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THE ENERGY CRITERIA FOR STABILITY OF STRUCTURES

by

Gerald A. Wempner

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ABSTRACT

The energy criteria of Trefftz and Koiter, for the critical load upon an elastic system and for the stability of the system at the critical load, are presented. The presentation employs geometrical interpretations and simple examples to exhibit the essential features of the criteria and the related behavior of the structural system.

INTRODUCTION

If the load upon a structure attains a critical value, the structure buckles. The buckling may entail a gradual, albeit excessive, deflection as the load exceeds the critical value. Otherwise, the buckling may mean an abrupt collapse, so-called snap-buckling. The former occurs if an elastic system is stable at the critical load and the latter if it is unstable. From a practical viewpoint, the snap-buckling is the more dangerous phenomenon. Moreover, the structure which exhibits snap-buckling is also sensitive to imperfections; that is, the snap-buckling of a real (imperfect) structure may occur at loads much less than the critical load of the ideal structure.

The phenomenon of snap-buckling is especially prevalent in thin shells and, curiously, the most efficient shell-like structures are the most susceptible to such catastrophic failure. Consequently, as our attention turns increasingly to thin shells, to reduce costs and weight and to achieve structural and esthetic aims, the questions of stability and imperfection sensitivity are paramount.

The determination of the critical load by a stationary criterion upon the potential energy was given by Trefftz[1] in 1933. However, the question of stability at the critical state and the subsequent behavior remained unanswered.

A most significant work on the questions of stability at the critical load, post-buckling behavior and imperfection sensitivity, was the thesis of W. T. Koiter[2] in 1945. By employing a variational approach and the criteria for a minimum of the potential energy, Koiter developed stationary conditions for stability at the critical load. In addition, he examined the effects of small geometrical imperfections and showed how such imperfection can drastically reduce the buckling loads upon real structures.

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Although Koiter's thesis is now available in an English translation[3], the rigorous character of the work seems to inhibit a widespread appreciation and usage. By appealing to various geometrical interpretations, the following presentation offers a simple introduction to the essential features and the consequences of Koiter's work. For simplicity, the ideas are developed here for the discrete mechanical system, but are readily extended to the continuous system.

PRINCIPLES OF STATIONARY AND MINIMUM POTENTIAL ENERGY

If a discrete conservative mechanical system has N degrees-of-freedom, the configuration is determined by N generalized coordinates q_i and the potential energy is a function of the coordinates:

$$V = V(q_1, \cdots, q_N). \qquad 1$$

Throughout our development, the forces are assumed continuous with continuous derivatives, and so, the potential is also assumed continuous with continuous derivatives.

The principle of virtual work asserts that a state $(\bar{q}_i)$ is a state of equilibrium if the potential V $(\bar{q}_i)$ is stationary; that is, a small displacement $(q_i - \bar{q}_i)$, produces no change of first-degree in the potential:

$$\Delta V_1 = \overline{\frac{\partial V}{\partial q_i}}(q_i - \bar{q}_i) = 0 \qquad 2$$

where the bar (—) signifies evaluation in the reference state $(\bar{q}_i)$

The motionless system is in a state of stable equilibrium if and only if the potential V is a proper minimum[4]; that is, the state $(\bar{q}_i)$ is a state of stable equilibrium if, and only if,

$$\Delta V \equiv V(q_1, \cdots, q_N) - V(\bar{q}_1, \cdots, q_N) > 0 \qquad 3$$

for all displacements $(q_i - \bar{q}_i)$, sufficiently small. The qualification, sufficiently small, is added to emphasize that we require only the local minimum. For example, a ball resting in a shallow valley is in a stable configuration, strictly speaking, but a small jolt may kick the ball over the adjoining hill and into a lower valley (a more stable position).

AN EXAMPLE OF STRUCTURAL INSTABILITY

The system depicted in Fig. 1 is composed of two rigid links $\overline{AB}$ and $\overline{BC}$, joined by a frictionless pin at B and constrained by a linear extensional spring k and torsional spring β. The extensional spring resists lateral displacement W with a force $F = kW$ and the torsional spring resists the relative rotation (2θ) with a couple $C = \beta(2\theta)$. The top A is constrained to move vertically while the bottom C is pinned to a fixed support. Consider the equilibrium of this system under the action of an axial force, P = constant, applied to the end A:

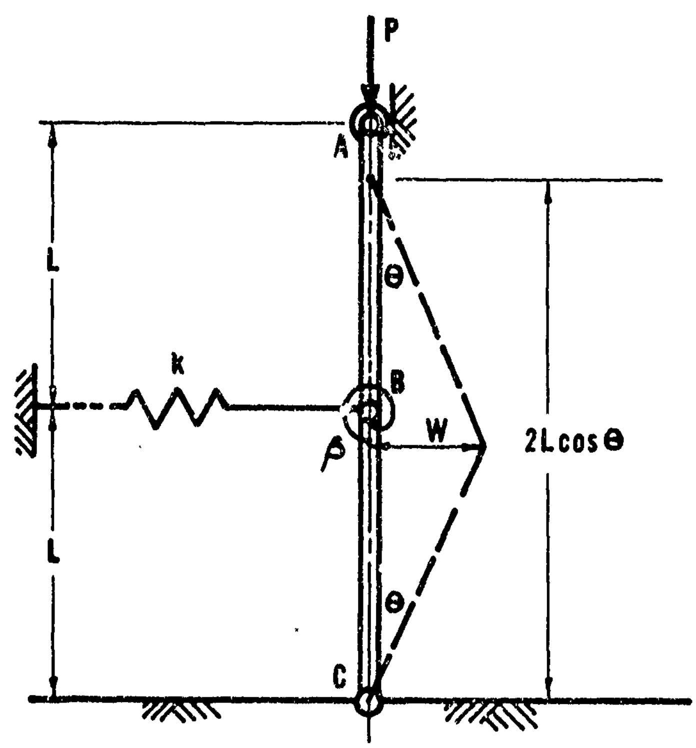


Fig. 1

Our system has one-degree-of-freedom. The configuration is determined by the kinematic variable θ and the total potential energy of the system is

$$V = \frac{kL^2}{2}\sin^2\theta + 2\beta\theta^2 + 2PL\cos\theta \qquad 4$$

By the principle of stationary potential energy, the system is in equilibrium if

$$kL^2\sin\theta\cos\theta + 4\beta\theta - 2PL\sin\theta = 0 \qquad 5$$

Evidently, $\theta = 0$ determines an equilibrium configuration as it satisfies (5) for all choices k, L, β, P. Now, we ask: Are there other solutions $\theta \neq 0$ which satisfy (5)? If so, we can divide (5) by 2L $\sin\theta$ and obtain

$$P = \frac{kL}{2}\cos\theta + \frac{2\beta}{L}\frac{\theta}{\sin\theta} \qquad 6$$

Some plots of (6) trace the solid lines in Fig. 2. The ordinate is the dimensionless load P/P_{cr} where

$$P_{cr} = \frac{kL}{2} + \frac{2\beta}{L} \qquad 7$$

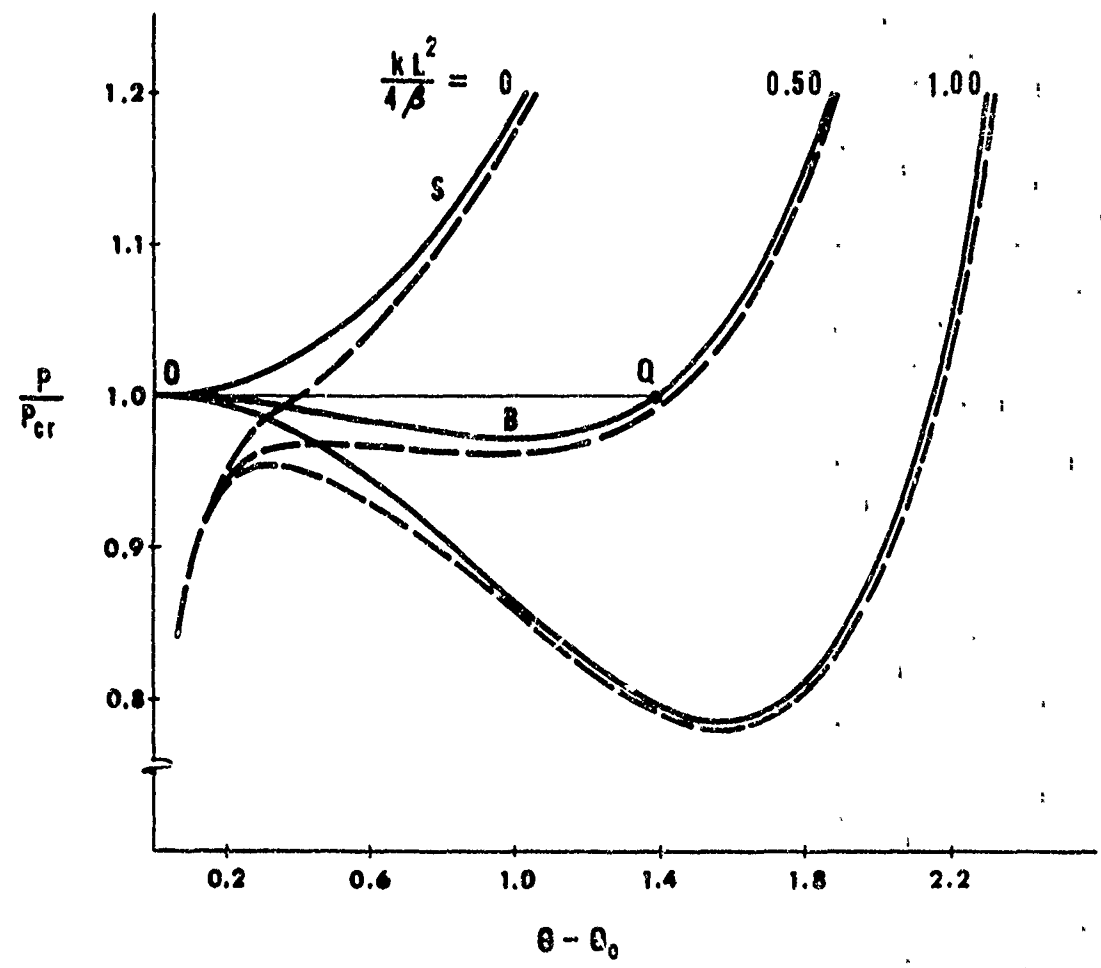


Fig. 2

At the point labeled 0 in Fig. 2 there is a bifurcation; one branch is the vertical line $\theta = 0$ and another branch is the curve of (6).

Let us consider the system when $kL^2 = 8\beta$ and examine its behavior as the load P is gradually applied. The load-deflection curve is the solid curve of Fig. 3; the dotted curves are energy-deflection curves for $P = P_{cr}$ and $P = 0.8\,P_{cr}$.

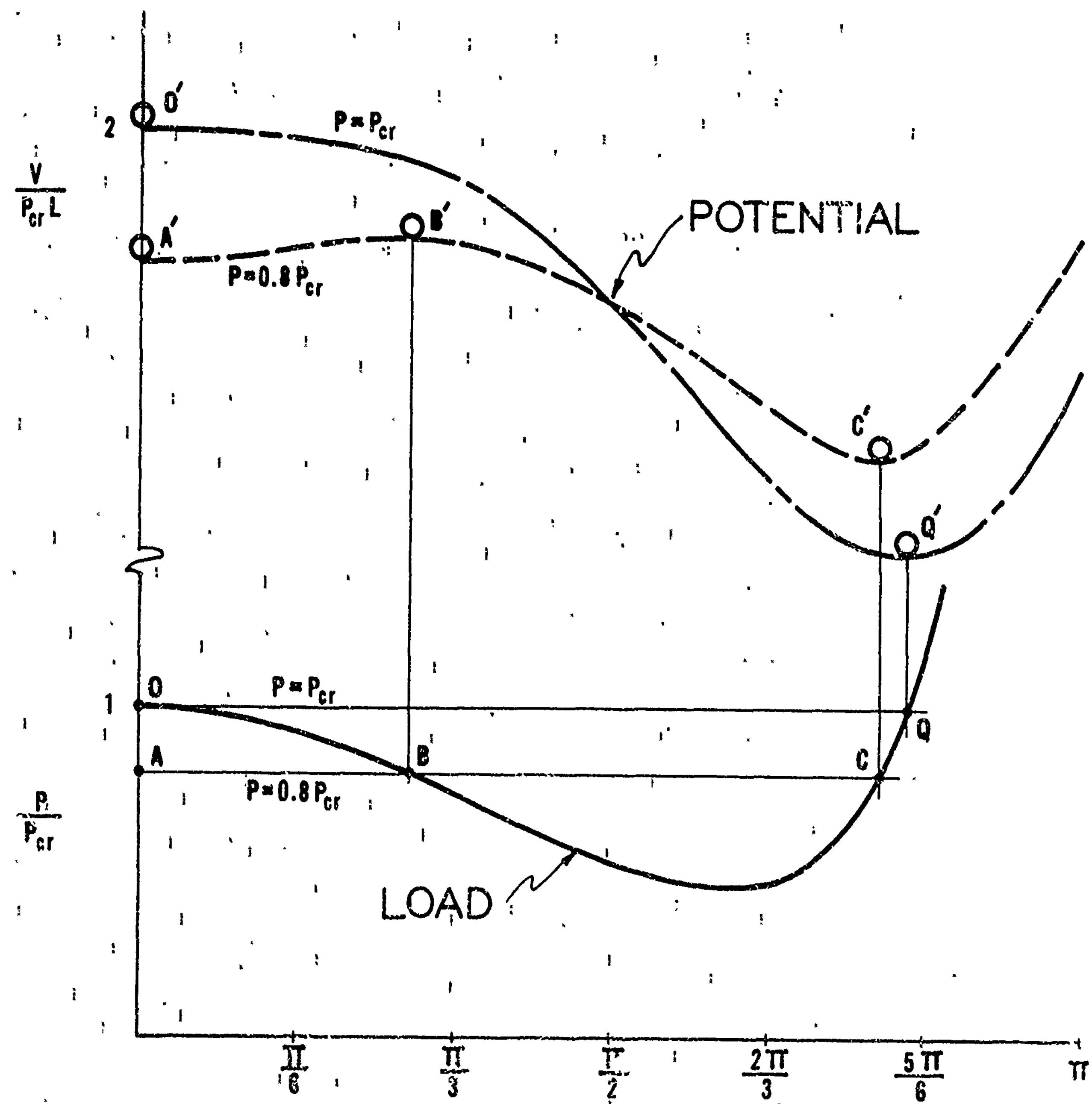


Fig. 3

The system can and does sustain the load in the straight configuration $\theta = 0$ until the load reaches the critical value $P = P_{cr}$. For loads $P > P_{cr}$ the configurations $\theta = 0$ are unstable. At the critical state 0, a slight additional load will cause the system to snap-thru to the configuration of point Q, because the total potential energy at Q is less than at 0; point Q' lies below point 0'. Moreover, every intermediate configuration has a lower energy level. In particular, the configurations adjacent to $\theta = 0$ have less potential energy and, consequently, the system tends to move from the straight configuration. This is apparent from the energy curve near 0' in Fig. 3.

Now consider the same system as the load is gradually applied until $P = 0.8\ P_{cr}$. The system sustains this load in the straight configuration of point A in Fig. 3. At this load the potential energy follows the curve A'B'C' with relative minima at A' and C' and a relative maximum at B'. The configurations of A and C are stable while that of B is unstable. If some energy were supplied to the system in configuration A, the system could be kicked over the energy hill A'B' into the valley at C' corresponding to C on the load-curve. The configuration of A is less stable than that of C because a slight disturbance can cause a violent snap-through from A to C.

Notice that the curve OS in Fig. 2 has a positive slope everywhere. It represents stable configurations. However, because the deflections increase rapidly with load, a structure of this kind may be unusable for loads beyond the critical value, i.e. $P > P_{cr}$. A column, for example, is said to buckle when the load exceeds the critical value of the bifurcation point.

At the critical load the unbuckled configuration ($\theta = 0$) may be stable or unstable; that is, the system may sustain additional load accompanied by a gradual increase in deflection (curve OS of Fig. 2) or it may snap abruptly to a severely deformed configuration at the slightest disturbance (curve OQ of Fig. 2). From a practical viewpoint the question of stability at the critical load is extremely important. In the present example, the question is easily resolved by examining the load-deflection function $P = P(\theta)$. However, we are expressly concerned with the energy criteria and so we examine the conditions for a minimum of potential energy.

In the neighborhood of the reference state, the potential V of (4) can be represented by the series expansion:

$$V = \left(\frac{kL^2}{2} + 2\beta - PL\right)\theta^2 + \frac{1}{6}\left(\frac{PL}{2} - kL^2\right)\theta^4 + \cdots \qquad \text{8a}$$

$$= V_2 + V_4 + \cdots \qquad \text{8b}$$

where V_N denotes the term of degree N in the variable θ. Notice that the odd powers are absent because the structure is symmetrical; equal deflections to the right or left produce the same change of potential. If $V_2 \not\equiv 0$, then V_2 dominates and, sufficiently near to the reference state $\theta = 0$, the stationary condition (5) can be replaced by

$$\frac{d(V_2)}{d\theta} = 2(\frac{kL^2}{2} + 2\beta - PL)\,\theta = 0 \qquad 9$$

The "equilibrium" condition (9) has a nontrivial solution $\theta \neq 0$, if and only if the parenthetical factor vanishes, that is, if the load has the critical value (7). The stationary condition upon the second-degree term V_2 is the criterion of Trefftz.

At the critical state, $V_2 = 0$ and

$$V = V_4 + \cdots$$

The system is stable according to the principle of minimum energy, if

$$V_4 > 0 \qquad 10a$$

In accordance with (7) and (8) the system is stable if

$$\frac{3kL^2}{4\beta} < 1 \qquad 10b$$

If the system of Fig. 1 is imperfect, say the linkage has an initial angle θ_0, then a lateral deflection W accompanies the initial increment of load. As the load increases, a plot of load versus deflection traces the dotted curve of Fig. 2 and approaches the solid trace of the perfect system. If the system exhibits the snap-through characteristic, then the load-deflection curve has a negative slope, as OB, and the crest of the actual (dotted) curve falls below the line $P = P_{cr}$. This suggests that structures exhibiting the snap-through phenomena are also sensitive to imperfections. Experimental evidence confirms our suspicions.

STABILITY OF A DISCRETE MECHANICAL SYSTEM

The essential features of the criterion[1] for a critical load and the criteria[2] for stability at the critical load are exhibited most clearly by a discrete mechanical system. The underlying concepts apply to a continuous system so that the criteria are readily extended. The characteristics and behavior of our system follow:

All loads upon the system are assumed to increase in proportion and, therefore, the magnitude is given by a positive parameter λ. A configuration of the system is defined by N generalized coordinates $q_i (i = 1, \cdots, N)$. As the loading parameter is increased from zero, the equilibrium states trace a path in a configuration-load space. For example, a system with two degrees of freedom (q_1, q_2) follows a path in the space ($q_1, q_2; \lambda$) of Fig. 4a or Fig. 4b.

The point P of Fig. 4a or Fig. 4b is a <u>critical state</u> characterized by the existence of neighboring states which are not uniquely determined by an increment of the load. At the critical point P of Fig. 4a the path OP forms two branches, PR and PQ. The branch PQ may ascend or descend, or the tangent $\hat{W}$ may be normal to the λ axis. At the point P of Fig. 4b, the smooth curve has a tangent $\hat{W}$ normal to the λ axis. Then, if ϵ denotes arc length along the path, the path PQ at P is characterized by the condition $d\lambda/d\epsilon = 0$; in words, the system tends to move from P with no increase of load.

The critical state of Fig. 4a occurs at a bifurcation point; two paths of equilibrium emanate from P. However, the path PR represents unstable paths which can not be realized. Actually, the system tends to move along PQ. If PQ is an ascending path, then additional loading is needed, and the system is said to be stable at the critical state. In actuality, a very slight increment is usually enough to cause an unacceptable deflection and the system is said to buckle. If the curve PQ is descending, the system collapses under the critical load λ^*.

The path of Fig. 4b is entirely smooth, but reaches a so-called limit point P. The state of P is again critical in the sense that the tangent $\hat{W}$ is normal to the λ axis. At P the system tends to move under the critical load λ^*. It tends to buckle, but it is theoretically stable if PQ is an ascending curve. It collapses if the path PQ descends.

By our remarks, instability is signaled by the advent of excessive deflections which are produced by a critical load λ^*. However, the stability of a conservative system can be characterized by an energy criterion: <u>The conservative mechanical system is in stable equilibrium if the potential energy is a proper minimum</u>, unstable if any adjacent state has a lower potential. Let us apply the energy criterion at the critical state:

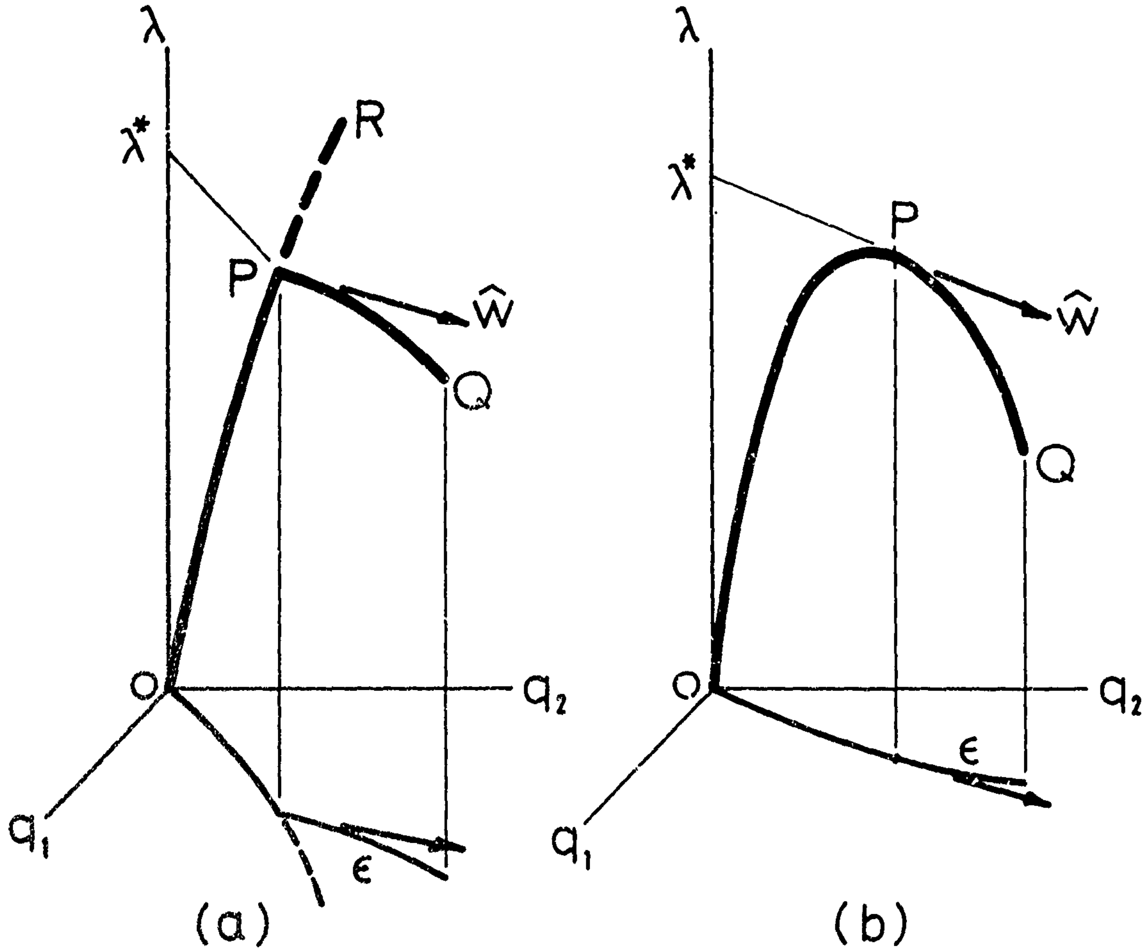


Fig. 4

TREFFTZ CRITERION FOR THE CRITICAL STATE

We presume that the potential energy can be expanded in a power series about the critical state. If $(q_i;\lambda)$ defines a state of equilibrium, $u_i \equiv \Delta q_i$ defines a displacement from the reference state, and V the change of potential caused by the displacement, then

$$V = A_i u_i + \frac{1}{2} A_{ij} u_i u_j + \frac{1}{3!} A_{ijk} u_i u_j u_k + \frac{1}{4!} A_{ijkl} u_i u_j u_k u_l + \dots \qquad 11$$

where

$$A_{ij\dots} = A_{ij\dots}(\lambda) \qquad 12$$

Since the state is a state of equilibrium, in accordance with the principle of virtual work,

$$A_i = 0 \qquad 13$$

If the quadratic term of (11) does not vanish identically, then it dominates for small enough displacement. It follows that the state is <u>stable if</u>

$$V_2(u_i) = \frac{1}{2}A_{ij}u_iu_j > 0 \qquad 14$$

The state is critical if

$$\frac{1}{2}A_{ij}u_iu_j = 0 \qquad 15$$

In words, the state is critical, if there exists one (or more), non-zero displacement(s) u_i which causes the quadratic term to vanish, i.e.

$$V_2(\bar{u}_i) = \frac{1}{2}A_{ij}\bar{u}_i\bar{u}_j = 0 \qquad 16$$

The displacement $\bar{u}_i$ is a buckling mode.

A minimum is characterized by a stationary condition. Here, the required minimum of $V_2(u_i)$ is determined by the stationary criterion of Trefftz: For an arbitrary variation δu_i,

$$\delta V_2 = A_{ij}u_i\delta u_j = 0 \qquad 17$$

It follows that the buckling mode $\bar{u}_i$ is a nontrivial solution of the equations:

$$A_{ij}\bar{u}_i = 0 \qquad 18$$

The homogeneous system has a non-trivial solution, if and only if the determinant of coefficient vanishes

$$\left| A_{ij}(\lambda) \right| = 0 \qquad 19$$

The least root of (19) determines the critical load λ^*.

Let

$$\bar{u}_i = \epsilon \overline{W}_i \qquad 20$$

where $\overline{W}_i$ are components of the unit vector in our N-dimensional space of q_i, i.e.

$$\overline{W}_i \overline{W}_i = 1 \qquad 21$$

The parameter ϵ measures the magnitude of an excursion from the critical state and $\overline{W}_i$ defines the direction of the buckling.

DIFFERENTIAL GEOMETRY OF A PATH FROM THE CRITICAL STATE

Let us consider a movement along the path emanating from the critical point P. In the plane of (q_1, q_2), we see a path as shown in Fig. 5.

In Fig. 5, ρ denotes the curvature of the path PQ at P, $\hat{W}$ is the unit tangent at P and $\hat{V}$ the unit normal. The displacement from P to Q can be expressed in the form $u = \xi\hat{W} + \eta\hat{V}$ or, if ϵ denotes arc-length along PQ,

$$u = \epsilon\frac{du}{d\epsilon} + \frac{1}{2}\epsilon\,\frac{2d^2u}{d\epsilon^2} + \cdots\cdots$$

$$= \epsilon\hat{W} + \frac{\epsilon^2}{2\rho}\hat{V} + \cdots\cdots$$

If we accept an approximation of second-degree in the arc-length ϵ then

$$\xi \doteq \epsilon, \quad \eta \doteq \frac{\epsilon^2}{2\rho}. \qquad 22a,b$$

In the N-dimensional space, as in the 2-dimensional space, one can define an arc-length ϵ along a path stemming from the critical state, i.e.

$$du_i du_i = d\epsilon^2$$

A component of the unit tangent is

$$\overline{W}_i = \frac{du_i}{d\epsilon} \qquad 23$$

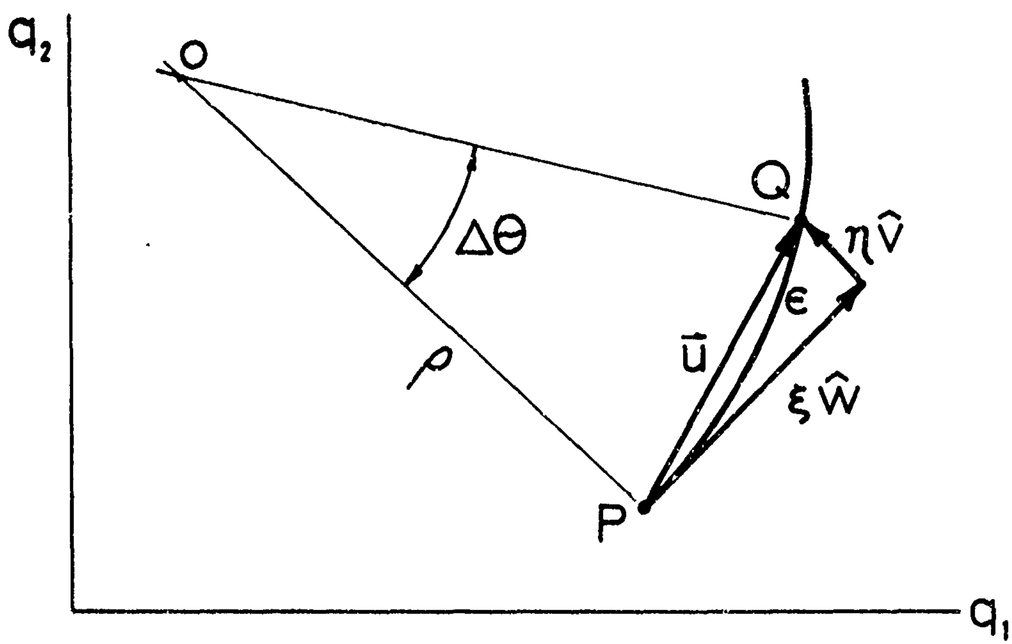


Fig. 5

A component of the unit normal is

$$\kappa V_i = \frac{d^2u_i}{d\epsilon^2} \qquad 24$$

The displacement along a small segment is

$$u_i = \frac{du_i}{d\epsilon}\epsilon + \frac{1}{2}\frac{d^2u_i}{d\epsilon^2}\epsilon^2 + \ldots \qquad 25$$

$$= \epsilon\overline{W}_i + \frac{\epsilon^2}{2}\kappa V_i + \ldots$$

Here V_i is normalized in the manner of (21).

KOITER'S CRITERIA FOR STABILITY AT THE CRITICAL LOAD

A small displacement from the critical state is given by (25). The buckling mode $\overline{W}_i$ is determined according to (18), (19) and (20). Now, we seek the normal V_i and curvature κ which determine the curved path of minimum change V. The change of potential follows from (11) and (25) and simplifies according to (13), (16) and (18):

$$V = \frac{\epsilon^3}{3!} A_{ij\kappa} \overline{W}_i \overline{W}_j \overline{W}_\kappa + \frac{\epsilon^4}{8} \kappa^2 A_{ij} V_i V_j + \frac{\epsilon^4}{4!} A_{ij\kappa l} \overline{W}_i \overline{W}_j \overline{W}_\kappa \overline{W}_l + \frac{\epsilon^4}{4} \kappa A_{ij\kappa} \overline{W}_i \overline{W}_j V_\kappa + O(\epsilon^5) \qquad 26$$

If $V_i = 0$ and ϵ is sufficiently small, the initial term of (26) is dominant. Since the sign of the initial (cubic) term can be positive or negative, depending on the sense of the displacement $\overline{W}_i$, a necessary condition for stability follows:

$$A_3 = \frac{1}{3!} A_{ij\kappa} \overline{W}_i \overline{W}_j \overline{W}_\kappa = 0 \qquad 27$$

If A_3 vanishes, as it usually does in the case of a symmetrical structure, then the sign of V rests with the terms of higher degree. If V is negative for one displacement V_i then the system is unstable. The minimum of (26) is stationary, i.e. $\delta(V) = 0$, for variations of V_i. The stationary conditions follow

$$A_{ij} \kappa V_j = - A_{ij\kappa} \overline{W}_j \overline{W}_\kappa + \ldots + O(\epsilon) \qquad 28$$

If the terms of higher degree are neglected, then equations (28) constitute a linear system in the displacement V_j. In accordance with (23) and (24), the solution $\overline{V}_i$ is to satisfy the orthogonality condition:

$$\overline{V}_i \overline{W}_i = 0 \qquad 29$$

If follows from (28) that

$$\kappa^2 A_{ij} \overline{V}_i \overline{V}_j = - \kappa A_{ij\kappa} \overline{W}_i \overline{W}_j \overline{V}_\kappa + \ldots + O(\epsilon) \qquad 30$$

The potential change corresponding to the displacement $u_i = \epsilon \overline{W}_i + \frac{\epsilon^2}{2} \kappa \overline{V}_i$ is obtained from (26) and simplified by means (27) and (30):

$$V = \epsilon^4 A_4 \qquad 31$$

where

$$A_4 = \frac{1}{4!} A_{ij\kappa l} \overline{W}_i \overline{W}_j \overline{W}_\kappa \overline{W}_l - \frac{\kappa^2}{8} A_{ij} \overline{V}_i \overline{V}_j \qquad 32$$

The system is stable if

$$A_4 > 0 \qquad 33a$$

The system is unstable if

$$A_4 < 0 \qquad 33b$$

In a system with one degree of freedom, $\overline{V}_i = 0$ and the final term of (32) vanishes.

EQUILIBRIUM STATES NEAR THE CRITICAL LOAD

In our preceding view of stability at the critical load λ^*, we examined the energy increment upon excursions from the critical state, but assumed that the load remained constant. Such excursions follow the path of minimum potential on a hyperplane ($\lambda = \lambda^*$) in the configuration-load space ($q_i;\lambda$). To trace a path of equilibrium from the critical state requires, in general, a change in the load. Let us now explore states of equilibrium near the reference state of equilibrium ($q_i;\lambda^*$). To this end, we assume that the potential $V(q_i;\lambda)$ can be expanded in a Taylor's series in the load λ, as well as the displacement u_i. Then, in place of (11), we have

$$V = (A_i u_i + \frac{1}{2} A_{ij} u_i u_j + \frac{1}{3!} A_{ij\kappa} u_i u_j u_\kappa \ldots) + (A'_i u_i + \frac{1}{2} A'_{ij} u_i u_j + \frac{1}{3!} A'_{ij\kappa} u_i u_j u_\kappa + \ldots)(\lambda - \lambda^*) + \ldots \qquad 34$$

Here the prime signifies a derivative with respect to the parameter λ and each of the coefficients (A_i, A'_i, etc.) is evaluated at the critical load.

Along a smooth path from the reference state in the configuration-load space, the "displacement" includes a component in the direction of λ, as well as the direction of q_i. In place of (25), we have

$$u_i = \epsilon u'_i + \frac{\epsilon^2}{2} \kappa V_i + \ldots \qquad 35a$$

$$(\lambda - \lambda^*) = \epsilon \lambda' + \frac{\epsilon^2}{2} \kappa \mu + \ldots \qquad 35b$$

Here, the vector (u'_i; λ') is the unit tangent and (V_i; μ) is the principal normal at (q_i; λ^*) of the path which traces equilibrium states in the space of configuration-load (q_i; λ).

Upon substituting (35a,b) into (34), we obtain

$$V = \epsilon(A_i u_i') + \epsilon^2(\frac{1}{2} A_{ij} u_i' u_j' + A_i' u_i' \lambda') + \dots \qquad 36$$

The principle of stationary potential energy gives the equations of equilibrium at the reference state:

$$A_i = 0 \qquad 37$$

In view of (37), the quadratic terms (ϵ^2) dominate (36). The stationary principle $\delta(V) = 0$, gives the equilibrium equations for states very near the reference state:

$$A_{ij} u_j' = -A_i' \lambda' \qquad 38$$

Now, the reference state is critical, if

$$A_i' \lambda' = 0 \qquad 39$$

In words, either $\lambda' = 0$, which implies the existence of an adjacent state at the same level of loading, and/or $A_i' = 0$, which holds, as (37), if the reference configuration is an equilibrium configuration for $\lambda \neq \lambda^*$. Then, the equilibrium equations of the neighboring state follow:

$$A_{ij} u_i' = 0 \qquad 40$$

Equations (40) are the equations (18) of the Trefftz condition (17). The solution of (40) is the buckling mode

$$u_i' = \overline{W}_i \qquad 41$$

Suppose, firstly, that $A_i' = 0$ in (39) and $\lambda' \neq 0$. Then, according to (37), (40) and (41), the potential of (34) and (36) takes the form:

$$V = \epsilon^3(A_3 + A_2' \lambda' + \dots) + 0(\epsilon^4) \qquad 42$$

where

$$A_3 = \frac{1}{3!} A_{ij\kappa} \overline{W}_i \overline{W}_j \overline{W}_\kappa \qquad 43$$

$$A'_2 = \frac{1}{2} A'_{ij} \overline{W}_i \overline{W}_j \qquad 44$$

We accept the indicated terms of (42) as our approximation and, therefore, set

$$\epsilon\lambda' = \lambda - \lambda^* \qquad 45$$

Our approximation of (42) follows:

$$V \doteq \epsilon^3 A_3 + \epsilon^2 A'_2 (\lambda - \lambda^*) \qquad 46$$

The principle of stationary potential provides the equation of equilibrium:

$$\frac{dV}{d\epsilon} = 3\epsilon^2 A_3 + 2\epsilon A'_2(\lambda - \lambda^*) = 0 \qquad 47a$$

or

$$\epsilon = -\frac{2A'_2}{3A_3}(\lambda - \lambda^*) \qquad 47b$$

The state is stable if the potential is a minimum, that is, if

$$\frac{d^2V}{d\epsilon^2} = 6\epsilon A_3 + 2A'_2(\lambda - \lambda^*) > 0 \qquad 48a$$

or, in accordance with (47b), the system is stable in the adjacent state if

$$-A'_2 (\lambda - \lambda^*) > 0 \qquad 48b$$

In accordance with (34), (40) and (41), the quadratic terms of V in the buckled mode follow:

$$V_2(\overline{W}_i) \equiv \frac{1}{2}\left[A_{ij} + A'_{ij}(\lambda - \lambda^*) + \frac{1}{2} A''_{ij}(\lambda - \lambda^*)^2\right] \overline{W}_i \overline{W}_j$$

Since $V_2(\overline{W}_i) = 0$ at the critical load, we expect that $V_2(\overline{W}_i) > 0$ at loads slightly less than the critical value and that $V_2(\overline{W}_i) < 0$ at loads slightly above the critical value. Therefore, we conclude that

$$A_2' < 0 \qquad 49$$

According to (49) the numerator of (47b) is always negative, but the denominator of (47b) is a homogeneous cubic in $\overline{W}_i$ and the sign is reversed by a reversal of the buckling mode. In this case, an adjacent state of equilibrium exists at loads above ($\lambda > \lambda^*$) or below ($\lambda < \lambda^*$) the critical value. In view of (49) and (48b), an equilibrium state above the critical load is stable and a state below is unstable.

Now, suppose that

$$\lambda' = A_3 = 0 \qquad 50$$

Then, in view of (37), (40), (41) and (50), the potential of (34) and (36) takes the form:

$$V = \epsilon^4 \left(\frac{\kappa^2}{8} A_{ij} V_i V_j + \frac{\kappa}{4} A_{ij\kappa} V_i \overline{W}_j \overline{W}_\kappa + \frac{1}{4!} A_{ij\kappa l} \overline{W}_i \overline{W}_j \overline{W}_\kappa \overline{W}_l + \ldots \right) + \epsilon^3 \mu \frac{\kappa}{2} \left(\underline{A_i' \overline{W}_i} \right.$$

$$\left. + \epsilon \frac{1}{2} A_{ij} \overline{W}_i \overline{W}_j + \ldots \right) + 0(\epsilon^5) \qquad 51$$

The underlined term of (51) dominates if $\mu \neq 0$ and if ϵ is sufficiently small. The term is odd in $\overline{W}_i$ and, therefore, always provides a negative potential change at any load $\lambda \neq \lambda^*$. A condition for the existence of stable states at noncritical values of load follows:

$$A_i' \overline{W}_i = 0 \qquad 52$$

However, the bucking mode $\overline{W}_i$ is independent of the coefficients A_i'. Therefore, equation (52) implies generally that

$$A_i' = 0 \qquad 53$$

Now, we accept the remaining terms indicated in (51) as our approximation. Also, in view of (50),

$$\epsilon^2 \frac{\kappa}{2} \mu \doteq \lambda - \lambda^* \qquad 54$$

Our approximation of (51) follows:

$$V \doteq \epsilon^4\left(\frac{\kappa^2}{8} A_{ij}V_iV_j + \frac{\kappa}{4} A_{ij\kappa} V_i\overline{W}_j\overline{W}_\kappa + \frac{1}{4!} A_{ij\kappa l}\overline{W}_i\overline{W}_j\overline{W}_\kappa\overline{W}_l\right) + (\epsilon^2 \frac{1}{2} A'_{ij}\overline{W}_i\overline{W}_j) \quad (\lambda - \lambda^*) \qquad 55$$

Again, we require a stationary potential for variations of the displacement V_{ij}. The equations of equilibrium follow:

$$\epsilon^2\kappa A_{ij}V_j = -\epsilon^2 A_{ij\kappa}\overline{W}_j\overline{W}_\kappa \qquad 56$$

Let $\kappa\overline{V}_i$ denote the solution of (56). Then, it follows that

$$\epsilon^2\kappa^2 A_{ij}\overline{V}_i\overline{V}_j = -\epsilon^2\kappa \ A_{ij\kappa}\overline{W}_j\overline{W}_\kappa\overline{V}_i \qquad 57$$

If the solution $\kappa\ \overline{V}_i$ and (57) are used in (55), then our approximation of the potential takes the form:

$$V = \epsilon^4 A_4 + \epsilon^2(\lambda - \lambda^*)A'_2 \qquad 58$$

where A'_2 is defined by (44) and

$$A_4 = \frac{1}{4!} A_{ij\kappa l}\overline{W}_i\overline{W}_j\overline{W}_\kappa\overline{W}_l - \frac{\kappa^2}{8} A_{ij}\overline{V}_i\overline{V}_j \qquad 59$$

The solution of (56) determines the unit vector $\overline{V}_i$ which renders V stationary, but still dependent upon the distance ϵ. The principle of stationary potential gives the equilibrium condition:

$$\frac{dV}{d\epsilon} = 4\epsilon^3 A_4 + 2\epsilon(\lambda - \lambda^*)A'_2 = 0 \qquad 60$$

or

$$\epsilon^2 = -\frac{A'_2}{2A_4} \quad (\lambda - \lambda^*) \qquad 61$$

According to (33a,b), (49) and (61), a stable adjacent state of equilibrium can exist only at loads above the critical value ($\lambda > \lambda^*$) and a state below the critical value is unstable.

Koiter provides rigorous arguments for the conditions (53) and (49) if the critical configuration is a stable equilibrium configuration for loads less than the critical value. For example, the two-dimensional system has equilibrium configurations which trace a line parallel to the λ axis, as shown in Fig. 6. The portion OP represents stable states, the bifurcation point P the critical state, PR represents unstable states of the reference configuration and PQ the stable postbuckled equilibrium states. Here, the principle of stationary energy in the critical configuration at any load, leads to the equation (53) and the principle of minimum energy in the stable states of OP ($\lambda < \lambda^*$) leads to the inequality (49).

If the cubic term of V does not vanish, then equilibrium states trace paths with slope λ' at the critical load, as shown in Fig. 6a. If the cubic term vanishes, then $\lambda' = 0$ and the equilibrium states trace paths as shown in Fig. 6b. In each figure, the solid lines are stable branches and the dotted lines are unstable.

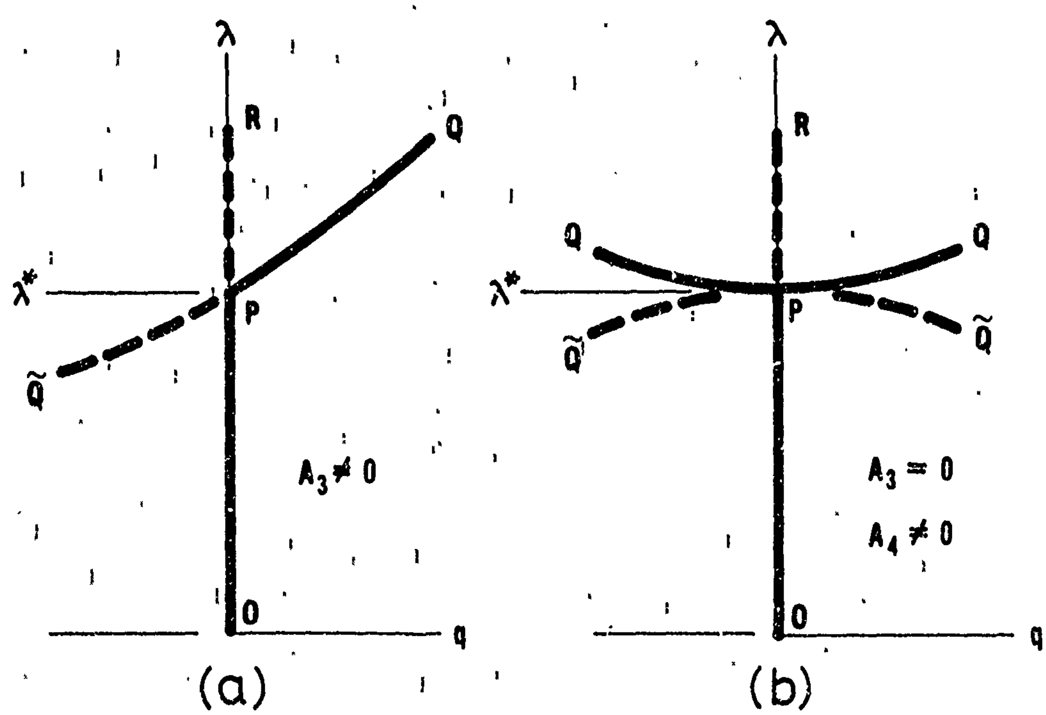


Fig. 6

Practically speaking, many structural systems display the instability patterns of Fig. 6, that is, the prebuckled configuration of the ideal structure is an equilibrium state under all loads. Notable examples are the column under axial thrust, the spherical or cylindrical shell under external pressure and the cylinder under uniform axial compression. Essentially, each retains its form until the load reaches the critical value and, then buckles. In the case of thin shells, initial imperfections cause pronounced departures from the initial form and often cause premature buckling ($\lambda < \lambda^*$).

Our analysis of stability at the critical load is limited. The reader should note, especially, that any of the various terms of the potential, e.g. V_2, V_4, may vanish identically. Then, further investigation, involving terms of higher degree, is needed.

EFFECT OF IMPERFECTIONS UPON THE BUCKLING LOAD

In the monumental work of Koiter[2], an important practical achievement was his assessment of the effect of geometrical imperfections upon the buckling load of an actual structure. Here, we outline the procedure and cite the principal results:

Under the conditions of dead loading upon a Hookean structure[2], the energy potential $\tilde{V}$ of the actual structure is expressed in terms of a displacement u_i from the critical state of the ideal structure and a parameter e which measures the magnitude of the initial displacements of the actual unloaded structure:

$$\tilde{V} = \left[\frac{1}{2} A_{ij}(\lambda)u_i u_j + \frac{1}{3!} A_{ijk}(\lambda)u_i u_j u_k + \frac{1}{4!} A_{ijkl}(\lambda)u_i u_j u_k u_l\right] + e\left[B_i(\lambda)u_i + \frac{1}{2} B_{ij}(\lambda)u_i u_j + \dots\right] \quad 62$$

Here, the linear terms in u_i vanish in the first bracket, because the reference configuration is an equilibrium configuration of the ideal structure (e = 0) at any load.

As before, the components u_i and $(\lambda - \lambda^*)$ are expanded in powers of the arc length ϵ along the ideal curve of Fig. 7. Here, we make an assumption that the initial deflection of the actual structure is nearly the buckling mode $\overline{W}_i$ of the ideal structure. Therefore, we have the approximation:

$$u_i \doteq \epsilon \overline{W}_i + \epsilon^2 \frac{K}{2} V_i \quad 63$$

$$(\lambda - \lambda^*) \doteq \epsilon\lambda' + \epsilon^2 \frac{K}{2}\mu \quad 64$$

In the present case, the second-order terms of (63) and (64) contain an unspecified parameter K, because these terms do not represent deviations from the tangent ($\epsilon\hat{W}$) along the ideal path of Fig. 7, but represent the displacement (**d**) which carries the system to the actual path as depicted in Fig. 7.

Upon substituting (63) and (64) into (62) and acknowledging (16), (18) and (20), we obtain

$$\widetilde{V} = \epsilon^3 \left[A_3 + A_2'\lambda'\right] + \epsilon^4 \left[\frac{1}{4!} A_{ijkl}\overline{W}_i\overline{W}_j\overline{W}_k\overline{W}_l + \frac{K^2}{8} A_{ij}V_iV_j + \frac{K}{4} A_{ijk}\overline{W}_i\overline{W}_jV_k + \frac{K}{4} A_{ij}'\overline{W}_i\overline{W}_j\mu + \frac{1}{2} A_{ij}''\overline{W}_i\overline{W}_j(\lambda')^2\right] + 0(\epsilon^5) + e\left[\epsilon B_i\overline{W}_i + \epsilon^2 B_i \frac{K}{2} V_i\right] \qquad 65$$

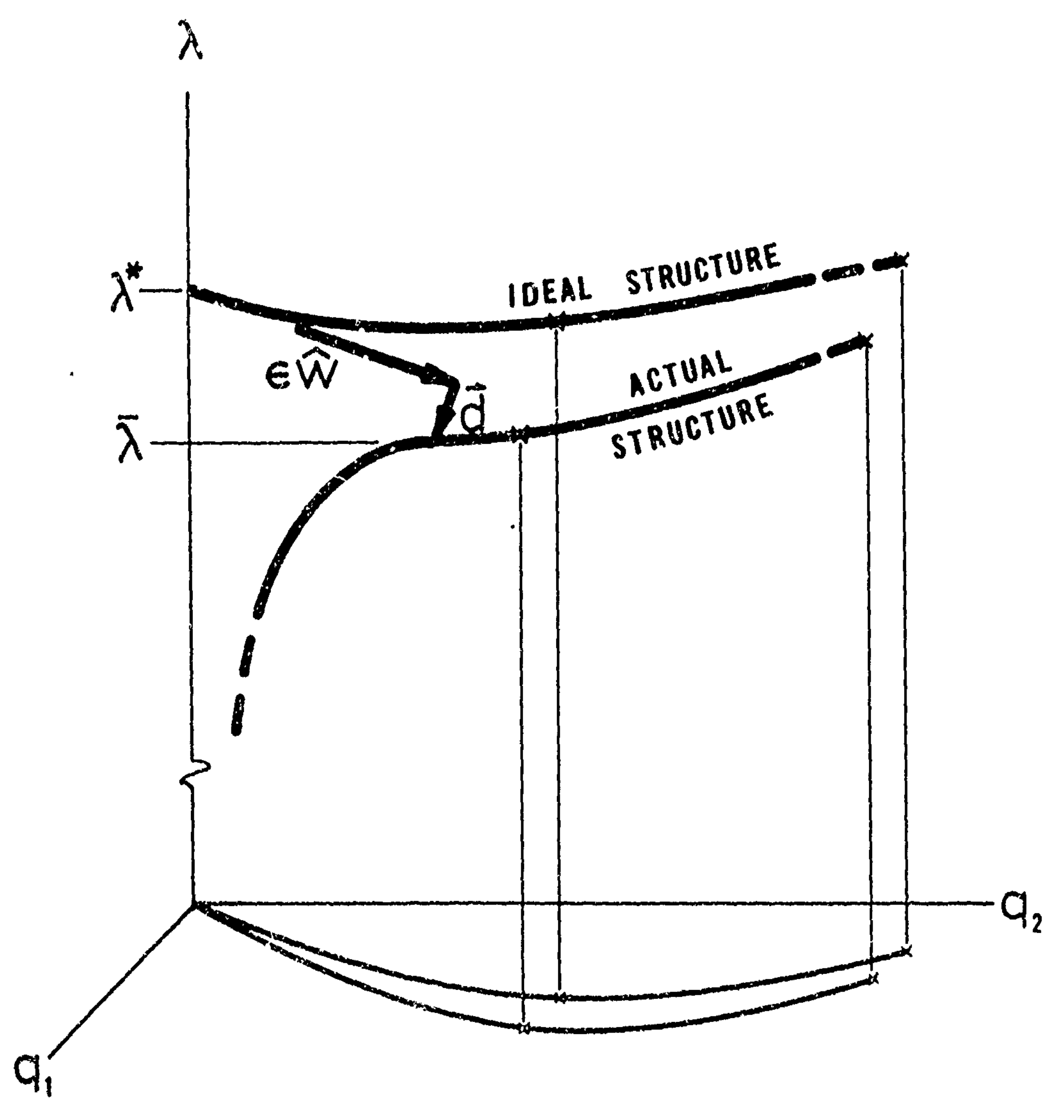


Fig. 7

where A_3 and A_2' are defined by (43) and (44), as before. Since the relative magnitudes of ϵ and e are unspecified, we must suppose that the terms $O(\epsilon^3)$ and $O(e\epsilon)$ dominate (65), if $A_3 \neq 0$ and $\lambda \neq 0$. Then, we have the approximation:

$$\tilde{V} \doteq \epsilon^3 A_3 + \epsilon^2 A_2'(\lambda - \lambda^*) + e\epsilon B_i \overline{W}_i \qquad 66$$

The stationary condition of equilibrium follows:

$$\frac{d\tilde{V}}{d\epsilon} = 3\epsilon^2 A_3 + 2\epsilon A_2'(\lambda - \lambda^*) + eB_i\overline{W}_i = 0 \qquad 67$$

Now, recall that $\tilde{V}$ is not the potential increment from the critical configuration of the actual structure but the potential referred arbitrarily to the critical configuration of the ideal structure. An equilibrium configuration of the actual structure is stable or unstable, respectively, if the potential is a minimum or a maximum; therefore, the critical load $\bar{\lambda}$ of the actual structure satisfies the conditions

$$\frac{d^2\tilde{V}}{d\epsilon^2} = 6\epsilon A_3 + 2A_2'(\lambda - \lambda^*) \qquad 68$$

$$> \quad 0 \Rightarrow \text{stability} \qquad 69a$$

$$< \quad 0 \Rightarrow \text{instability} \qquad 69b$$

Observe that the distinction between stability and instability of a post-buckled state rests upon the same conditions, (69a) and (69b), as the ideal structure [see equation (48a)] and that the conditions are independent of the imperfection parameter e.

If $A_3 \neq 0$, then the imperfect structure deflects and reaches a critical state of equilibrium when (67) is satisfied and (68) vanishes. If $(\lambda - \lambda^*)$ is eliminated from the two equations, then

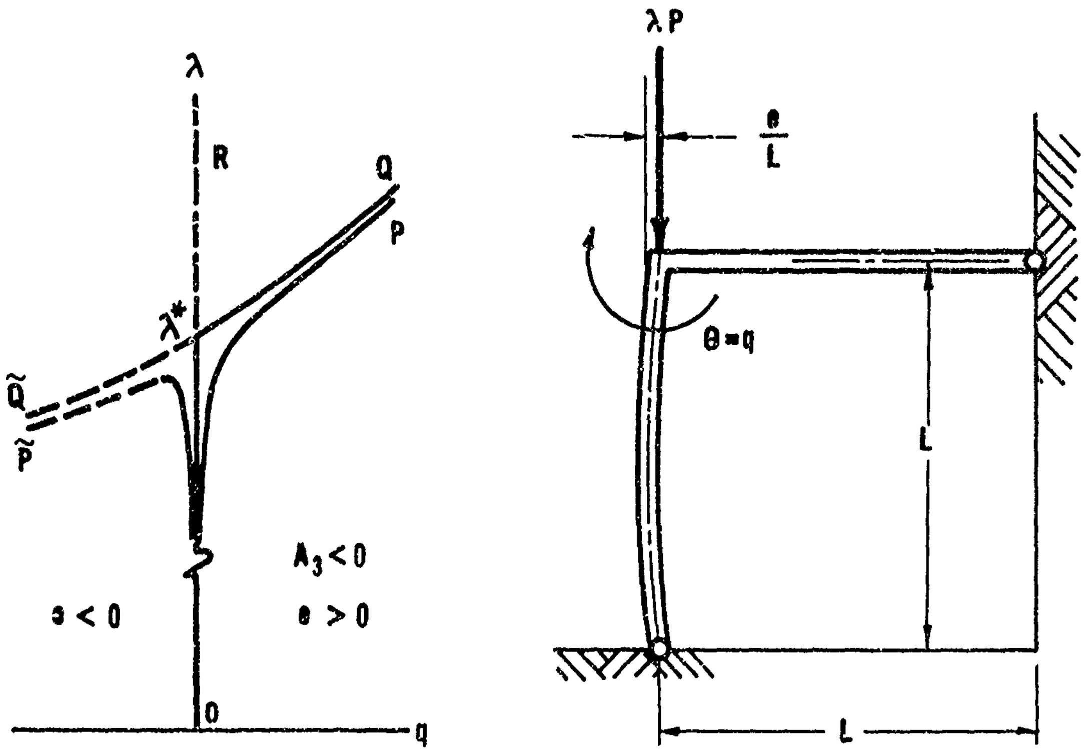
$$e = \frac{3\epsilon^2 A_3}{B_i\overline{W}_i} \qquad 70$$

The sign of the sum $B_i\overline{W}_i$ is arbitrary, since a change of sign is effected by redefining the parameter e. Therefore, we can choose e so that $B_i\overline{W}_i$ has the opposite sign of A_3. Then the condition (70) for a critical state is fulfilled only if $e < 0$. From our observations, we know that an imperfect structure tends to buckle in a preferred direction, depending upon the character of the geometrical deviations. In the present case, if $A_3 < 0$, a critical state occurs only if $e < 0$. A plot of load versus deflection is depicted in Fig. 8a; here, a negative value e produces buckling in a negative mode ($eW < 0$) according to the curve $O\tilde{P}$, whereas the positive value e produces only stable states along the path OP.

A real structure which behaves in the manner of Fig. 8a is the frame of Fig. 8b. If the vertical strut is bent to the right or left the imperfection parameter e is negative or positive, respectively. The rotation θ of the joint serves as a generalized coordinate($\theta = q$) and plots of load versus rotation take the forms of Fig. 8a. The frame under eccentric loading has been studied experimentally by Roorda[5] and theoretically by Koiter[6]. The latter computations show remarkable agreement with the former experimental results.

Now, let us turn to the structure of Fig. 6b, characterized by the conditions

$$A_3 = \lambda' = 0 \qquad (71)$$



(a) (b)

Fig. 8

Now, terms $0(\epsilon^3)$ are absent from the potential of (65). The latter must be stationary with respect to the displacement V_i; for equilibrium,

$$\epsilon^2 K A_{ij} V_j = -\epsilon^2 A_{ij\kappa} \overline{W}_j \overline{W}_\kappa - 2eB_i \qquad 72$$

If $K\overline{V}_i$ denote the solution of (72), then

$$\epsilon^2 K A_{ij\kappa} \overline{W}_j \overline{W}_\kappa \overline{V}_i = -\epsilon^2 K^2 A_{ij} \overline{V}_i \overline{V}_j - 2eKB_i \overline{V}_i \qquad 73$$

In accordance (64), (71) and (73), our approximation of the stationary value of (65) follows:

$$\widetilde{V} = \epsilon^3 \left[\epsilon A_4 + A_2'(\lambda - \lambda^*) \right] + e\epsilon B_i \overline{W}_i \qquad 74$$

As before, the potential $\widetilde{V}$ is still dependent upon the distance ϵ. The stationary condition of equilibrium follows:

$$\frac{d\widetilde{V}}{d\epsilon} = 4\,\epsilon^3 A_4 + 3\epsilon^2 A_2'(\lambda - \lambda^*) + eB_i \overline{W}_i = 0 \qquad 75$$

The stability of equilibrium depends upon the second derivative as follows:

$$\frac{d^2\widetilde{V}}{d\epsilon^2} = 12\epsilon^2 A_4 + 6\epsilon A_2'(\lambda - \lambda^*) \qquad 76$$

$$> 0 \Rightarrow \text{stability} \qquad 77a$$

$$< 0 \Rightarrow \text{instability} \qquad 77b$$

Again, the critical state of equilibrium is characterized by vanishing of the first derivative (75) and second derivative (76). The elimination of $(\lambda - \lambda^*)$ yields the result:

$$e = \frac{2\epsilon^3 A_4}{B_i \overline{W}_i} \qquad 78$$

Again, we note that the definition of e and the sign of $\overline{W}_i$ are arbitrary and, therefore, we assume that $\overline{W}_i$ renders the sum $B_i\overline{W}_i > 0$. Then, the condition (78) for a critical load is attained if $A_4 > 0$, $e < 0$, in keeping with (33b). Now, the structure also exhibits instability at the critical load if the sign of the parameter e and the buckling mode are both reversed. A plot of load versus deflection is depicted in Fig. 9. The linkage of Fig. 1 and the curves of Fig. 2 exemplify such structural systems. The structure is symmetrical and can buckle in either direction depending upon the character of the geometrical deviation. In either case, it is stable or unstable depending upon the sign of the constant A_4. The critical load $\bar{\lambda}$ of the actual structure may be much less than the critical load λ^* of the ideal structure.

If the distance ϵ is eliminated from (67) and (68) = 0, or if ϵ is eliminated from (75) and (76) = 0, the critical load $\lambda = \bar{\lambda}$ is expressed in terms of the imperfection parameter (-e):

$$\bar{\lambda} = \lambda^* - \left[n(-e)(B_i\overline{W}_i)(-A_n)^{n-2} \right]^{\frac{1}{n-1}} (-A_2')^{-(n-1)} \qquad 79$$

Here n = 3 if $A_3 < 0$, n = 4 if $A_3 = 0$ and $A_4 < 0$, and $e < 0$. A plot of the actual buckling load versus the imperfection parameter has the appearance of Fig. 10. Since the curve is tangent to the axis $\bar{\lambda}$ at e = 0, small imperfections can cause considerable reduction of the buckling load.

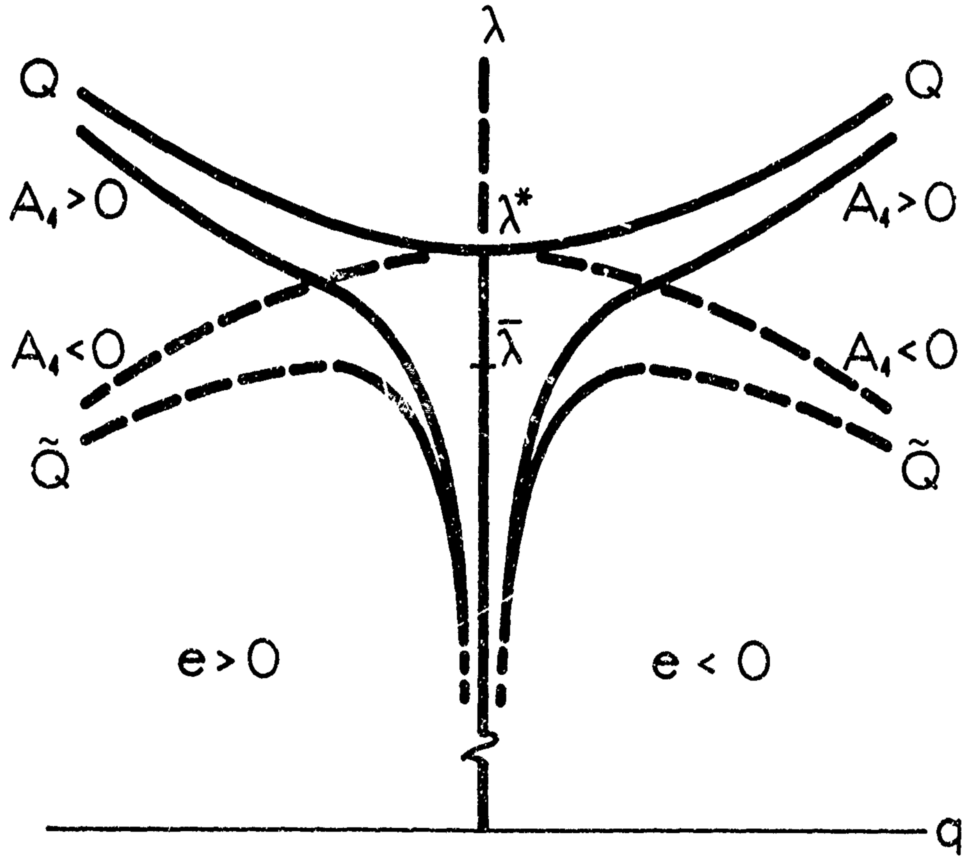


Fig. 9

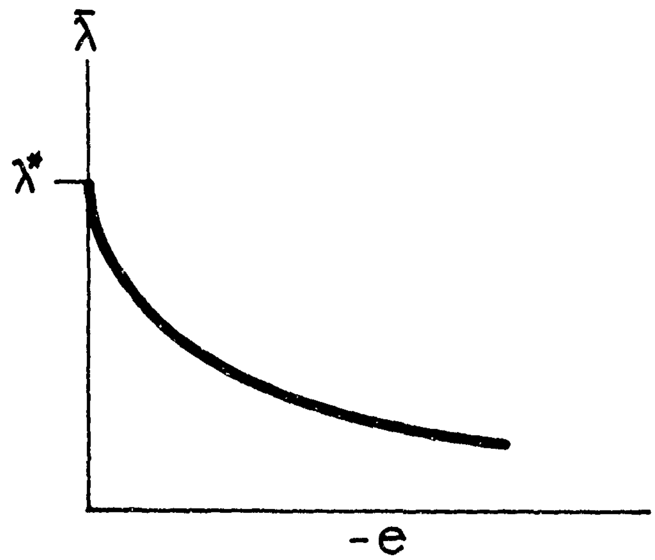


Fig. 10

THEORY FOR CONTINUOUS BODIES

The stationary criteria [1,2] were originally given for continuous bodies. The concepts are the same, but the mathematical form is altered: Our discrete displacements u_i are replaced by the continuous field $u_i(x_i)$ and the sums of (11) by integrals of the continuous field. The stationary conditions of Trefftz[1] and Koiter[2] apply to the integrals of corresponding degree, e.g. Trefftz criteria is the stationary condition upon the homogeneous functional of second degree which replaces the quadratic function V_2 of (14).

Koiter's theory has contributed most significantly to our understanding of instability, snap-buckling and imperfection sensitivity of thin shells. The reader is referred to the work of Koiter[7]; Budiansky and Hutchinson[8], Hutchinson[9], Hutchinson and Amazigo[10], and B. Budiansky[11].

Finally, we note that the continuous structure can always be approximated by a discrete system whereupon the foregoing equations are applicable. In particular, a shell can be subdivided into finite elements, the deformation can be approximated by interpolation, and then, the stationary criteria can be applied to the discrete model in the manner of Rayliegh-Ritz. The success of such methods will depend greatly on the complexity of the continuous body, the buckling mode

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