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A NATURAL INITIAL SOLUTION AND SENSITIVITY ANALYSIS OF THE OPTIMUM SOLUTION IN LINEAR PROGRAMMING PROBLEMS

By

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13. ABSTRACT

The author discusses a primal-dual algorithm used to obtain the optimum solution to the general linear programming problem and uses the method in the sensitivity analysis of the optimum solution to the primal problem.

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An increase in the initial number of unknowns at the expense of artificial variables can be avoided when constructing an initial solution for the general linear programming problem if a natural initial solution and the optimization are carried out in accordance with primal and dual criteria. For the sensitivity analysis of the optimal solution in the primal problem with regard to changes in the free terms which enter the constraints one must remember the manner in which the initial solution was constructed.

1. The Natural Initial Solution

By a natural initial solution of the general linear programming problem, we mean a basic solution which consists only of the initial unknown variables and those additional variables which are used to convert inequalities into equalities.

A natural initial solution can be constructed and used in the following manner.

When inequalities of the type

$$0 < b \leq \sum_{i=1}^{n} a_i x_i$$

are included in the initial simplex matrix without using an artificial variable but in the form of an inequality with the appropriate sign

$$-b \geq -\sum_{i=1}^{n} a_i x_i$$

we apply an algorithm in which optimization takes place in accordance with the primal and the dual criteria. According to S. Gass [1] there exist sufficiently effective methods of solving the general linear programming problem which are based on a combined usage of the general simplex operation and the dual method (p. 178). Among the bibiographical references given by Gass, there is no reference to the work of Geisen which deals with the composite simplex operation [2].

To avoid the artificial variables in the equations while constructing an initial basic solution, one can apply together with the simplex algorithm the method of successive eliminations. In fact optimization according to the dual criterion removes negative free terms which can occur during the construction of the initial basic solution and contradictory and linearly dependent equality constraints are detected in the course of the elimination.

I. Kalikhman on pages 264-269 of his text [3] applies in an example together with the general simplex method the method of successive elimination in the construction of an initial basic solution. Unfortunately, he does not warn the reader about the possibility of the difficulties which can arise and which have been described above, namely, the negativity of the free terms and the linear dependence of linear forms.

2. Matrix Representation of the Solution Process

We represent the above solution process in matrix form. We will denote the coefficient matrix of the objective function and the constraints of the linear programming problem in the following form:

$$A = \begin{vmatrix} \bar{P}_{q\times 1} & \bar{E}_q & \bar{F}_{q\times(m-q)} & \bar{B}_{q\times m} & \bar{D} \\ \bar{\bar{P}}_{(m-q)\times 1} & \bar{\bar{O}}_{(m-q)\times q} & \bar{\bar{F}}_{m-q} & \bar{\bar{B}}_{(m-q)\times m} & \bar{\bar{D}} \end{vmatrix} \quad (1)$$

or

$$A = \begin{vmatrix} \bar{P} & \bar{E} & \bar{F} & \bar{B} & \bar{D} \\ \bar{\bar{P}} & \bar{\bar{O}} & \bar{\bar{F}} & \bar{\bar{B}} & \bar{\bar{D}} \end{vmatrix}.$$

Here the elements of the first row of the block matrix A consist of the coefficients of the objective function and former inequalities which have been converted into equalities, and the elements of the second row are the coefficients of the equalities. Here P is the column of free terms, E is the identity matrix, and O is the null matrix, F are columns which are transformed into unit columns when constructing the natural initial solution, B are columns which enter the optimum basis[1]; and D are the remaining col-

[1] Some columns of matrix B can also reappear in matrices E and F.

umns.

To obtain a natural initial solution it suffices to compute QA, where

$$Q = \begin{pmatrix} \overline{E} & -\overline{F}\overline{\overline{F}}^{-1} \\ \overline{\overline{O}} & \overline{\overline{F}}^{-1} \end{pmatrix}. \tag{2}$$

In fact,

$$QA = \begin{pmatrix} \overline{P} - \overline{F}\overline{\overline{F}}^{-1}\overline{\overline{P}} & \overline{E} & \overline{O} & \overline{B} - \overline{F}\overline{\overline{F}}^{-1}\overline{\overline{B}} & \overline{D} - \overline{F}\overline{\overline{F}}^{-1}\overline{\overline{D}} \\ \overline{\overline{F}}^{-1}\overline{\overline{P}} & \overline{\overline{O}} & \overline{\overline{E}} & \overline{\overline{F}}^{-1}\overline{\overline{B}} & \overline{\overline{F}}^{-1}\overline{\overline{D}} \end{pmatrix} \tag{3}$$

or

$$QA = (\hat{P}_{m\times 1} \quad \hat{E}_m \quad \hat{B}_m \quad \hat{D}). \tag{4}$$

In practice multiplication by Q takes place through $m-q$ eliminations of the unknowns.

To obtain the optimum solution, we multiply the matrix QA by the matrix $\hat{B}_m^{-1}$, i.e. we carry out a simplex transformation

$$\hat{B}^{-1}QA = (\hat{B}^{-1}\hat{P} \quad \hat{B}^{-1} \quad \hat{E} \quad \hat{B}^{-1}\hat{D}), \tag{5}$$

where the column $\hat{B}^{-1}\hat{P}$ is the optimum solution.

3. Sensitivity Analysis of the Optimum Solution Using a Natural Initial Solution

When applying a natural initial solution on the basis of the simplex matrix (5), in general, one cannot obtain immediately the optimal solution to the dual problem, as mentioned by Kalikhman [3]. To achieve this he proposes two methods. The first of these requires, in addition to solving the initial problem, the solution of a system of m-1 linear equations in m-1 unknowns, and the second requires the inversion of a matrix of order m-1.

We will show a method for obtaining the solution for the dual problem with a natural initial solution which requires considerably less effort than the methods proposed by Kalikhman.

In the case of the general simplex method, a sensitivity analysis of the solution to the primal problem can be carried out with the aid of the dual problem. We will investigate the question how the changes in the free terms of the initial system of constraints affect the optimal solution of the primal problem when the composite simplex method with a natural initial solution is used. Clearly the vector increment ΔP of the column of free terms can only change the first column of matrices 1, 3, 4, and 5. We decompose ΔP into a sum of two terms:

$$\Delta P = \bar{\Delta} P + \Delta P,$$

where

$$\bar{\Delta} P = \begin{pmatrix} \Delta\bar{P} \\ O \end{pmatrix} \quad \text{and} \quad \Delta P = \begin{pmatrix} \bar{O} \\ \Delta\bar{P} \end{pmatrix}. \tag{6}$$

First we investigate the effects of the increment $\bar{\Delta}P$, i.e. the effects of the changes in the free terms of the inequalities. Since the first column of matrix A assumes now the form

$$\begin{pmatrix} \bar{P} + \Delta\bar{P} \\ P \end{pmatrix},$$

the first column of QA will be

$$\begin{pmatrix} \bar{P} + \Delta\bar{P} - \bar{F}F^{-1}P \\ \bar{\bar{F}}^{-1}\bar{\bar{P}} \end{pmatrix}$$

or, according to (3) and (4)

$$\hat{P} + \bar{\Delta}P.$$

For the first column of matrix $\hat{B}^{-1}QA$ we shall now have $\hat{B}^{-1}\hat{P} + \hat{B}^{-1}\bar{\Delta}P$, i.e. the optimum solution $\hat{B}^{-1}\hat{P}$ has an increment

$$\hat{B}^{-1}\Delta P,$$

which is easily computed on the basis of (5) and (6).

Second, we will investigate the effect of the increment P, i.e. the effect of changes in the free terms of the equalities on the optimum solution to the problem. Since the first column of matrix A has now the form

$$\begin{pmatrix} \bar{P} \\ P + \Delta P \end{pmatrix},$$

the first column of matrix QA will be

$$\begin{pmatrix} \bar{P} - \bar{F}\bar{F}^{-1}(P + \Delta P) \\ \bar{F}^{-1}(P + \Delta P) \end{pmatrix} \tag{7}$$

or, according to (2), (3), and (4), $\hat{P} + Q\Delta P$, where

$$Q\Delta P = \begin{pmatrix} -\bar{F}F^{-1}\Delta P \\ F^{-1}\Delta \bar{P} \end{pmatrix}. \tag{8}$$

Instead of the first column of matrix $\hat{B}^{-1}QA$ we now have $\hat{B}^{-1}\hat{P} + \hat{B}^{-1}Q\Delta P$, i.e. the optimum solution $\hat{B}^{-1}\hat{P}$ has an increment

$$\hat{B}^{-1}Q\Delta P.$$

In this increment we have the factor Q, which does not appear as an element in the final matrix (5). To obtain Q we must retain the transformations which were carried out in constructing the natural initial solution. Because of (2) and (6) we need only retain the second column of the matrix.

If we augment the matrix A by the column[1]

$$\begin{pmatrix} \bar{O} \\ \bar{\bar{E}} \end{pmatrix}, \tag{9}$$

then the second column of the matrix Q is added to the matrix QA:

$$\begin{pmatrix} -\bar{F}\bar{\bar{F}}^{-1} \\ F^{-1} \end{pmatrix}. \tag{10}$$

We will show that one need not retain (10). We partition the matrix $\hat{B}^{-1}$ into the blocks:

$$\hat{B}^{-1} = \begin{pmatrix} \bar{G} & \bar{H} \\ \bar{\bar{G}} & \bar{\bar{H}} \end{pmatrix}. \tag{11}$$

On the basis of (10) and (11) the column $\hat{B}^{-1}QA$ which is added to the matrix has the following form:

$$\begin{pmatrix} -\bar{G}\bar{F}\bar{\bar{F}}^{-1} + \bar{H}F^{-1} \\ -G\bar{\bar{F}}F^{-1} + \bar{\bar{H}}\bar{\bar{F}}^{-1} \end{pmatrix}. \tag{12}$$

1 The initial number of unknowns is not increased in practice. Since the identity matrix of the initial basis is usually not retained in explicit form, we exclude in the course of constructing a natural initial solution the same number of columns as the number of columns added.

We easily compute the increment $\hat{B}^{-1}Q\bar{\Delta}P$ with the aid of (12). In fact, from (8) and (11)

$$\hat{B}^{-1}Q\bar{\bar{\Delta}}P = \begin{Bmatrix} (-\bar{G}\bar{F}F^{-1} + \bar{H}F^{-1})\Delta P \\ (-\bar{G}\bar{F}F^{-1} + \bar{H}\bar{F}^{-1})\Delta P \end{Bmatrix}.$$

We will show further how the solution to the dual problem is obtained.

The numbers in the first row[1] of the submatrices $\vec{G}$ and $(\bar{H} - \bar{G}\bar{F})\bar{F}^{-1}$ of the finite matrix (5), augmented by column (12), are the optimum values of the dual variables, where the first ones correspond to inequalities and the remaining ones to equalities.

Bibliography

1. Gass, S., Lineinoe programmirovanie (metody i prilozheniya) [Linear Programming (Methods and Applications)], Moscow, 1961.

2. Giesen, G., Unternehmungsforschung 5, No. 3, 1961, p. 137.

3. Kalikhman, I. L., Lineinaya algebra i programmirovaniye [Linear Algebra and Programming], Moscow, 1967.

1 We assume that the coefficients of the objective function appear in the first row of matrix A.