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DISTRIBUTION-FREE INTERVAL ESTIMATION OF  
THE LARGEST  $\alpha$  QUANTILE

BY

M. HASEEB RIZVI and K. M. LAL SAXENA

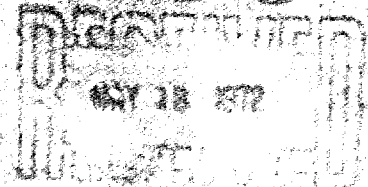
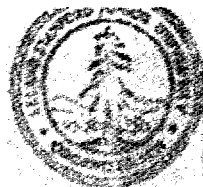
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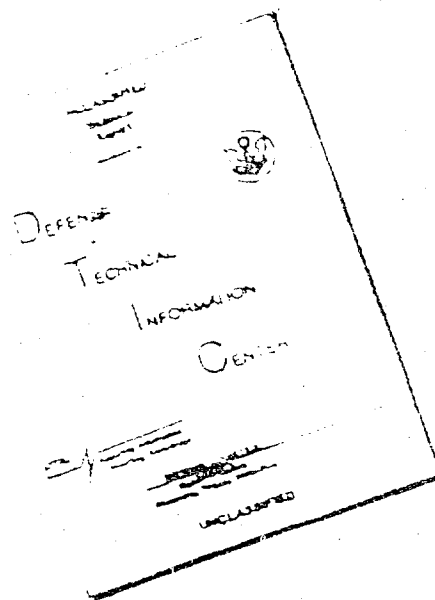
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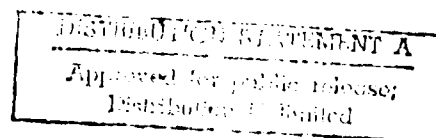
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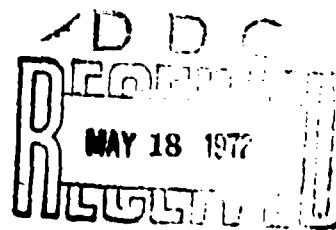
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DISTRIBUTION-FREE INTERVAL ESTIMATION OF  
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by

M. Haseeb Rizvi and K. M. Lal Saxena  
Stanford University and University of Nebraska

1. Introduction and Formulation of the Problem

Ordering of several unknown parameters is a problem of wide practical applications. Saxena and Tong [3] and Saxena [2] have recently constructed confidence intervals for the largest location and scale parameters respectively. This paper deals with the distribution-free interval estimation of the largest  $\alpha$ -quantile of several continuous distributions.

Consider  $k(\geq 1)$  distributions with unknown continuous cdfs  $F_i$ ,  $i=1, \dots, k$ . Let  $x_\alpha(F_i)$  denote the unique  $\alpha$ -quantile ( $0 < \alpha < 1$ ) of  $F_i$ . If  $x_\alpha(F_i)$  is not unique, it can be defined to be so in an obvious manner. Define  $\theta = \max_{1 \leq i \leq k} x_\alpha(F_i)$ . For a specified constant  $\gamma$ , a random interval  $I$  is desired such that

$$(1) \quad \inf_{\Omega} P(\theta \in I) \geq \gamma$$

where  $\Omega$  denotes the set of all possible  $k$ -tuples  $(F_1, F_2, \dots, F_k)$ . Such an interval  $I$ , based on order statistics of random samples of equal sizes from each  $F_i$ , is proposed below.

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## 2. Proposed Procedure and Its Probability of Coverage

Independent random samples of common size  $n$  are taken from each of the  $k$  distributions. Let  $Y_{r,i}$  denote the  $r^{\text{th}}$  order statistic from  $F_i$  and  $Y_r = \max_{1 \leq i \leq k} Y_{r,i}$  for  $r=1, \dots, n$ ; define  $Y_0 = -\infty$  and

$Y_{n+1} = +\infty$ . For  $s < t$ , consider the random interval  $I_0 = (Y_s, Y_t)$  and assert that  $\theta \in I_0$ , where  $s$  and  $t$  are chosen so as to satisfy (1).

With  $G_r(p)$  denoting the incomplete beta function

$$(2) \quad G_r(p) = r \binom{n}{r} \int_0^p u^{r-1} (1-u)^{n-r} du = \sum_{j=r}^n \binom{n}{j} p^j (1-p)^{n-j},$$

the cdf of  $Y_{r,i}$  is given by  $G_r(F_i(y))$ . We will adopt the convention that  $G_0(\cdot) \equiv 1$  and  $G_{n+1}(\cdot) \equiv 0$ . The probability of coverage of  $\theta$  by  $I_0$  is then

$$\begin{aligned} (3) \quad P(\theta \in I_0) &= P(Y_s \leq \theta) - P(Y_t \leq \theta) \\ &= P(Y_{s,i} \leq \theta, i=1, \dots, k) - P(Y_{t,i} \leq \theta, i=1, \dots, k) \\ &= \prod_{i=1}^k G_s(F_i(\theta)) - \prod_{i=1}^k G_t(F_i(\theta)). \end{aligned}$$

We know that  $F_i(\theta) \geq \alpha$  for  $i=1, \dots, k$  with equality for at least one  $i$ . Hence without any loss of generality we assume that  $F_k(\theta) = \alpha$ . Thus (3) becomes

$$(4) \quad P(\theta \in I_0) = G_s(\alpha) \prod_{i=1}^{k-1} G_s(F_1(\theta)) - G_t(\alpha) \prod_{i=1}^{k-1} G_t(F_1(\theta)) .$$

### 3. Minimization of the Probability Coverage

For one-sided random intervals the minimization over  $\Omega$  of  $P(\theta \in I_0)$  is given by Theorem 1. The proof of the theorem follows from considerations of (4) and noting that  $G_0(\cdot) \equiv 1$ ,  $G_{n+1}(\cdot) \equiv 0$  and  $\alpha \leq F_1(\theta) \leq 1$  for each  $i$ . The details are omitted.

#### Theorem 1

(a) For  $s > 0$ ,  $t = n + 1$ , that is, with  $I_0 = (Y_s, \infty)$ ,

$$(5) \quad \inf_{\Omega} P(\theta \in I_0) = G_s^k(\alpha);$$

and (b) for  $s = 0$ ,  $t < n + 1$ , that is, with  $I_0 = (-\infty, Y_t)$ ,

$$(6) \quad \inf_{\Omega} P(\theta \in I_0) = 1 - G_t(\alpha) .$$

Note that for any  $s < t$ ,  $G_s(x) > G_t(x)$ . Therefore in (a) of Theorem 1, we choose  $s$  to be the largest integer such that the right side of (5) exceeds  $\gamma$  of requirement (1) with  $0 < \gamma < \{1 - (1 - \alpha)^n\}^k$ . Further in (b) of Theorem 1, we choose  $t$  to be the smallest integer such that the right side of (6) exceeds  $\gamma$  with  $0 < \gamma < 1 - \alpha^n$ . It is clear from the above upper bounds on  $\gamma$  that, for fixed  $k$  and  $\alpha$ ,  $I_0$  can satisfy (1) for any value of  $\gamma$  between 0 and 1, provided  $n$  is taken large enough.

Next we consider the minimization over  $\Omega$  of  $P(\theta \in I_0)$  for two-sided random intervals.



Theorem 2 (two-sided intervals)

For  $0 < s < t < n + 1$ ,

$$(7) \quad \inf_{\Omega} P(\theta \in I_{\alpha}) = \min(G_s(\alpha) - G_t(\alpha), G_s^k(\alpha) - G_t^k(\alpha)) .$$

Proof

Since  $P(\theta \in I_{\alpha})$ , given by (4), involves  $F_i$ 's evaluated at  $\theta$  (constant) and  $F_i(\theta) \geq \alpha$  for  $i=1, \dots, k-1$ , we can write  $F_i(\theta) = \alpha + \delta_i$ , where  $0 \leq \delta_i \leq 1 - \alpha$ . This enables us to reparametrize (4) as a function of the  $\delta_i$ 's. Consequently the problem of minimization of (4) over  $\mathcal{H} = \{(F_1, \dots, F_k): F_i \text{ is continuous for each } i\}$  is reduced to its minimization over  $\{(\delta_1, \dots, \delta_{k-1}): 0 \leq \delta_i \leq 1 - \alpha, i=1, \dots, k-1\}$ .

We have

$$(8) \quad P(\theta \in I_{\alpha}) = G_s(\alpha) \prod_{i=1}^{k-1} G_s(\alpha + \delta_i) - G_t(\alpha) \prod_{i=1}^{k-1} G_t(\alpha + \delta_i) \\ = J(\delta_1, \dots, \delta_{k-1}), \text{ say .}$$

For some  $j$ , fix  $\delta_1, \dots, \delta_{j-1}, \delta_{j+1}, \dots, \delta_{k-1}$  and consider  $\partial J / \partial \delta_j$ .

Using (2), we define

$$g_r(p) = \frac{d}{dp} G_r(p) = r \binom{n}{r} p^{r-1} (1-p)^{n-r}, \quad 0 \leq p \leq 1,$$

and observe that  $g_t(p)/g_s(p)$  is increasing in  $p$  for  $t > s$ .

Let

$$A = \prod_{\substack{i=1 \\ i \neq j}}^{k-1} G_s(\alpha + \delta_i), \quad B = \prod_{\substack{i=1 \\ i \neq j}}^{k-1} G_t(\alpha + \delta_i), \quad A \geq B.$$

Then from (8) we obtain

$$\frac{\partial J}{\partial \delta_j} = AG_s(\alpha)g_s(\alpha + \delta_j) \left[ 1 - \frac{BG_t(\alpha)g_t(\alpha + \delta_j)}{AG_s(\alpha)g_s(\alpha + \delta_j)} \right].$$

Since  $g_t(\alpha + \delta_j)/g_s(\alpha + \delta_j)$  is increasing in  $\delta_j$ , it follows that the expression inside the brackets is decreasing in  $\delta_j$ . Hence we conclude that  $\partial J/\partial \delta_j$  has either the same sign at every value of  $\delta_j$  or at most one change of sign from positive to negative and consequently  $\inf_{\delta_j} J$  is either at  $\delta_j = 0$  or at  $\delta_j = 1 - \alpha$ . This conclusion is valid for every other  $j$ . Therefore infimum of  $J$  is achieved when a certain number  $m$  of  $\delta_j$ 's are zero and the rest equal to  $1 - \alpha$ . Define

$$(9) \quad H_m = H_m(s, t) = G_s^{m+1}(\alpha) - G_t^{m+1}(\alpha), \quad 0 \leq m \leq \infty.$$

Differentiating  $H_m$  with respect to  $m$ , and noting that  $G_s(\alpha) > G_t(\alpha)$ , it is seen that  $\partial H_m/\partial m$  either has the same sign for every value of  $m$  or has at most one change of sign from positive to negative. Hence

$$(10) \quad \min_{m=0, 1, \dots, k-1} H_m = \min(H_0, H_{k-1}).$$

This proves the theorem.

Note that for fixed  $k$  and  $\alpha$ ,  $I_0$  can satisfy (1) provided  $\gamma$  lies between 0 and

$$\min_{r=1, k} ([1-(1-\alpha)^n]^r - \alpha^{nr}) = p(k, \alpha, n), \text{ say.}$$

Clearly,  $p(k, \alpha, n)$  can be made arbitrarily close to 1 by taking  $n$  large enough.

#### 4. An Optimum Two-Sided Random Interval

For specified  $k, \alpha, \gamma$  and  $n$ , the choice of integers  $s$  and  $t$ , such that the two-sided random interval  $I_0 = (Y_s, Y_t)$  satisfies (1), may not be unique unless some criterion for an optimum choice is introduced. For the case  $k=1$ , Wilks [4] proposed an optimality criterion that requires the ranks  $s$  (of  $Y_s$ ) and  $t$  (of  $Y_t$ ) to be as close together as possible. Extending this criterion to  $k \geq 1$ , we would be interested in choosing  $s$  and  $t$  so that the rank-difference  $(t-s)$  is minimized for a preassigned  $\gamma$ . For this purpose we present the following algorithm. Let  $c$  be a positive integer less than  $n$  and consider  $I_0 = (Y_s, Y_{s+c})$ . Denote by  $Q(s, c)$  the infimum of  $P\{Y_s < \theta < Y_{s+c}\}$  over  $\Omega$  as given by (7). For every fixed  $c$ , let  $s_0(c)$  be that value of  $s$  for which

$$Q(s_0(c), c) = \max_{1 \leq s \leq n-c} Q(s, c).$$

Now choose the smallest  $c$ , call it  $c_0$ , such that

$$Q(s_0(c_0), c_0) \geq \gamma .$$

Then the optimum choice of the random interval satisfying the requirement (1) is  $(Y_s, Y_{s+c})$  with  $c = c_0$  and  $s = s_0(c_0)$ .

For moderate values of  $n$ , say  $n \leq 50$ , the above algorithm can be carried out in an easy manner using the incomplete beta function tables or the more readily available binomial tables for smaller values of  $n$  in view of (2). For example, when  $k=4$ ,  $\alpha=0.5$ ,  $\gamma=0.90$ , and  $n=25$ , we obtain  $s=8$  and  $t=17$ . In this illustration, it is interesting to note that even for  $k=1,2,3$  and the same values of  $\alpha, \gamma$  and  $n$ , we obtain  $s=8$  and  $t=17$ .

#### 5. Large Sample Approximations

For large  $n$ , using normal approximation to binomial in (2), with  $\Phi(\cdot)$  denoting the standard normal cdf, we obtain

$$(11) \quad G_r(\alpha) \approx \Phi((-r+n\alpha)/(n\alpha(1-\alpha))^{1/2}) .$$

For one-sided intervals of Theorem 1, using (11), in the case (a) we take  $s$  to be the largest integer such that

$$s \leq n\alpha + (n\alpha(1-\alpha))^{1/2} \Phi^{-1}(1-\gamma^{1/k}),$$

and in the case (b) we take  $t$  to be the smallest integer such that

$$t \geq n\alpha + (n\alpha(1-\alpha))^{1/2} \Phi^{-1}(\gamma) .$$

For the two-sided optimum random interval described in Section 4, using (11), we have

$$(12) \quad Q(s, c) \approx \min(\Phi(d-x) - \Phi(-x), \Phi^k(d-x) - \Phi^k(-x)),$$

where

$$d = c(n\alpha(1-\alpha))^{-1/2}, \quad x = (n\alpha s)(n\alpha(1-\alpha))^{-1/2}.$$

Dudewicz and Tong [1] show that for any given  $d$ , the value of  $x$ , say  $x_0$ , that maximizes the right side of (12) is either the value for which the two terms within braces of (12) are equal or the value which maximizes the second of these two terms. Table 1 of [1] gives  $x_0$  for  $k = (10, 2, 4)$  and  $d = (0.1)^k$ . For  $k=1$ , obviously  $x_0 = d/2$ ; also for  $k=2$ ,  $x_0 = \frac{d}{2}$  as shown in [1]. Thus for  $k=1, 2$  and  $n$  large, from (12) it is seen that  $s$  and  $c$  will be equidistant from  $n\alpha$  on either side. Table 2 of [1] gives coverage probability (12) evaluated at  $x_0$  of Table 1. We consider once again the example of Section 4 to illustrate the use of the tables of [1] for adapting our algorithm to large sample sizes. For  $k=6$ ,  $\alpha=0.5$ ,  $\gamma=0.00$ , and  $n=25$ , Table 2 gives  $d=3.4$ . Using  $d=3.4$  we obtain from Table 1,  $x_0 = 1.452$ . Now from (12) with  $x = x_0$  we obtain  $s=8.77$  and  $c=8.77$ . These values of  $s$  and  $c$  are then the optimum values  $s_0(c_0)$  and  $c_0$  respectively of Section 4. Since these optimum values have to be integers, we round them off as  $s_0(c_0) = 8$  and  $c_0 = 9$ . Thus in this example we obtain the random interval obtained previously.

The goodness of the large sample approximation considered above is directly related to the well known convergence of binomial to normal

For commonly used values of  $\alpha$  it is felt that this approximation is adequate for sample sizes larger than 50.

In conclusion it should be pointed out that the problem of interval estimation of the smallest  $\alpha$ -quantile  $\theta = \min_{1 \leq i \leq k} x_{\alpha}(F_i)$  can be handled in a manner analogous to the discussion of this paper by considering the random interval  $(Y_s^*, Y_r^*)$ ,  $s < r$ , where  $Y_r^* = \min_{1 \leq i \leq k} Y_{r,i}$ .

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- [4] Wilks, S. S., Mathematical Statistics, John Wiley and Sons, New York, 1967.