.0876	.29
BXMXF	1	.1081	.06	.1081	.36
BXMXN	2	.0420	.03	.0210	.07
BXFXN	2	.6837	.41	.3418	1.13
MXFXN	2	.5725	.34	.2862	.94
BXMXFXN	2	1.0740	.64	.5370	1.77
Error (within cell)	72	21.8488	13.02	.3035	
Total	95	167.7488	100.00	124.2811	

* $p < .01$

Table 18

Analysis of Variance Summary Table for R_1

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	5.6983	29.66	5.6983	1263.56 *
Model (M)	1	2.3047	11.99	2.3047	511.05 *
Number of Factors (F)	1	2.8079	14.61	2.8079	622.63 *
Sample Size (N)	2	.9883	5.14	.4941	109.57 *
BXM	1	2.0249	10.54	2.0249	499.01 *
BXF	1	2.3369	12.16	2.3369	518.19 *
BXN	2	.6468	3.37	.3234	71.71 *
MXF	1	.7764	4.04	.7764	172.16 *
MXN	2	.0143	.07	.0071	1.59
FXN	2	.2858	1.49	.1429	31.69 *
BXMXF	1	.6670	3.47	.6670	147.90 *
BXMXN	2	.0231	.12	.0115	2.56
BXFXN	2	.1906	.99	.0953	21.13 *
MXFXN	2	.0555	.29	.0278	6.15 *
BXMXFXN	2	.0690	.36	.0345	7.65 *
Error (within cell)	72	.3247	1.69	.0045	
Total	95	19.2141	100.00	17.7572	

* $p < .01$

Table 19

Analysis of Variance Summary Table for R_2

Source	Degrees of Freedom	Sum of Squares	Percent of Total SS	Mean Square	F
Level of B(B)	1	5.2793	26.73	5.2793	611.90 *
Model (M)	1	2.5712	13.02	2.5712	298.01 *
Number of Factors (F)	1	2.8584	14.47	2.8584	331.30 *
Sample Size (N)	2	1.1056	5.60	.5528	64.07 *
BXM	1	2.1929	11.10	2.1929	254.17 *
BXF	1	2.1704	10.99	2.1704	251.56 *
BXN	2	.6131	3.10	.3065	35.53 *
MXF	1	1.0054	5.09	1.0054	116.53 *
MXN	2	.0010	.01	.0005	.06
FXN	2	.3030	1.53	.1515	17.56 *
BXMXF	1	.8246	4.18	.8246	95.58 *
BXMXN	2	.0018	.01	.0009	.10
BXFXN	2	.1523	.77	.0762	8.83 *
MXFXN	2	.0159	.08	.0080	.92
BXMXFXN	2	.0345	.17	.0172	2.00
Error (within cell)	72	.6212	3.15	.0086	
Total	95	19.7505	100.00	18.0243	