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ON CONFIDENCE BOUNDS FOR SYSTEM RELIABILITY WHEN THE COMPONENTS HAVE EXPONENTIALLY DISTRIBUTED LIVES

by

Sam C. Saunders

The Boeing Company
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ABSTRACT

One of the problems that has been of concern for a number of years is the following: Suppose that there exists a series system of m exponentially lived components and we ask that the system be qualified for service by demonstrating a reliability that equals, or exceeds, a specified value $1 - \alpha$, with a confidence level at least $1 - \gamma$ (Here α and γ are small). This paper addresses the question of how this can be demonstrated using only data on the reliability of the components, where the reliability design goals for each component must be apportioned, as well as the confidence levels at which the design goals must be met in order to satisfy the overall system reliability requirement. Another question that is addressed is the calculation of the probability that the system reliability exceeds its goal. The purpose of the paper is to make a comparison between competing theories for the above situation when the components have exponential life lengths.

KEY WORDS

Reliability

Component Failure

Confidence Bounds

Systems Reliability

Asymptotic Model

1.0 INTRODUCTION

One of the problems that has been of concern for a number of years is the following: Suppose we have a series system of m exponentially lived components and we ask that the system be qualified for service by demonstrating a reliability that equals, or exceeds, a specified value $1 - \alpha$, with a confidence level at least $1 - \gamma$ (Here α and γ are small.). How can this demonstration be accomplished using only data on the reliability of the components? Since for all $t > 0$

$$R(t) = \prod_{i=1}^{m} R_i(t) \quad ,$$

where

$$R_i(t) = e^{-\lambda_i t} \quad \text{for some} \quad \lambda_i > 0 \quad ,$$

is the equation for the system reliability in terms of the reliability of the components, the question becomes that of apportioning the reliability design goals for each component, as well as the confidence levels at which the design goals must be met, in order to satisfy the overall system reliability requirement. This particular problem has been around in this form for at least twenty-five years (since the days of the V1). See [2].

Another problem, which is related to the one preceeding, arises at a somewhat later time during system development. If we have estimates $\tilde{R}_i$ based on the performance data of the i^{th} component such that for a fixed $t > 0$

$$P\left[R_i(t) \geq \tilde{R}_i(t)\right] = 1 - \alpha_i \qquad i = 1, \cdots, m$$

then with what confidence, i.e., as what function of the confidence levels α_i, can we say that

$$P\left[\prod_{i=1}^{m} R_i(t) \geq \prod_{i=1}^{m} \tilde{R}_i(t)\right] .$$

That is, we ask what is the probability that the system reliability exceeds its goal?

An additional complication which arises is that the number of failures of each component tested is usually quite small. This is endemic since a large number of failures would not be tolerated during a short test period, for if it were to occur, that component would be redesigned. Furthermore, lengthy testing in realistically simulated usage for highly specialized and complex systems is just too expensive. An archtypical example of this type of situation is the components for the Saturn missile system.

For over a decade there have been "solutions" presented to this problem in terms of approximate distributions for the true confidence level. However, none of them have been completely accepted because of their requiring either extensive tables or much computation for their implementation. But in addition, even if this were not the case, there would be the nagging uncertainty, since the exact solutions are unknown, as to how good the approximation really is.

One suggestion that has arisen in this partial vacuum has been the Bayesian method of obtaining confidence intervals which postulates that the failure rates [illegible] are random variables with [illegible]

Such an assumption does yield an exact (Bayesian) confidence level, but whether or not such a model is appropriate for all situations (or even any) is a matter of some controversy. However, this procedure, which necessarily assumes prior information summarized in a particular way, often may yield tighter confidence bounds than those obtained as the inverse of a statistical test (see [14]). Another concept utilizes a fiducial interpretation of confidence bounds (see [4]). The catch is whether such restrictive assumptions apply to the case at hand.

The applicability of the asymptotic distributions of certain statistics (such as the maximum likelihood estimates or the log-likelihood ratio) to this problem has been recently extended both by broadening the class of structure functions and the type of data on performance which can be considered. This was accomplished in a series of papers.

Approximate confidence intervals for the reliability of any system (or structure) which can be represented by a monotone Boolean function of Bernoulli variates were first obtained using the asymptotic distribution of the likelihood ratio in [11]. In this study component failure data were the outcomes of a number of Bernoulli trials of the success or nonsuccess of the component. Later similar asymptotic results were obtained for all coherent systems under the assumption that the components were exponentially lived and failure data were available for each component (see[9]). Both of these papers were extensions of the results of Madansky [7] for series, parallel and series-parallel systems with binomial data on the reliability of each component and the use of the asymptotic distribution of the log-likelihood ratio to compute the confidence bounds. Also, these papers depended upon the precise definition of coherent system, given, e.g.,

in [3], and the theory of confidence regions involving constraints presented in [8]. A comparison between the results of [11] and the use of asymptotic normality of maximum likelihood estimates was accomplished in [10] for the case of binomial data on the components' reliability. What we now undertake is to make a comparison between competing theories for the situation when the components have exponential life lengths.

Recent publications of Sarkar [13] and Lieberman and Ross [6] give solutions to the problem of exact confidence bounds for series systems which have independent exponentially distributed component lives. The latter method appears to be not only elegant but the equal in computational simplicity to any method proposed hithertofore and superior in many cases because of its high utilization of the data and its consequent precision of the resulting confidence interval. We shall include this method, a suitable generalization to arbitrary systems, in our comparisons.

2.0 THE ASYMPTOTIC RESULTS

Suppose we have a coherent structure function Φ of order m from which we obtain a system with components which have independent exponentially distributed lines $T_1, \cdots, T_m$. The reliability of the system is, at any time $t > 0$,

$$(2.1)\quad R_\Phi(t) = E\Phi(\{T_1 > t\}, \cdots, \{T_m > t\})$$

$$= \sum_{y_1=0}^{1} \cdots \sum_{y_m=0}^{1} \Phi(y_1, \cdots, y_m) \prod_{i=1}^{m} (1-e^{-\lambda_i t})^{1-y_i} e^{-\lambda_i y_i t}$$

where $\{\pi\}$ is the indicator of the relation π being one if true and zero otherwise, and λ_i is the failure rate of T_i.

In certain cases when we consider the reliability only at a fixed time, it may be preferable to consider Equation (2.1), which is an expectation involving exponential component life lengths, as some function, say $h_\Phi(\lambda)$, of the vector of component failure rates $\lambda = (\lambda_1, \cdots, \lambda_m)$ with λ_i the failure rate of the i^{th} component.

Those asymptotic procedures which have been proposed depend upon the sample size for data on each component being large and thus, for proper usage, required a determination of when the sample sizes are large enough for the asymptotic result to be applied. Assuming the condition

$$(2.2)\quad \sum_{i=1}^{m} \frac{1}{n_i} \to 0$$

where n_i is the number of failures observed for the i^{th} component during testing, we have the following two well-known results:

If $\hat{\lambda}$ denotes the vector of maximum likelihood estimates of λ, then asymptotically, in law, we have

$$\frac{h(\hat{\lambda})-h(\lambda)}{\sqrt{\text{var } h(\hat{\lambda})}} \sim \mathcal{N}(0,1) \quad . \tag{2.3}$$

If $\tilde{\lambda}(\delta)$ is the vector solution of the maximization of the log-likelihood subject to the restraint $\ln h(\lambda) = \delta$ and we define

$$\tilde{H}(\cdot) = h[\tilde{\lambda}(\cdot)] \quad , \quad \Lambda(\delta) = \int_0^{\delta} x d\tilde{H}(x)$$

then asymptotically, in law,

$$-2\Lambda(\tilde{H}^{-1}[h(\lambda)]) \sim \chi_1^2 \quad . \tag{2.4}$$

It follows from Wilks' theorem (see [15] p. 419), that (2.3 is equivalent with (2.4) under assumption (2.2) because

$$-2\Lambda\tilde{H}^{-1}h(\lambda) \simeq \left[\frac{h(\hat{\lambda}) - h(\lambda)}{|\tilde{H}'(0)|}\right]^2 \tag{2.5}$$

A question of practical interst is whether either one converges rapidly enough to be usable for the sample sizes which are available. To explore sufficient conditions for convergence requires not only some mathematical analysis but some Monte Carlo simulation as well. We also note that both of the proceeding results are stated for a fixed time which is omitted from the notation. Presumably the conditions for convergence could be altered by the choice of a different time constant.

3.0 SYSTEM FAILURES GENERATED FROM COMPONENT FAILURE DATA

Suppose we have $n_i \geq 1$ independent observations of T_i, say

$$t_{i1}, t_{i2}, \cdots, t_{in_i} \quad \text{for } i = 1, \cdots, m \quad . \tag{3.1}$$

We wish to use this data on the components to "simulate" the systems behavior and either to construct an estimate of the structural reliability R as a function of time t or to estimate the reliability h at a fixed time. To do so we utilize a straightforward extension of the procedure given by Lieberman and Ross for series systems in [6].

Take the first observation t_{i1} for $i = 1, \cdots, m$ from each sample and let

$$x_1 = \max \{t: \Phi(t_{11} > \tau , \cdots, t_{m1} > \tau) = 1\} \tag{3.2}$$

be the longest time the system lives with the specified life lengths for each component. We may write this more conveniently. Let $P_1, \cdots, P_r$ denote the r minimal paths of the structure Φ. Then for components with random life lengths $T_1, \cdots, T_m$, it is known that we can write (see [3]) the system life as

$$X = \tau(T_1, \cdots, T_m) = \max_{j=1}^{r} \min_{i \in P_j} T_i \tag{3.3}$$

and

$$R(t) = P[\tau(T_1, \cdots, T_m) > t] \quad \text{for } t > 0 \quad .$$

Thus in terms of system life and the observations t_{i1} for $i = 1,\cdots,m$ we may rewrite (3.2) as

$$x_1 = \max_{j=1}^{r} \min_{i \in P_j} t_{i1} \quad .$$

On the basis of the data on the components a system failure x, has been generated which clearly has the survival distribution R.

We now generate for $i = 1,\cdots,m$

$$t_{i,1}^{(2)} = \begin{cases} t_{i,2} & \text{if } t_{i1} \leq x_1 \\ t_{i,1} - x_1 & \text{if } t_{i1} > x_1 \end{cases}$$

and for $j \geq 2$

$$t_{i,j}^{(2)} = \begin{cases} t_{i,j+1} & \text{if } t_{i,1} \leq x_1 \text{ and } j \leq n_i - 1 \\ t_{i,j} & \text{if } t_{i,1} > x_1 \text{ and } j \leq n_i \end{cases} .$$

We have for $i=1,\cdots,m$ a new sample of mutually independent observations, each with the same distribution as T_i, namely

$$t_{i,1}^{(2)}, \quad t_{i,2}^{(2)},\cdots,t_{i,n_2^{(2)}}^{(2)}$$

where $n_i^{(2)}$ is defined by

$$n_i^{(2)} = \begin{cases} n_i - 1 & \text{if } t_{i1} \leq x_1 \\ n_i & \text{if } t_{i1} > x_1 \end{cases} .$$

We proceed to utilize this data to obtain

$$x_2 = \max_{j=1}^{r} \min_{i \in P_j} t_{i,1}^{(2)}$$

which again has survival distribution R. Let

$$t_{i,1}^{(3)} = \begin{cases} t_{i,2}^{(2)} & \text{if } t_{i,1}^{(2)} \leq x_2 \\ t_{i,1}^{(2)} - x_2 & \text{if } t_{i,1}^{(2)} > x_2 \end{cases}$$

and for $j \geq 2$

$$t_{i,j}^{(3)} = \begin{cases} t_{i,j+1}^{(2)} & \text{if } t_{i,1}^{(2)} \leq x_2 \text{ and } j \leq n_i^{(2)} - 1 \\ t_{i,j}^{(2)} & \text{if } t_{i,1}^{(2)} > x_2 \text{ and } j \leq n_i^{(2)} \end{cases}$$

Combining the equations, we find explicitly that

$$t_{i,1}^{(3)} = \begin{cases} t_{i,3} & \text{if } t_{i,1} \leq x_1 \ , \ t_{i,2} \leq x_2 \\ t_{i,2} & \text{if } t_{i,1} > x_1 \ , \ t_{i,1} \leq x_1 + x_2 \\ t_{i,2} - x_2 & \text{if } t_{i,1} \leq x_1 \ , \ t_{i,2} > x_2 \\ t_{i,1} - x_1 - x_2 & \text{if } t_{i,1} > x_1 \ , \ t_{i,1} > x_1 + x_2 \ . \end{cases}$$

These cases correspond, respectively, to the situation when after two system simulator trials the i^{th} component had failed twice, had not

failed the first time but did fail the second, had failed the first time but not the second and lastly had not failed either time.

We now define the induction step: Given $k \geq 1$ and a set of data $t_{i,1}^{(k)},\cdots,t_{i,n_i(k)}^{(k)}$ for $i = 1,\cdots,m$ where $n_i^{(k)} \geq 1$, we set

$$x_k = \max_{j=1}^{r} \min_{i\epsilon P_j} \left| t_{i,1}^{(k)} \right|$$

and generate a new set of data for $k+1$ by

$$t_{i,1}^{(k+1)} = t_{i,2}^{(k)} + \left(t_{i,1}^{(k)} - t_{i,2}^{(k)} = \chi_k\right) t_{i,1}^{(k)} > x_k$$

and for $j = 2,\cdots,n_i^{(k+1)}$ where $n_i^{(k+1)} = n_i^{(k)} - t_{i,1}^{(k)} > x_k$

$$t_{i,j}^{(k+1)} = t_{i,j+1}^{(k)} + \left(t_{i,j}^{(k)} - t_{i,j+1}^{(k)}\right) t_{i,1}^{(k)} > x_k \quad .$$

We begin the induction by setting

$$t_{i,j}^{(1)} = t_{i,j} \qquad i = 1,\cdots,m$$

$$j = 1,\cdots,n_i^{(1)} = n_i \quad .$$

Thus we calculate from the data a sample, label it, $x_{11},\cdots,x_{1,k_1}$ where each $x_{1,i}$ is independent with survival distribution k and

$$k_1 = \max \left\{ j\colon \min\left(n_1^{(j)}, n_2^{(j)}, \cdots, n_m^{(j)}\right) = 0 \right\}$$

and we note that

$$\min(n_1,\cdots,n_m) \le k_1 \le \sum_{i=1}^{m} n_i - m + 1 \quad .$$

Thus as an unbiased estimate of $R(t)$ consider

$$R_1(t) = \frac{1}{k_1} \sum_{j=1}^{k_1} \{t < x_{1j}\} \quad . \tag{3.4}$$

However, the sample values $x_{1,1},\cdots,x_{1,k_1}$ depend upon the first arrangement of the data $(t_{i,1},\cdots,t_{i,n_i})$ for $i = 1,\cdots,m$. There are all together $p = \prod_{i=1}^{m} (n_i!)$ possible permutations of these sample values. Since each arrangement of the values is equally likely, in order to utilize more of the information available in the sample we consider the estimate obtained by averaging over all of these equally likely permutations, namely

$$U(t) = \frac{1}{p} \sum_{j=1}^{p} \tilde{R}_j(t) \quad \text{for} \quad t > 0 \tag{3.5}$$

where $j = 1,\cdots,p$ denotes one indexing of the p permutations and the resulting estimate of $R(t)$ defined similarly to $\tilde{R}_1(t)$.

As a matter of practical interest, this estimate cannot be usable unless the number of failures n_i for $i = 1,\cdots,m$ are all rather small. Fortunately for the applications intended in the estimation of high reliability, this is usually the case.

4.0 THE DISTRIBUTION OF THE SIMULATED SAMPLE SIZE

We now wish to evaluate the probability of a given subset of m components having failed by the time the system has failed. Let $A \subset \{1,2,\cdots,m\}$ be a set of indices. The probability that exactly this set of components indexed by A will be failed when the system fails is given by

$$f_{\Phi}(A) = P\left[\max_{j=1}^{r} \min_{i \in P_j A} T_i \leq X\right] \tag{4.1}$$

where X is the system life defined in Equation (3.3). To see this note that all, and only, the components indexed by A must have failed and these failures must have effected the failure of each minimal path, i.e.,

$$\min_{i \in P_j A} T_i \leq \min_{i \in P_j} T_i$$

The reverse inequality is always true, and a moment's reflection verifies (4.1).

This discrete distribution on the set of subsets of $\{1,2,\cdots,m\}$ is rather complicated to express in general. We now consider the special case of a series system.

In the case of a series system there is only one minimal path, namely the entire set of components. It then follows that $X = \min_{i-1}^{m} T_i$. Elementary calculation shows that $f(A) = 0$ unless A equals a singleton set, in which case

$$f(\{j\}) = P\left[\min_{i=1}^{m} T_i \geq T_j\right]$$

$$= \int_0^\infty \left[e^{-\Sigma_j \lambda_i t}\right] \lambda_j e^{-\lambda_j t} dt \quad ,$$

where Σ_j indicates the summation from $i = 1,\cdots,m$ with $i \neq j$,

$$(4.2) \qquad f(\{j\}) = \frac{\lambda_i}{\sum_1^m \lambda_i} = p_j \quad \text{for} \quad j = 1,\cdots,m \quad .$$

Thus for a series system we have a multinomial probability distribution that any one of the m components will fail and be the one causing system failure. The probability that the j^{th} component will fail is p_j for $j = 1,\cdots,m$ as given in (4.2).

We recall that if the probability of success is p for each trial, then the probability that the r^{th} success occurs at trial number $r + k$ is given by the negative Binomial distribution

$$(4.3) \qquad \binom{k+r-1}{k} p^r (1-p)^k \qquad k = 0,1,2,\cdots,$$

This is the probability that exactly k failures preceed the r^{th} success. Consider now the case for $m = 2$ as treated in [6].

Let p_1 be the probability of the occurrence of event E_1 and $p_2 = 1-p_1$ be the probability of occurrence of event E_2 (Here $E_1 \cup E_2$ is the sure event). Then the probability that the n_1^{st} occurrence of E_1 or the n_2^{nd} occurrence of E_2 occurs at the k^{th} trial is

$$(4.4) \qquad g_2(k) = \binom{k-1}{n_1-1} p_1^{n_1} p_2^{k-n_1} + \binom{k-1}{n_2-1} p_2^{n_2} p_1^{k-n_2}$$

where $\min(n_1,n_2) \leq k \leq n_1 + n_2 - 1$.

This formula agrees with that given in [6].

We now generalize to an arbitrary m. The probability that in $(k-1)$ trials E_1 occurs k_1 times, E_2 occurs k_2 times, etc., is

$$\frac{(k-1)!}{\prod_{i=1}^{m} (k_i)!} \prod_{i=1}^{m} p_i^{k_i} \qquad \text{where } \sum_{i=1}^{m} k_i = k - 1 \ .$$

We specify that the n_1^{st} occurrence of E_1 occurs at k^{th} trial. We let $k = n_1 - 1$ in the formula above and multiply by p_1, the probability that E_1 occurs, to obtain

$$\frac{(k-1)!}{(n_1-1)! \prod_{2}^{m} (k_i)!} p_1^{n_1} \prod_{2}^{m} p_i^{k_i} \qquad \text{where } \sum_{2}^{m} k_i = k - n_1 \ .$$

Thus the probability that on the k^{th} trial we have any one of the events "the n_j^{th} occurrence of E_j for the first time but less than n_i occurrences of E_i for $i \neq j$." is

$$(4.5) \qquad g_m(k) = \sum_{j=1}^{m} \sum_{\Lambda_{j,k}} \frac{(k-1)!\, p_j^{n_j}}{\Gamma(n_j) \prod_j k_i!} \prod_j p_2^{k_i}$$

where $\prod_j = \prod_{\substack{i=1 \\ i \neq j}}^{m}$ and

$$\min(n_1,\cdots,n_m) \leq k \leq \sum_1^m n_i - m + 1 \quad .$$

Also

$$(4.6)\quad \Lambda_{j,k} = \left\{(k_1,k_2,\cdots,k_m): \quad \sum_1^m k_i=k,\ k_j=n_j,\ k_i \leq m_i-1 \text{ for } i\neq j\right\}$$

with the convention that summing over the vacuous set gives zero. We see that $\Lambda_{j,n} = \emptyset$ unless $k \geq n_j$.

We would like to use this distribution to obtain information about the size of the simulated system sample. We first consider the special case of the system sample when $m = 2$, the density of which has been previously given in Equation (4.4). Before we begin this study we set out several identities which we will use subsequently.

We now quote two identities: Equations 26.5.26, 26.5 p.945, Reference [1]. For integers $0 \leq a \leq n$, and $0 \leq p \leq 1$

$$(4.8)\quad \sum_{j=a}^n \binom{n}{j} p^j q^{m-a} = I_p(a,n-a+1)$$

$$(4.9)\quad \sum_{j=a}^n \binom{n+j-1}{n-1} p^n q^j = I_q(a,n)$$

where $q = 1 - p$ and $I_p(n,m)$ is the Incomplete Beta function defined by

$$(4.10)\quad I_p(n,m) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^p x^{n-1}(1-x)^{m-1}dx \quad .$$

From this it is easily checked that

$$I_p(n,m) + I_q(m,n) = 1 \quad . \tag{4.11}$$

We now prove

Lemma 1: For integers $m,n > 0$

$$\sum_{j=o}^{m-1} \binom{n+j-1}{n-1} p^n q^j = I_p(n,m) \quad . \tag{4.12}$$

Proof: It follows from (4.8) by setting $a = 0$ that

$$1 = I_p(0,n+1) \quad \text{for any } n \geq 0 \tag{4.13}$$

Thus by setting $a = 0$ in (4.9) we realize that for $1 \leq m \leq n$

$$1 = \left[\sum_{j=o}^{m-1} + \sum_{j=m}^{n}\right] \binom{n+j-1}{n-1} p^n q^j = I_q(m,n) + I_p(n,m) \quad .$$

The last equality is from (4.11). We subtract the identity in (4.9), with $a = m$, from both sides to obtain (4.12).||

Corollary: For integers $m \geq n > 0$

$$\sum_{k=n}^{m} \binom{k-1}{n-1} p^n q^{k-n} = I_p(n,m-n+1) \tag{4.14}$$

Proof: In (4.12) make the change of index $R = j + n$ and let m change to $m - n + 1$.||

We now make use of these preliminary results to obtain theorem: The distribution of K_2, the series system sample size for structures of order

2 is given, for $1 \le k \le n_1 + n_2 - 1$, by

$$G_2(k) = \sum_{i=1}^{2} I_{p_i}(n_i, k-n_i+1) \quad . \tag{4.15}$$

Here we use the convention from (4.13) that $I_p(n,a) = 0$ for integers $a < 1 \le n$. Letting $n_\cdot = n_1 + n_2$, the expected value is

$$EK_2 = \sum_{i=1}^{2} \frac{n_i}{p_i} I_{p_i}(n_i+1, n_\cdot - n_i) \quad . \tag{4.16}$$

Proof: From (4.4) we have that

$$G_2(k) = \sum_{j=1}^{k} g_2(j) = \sum_{j=n_1}^{k} \binom{j-1}{m_1-1} p_1^{n_1} p_2^{j-n_1} + \sum_{j=n_2}^{k} \binom{j-1}{m_2-1} p_2^{n_2} p_1^{j-n_2} \quad .$$

Now we apply Equation (4.14) to obtain (4.15). But also

$$EK_2 = \sum_{k=1}^{n_\cdot - 1} k g_2(k) = \sum_{i=1}^{2} n_i \sum_{k=1}^{n_\cdot - 1} \binom{k}{n_i} p_i^{n_i} (1-p_i)^{k-n_i}$$

to which we also apply (4.14) to obtain (4.16). ||

We now obtain a generalization of this result for the case of a series structure of order m. We express a certain probability which is given in terms of the multinomial density as an integral over a certain set of the Dirichlet density. In this regard, see Wilks [15], pp. 177-178.

Let S_m denote the simplex of dimension m defined by

$$S_m = \left\{ (x_1, \cdots, x_m): \ 0 \le x_i \le 1 \ \text{ for } i = 1, \cdots, m, \ \sum_{i=1}^{m} x_i \le 1 \right\} \quad .$$

Lemma 2: For all $m \geq 2$ and all $j = 1,\cdots,m$

$$A_j(m) = \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_{j-1}=0}^{n_{j-1}-1} \sum_{k_{j+1}=0}^{n_{j+1}-1} \cdots \sum_{k_m=0}^{n_m-1} \frac{\Gamma(k_1+\cdots+k_{j-1}+n_j+k_{j+1}+\cdots+k_m)}{\Gamma(n_j)\prod_j k_i!} p_j^{n_j} \prod_j p_i^{k_i}$$

$$= \frac{\Gamma(n.)}{\prod_{i=1}^{m}\Gamma(n_i)} \int\cdots\int_{T_j(m)} x_1^{n_1-1} x_2^{n_2-1} \cdots x_{m-1}^{n_{m-1}-1} (1-x_1-\cdots-x_{m-1})^{n_m-1} dx_1\cdots dx_{m-1}$$

where $T_j(m)$ for $j < m$ is the subset of the $m-1$ dimensional simplex S_{m-1} whose m vertices are the points $(p_1,\cdots,p_{m-1})$ $(0,\cdots,0)$ and $(\delta_{i,1},\cdots,\delta_{i,m-1})$ for $i = 1,\cdots,m-1$ but $i \neq j$. Here δ_{ij} is the Kronecker delta and $T_m(m) = S_{m-1} - \bigcup_{j=1}^{m-1} T_j(m)$.

Proof: We shall give a proof by induction but in fact supply only the induction step from $m = 3$ to $m = 4$. The general proof may be obtained merely by writing out the expression for arbitrary m and following the steps in the same order. We have proved in Lemma 1 that for $m = 2$,

$$A_1(2) = \sum_{k_2=0}^{n_2-1} \binom{k_2+n_1-1}{n_1-1} p_1^{n_1}(1-p_1)^{k_2} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_o^{p_1} x^{n_1-1}(1-x)^{n_2-1} dx \quad .$$

We now assume the truth of our assertion for $m = 3$, namely

$$A_1(3) = \sum_{k_2=0}^{n_2-1} \sum_{k_3=0}^{n_3-1} \frac{\Gamma(k_2+k_3+n_1)}{\Gamma(n_1)k_2!k_3!} p_1^{n_1} p_2^{k_2} p_3^{k_3}$$

$$= \frac{\Gamma(n_1+n_2+n_3)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \int_o^{p_1} \int_{\frac{p_2 x_1}{p_1}}^{1-\frac{q_2 x_1}{p_1}} x_1^{m_1-1} x_2^{m_2-1} (1-x_1-x_2)^{m_3-1} dx_2 dx_1 \quad .$$

We now undertake the case for $m = 4$

$$A_1(4) = \sum_{k_2=0}^{n_2-1} \sum_{k_3=0}^{n_3-1} \sum_{k_4=0}^{n_4-1} \frac{\Gamma(k_2+k_3+k_4+n_1)}{\Gamma(n_1) \prod_{i=2}^{4} k_i!} p_1^{n_1} p_2^{k_2} p_3^{k_3} p_4^{k_4} \quad .$$

By altering the order of summation we obtain

$$A_1(4) = \sum_{k_4=0}^{m_4-1} \frac{\Gamma(n_1+k_4) p_1^{n_1} p_4^{k_4}}{\Gamma(n_1) k_4! (p_1+p_4)^{n_1+k_4}} \sum_{k_2=0}^{n_2-1} \sum_{k_3=0}^{m_3-1} \frac{\Gamma(k_2+k_3+k_4+n_1)}{\Gamma(n_1+k_4) k_2! k_3!} (1-p_2-p_3)^{n_1+k_4} p_2^{k_2} p_3^{k_3}$$

Applying the identity given by $A_1(3)$ with $n_1 = n_1 + k_4$, $p_1 = 1 - p_2 - p_3$, we obtain

$$= \sum_{k_1=0}^{n_4-1} \frac{\Gamma(n_1+n_2+n_3+k_4)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3) k_4!} \frac{p_1^{n_1} p_4^{k_4}}{(p_1+p_4)^{n_1+k_4}} \int_0^{p_1+p_4} dx_1 \int_{\frac{p_2 x_1}{p_1+p_4}}^{1-\frac{q_2 x_1}{p_1+p_4}} dx_2 x_1^{n_1+k_4-1} x_2^{n_2-1} (1-x_1-x_2)^{n_3-1}$$

Make the change of variable $y_1 = \frac{x_1}{p_1+p_4}$, $y_2 = x_2$ resulting in

$$= \int_0^1 dy_1 \int_{p_2 y_1}^{1-q_2 y_1} dy_2 \frac{p_1^{n_1}}{\prod_1^3 \Gamma(n_i)} \left\langle \sum_{k_1=0}^{n_4-1} \frac{\Gamma(n_0-n_4+k_4)}{(k_4)!} (p_4 y_1)^{k_4} \right\rangle y_1^{n_1-1} y_2^{n_2-1} [1-y_1(p_1+p_4)-y_2]^{n_3-1}$$

Using the identity for $A_1(2)$ in the angular brackets, we have

$$\left\langle \ldots \right\rangle = \frac{\Gamma(n_.)}{\Gamma(n_4)} \frac{1}{(1-p_4 y_1)^{n_1+n_2+n_3}} \int_0^{1-p_4 y_1} t^{n_1+n_2+n_3-1} (1-t)^{n_4-1} dt$$

Make the transformation from (y_1,y_2,t) into (x_1,x_2,x_3) by

$$x_1 = \frac{p_1 y_1 t}{1-p_4 y_1} \quad , \quad x_2 = \frac{t y_2}{1-p_4 y_1} \qquad x_3 = \frac{t[1-y_1(p_1+p_4)-y_2]}{1-p_4 y_1}$$

which has inverse

$$y_1 = \frac{x_1}{p_1 t+p_4 x_1} \quad , \quad y_2 = \frac{p_1 x_2}{p_1 t+p_4 x_1} \quad , \quad t = x_1+x_2+x_3$$

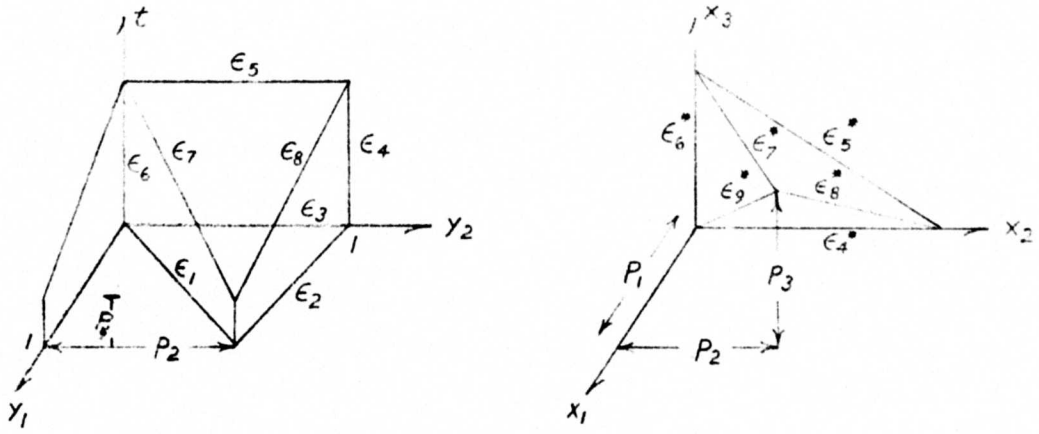
and Jacobian

$$J\left(\frac{y_1,y_2,t}{x_1,x_2,x_3}\right) = \frac{p_1^2 t}{(p_1 t+p_4 x_1)^3} \quad .$$

Substituting into the integral we obtain as the new integrand

$$x_1^{n_1-1} x_2^{n_2-1} x_3^{n_3-1} (1-x_1-x_2-x_3)^{n_4-1} \quad .$$

The original and transformed regions of integration can be pictured:



Each edge ε_i is transformed into the edge ε_i^* for $i = 4,\cdots,9$ with $\varepsilon_1,\varepsilon_2,\varepsilon_3$ being transformed into the origin $(0,0,0)$. An elementary calculation verifies this. It can be seen that the tetrahedron pictured is $T_1(4)$ as claimed.||

From (4.5) and (4.6) we have

$$G_m(n) = \sum_{k=1}^{n} \sum_{j=1}^{m} \sum_{\Lambda_{j,k}} \frac{\Gamma(k)p_j^{n_j}}{\Gamma(n_j)\Pi_j(k_i)!} \prod_j p_i^{k_i} .$$

By examining (4.6) we see that for fixed j the $\Lambda_{j,k}$ are disjoint for different k. As a consequence we define

$$\chi_{j,n} = \bigcup_{k=1}^{n} \Lambda_{j,k} = \left\{(k_1,\cdots,k_m): \sum_1^m k_i \le n,\ k_j = n_j,\ k_i \le n_i - 1 \text{ for } i \ne j\right\}$$

and then we write

$$G_m(n) = \sum_{j=1}^{m} \sum_{\chi_{j,n}} \frac{\Gamma(k_1+\cdots+k_{j-1}+n_j+\cdots k_m)}{\Gamma(n_j)\Pi_j(k_i)!} \prod_j p_2^{k_i} p_j^{n_j} .$$

If we alter our notation to indicate explicitly the dependence upon parameters, in accord with precedence, we define for any $T \subset S_m$

$$I_T(n_1,\cdots,n_m) = \frac{\Gamma(n.)}{\prod_1^m \Gamma(n_i)} \int\cdots\int_T \prod_{i=1}^{m-1} x_i^{n_i-1}(1-x_1-\cdots-x_{m-1})^{n_m-1} dx_1\cdots dx_{m-1}$$

Then in this notation we have

$$A_j(m) = I_{T_j(m)}(n_1,\cdots,n_m) \quad .$$

We may now check that we have a bona fide distribution function since

$$G_m(n.-m+1) = \sum_{j=1}^{m} I_{T_j(m)}(n_1,\cdots,n_m) = I_{\cup T_j(m)}(n_1,\cdots,n_m) = 1 \quad .$$

since $\cup T_j(m) = S_{m-1}$ and by the corresponding property of the Dirichlet distribution we have unity.

We now evaluate the expected sample size

$$EK_m = \sum_{R=1}^{n.-m+1} kg_m(k) = \sum_{k=1}^{n.-m+1} \sum_{j=1}^{m} \sum_{\Lambda_{j,k}} \frac{k!p_i^{n_j}}{\Gamma(n_j)\Pi_j(k_i)!} \Pi_j p_i^{k_i} \quad .$$

By comparison with Lemma 2, we see after rearranging summations,

$$EK_m = \sum_{j=1}^{m} \frac{n_j}{p_j} I_{T_j(m)}(n_1,\cdots,n_{j-1},n_j+1,\cdots,n_m) \quad .$$

The above result, while interesting, has the disadvantage of not being directly computable since tables of the Dirichelet distribution are not available like those of the Incomplete Beta function. However, an approximation can be found.

Define $p_{ij} = \frac{p_i}{1-p_j}$ for $i \neq j$ and note $\sum_{\substack{i=1 \\ i\neq j}}^{m} p_{ij} = 1$. Then

$$(4.17) \qquad g_m(k) = \sum_{j=1}^{m} \binom{k-1}{n_j-1} p_j^{n_j} (1-p_j)^{k-n_j} \sum_{\Lambda_{j,n}} \frac{(k-n_j)!}{\Pi_j (k_i)!} \prod_{\substack{i=1 \\ i=j}}^{m} p_{ij}^{k_i}$$

is another formula for the density. In this form we see the second summation is unity for k sufficiently small. Thus we have an approximation and upper bound.

$$g_m(k) \leq \sum_{j=1}^{m} \binom{k-1}{n_j-1} p_j^{n_j} (1-p_j)^{k-n_j} \quad \min n_i \leq R \leq n_\cdot - m + 1$$

and proceeding exactly as in the case $m = 2$ we find

$$EK_m \leq \sum_{j=1}^{m} \frac{n_j}{p_j} I_{p_j}(n_j+1,\ n_\cdot - m + 2 - n_j) \quad .$$

We now examine the accuracy of this approximation for the case $m = 3$. Let

$$g_3(k) = \sum_{j=1}^{3} \binom{k-1}{n_j-1} p_j^{n_j} (1-p_j)^{k-n_j} H_{j,k}$$

where the definition of $H_{j,k}$ may be inferred from Equation (4.17) above. We shall calculate $H_{1,k}$ only and obtain the others by symmetry. Assume that $k \geq n_1$ then by definition

$$H_{1,k} = \sum_{\Lambda_{1,n}} \frac{(k-n_1)!}{k_2! k_3!} p_{21}^{k_2} p_{31}^{k_3} = \sum_{j=v_k}^{u_k} \binom{k-n_1}{j} p_{21}^{j} p_{31}^{k-n_1-j}$$

where $u_k = \min (n_2 - 1,\ k - n_1)$, $v_k = \max (k - n_1 - n_3 + 1, 0)$, and we note $v_k \leq u_k$ for $n_1 \leq k \leq n_1 + n_2 + n_3 - 2$. Since $u_k \leq k - n_i$, we have

$$H_{1,k} = \sum_{j=v_k}^{k-n_1} \binom{k-n_1}{j} p_{21}^{j} p_{31}^{k-n_1-j} - \sum_{j=u_k+1}^{k-n_1} \binom{k-n_1}{j} p_{21}^{j} p_{31}^{k-n_1-j}$$

Using the identity (4.8) we can write

$$H_{1,k} = I_{p_{21}}(v_k,\ k - n_1 - v_k + 1) - I_{p_{21}}(u_k + 1,\ k - n_1 - u_k)$$

Using the symmetry of the Incomplete Beta function and the notation $x^+ = \max(x,0)$ allows us to obtain

$$H_{1,k} = I_{p_{21}}\left((k - n_1 - n_3 + 1)^+,\ n_3\right) + I_{p_{31}}\left((k - n_1 - n_2 + 1)^+,\ n_2\right) - 1$$

where we recall the Equation (4.13). Thus by symmetry we see that $H_{jk} = 1$ if $k \leq \min\left|n_i - n_j - 1:\ \text{for}\ i \neq j\right|$.

5.0 THE LIKELIHOOD RATIO AND THE SERIES SYSTEM

For any coherent structure ϕ with associated reliability function h, there exists a series system, the reliability of which bounds the system reliability h from below, i.e.,

$$h(\lambda) \geq \exp -\left|\sum_{j=1}^{m} \omega_j \lambda_j\right| \tag{5.1}$$

where ω_j is a known weighting factor determined from h (or ϕ) for λ_j.

Because of this fact, we will in this section consider only the case $\ell n\ h(\lambda) = -\Sigma\omega_j\lambda_j$ and obtain a lower confidence bound. We think this is the most important case.

We will follow the general results which have been presented in an earlier publication [9]. However, as a convenience, we reparametrize by imposing the restriction in the form that the $\ell n\ h(\lambda)$ is constant, and equate

$$\frac{\partial}{\partial\lambda_j}\left[L^*(\lambda) - \delta\ \ell n\ h(\lambda)\right] = 0 \quad .$$

Thus we obtain, instead of the general equation (1.3), as given in [9].

$$\frac{n_j}{\lambda_j} - t_j = -\delta\omega_j \tag{5.2}$$

where t_j is the total list time for the j^{th} component and n_j is the total number of failures

$$\tilde{\lambda}_j(\delta) = \frac{n_j}{t_j - \delta\omega_j} \qquad j = 1,\cdots,m \quad .$$

Thus in this particular case it is unnecessary to use the general contractive map theorems in order to calculate $\tilde{\lambda}(\delta) = (\tilde{\lambda}_1(\delta),\cdots, {}_m(\delta)$. For this special case we find, using the definition of Λ given in [9]

$$\Lambda(\delta) = \sum_{j=1}^{m} s_j\left[1 - \frac{\tau_j}{\tau_j - \delta} + \ell n\left(\frac{\tau_j}{\tau_j - \delta}\right)\right]$$

where $\tau_j = t_j/\omega_j$ for $j = 1,\cdots,m$. In accord with finding only a lower confidence bound, we want to solve for $x > 0$ such that

$$\Lambda(x) = -\tfrac{1}{2}\,\chi^2_\gamma(1) \quad , \tag{5.3}$$

where $\chi^2_\gamma(1)$ is the $100\gamma^{th}$ percentile of the Chi-square distribution with one degree of freedom. Call the solution of (5.3) the value δ^+_γ. Then

$$h\left[\tilde{\lambda}(\delta^+_\gamma)\right] = \exp\left\{-\sum_{j=1}^{m} \frac{n_1}{\tau_j - \delta^+_\gamma}\right\}$$

is an (approximate) lower bound for $h(\lambda)$ of level $\frac{1+\gamma}{2}$, rather than of level γ, since we are obtaining only a one-sided confidence bound.

We now exhibit a practical method for the determination of δ^+. For a given γ, $0 < \gamma < 1$ there exists an $x > 0$ such that $f(x) = 0$ where

$$f(x) = -\Lambda(x) - \tfrac{1}{2}\,\chi^2_\gamma(1)$$

$$(5.4) \qquad = \sum_{i=1}^{m} m_i \left[\frac{\tau_i}{\tau_i - x} - \ell n \left(\frac{\tau_i}{\tau_i - x} \right) \right] - \sum_{1}^{m} m_i - \tfrac{1}{2} \chi^2_{\gamma}(1) \quad .$$

The solution of $f(x) = 0$ must be unique since

$$f'(x) = x \sum_{i=1}^{m} \frac{s_i}{(\tau_i - x)^2} > 0$$

whenever $0 < x < \min(\tau_1, \cdots, \tau_m) = \tau_{(1)}$, which is a more stringent condition than that given in [9]. But note that

$$f''(x) = \sum_{i=1}^{m} s_i \frac{\tau_i + x}{(\tau_i - x)^3} > 0$$

for $0 < x < \tau_{(1)}$.

Thus we see that both f', f'', which are continuous, do not vanish for $0 < x < \tau_{(1)}$. These are sufficient conditions that the Newton iteration procedure, namely

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} \qquad k = 1, 2, \cdots,$$

will converge to the value δ^+_{γ}.

We now state the

Theorem : Let the data (t_i, m_i) $i = 1, \cdots, m$ be given, where t_i represents the total test time for the i^{th} component and $n_i \geq 1$, represents the number of failures of that component during that time. If ω_i is the multiplicity of the i^{th} component in any structure and $\tau_i = t_i/\omega_i$, then

$$\exp\left\{-\sum_{j=1}^{m}\frac{s_j}{(\tau_j/t)-\delta_\gamma^+}\right\} \quad \text{for } 0 < t < \tau_{(1)}/\delta_\gamma^+$$

(5.5)

$$0 \quad \text{for } t \geq \tau_{(1)}/\delta_\gamma^+$$

is a lower confidence bound of level $\frac{1+\gamma}{2}$, where $0 < \gamma < 1$ is preselected, ior the system reliability for all time $t > 0$. The quantity δ_γ^+ is the solution of the equation $f(x) = 0$ and f was defined in equation (5.4).

Proof. We have only to remark that the selection of a time scale in terms of mission length was arbitrary so that the bound derived previously can be used for the reliability at all times.||

6.0 A NUMERICAL ILLUSTRATION

As a numerical illustration of the behavior of the previously outlined procedure, we shall study its performance on a specified series system with ten components. Take $m = 10$. We shall assume that $\lambda_j = 1$ for $j = 1,\cdots,10$. Thus the true reliability of the system is, by e^{-10t} for $t > 0$. We also assume that $n_j = \omega_j = 1$ and thus $\tau_j = t_j$ for $j = 1,\cdots,m$.

We will now generate 10 exponential variates, $t_1,\cdots,t_m$, each with unit mean and solve the equation $f(x) = 0$ where

$$f(x) = \sum_{i=1}^{m} \left[\frac{t_i}{t_i - x} - \ln\left(\frac{t_i}{t_i - x}\right)\right] - C_\gamma \tag{6.1}$$

and C_γ is a constant determined by the nominal level of the confidence bound.

We find that for $\gamma = .95$, $C_{.95} = 11.353$. Call the solution of (6.1) the value δ_γ^+. The lower bound for the reliability is by (2.5)

$$\exp\left\{-\sum_{i=1}^{m}\left(\frac{1}{t_i/t - \delta_\gamma^+}\right)\right\}, \quad \text{for } 0 < t < \delta_\gamma^+ t_{(1)} , \tag{6.2}$$

and the confidence level is 97.5.

Generating forty independent observations of an exponential variate with unit mean resulted in the following four samples:

(1)	(2)	(3)	(4)
1.305	1.889	.2331	3.258
1.346	3.864	.8881	.1848
1.483	.7789	.04059	.7594
.2931	.4825	.8329	2.420
.2282	1.171	.1948	.8638
.3489	.5642	1.332	.5307
1.342	.3525	2.105	.06695
.7171	.9037	1.144	.8239
.3397	.5590	1.203	.2452
1.126	.5445	.07427	.2306

Solving (6.1) by machine program and calculating (6.2) for each set of observations yielded the confidence bounds which are summarized visually in Figure 1.

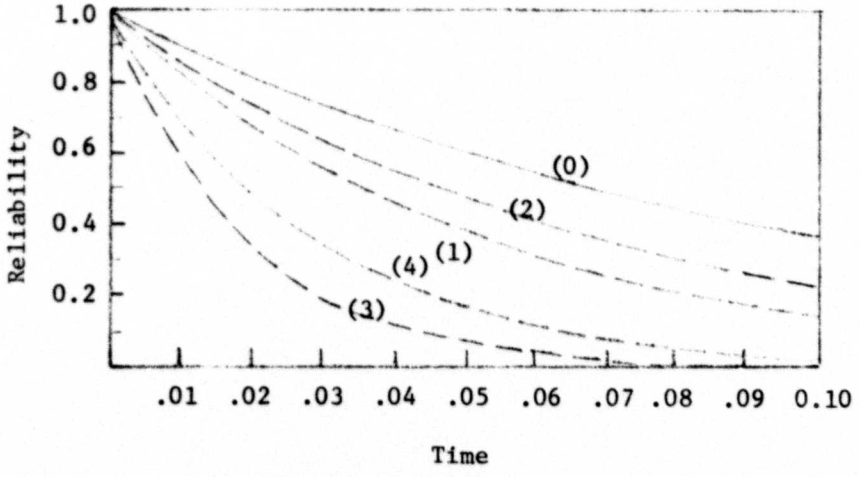


Figure 1. 97.5% lower confidence bounds on the reliability of a 10 component system based on independent samples.

(0) is the true reliability e^{-10t} for $t > 0$

(i) is the confidence bound based on sample (i) for $i = 1,2,3,4$.

As a further illustration of the stochastic behavior of this method, we shall repeat the entire procedure two thousand times, but instead of calculating the entire lower bound function we shall only compute (3.2) at $t = .01$. Then we shall make a frequency histogram of the values of the reliability bounds for the specified value $t = .01$. The twenty percentile points, successive differences being .05, of the empiric distribution are:

Percentile	Percentile Point
.05	0.106621
.10	0.297174
.15	0.417770
.20	0.492659
.25	0.552798
.30	0.596595
.35	0.632835
.40	0.665383
.45	0.689521
.50	0.707729
.55	0.730936
.60	0.750021
.65	0.767430
.70	0.782541
.75	0.798792
.80	0.815686
.85	0.833291
.90	0.851064
.95	0.877483
1.00	0.940218

A graph of the empiric distribution is given in Figure 2. Note that the true reliability of $e^{-.1} = .904837$ slightly exceeded the nominal 97.5 percentile. In fact, the actual count was 1979 values less than $e^{-.1}$ out of the 2,000 observations sampled. This was approximately the 99th percentile.

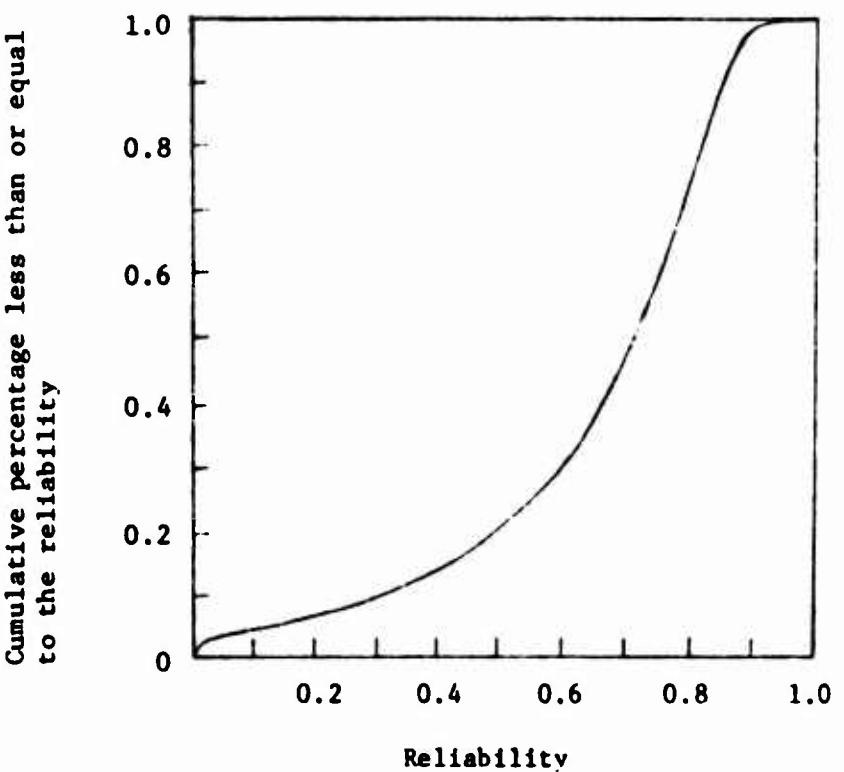


Figure 2. Empiric distribution of the 97.5 percent lower confidence bound using 2,000 observations when the true reliability is $e^{-.1} = .905$.

Thus we see the stated confidence level of 97.5 percent appears to be, in this instance, slightly conservative. To check this we repeated this entire experiment a second time and we observed 1,990 out of 2,000 observations of the reliability bound were less than $e^{-.1}$, which increases our suspicion of a slightly conservative tendency for the level of confidence in this case.

The determination of a confidence bound using a procedure which is valid when a large number of life length observations have been made for each component when, in fact, only one observation has been taken of each component life length is about as far from the conditions of known validity as possible, and it leaves the exact confidence level open to question.

As a comparison, we also construct the lower 97.5% confidence bound for the system reliability of a 10 component series system, each component of which has the same reliability using ten observations of each component life length. The much improved result is presented in the graph of Figure 3. A table of the values of the lower 97.5% confidence bound on the same system reliability but with twenty observations on each component is given below:

Time	True Reliability	Lower Confidence Bound
0	1.000	1.000
.2	.819	.815
.4	.670	.663
.6	.549	.538
.8	.449	.435
1.0	.368	.351
1.2	.301	.283
1.4	.247	.227
1.6	.202	.182
1.8	.165	.146
2.0	.135	.116

The improvement is obvious.

While such examples as these can never prove anything in general about the adequacy of this approximation, they do not indicate that it cannot be used, with suitable caution, in those cases where the exact distribution is unknown.

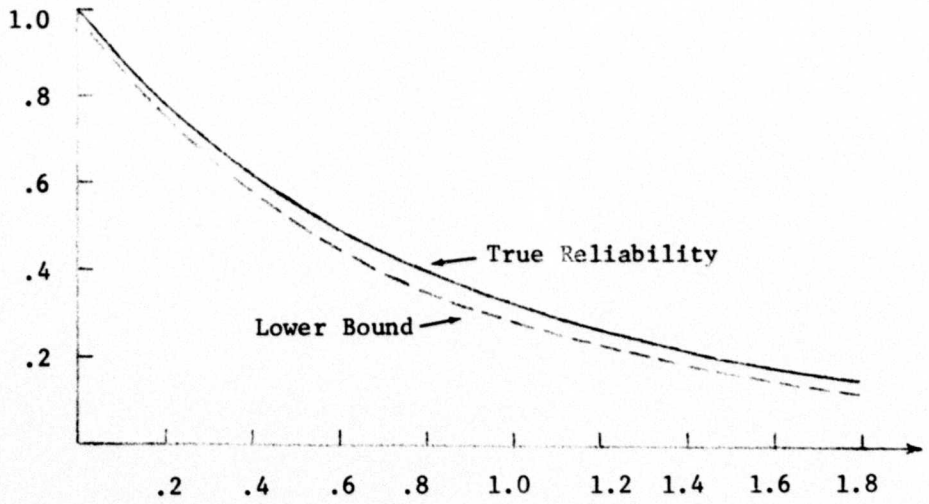


Figure 3. A lower 97.5% Confidence Bound on the Reliability of a 10 Component Series System With 10 Observations on Each Component.

7.0 SOME PROBLEMS IN DETERMINING THE ASYMPTOTIC DISTRIBUTION

It is clear that for each fixed t, the stochastic variable $U(t)$, as defined in (3.5), is a U-statistic in the sense of Haeffding [5]. In a way we are returning to a suggestion made by Joan Rosenblatt in [12] concerning the use of a U-statistic to compute reliability bounds. However, we are now combining such statistics with special properties of the exponentially distributed component data and more recent results concerning the structure function.

Clearly, $U(t)$ is unbiased for $R(t)$ for each $t > 0$. Moreover, U is asymptotically a Gaussian Random element (in the appropriate Banach space), since for every element V the inner product

$$\int_0^\infty V(t)dU(t) = -\frac{1}{p}\sum_{i=1}^{p}\frac{1}{K_i}\sum_{j=1}^{K_i} V(x_{ij})$$

is asymptotically a normal random variable under mild conditions as $p \to \infty$.

Let us consider the case where we have been given the vectors of failure data as defined in (3.1). Call it ξ_i. We can define an estimate

$$U^*(t) = \frac{1}{\nu}\sum_{j=1}^{\nu}\tilde{R}_j(t)$$

where the index $j = 1,\cdots,\nu$ is a number arbitrarily chosen and may be less than p and thereby not cover all possible promulations of the elements in each component of

$$\Xi = (\xi_1,\cdots,\xi_m) \quad .$$

This less restrictive estimate U^* is constructed by picking ν such permutations independently, from the $p = \prod_{i=1}^{m} (n_i!)$ distinct permutations which are possible, with identically uniform chances.

Obtaining the estimate U^* is computationally feasible for ν of moderate size, while the computation time necessary to obtain all p permutations for U may not be realistic if the sample sizes n_i, or the system order m, are large. Of course U^* is unbiased, as in U. However, we indicate one other reason that U^* may be preferable.

Consider the covariance of U. Since

$$EU(t)U(t+s) = \frac{1}{p^2} \sum_{i,j=1}^{p} E\tilde{R}_i(t)\tilde{R}_j(t+s)$$

we see that this computation reduces to

$$E\tilde{R}_k(t)\tilde{R}_\ell(t+s) = E\,\frac{1}{K_k K_\ell} \sum_{i=1}^{K_k} \sum_{j=1}^{K_\ell} \{t < x_{ik}\}\{t + 0 < x_{j\ell}\} \quad .$$

But now not only do the sample sizes u_j depend upon the particular j^{th} permutation chosen, but so do the actual system failure times x_{ij}. Thus the evaluation would depend upon a functional relationship being determined between these random variables and the index of the permutation enumeration. Whether or not the substitution of U^* for U will eliminate this difficulty is not known at the present time, but it is under investigation.

As an illustration of the type of estimates that this method gives, we exhibit the graphs in Figure 4 of two estimates of the system reliability of

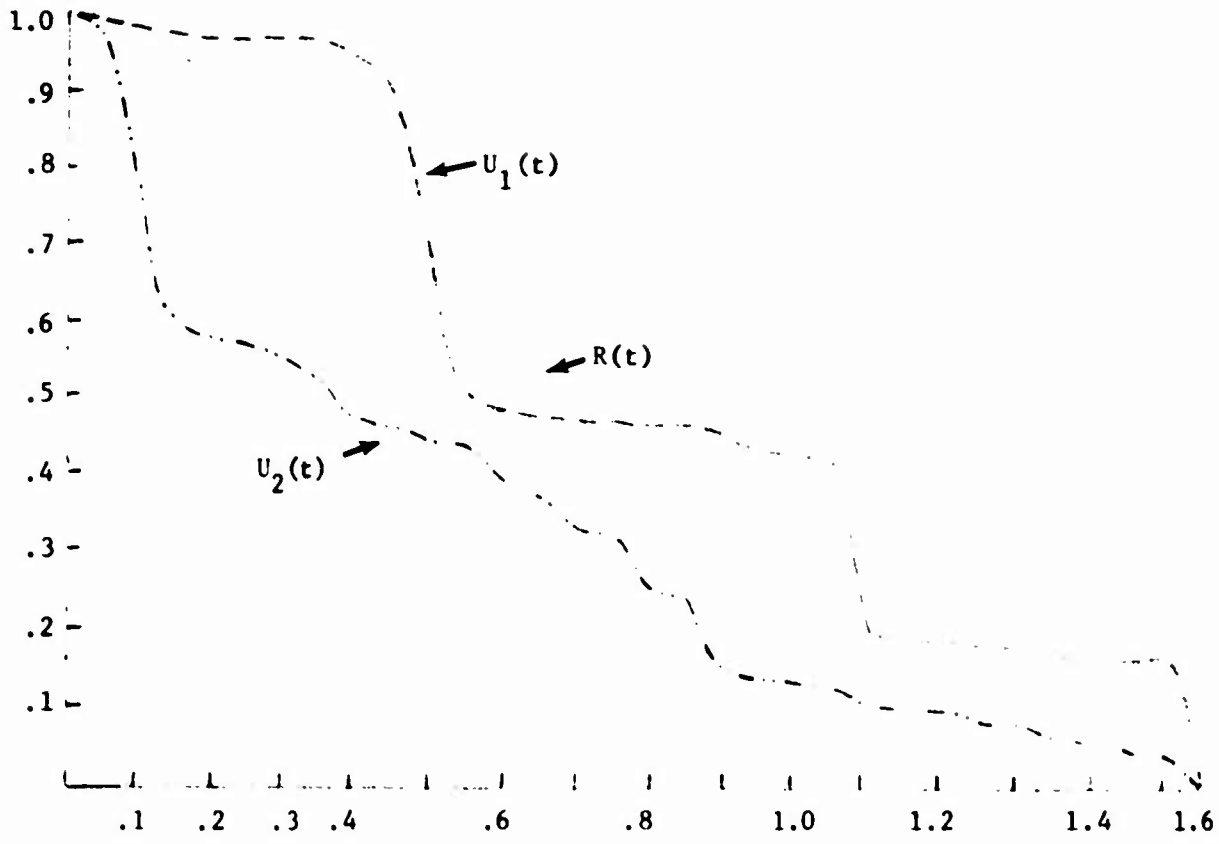


Figure 4. Two different reliability estimates $U_1(t)$ and $U_2(t)$, each based on 96 permutations of three sets of sample data of sizes 2, 4 and 2, respectively, for each component, for the "2 out of 3" three component system with reliability given by $R(t) = e^{-2t}(3-2e^{-t})$.

a simple two-out-of-three system, each estimate based on the generation of data from the independent exponential observations of the reliability of each component. The data for these two estimates are as follows:

Observations Used in U_1

Component 1	2.45413, 1.71585
Component 2	.49114, 1.52854, .52064, 1.06873
Component 3	.31766, 2.13006

Observations Used in U_2

Component 1	1.54401, .39385
Component 2	.13779, .89004, .07049, 1.62566
Component 3	.75851, .12614

All component data were machine generated with a true exponential distribution with unit mean. The set of 96 possible permutations were listed by the machine and each empirical distribution combined in the manner indicated in the equations.

In conclusion, it is clear that this method certainly improves the system reliability estimated with a small amount of data. That we cannot yet determine the covariance in order to apply the asymptotic normal theory and obtain confidence bounds is regrettable, but doubtless this defect will be overcome soon.

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REFERENCES

[1] Abramowitz, M., and Stegun, I.A., Eds., "Handbook of Mathematical Functions", Dover Publications, Inc., New York, 1965

[2] Bazousky, Igor, "Reliability Theory and Practice", Prentice Hall, Inc., Englewood Cliffs, New Jersey, 275-276, 1961

[3] Esary, J. D. and Proschan, F., "A Reliability Bound for Systems of Maintained and Interdependent Components", Journ. of Amer. Statist. Assoc. 65, 329-338, 1970

[4] Grubbs, F. E., "Approximate Fiducial Bounds for the Reliability of a Series System for Which Each Component has an Exponential Time-to-Fail Distribution", Technometrics 13, 865-872, 1971

[5] Hoeffding, W., "A Class of Statistics With Asymptotically Normal Distribution", Amer. Math. Stat. 19, 293-325, 1948

[6] Lieberman, Gerald J. and Ross, Sheldon M., "Confidence Intervals for Independent Exponential Series Systems", Journ. of Amer. Statist. Assoc. 66, 837-840, 1971

[7] Madansky, Albert, "Approximate Confidence Limits for the Reliability of Series and Parallel Systems", Technometrics 4, 495-503, 1965

[8] Madansky, Albert and Olkin, Ingram, "Approximate Confidence Regions for Constraint Parameters", Multivariate Analysis, Vol. 2, pp. 261-286, Academic Press, Inc., New York 1969

[9] Myhre, J. M. and Saunders, S. C., "Approximate Confidence Limits for Complex Systems with Exponential Component Lives", Ann. Math. Statist., 42, 342-348, 1971

[10] Myhre, J. M. and Saunders, S. C., "Comparison of Two Methods of Obtaining Approximate Confidence Intervals for System Reliability", Technometrics 10, 37-49, 1968

[11] Myhre, J. M. and Saunders, S. C., "On Confidence Limits for the Reliability of Systems", Ann. Math. Statist. 39, 1968

[12] Rosenblatt, Joan R., "Confidence Limits for the Reliability of Complex Systems", in Statistical Theory of Reliability, Ed. Marvin Zelen, The University of Wisconsin Press, 115-137, 1963

[13] Sarkar, T. K., "An Exact Confidence Bound for the Reliability of a Series System Where Each Component has an Exponential Time to Failure Distribution Technometres 13, 535-546, 1971

[14] Springer, M. D. and Thompson, W. E., "Bayesian Confidence Limits for Reliability of Redundant Systems When Tests are Terminated at First Failure", Technometrics 10, 29-36, 1968

[15] Wilks, Samuel S., "Mathematical Statistics", John Wiley and Sons, Inc., New York 1962

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13. ABSTRACT

One of the problems that has been of concern for a number of years is the following: Suppose that there exists a series system of n exponentially lived components and we ask that the system be qualified for service by demonstrating a reliability that equals, or exceeds, a specified value $1 - \alpha$, with a confidence level at least $1 - \gamma$ (Here α and γ are small). This paper addresses the question of how this can be demonstrated using only data on the reliability of the components, where the reliability design goals for each components must be apportioned, as well as the confidence levels at which the design goals must be met in order to satisfy the overall system reliability requirement. Another question that is addressed is the calculation of the probability that the system reliability exceeds its goal. The purpose of the paper is to make a comparison between competing theories for the above situation when the components have exponential life lengths.

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