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DERIVATIVES AND DOUBLE DIFFERENCES

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DERIVATIVES AND DOUBLE DIFFERENCES

L. C. Young

§1. Introduction. Let $F(t)$ be a continuous real, complex or Banach-valued function defined for small real t, let Δ_h be the difference operator defined by $\Delta_h F(t) = F(t+h) - F(t)$, and let $\Delta_{hk} = \Delta_h \Delta_k$ be similarly defined by writing

$$(1.1) \qquad \Delta_{hk} F(t) = F(t+h+k) - F(t+h) - F(t+k) + F(t) .$$

We write further $\Delta_h^* = -\Delta_h \Delta_{-h}$, so that

$$(1.2) \qquad \Delta_h^* F(t) = F(t+h) + F(t-h) - 2F(t) ,$$

and $\Delta_{hk}^* = \Delta_h^* \Delta_k^*$, so that

$$(1.3) \qquad \Delta_{hk}^* F(t) = \begin{array}{l} F(t+h+k) + F(t+h-k) + F(t-h+k) + F(t-h-k) \\ -2F(t+h) - 2F(t+k) - 2F(t-h) - 2F(t-k) + 4F(t) . \end{array}$$

A pair of functions $\varphi(u)$, $\psi(u)$ defined for small $u \geq 0$ will be termed estimate functions of orders n, m if φ is non-negative and Borel measurable, ψ continuous and monotone increasing from the value $\psi(0) = 0$, provided they satisfy the following conditions:

(1.4) For $0 < \theta < 1$, we have $\varphi(\theta u) \geq (\frac{1}{2}\theta)^n \varphi(u)$, $\psi(\theta u) \geq (\frac{1}{2}\theta)^m \psi(u)$,

(1.5) For small $h > 0$, we have $\int_0^h u^{-n} \varphi(u)\, d\psi(u) < \infty$.

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For simplicity, we suppose in the sequel (1. 5) to hold for $h = 1$, so that φ, ψ are defined on the unit interval; further we confine ourselves to the orders $n = m = 1$ and $n = m = 2$. Moreover F will be supposed defined for $|t| \leq 3$, so that symbols such as $F(t + h + k)$ have a meaning for $|t|$, $|h|$, $|k|$ all ≤ 1. Finally K will denote an absolute constant, whose value may depend on the context.

One of the consequences, drawn in [1], from an already much used algorithm, was that if, for all small t, h, k, F satisfies

(1. 6) $$|\Delta_{hk} F(t)| \leq \varphi(h)\, \psi(k)$$

where φ, ψ are estimate functions of order 1, then, for all small h, the derivative $F'(0)$ satisfies

(1. 7) $$|F'(0) - h^{-1} \Delta_h F(0)| \leq K \int_0^{|h|} u^{-1} \varphi(u) d\psi(u) ,$$

and, in particular, exists. In this note we consider two analogues of (1. 6), namely

(1. 8) (a) $$|\Delta^*_{hk} F(t)| \leq \varphi^*(h)\, \psi^*(k) ,$$

(1. 8) (b) $$\int_0^1 |\Delta_{hk} F(t)|^2 dt \leq \varphi^*(h)\, \psi^*(k) ,$$

where φ^*, ψ^* are estimate functions of order 2. In (1. 8) (b) we can set $h = k$, $\varphi^* = \psi^*$ if we wish, the conclusions will be the same. We show that (1. 8) (a) implies

(1.9) (a) $$|F''^{*}(0) - h^{-2} \Delta_h^{*} F(0)| \leq K \int_0^{|h|} u^{-2} \varphi^{*}(u)\, d\psi^{*}(u) \quad ,$$

where F''^{*} denotes the Schwartz generalized second derivative

$$F''^{*}(t) = \lim_{h \to 0} h^{-2} \Delta_h^{*} F(t) \; ;$$

it is understood that the existence of this quantity is part of the assertion (1.9) (a). In (1.8)(b) F will be supposed periodic and real or complex-valued; we can drop the continuity assumption and suppose instead that F is square integrable in its period; our corresponding result is then

(1.9)(b) $$\int_0^1 |F'(t) - h^{-1} \Delta_h F(t)|^2 dt \leq K \int_0^{|h|} u^{-2} \varphi^{*}(u) d\psi^{*}(u) \quad ,$$

and in particular, for $h = 1$,

(1.9)(c) $$\int_0^1 |F'(t)|^2 dt \leq K \int_0^1 u^{-2} \varphi^{*}(u)\, d\psi^{*}(u) \quad ;$$

these assertions similarly include the existence of F', but this time as a limit in L^2.

§2. Estimate functions of orders n, m. Let $\varphi(u)$, $\psi(u)$ be defined for $0 \leq u \leq 1$, and suppose φ non-negative and Borel measurable, ψ continuous and monotone increasing from the value $\psi(0) = 0$; further suppose (1.4) satisfied. We denote by $\{h\}$ a decreasing sequence h_ν $(\nu = 1, 2, \ldots)$ with limit 0, and we suppose its terms less than an initial number $h = h_0 \leq 1$. We write (for fixed n)

$$
(2.1)\qquad \begin{cases} S_{\{h\}} = \sum_{\nu=1}^{\infty} (h_\nu)^{-n} \varphi(h_\nu)\, \psi(h_{\nu-1}) , \\ S_h \quad = \int_0^h u^{-n} \varphi(u)\, d\psi(u) , \\ S \quad = S(1) . \end{cases}
$$

We say further that $\{h\}$ is subject to condition $C(1, m)$ if, for each ν , the ratio $h_{\nu-1}/h_\nu$ is an integer expressible as a power of 2, and to condition $C(2, m)$ if

$$
2\psi(h_\nu) \leq \psi(h_{\nu-1}) \leq 2^{2m+1} \psi(h_\nu) .
$$

In accordance with (1.5), φ, ψ are estimate functions of orders n, m if and only if $S < \infty$. The following lemma provides alternative forms of this condition:

Lemma (2.2). (i) The following conditions are all equivalent: (a) $S < \infty$; (b) $S_{\{h\}} < \infty$ for some $\{h\}$; (c) $S_{\{h\}} < \infty$ for some $\{h\}$ subject to $C(1, n)$ and $C(2, n)$. (ii) We have, for any $\{h\}$,

$$
(2.3)\qquad S(h_0) \leq 2^n S_{\{h\}} ;
$$

and moreover, for any $\{h\}$ subject to $C(1, m)$ and $C(2, m)$

$$
(2.4)\qquad (h_1)^{-n} \varphi(h_1)\, \psi(h_1) \leq 2^{n+2m+1} S(h_1) ,
$$

$$
(2.5)\qquad S_{\{h\}} \leq 2^{n+4m+2} S(h_1) .
$$

Proof: We write for short

$$s_\nu = \int_{h_\nu}^{h_{\nu-1}} u^{-n} \varphi(u)\, d\psi(u); \quad s_\nu^* = (h_\nu)^{-n} \varphi(h_\nu)\, \psi(h_{\nu-1}) .$$

By (1.4) for φ,

$$s_\nu \leq 2^n (h_\nu)^{-n} \varphi(h_\nu)[\psi(h_{\nu-1}) - \psi(h_\nu)] ,$$

$$2^n s_{\nu+1} \geq (h_\nu)^{-n} \varphi(h_\nu)[\psi(h_\nu) - \psi(h_{\nu+1})] .$$

From the first of these relations we derive $s_\nu \leq 2^n s_\nu^*$, and so (2.3). From the second, by successively using the first and the second inequality of C(2, m) with $\nu + 1$ in place of ν, and then the second inequality of C(2, m) itself, we find that

$$2^n s_{\nu+1} \geq (h_\nu)^{-n} \varphi(h_\nu)\, \psi(h_{\nu+1}) \geq (h_\nu)^{-n} \varphi(h_\nu)\, \psi(h_\nu) \cdot 2^{-2m-1} \geq s_\nu^*\, 2^{-4m-2} .$$

Evidently (2.4) and (2.5) follow from these relations.

To establish (i) it will now suffice to show that there exists an $\{h\}$ subject to C(1, m) and C(2, m). For this purpose, we attach to any h in $0 < h \leq 1$ and h^* and an $\hat{h}$. Here $\hat{h}$ is defined as the smallest number of the form $h2^{-\mu}$ $(\mu = 0, 1, 2, \ldots)$ such that $\hat{h} \geq h^*$; and h^* is defined so that $\psi(h) = 2^{2m+1}\psi(h^*)$. For definiteness we take h^* to be the smallest such number unless $\psi(h) = 0$, in which case we set $h^* = h/2$. Then

$$\psi(\hat{h}) \geq \psi(h^*) = 2^{-2m-1}\psi(h) ;$$

moreover $\hat{h} < 2h^*$, and so by (1.4) for ψ we find that

$$\psi(\hat{h}) \leq (2\hat{h}/h^*)^m \psi(h^*) \leq 4^m \psi(h^*) = \frac{1}{2}\psi(h) \ .$$

Here $\hat{h} \neq h$, so that $\mu \neq 0$. Hence by setting $h_0 = h$, $h_1 = \hat{h}$ and generally $h_\nu = \hat{h}_{\nu-1}$, we define by induction a sequence $\{h\}$ subject to $C(1,m)$ and $C(2,m)$. This completes the proof.

§3. <u>The algorithm for $n = m = 2$</u>. We shall base our algorithm on an elementary consequence of the geometric series identity. Let $z = e^{ih}$, $N = 2M$ and

$$P = \sum_{m=0}^{M-1} (e^{i(m+\frac{1}{2})h} + e^{-i(m+\frac{1}{2})h}) \ ,$$

so that

$$P = z^{-(N-1)/2}(1 + z + \ldots + z^{N-1}) = z^{-(N-1)/2} \cdot \left(\frac{z^N - 1}{z - 1}\right) \ .$$

Then

$$e^{iNh} + e^{-iNh} - 2 - N^2(e^{ih} + e^{-ih} - 2) = z^{-N} \cdot (z^N - 1)^2 - N^2 z^{-1}(z-1)^2$$

$$= z^{-1}(z-1)^2 (P^2 - N^2) = z^{-1}(z-1)^2 (P - 2M)(P + 2M)$$

$$= (e^{ih} + e^{-ih} - 2)(P - 2M)(P + 2M) \ .$$

If we multiply by e^{it} this shows that for the function $F(t) = e^{it}$ we have the identity

$$(\Delta^*_{(Nh)} - N^2 \Delta^*_h) F(t) = \Delta^*_h \sum_{m=0}^{M-1} \Delta^*_{[(m+\frac{1}{2})h]} \sum_{\mu=0}^{M-1} \{F(t+(\mu+\tfrac{1}{2})h) + F(t-(\mu+\tfrac{1}{2})h) + 2F(t)\} \ ,$$

or equivalently, after division by N^2h^2,

$$(3.1)\quad a^*(N,h) = h^{-2}M^{-2} \sum_{\substack{0\le m\le M-1\\ 0\le \mu\le M-1}}\sum \Delta^*_{h,[(m+\frac{1}{2})h]}\, \tfrac{1}{4}\{F(t+(\mu+\tfrac{1}{2})h) + F(t-(\mu+\tfrac{1}{2})h) + 2F(t)\},$$

where

$$a^*(N,h) = (Nh)^{-2}\Delta^*_{(Nh)} F(t) - h^{-2}\Delta^*_h F(t) .$$

The identity (3.1) is valid for an arbitrary continuous function $F(t)$; for we need only establish it at $t = 0$, and we may suppose $F(t)$ a Fourier polynomial on $(-\pi, \pi)$, by passage to the limit, since this is a larger interval than that of the arguments of F that are present. By linearity in F we may suppose F of the form $e^{i\nu t}$ without loss of generality; for $\nu = 0$ there is nothing to prove, while the case $\nu \neq 0$ reduces by change of scale to $\nu = 1$, and this has been treated above.

From (3.1) we deduce, for an F subject to (1.8) (a)

$$(3.2)\quad a^*(N,h) \le \varphi^*(h)\, \psi^*(Nh)/h^2 .$$

As in [1], our algorithm first uses (3.2) to form Schwartz second derivatives along certain sequences. We write for short $F^*_{\{h\}}$ for the limit of $(h_\nu)^{-2}\Delta^*_{h_\nu} F(0)$ as h_ν describes the sequence $\{h\}$ and in particular F^*_h for $F^*_{\{h\}}$ when $h_\nu = h2^{-\nu}$. We shall see that these limits exist. We first take an $\{h\}$ subject to $C(1,2)$ and $C(2,2)$ of the preceding section (with $n = 2$). Setting $h_{\nu-1} = N_\nu h_\nu$, we have

$$-F^*_{\{h\}} = -h_0^{-2}\,\Delta^*_{h_0} F(0) + \sum_{\nu=1}^{\infty} a^*(N_\nu, h_\nu) \ ,$$

so that, by (3.2) and Lemma (2.2), $F^*_{\{h\}}$ then exists and satisfies

$$|(h_0)^{-2}\Delta^*_{h_0} F(0) - F^*_{\{h\}}| \leq 2^{12}\, S(h_1) \ . \tag{3.3}$$

We now consider the binary sequence $h^*_\mu = h\,2^{-\mu}$ $(\mu = 0, 1, \ldots)$ with $h = h_0$. Choosing any $h^* = h^*_\mu$, we determine ν so that $h_\nu \leq h^* < h_{\nu-1}$, and we set $h^* = Nh_\nu$. Then N is a power of 2, and excluding the trivial case $N = 1$ we see that the expression

$$(h^*)^{-2}\,\Delta^*_{h^*} F(0) - (h_\nu)^{-2}\,\Delta^*_{h_\nu} F(0) = a^*(N, h_\nu)$$

is in norm at most $(h_\nu)^{-2}\,\varphi^*(h_\nu)\,\psi^*(Nh_\nu)$, and therefore at most $(h_\nu)^{-2}\,\varphi^*(h_\nu)\,\psi^*(h_{\nu-1})$. This last quantity tends to 0 as $\nu \to \infty$. Thus for an arbitrary $h_0 = h$ in $(0,1)$ there follows the existence of F^*_h and its equality with $F^*_{\{h\}}$, so that (3.3) implies

$$|h^{-2}\,\Delta^*_h F(0) - F^*_h| \leq 2^{12}\, S(h) \ . \tag{3.4}$$

We next show that F^*_h is independent of h. First consider the difference $F^*_h - F^*_k$ where k is a rational multiple p/q of h. We write $h = q\delta$, $h_\nu = q\delta_\nu$. Then

$$(h_\nu)^{-2}\Delta^*_{\delta_\nu} F(0) - (\delta_\nu)^{-2}\Delta^*_{\delta_\nu} F(0) = a^*(q, \delta_\nu) \ ,$$

where the right-hand side, by (3.2), (1.4) for ψ, and (2.4), is in norm at most

$$(\delta_\nu)^{-2}\varphi^*(\delta_\nu)\psi^*(q\delta_\nu) \le 4q^2(\delta_\nu)^{-2}\varphi^*(\delta_\nu)\psi^*(\delta_\nu) \le 2^9 q^2 S(\delta_\nu) .$$

By making $\nu \to \infty$, we derive $F_h^* - F_\delta^* = 0$; similarly $F_k^* - F_\delta^* = 0$, and therefore $F_k^* - F_h^* = 0$. Finally, if in (3.4) we substitute for h a rational k tending to an irrational h, the number $\lambda = F_k^*$ must be independent of k, h and satisfy by continuity of F the inequality

$$|h^{-2}\Delta_h^* F(0) - \lambda| \le 2^{12} S(h) . \tag{3.5}$$

Making $h \to 0$ we derive $\lambda = F''*(0)$, and (3.5) then becomes (1.9)(a). This establishes (1.9)(a) with $K = 2^{12}$.

4. <u>The L^2 norm inequality for a derivative</u>. We now write g for the continuous real or complex-valued function of period 1, which was denoted by F in (1.8)(b); or more precisely, we can now drop for g the continuity assumption, and suppose instead that $g \in L^2$. We begin by proving (1.9)(c), which is now

$$\int_0^1 |g'(t)|^2 dt \le K \int_0^1 u^{-2}\varphi^*(u) d\psi^*(u) . \tag{4.1}$$

For this purpose we shall apply (1.9)(a) to the function

$$F(t) = \int_0^1 g(t+u)\bar{g}(u) du ,$$

where $\bar{g}$ is the complex conjugate of g. On account of periodicity of g

and $\bar{g}$, we find easily that

$$\Delta^*_{hk} F = \int_0^1 \Delta_{hk}\, g(t+u)\, \Delta_{hk}\, \bar{g}(u)du ,$$

since, in any of the products of the form $g(a+t+u)\,\bar{g}(b+u)$, which appear when we write out the integrand in full, the range of integration can be taken, by periodicity, to be any interval of length 1. By Schwarz's inequality, combined with the same periodicity argument, we have further

$$(4.2) \qquad \left|\int_0^1 \Delta_{hk}\, g(t+u)\, \Delta_{hk}\, \bar{g}(u)du\right| \le \int_0^1 |\Delta_{hk}\, g(u)|^2\, du ,$$

so that

$$|\Delta^*_{hk} F(t)| \le \int_0^1 |\Delta_{hk}\, g(u)|^2\, du \le \varphi^*(h)\, \psi^*(k) ,$$

i.e. F is subject to (1.8)(a).

On the other hand, our periodicity argument allows us to write u for $u+h$ in the second and fourth terms of the expression

$$g(u)\,\bar{g}(u) + g(u+h)\,\bar{g}(u+h) - g(u+h)\,\bar{g}(u) - g(u)\,\bar{g}(u+h)$$

when we integrate this form 0 to 1. Hence, dividing by h^2, we find that

$$(4.3) \qquad \int_0^1 h^{-2}|g(u+h) - g(u)|^2\, du = -\Delta^*_h F(0)/h^2 .$$

Hence (1.9)(a) shows that the left-hand side differs from its limit as $h \to 0$ by at most $K S^*(h)$, where

$$S^*(h) = \int_0^h u^{-2}\, \varphi^*(u)\, d\psi^*(u) ;$$

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by taking $h = 1$ we obtain (4.1), provided we can identify $g'(u)$ with the L^2 limit of $h^{-1}\Delta_h g(u)$.

This final identification and the proof of (1.9)(b), i.e. of

$$\int_0^1 |[g(u+h) - g(u)]/h - g'(u)|^2 du \le KS^*(h) \quad , \tag{4.4}$$

we shall obtain by considering in place of g the function $\hat{g}$ given by

$$\hat{g}(t) = \frac{1}{ab}\int_0^a du \int_0^b dv\{g(t+u) - g(t+v)\}$$

where $0 < b < a < 1$. We write also

$$\hat{\psi}(u) = \psi^*(u) \ (u \le a), \quad \hat{\psi}(u) = \psi^*(a) \ (a \le u) \quad ,$$

so that

$$\int_0^1 u^{-2} \varphi^*(u)\, d\hat{\psi}(u) = S^*(a) \quad .$$

Arguing as in (4.2), we estimate the quantity

$$\int_0^1 |\Delta_{hk} \hat{g}(t)|^2 dt = \int_0^1 dt \left|\frac{1}{ab}\int_0^a du \int_0^b dv\, \Delta_{h,k,u-v}\, g(t+v)\right|^2 \quad ; \tag{4.5}$$

it cannot exceed the mean value

$$\frac{1}{ab}\int_0^a du \int_0^b dv$$

of the expression

$$\int_0^1 |\Delta_{h,k,u-v}\, g(t+v)|^2 dt$$

by Schwarz's inequality, and this is less than or equal to both

$$\int_0^1 |2\Delta_{hk}\, g(t)|^2\, dt \quad \text{and} \quad \int_0^1 |2\,\Delta_{h,\,u-v}\, g(t)|^2\, dt \ ,$$

and therefore to both $4\varphi^*(h)\,\psi^*(k)$ and $4\varphi^*(h)\,\psi^*(a)$.

Consequently (4.5) is at most $4\varphi^*(h)\,\hat{\psi}(k)$, so that $\hat{g}$ satisfies an integrated square double difference inequality similar to (1.8)(b) but with $4\hat{\psi}$ in place of ψ^* . Since $\hat{g}$ has a continuous derivative, and therefore a derivative in the sense of a limit in L^2, (4.1) applies to it, and we find

$$(4.6) \qquad \int_0^1 |\hat{g}'(t)|^2\, dt = \int_0^1 |a^{-1}\,\Delta_a\, g - b^{-1}\,\Delta_b\, g|^2\, dt \le KS(a) \ .$$

By making $a, b \to 0$ (where $0 < b < a$) we see that g has in the L^2 sense a derivative in L^2: strictly this is a right-hand derivative, the L^2 limit of $h^{-1}\Delta_h g$ as $h \to +0$, however the left-hand derivative is seen at once to coincide with it, since we have, if $\|\ \|$ denotes the L^2 norm,

$$\|h^{-1}\,\Delta_h\, g - (-h)^{-1}\,\Delta_{-h}\, g\|^2 = \|h^{-1}(\Delta_h + \Delta_{-h})g\|^2 = \|h^{-1}\Delta^*\, g\| \le h^{-2}\,\varphi^*(h)\,\psi^*(h)\ ,$$

which is at most $2^7\, S^*(h)$ by (2.4). This proves (4.1), and we then obtain (4.4) from (4.6) by making $b \to 0$ and setting $h = a$.

5. Remarks. (a) The argument leading to (4.2) shows similarly that

$$\Delta^*_{hh}\, F = \int_0^1 (\Delta_{hh}\, g(t+u))\,\Delta_{kk}\,\bar{g}(u)du \ ,$$

$$\int_0^1 |\Delta_{hk}\, g(u)|^2\, du = \int_0^1 \Delta_{hh}\, g(u)\,\Delta_{kk}\,\bar{g}(u)du \le \|\Delta_{hh}\, g\| \cdot \|\Delta_{kk}\, g\| = \|\Delta^*_h g\| \cdot \|\Delta^*_k g\| \ .$$

Thus we only need (1.8)(b) with $h = k$, provided that we set $\varphi^* = \psi^*$. In that case the integral whose convergence is required becomes

$$(5.1) \qquad \int_0^h u^{-3} \varphi^*(u) \psi^*(u)\, du ,$$

For $\varphi^* \neq \psi^*$ this is a stronger condition than the convergence of

$$S^*(h) = \int_0^h u^{-2} \varphi^*(u) d\psi^*(u) .$$

In this symmetrical case $h = k$, $\varphi^* = \psi^*$ of (1.8)(b), Marcinkiewicz and Zygmund [2] and Stein and Zygmund [3] have studied pointwise differentiability almost everywhere of F.

(b) Our proofs make clear that the inequalities (1.8)(a) and (1.8)(b) are needed only for $0 < h < k < Nh$, where N is an even integer depending on h such that, with $m = 2$,

$$(5.2) \qquad 2\psi(h) \leq \psi(Nh) \leq 2^{2m+1} \psi(h) .$$

This corresponds to the second condition $C(2, m)$ in Section 2. For instance if $\psi(h) = h^\alpha$ we can take for N a constant even integer such that $2^{1/\alpha} < N \leq 2^{5/\alpha}$. Here $0 < \alpha \leq 2$ is of course required by monotonicity of ψ and by (1.4) for ψ with $m = 2$, since that inequality, where θ is arbitrary in $0 < \theta < 1$ would require, for an $\alpha > 2$, $\theta^\alpha \geq \frac{1}{4} \theta^2$ i.e. $\theta \geq 2^{-2/(\alpha-2)}$, contrary to hypothesis. On the other hand, if ψ tends to 0 more slowly than any power, N may vary with h and tend to infinity.

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