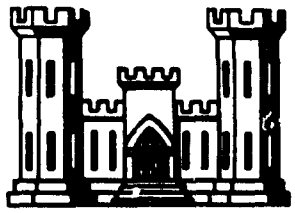

Research Note ETL-RN-71-3

THE DERIVATION AND POTENTIAL OF NEW FILTER EQUATIONS FOR NUMERICAL WEATHER PREDICTION

by

Hans Baussus von Luetzow

December 1971

U.S. ARMY ENGINEER TOPOGRAPHIC LABORATORIES
FORT BELVOIR, VIRGINIA

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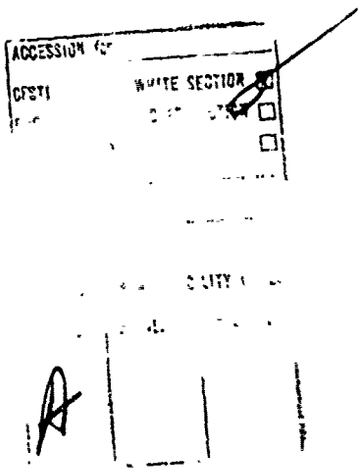

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This research presents two initialization methods in the form of optimal filter equations suitable for adiabatic and diabatic numerical weather prediction and an optimal system of equations for numerical weather prediction involving four prognostic equations and one diagnostic equation. The first method commensurable with the hydrostatic assumption is a practical generalization of Fjortoft's results and does not suffer from some weaknesses of the balance equation—particularly in the anticyclonic case analyzed by Elsaesser. It also significantly improves the effective prediction time scale and can readily replace computer programmed models based on the balance equation with a resulting improvement of global weather prediction. The second method consisting of three filter equations in the (x, y, z, t)-system does not presuppose hydrostatic equilibrium and does not encounter the problem of hyperbolicity, and is therefore applicable to smaller scales. The third filter equation may simultaneously replace the hydrostatic one within an optimal prognostic system containing the horizontal equations of motion, the continuity, and the thermodynamic equation in their invariant forms. The relaxation procedure for solving the nonhydrostatic filter differential equations is no more difficult than the numerical solution of the general balance equation in conjunction with the ω-equation. Finally, the study throws new light on problems such as ergodicity, long range predictability, gridscale diffusion, surface friction, and boundary conditions.

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Prepared by

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SUMMARY

This research presents two initialization methods in the form of optimal filter equations suitable for adiabatic and diabatic numerical weather prediction and also an optimal system of equations for numerical weather prediction involving four prognostic equations, the optional mixing-ratio continuity equation, and one diagnostic equation.

The first initialization method commensurable with the hydrostatic assumption is a practical generalization of Fjortoft's results and does not suffer from some weaknesses of the balance equation—particularly in the anticyclonic case analyzed by Elsaesser. It also significantly improves the effective prediction time scale and can readily replace computer programmed models based on the balance equation with a resulting improvement of global weather prediction.

The second initialization method consisting of three filter equations in the (x, y, z, t)-system does not presuppose hydrostatic equilibrium and does not encounter the problem of hyperbolicity and is, therefore, applicable to smaller scales. The third filter equation may simultaneously replace the hydrostatic one within an optimal prognostic system containing the horizontal equations of motion, the continuity, and the thermodynamic equation in their invariant forms. The relaxation procedure for solving the nonhydrostatic filter differential equations is no more difficult than the numerical solution of the general balance equation in conjunction with the ω-equation.

Finally, the study throws new light on problems such as ergodicity, long-range predictability, gridscale diffusion, surface friction, and boundary conditions.

FOREWORD

This research was started in 1965 and is a personal effort except for editing, typing, and printing. Some of its results were already available in 1966. In the meantime, the author tried to perfect the analysis and to put it in the right perspective. It is of significance that the filter methods developed for the hydrostatic and less restricted (x,y,z,t)-system associated with a new set of prognostic-diagnostic equations are not only new and optimal but have a useful purpose and include operativeness as well. The author would like to thank Colonel John R. Oswalt, Jr., who endorsed the preparation and preliminary publication of this work in the form of a comprehensive USAETL Research Note.

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THE DERIVATION AND POTENTIAL OF NEW FILTER EQUATIONS FOR NUMERICAL WEATHER PREDICTION*

1. Introduction. The Global Atmospheric Research Program (GARP) sponsored by the International Council of Scientific Unions and the World Meteorological Organization is aimed during the next decade at gaining additional insight about the physics of the atmosphere so that it becomes feasible to predict the weather worldwide for periods of at least 2 weeks.

The first requirement to meet this goal is the establishment of a worldwide observation system involving advanced satellites providing for interrogating, recording, and locating on a realtime basis and for measuring of cloud cover. The satellites will carry passive microwave radiometer, occultation and refraction sensors for the determination of vertical profiles of temperature, water vapor, and density from which constant pressure heights could be determined.[1]

The second requirement is to acquire a better understanding of some physical processes, such as the flow of heat and moisture near the tropical waters, which have to be incorporated in a long-range atmospheric prediction scheme. The processes of air-sea interaction and related research efforts are the subject of the Barbados Oceanographic and Meteorological Experiment (Bomex)[2] which is the first experiment of the United States portion of GARP.

The third requirement is the development of computers capable of processing the tremendous amount of data and simulating the extremely complicated atmospheric generation process. The Bomex report states that a computer now being designed is expected to have 100 to 1000 times the capacity of present models.

Finally, more sophisticated prediction models, including nonadiabatic and frictional effects, some of these at least in a parametric form, will have to be

*Parts of Paragraphs 4 and 7 of this research were presented at the 52nd Annual Meeting of the American Geophysical Union, Washington, D.C., April 13, 1971.

[1] H. E. Newell: "NASA's Space Science and Applications Program (Meteorology)," National Aeronautics and Space Administration, Washington, D. C., April 1967.

[2] "News from BOMEX, Barbados Oceanographic and Meteorological Experiment," U.S. Interagency Release, Washington, D.C., March 1969.

developed. As Charney stated in 1951[3] and since proved by others, condensation phenomena are the simplest to introduce although much remains to be done in this respect. The same is true for long-wave radiational effects which are prognostically interrelated with humidity forecasts. Very difficult to incorporate is, of course, the turbulent transfer of momentum and heat.

Within the context of BOMEX, a "Hemispheric Model Study" is to be conducted by Pandolfo, while Charney acts as principal investigator of a "Theory of Large-Scale Atmospheric and Oceanic Processes."

As to the utility of improved, short-range predictions including those of precipitation and of useful long-range forecasting, it should be mentioned that the National Academy of Sciences – National Research Council[4] estimated the potential savings resulting therefrom as approaching 2½ billion dollars annually for the United States alone.

As Thompson[5, 6] and Novikov[7] have pointed out, and recognized again in the panel discussion on atmospheric predictability during the Golden Anniversary Meeting of the American Geophysical Union in April 1969, and also evident in Sections 4 through 7, herein, the initial state of the atmosphere is of considerable importance for long-range and sometimes even for short-range predictions in cases of rapid development. F.G. Shuman states: "We are aware of many weaknesses in our initialization procedures, principally the lack of the full meteorological balance between the winde and pressure field. This problem, however, is perhaps even more difficult than those encountered in developing the prediction model. The balance problem should properly be approached as part of the analysis problem. At NMC (National Meteorological Center) we are making a start on the problem, but it will be some time before we have a general solution to it. We expect to gain a great deal from other's work on the problem."[8]

After a critique of present initialization schemes, this report presents more powerful initialization methods in the form of optimal filter equations for numerical weather prediction which do not suffer from some essential weaknesses of the

[3]J.G. Charney: "Dynamic Forecasting by Numerical Process," *Compendium of Meteorology*, American Meteorological Society, Boston, Massachusetts, 1951.

[4]National Academy of Sciences – National Research Council: "Useful Applications of Earth-Oriented Satellites," *Meteorology*, National Academy of Sciences, Washington, D.C. 1969.

[5]P.D. Thompson: *Numerical Weather Analysis and Prediction.* The MacMillan Company, New York, 1961.

[6]P.D. Thompson: *Uncertainty of Initial State as a Factor in the Predictability of Large-Scale Atmospheric Flow Patterns*, Tellus, 1957.

[7]E.A. Novikov: "Toward a Formulation of the Problem of the Predictability of Synoptic Processes," U.S. Joint Publications Research Service, New York, August 1960.

[8]F.G. Shuman: "A Multi-Level Primitive Equation Model," U.S. Department of Commerce, Environmental Sciences Services Administration, Weather Bureau, National Meteorological Center, November 1965.

so-called balance equation, i.e., they are more accurate, applicable to relatively smaller scales, and suitable to include convection in case of the general nonhydrostatic filter process. The new filter equations, one of which simultaneously provides a diagnostic equation for an ultimate prognostic system, can also be profitably used for re-initialization and are fundamental pertaining to the inclusion of nonadiabatic processes through the use of statistical and parametric procedures. The general nonhydrostatic filter equations also offer an interesting and more satisfying approach as far as the use of numerical relaxation is concerned. Furthermore, the analysis throws some new light on the significance of initialization for long-range predictability, and the use of frictional terms and boundary conditions. Finally, the new equations offer advantages with reference to the computation of winds from worldwide data obtained through the use of advanced satellites.

2. Thompson's Generalized Filter Approximations, the Balance Equation, and Related Iteration Methods. Thompson[9] showed that it is necessary and sufficient to filter gravity waves out if the total time derivative of the divergence is omitted from the "divergence" equation obtained by applying the horizontal divergence operator on the equations of motion. This results in the diagnostic relationship

$$\nabla u \cdot \frac{\partial V}{\partial x} + \nabla v \cdot \frac{\partial V}{\partial y} - \vec{k} \cdot \nabla \times fV + \Delta^2 \Psi = 0 \qquad (1)$$

where ∇ is the horizontal nabla operator; V, the horizontal velocity vector; $\vec{k}$ a vector directed vertically upward; f, the Coriolis parameter; Δ^2, the 2-dimensional Laplace operator; and Ψ, the isentropic stream function in the $(x, y, \theta; t)$ – system where θ is the potential temperature. Equation (1) is valid for adiabatic, nonviscous flow and, of course, under the hydrostatic assumption.

Omission of $\frac{dD}{dt}$, with D as divergence in eq. (1), allows the establishment of the diagnostic ω-equation:

$$\frac{1}{\theta} \nabla^2 \frac{\omega}{\rho} - \frac{\partial}{\partial \theta}\left(K \frac{\partial \theta}{\partial p} \frac{\partial \omega}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\nabla \Psi \cdot \nabla^2 V\right) - \nabla^2\left(\frac{\partial V}{\partial \theta} \cdot \nabla \Psi\right)$$

$$+ 2\frac{\partial R}{\partial \theta} + \frac{\partial B}{\partial \theta} - \frac{\partial}{\partial \theta}\left(K \frac{\partial \theta}{\partial p} \frac{\partial V}{\partial \theta} \cdot \nabla p\right) . \qquad (2)$$

Here, it is $\omega = \frac{dp}{dt}$, ρ the density, and p the pressure, while K, R, and B are specific functions free of time derivatives. In order to obtain eq. (2), use has to be made of the vorticity equation and the continuity equation in isentropic coordinates.

[9] P. D. Thompson, "Numerical Weather Analysis. . . ."

Since, according to Helmholtz' theorem, the horizontal vector, V, may be expressed as the sum of an irrotational and a nondivergent vector so that

$$V = \nabla\chi + \vec{k} \times \nabla\psi ,$$

$$\text{curl } V = \vec{k}\,\Delta^2\psi \quad \text{and} \quad \text{div } V = \Delta^2\chi = \frac{\partial\theta}{\partial p}\left(\frac{\partial\omega}{\partial\theta} + \frac{\partial V}{\partial\theta}\cdot\nabla p\right) .$$

Under hydrostatically stable conditions, eq. (1) and (2) may thus be solved for ψ and χ. For prognostic purposes, either the vorticity equation together with eq. (1) or (2) or the primitive equations together with the continuity equation

$$\frac{d}{dt}\frac{\partial p}{\partial\theta} = -\frac{\partial p}{\partial\theta}\,\text{div } V$$

might be used. In the latter case, instabilities might occur under certain conditions.

A simplification of eq. (1) as well as the starting equation for an iterative process involving eq. (1) and (2) is the so-called balance equation

$$f\Delta^2\psi + 2\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + \nabla\psi\cdot\nabla f = \Delta^2\Psi \qquad (3)$$

which, according to Rellich's theorem[10] and as demonstrated by Bolin,[11] is of the elliptic type if

$$\frac{\Delta^2\Psi}{f} - \frac{1}{f}\nabla f\cdot\nabla\psi > -\frac{f}{2} . \qquad (4)$$

This condition is, according to Bolin,[11] also necessary to solve eq. (2) as an elliptic partial differential equation under the assumption of a hydrostatically stable atmosphere.

Apart from the fact that eq. (4) may not be fulfilled in the case of strong anticyclones, Bolin[11,12] has pointed out that the most serious approximation in eq. (1) is neglecting the time-dependent term $\frac{d}{dt}\text{div } V$ through which all gravity-inertia oscillations are eliminated and that the filtering approximation is only valid for $\text{div } V \ll \zeta = \Delta^2\psi$. As mentioned by Phillips,[13] fast moving and nongeostrophic gravity waves have large, horizontal divergences ($\text{div } V \sim \zeta$), and, during convection,

[10] F. Rellich: "Zur ersten Randwertaufgabe bei Monge-Ampereschen Differentialgleichungen vom elliptischen Typus; differentialgeometrische Anwendungen," *Math. Ann.*, Vol. 107, 1933.

[11] B. Bolin: "Numerical Forecasting with the Barotropic Model," *Tellus* VII, 1955, 1.

[12] B. Bolin: "An Improved Barotropic Model and Some Aspects of Using the Balance Equation for Three-dimensional Flow," *Tellus* VIII, 1956, 1.

[13] N. A. Phillips: *Numerical Weather Prediction, Advances in Computer Control (or Techniques)*, Vol. 1, Academic Press, New York, 1960.

$\text{div } V > \zeta$.

Although Cressman[14] and other authors attributed a considerable improvement in numerical forecasts to the use of the balance eq. (3), Elsaesser[15] has come to the conclusion that its linearized version appears to be the optimum, particularly in view of recovery of the pressure height (if Ψ is replaced by ϕ in the (x, y, p) – version of eq. (3)), less computation time and the necessity of elliptizing in case eq. (4) does not hold. Elsaesser explains the differences with respect to previous investigations to the near hemispheric scope of his study and the inclusion of large areas of strong anticyclonic shear equatorward of the polar jet stream.

Hinkelmann[16] has demonstrated that the stream function obtained through eq. (3) fails during strong anticyclonic activity, and a better result would be achieved by applying the geostrophic stream function. Monin[17] has emphasized that eq. (3) would be equivalent to an equation in which the quadratic terms would read $\frac{\omega}{f^2}(\phi_{xx}\,\phi_{yy} - \phi^2_{xy})$ as far as the scale theory is concerned. Arnason[18], who neglects their vertical advection terms in the equation of motion, arrives at similar and practically equivalent results.

A method of initialization using finite difference techniques and employing the balance condition $\frac{\partial}{\partial t}\,\text{div } V = 0$ has been developed by Miyakoda and Moyer[19]. It is simpler from the computational point of view than the classical method of solving the balance equation together with the ω-equation, and numerical results obtained through the use of a simplified model agree rather well except for a slight displacement of the vertical velocity pattern. Moreover, this method would be relatively more advantageous if the initial vertical velocity had an insignificant influence on the meteorological evolution. This is, however, only true in case of weak development and also due to the use of the balance equation (as will be shown later). Better filter equations are indeed necessary for forecasts exceeding 5 days after which, according to the experience gained by Mintz and Krishnamurti, as quoted by the authors, rather exact initial conditions become important.

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[14] G. P. Cressman: "A Study of Numerical Forecasting Errors," *Monthly Weather Review*, July 1957.

[15] H. W. Elsaesser: "Comparative Test of Wind Laws for Numerical Weather Prediction," *Monthly Weather Review*, Vol. 92, No. 5, 1968.

[16] K. Hinkelmann: "Ueber die Einbeziehung divergenter Windkomponenten in Vorhersagemodelle," *Berichte des deutschen Wetterdienstes* Nr. 38 (Band 5) Bad Kissingen, 1957.

[17] A. S. Monin: "Izmenenila davlenila v baroklinnoi atmosfere Izv. Akad. Nauk," *ser. geofiz.*, no. 4, 1958.

[18] G. Arnason: "Higher-Order Geostrophic Wind Approximations, "*Monthly Weather Review*, Vol. 90, No. 5, May 1962.

[19] K. Miyakoda and R. W. Moyer: "A Method of Initialization of Dynamical Weather Forecasting," Geophysical Fluid Dynamics Laboratory, Environmental Sciences Services Administration, Washington, D.C., October 1966.

According to Phillips,[20] noise, through an initial divergence implied by the usual geostrophic theory, is greatly suppressed if the initial data is used for the primitive equations. However, the suppression of high-frequency, gravity-inertia waves will only be very effective if the motion is not strongly ageostrophic: "The extent to which the method will prove useful in low latitudes, or in high latitudes when div V approaches f in magnitude, can perhaps be answered only by experiment."[20]

A method (similar to the method of Miyakoda and Moyer) by Nitta and Hovermale[21] involves an actual iteration of forward and backward forecasts around the initial time with the Euler-backward time difference and yields acceptable rotational but unsatisfactory divergent wind components. According to Nitta and Hovermale, "the question remains unanswered whether or not the technique in its present design is accurate enough to be of practical value.... Perhaps, further diagnosis ... will lend some insight in this direction."

This diagnosis is readily available since the method corresponds to the filter conditions

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{t_0=0} = \left(\frac{\partial^2 v}{\partial t^2}\right)_{t_0=0} = 0 \ .$$

However, this filtering leaves the factorial term $\frac{\partial \omega}{\partial t}$ in the equations of motion undetermined so that in addition $\frac{\partial \omega}{\partial t} = 0$ has to be postulated which is essentially equivalent to $\left(\frac{\partial}{\partial t}\ \frac{\partial \omega}{\partial p}\right)_{t_0=0} = 0$ and, thus, slightly less satisfactory than the balance equation together with the corresponding ω-equation.

The ω-equation corresponding to the balance equation has been investigated by many authors. Krishnamurti[22] uses the vorticity equation including frictional terms together with the balance equation and the ω-equation in the (x, y, p, t)-system for diagnostic studies of weather systems with a term for latent heat included. Vukovich and Chow[23] employ the balance ω-equation in order to estimate the effect of long-term diabatic heating. Considerable experience in this regard has also been gained by Smagorinsky and collaborators. (Some of their results will be discussed in Section 5.)

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[20] N. A. Phillips: "On the Problem of Initial Data for the Primitive Equations," *Tellus*, Vol. 12, May 1960.

[21] T. Nitta, and J. B. Hovermale: "On Analysis and Initialization for the Primitive Forecast Equations," *Technical Memorandum WBTM NMC–42*, National Meteorological Center, Suitland, Maryland, October 1967.

[22] T. N. Krishnamurti: "Diagnostic Studies of Weather Systems of Low and High Latitudes (Rossby Number 1)," Department of Meteorology, University of California, Los Angeles, California, 30 Nov. 1964 – 30 Nov. 1966.

[23] F. M. Vukovich and Chi F. Chow: "Research in Numerical, Dynamical, and Operational Weather Forecast Procedures," Research Triangle Institute, Research Triangle Park, North Carolina, April 1968.

In conclusion, it should be emphasized that the shortcomings of the balance equation, i.e., $\left(\frac{d}{dt}\,\mathrm{div}\,V\right)_{t_o=0} = 0$ and $\Delta^2\phi + \frac{f^2}{2} > 0$ where ϕ denotes the geopotential, are necessarily reflected in the more explicitly named balance ω-equation. Superior filter equations which do not suffer under these restrictions will simultaneously result in an overall improvement of the initialization process and, thus, in better short- and long-range forecasts, as well as in the elimination of high-frequency meteorological noise and in stabilization.

3. **Fjortoft's Filter Equations.** The application of the filter conditions

$$\left(\frac{d^2u}{dt^2}\right)_{t_o=0} = \left(\frac{d^2v}{dt^2}\right)_{t_o=0} = 0 \tag{5}$$

with reference to the equations of motion in the (x, y, p, t)-system by Fjortoft[24] leads to two prognostic filter equations with the local time derivatives $\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x}$ and $\frac{\partial}{\partial t}\frac{\partial\phi}{\partial y}$ in which ϕ designates the geopotential. Together with the adiabatic thermodynamic equation involving $\frac{\partial}{\partial t}\frac{\partial\phi}{\partial p}$ and the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} = 0$ where $\omega = \frac{dp}{dt}$, these are four equations for the unknowns u, v, ω, and $\frac{\partial\phi}{\partial t}$. As the ellipticity criterion under which a solution with a suitable relaxation factor σ of the same sign can be achieved, Fjortoft lists the conditions

$$\left(f^2 + \phi_{xx}\right)\left(f^2 + \phi_{yy}\right) - \phi_{xy}^2 + \left(2f^2 + \Delta^2\phi\right)\left(\phi_{pp} - F\phi_p\right) - \left(\nabla\phi_p\right)^2 > 0 \tag{6}$$

where $F = \frac{R}{100}\,\frac{\partial}{\partial p}\left(\frac{p}{100}\right)^{\frac{R}{c_p}-1}$ and

$$\begin{vmatrix} f^2 + \phi_{xx} & \phi_{xy} & \phi_{xp} \\ \phi_{xy} & f^2 + \phi_{yy} & \phi_{yp} \\ \phi_{xp} & \phi_{yp} & \phi_{pp} - F\phi_p \end{vmatrix} > 0\,. \tag{7}$$

Approximately, the conditions may be stated as $f^2 + \Delta^2\phi > 0$ which means a considerable improvement in the anticyclonic case when compared with $\frac{f^2}{2} + \Delta^2\phi > 0$ applicable in

[24] R. Fjortoft: "A Numerical Method of Solving Certain Partial Differential Equations of Second Order," Fremlagt i Videnskaps-Mote, February 1962.

connection with the balance equation, and $\phi_{pp} - F\phi_p > 0$ or $\sigma > 0$ with σ as the static stability.

Because Fjortoft has not developed superior diagnostic filter equations based on the same filter conditions, his method does not appear to have been practically utilized.

As to the application of a relaxation factor of changing sign in the hyperbolic case, no useful results may be obtained since the simple model of the continuity equation does not, or not sufficiently, hold and should, therefore, not be adapted to the relatively more sophisticated thermodynamic equation. Apart from the fact that no accurate results can be expected, particularly in the case of pronounced hyperbolicity, the convergence of the relaxation process has been extremely slow according to Elsaesser.[25]

4. Optimal Filter Equations in the (x, y, p, t)-System. For the sake of simplicity, we omit in the following paragraph the use of map scale factors as has been done by Fjortoft. Inclusion of these factors does not, however, present any fundamental difficulties. Moist-adiabatic and frictional terms are also neglected at present. Thus, we proceed from the following system of equations which is familiar to the reader:

$$\frac{du}{dt} = -g\frac{\partial z}{\partial x} + fv \tag{8}$$

$$\frac{dv}{dt} = -g\frac{\partial z}{\partial y} - fu \tag{9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{10}$$

$$\frac{dT}{dt} - \frac{RT}{c_p p}\omega = 0 \tag{11}$$

or alternatively, with $\frac{\partial z}{\partial p} = \frac{1}{g\rho} = \frac{RT}{gp}$ and $\sigma = \frac{\partial^2 z}{\partial p^2} + \frac{1}{\chi p}\frac{\partial z}{\partial p}$

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial p} + u\frac{\partial}{\partial x}\frac{\partial z}{\partial p} + v\frac{\partial}{\partial y}\frac{\partial z}{\partial p} + \sigma\omega = 0\,. \tag{12}$$

After applying the filter operations of eq. (5), we arrive at

$$g\frac{d}{dt}\frac{\partial z}{\partial x} - \frac{d}{dt}(fv) = 0$$

[25] H. W. Ellsaesser.

$$g\frac{d}{dt}\frac{\partial z}{\partial y} + \frac{d}{dt}(fu) = 0$$

and more explicitly at

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial x} + u\frac{\partial}{\partial x}\frac{\partial z}{\partial x} + v\frac{\partial}{\partial y}\frac{\partial z}{\partial x} + \omega\frac{\partial}{\partial p}\frac{\partial z}{\partial x} - \frac{v}{g}\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right)$$

$$+\frac{f}{g}\left(g\frac{\partial z}{\partial y} + fu\right) = 0 \qquad (13)$$

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial y} + u\frac{\partial}{\partial x}\frac{\partial z}{\partial y} + v\frac{\partial}{\partial y}\frac{\partial z}{\partial y} + \omega\frac{\partial}{\partial p}\frac{\partial z}{\partial y} + \frac{u}{g}\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right)$$

$$+\frac{f}{g}\left(-g\frac{\partial z}{\partial x} + fv\right) = 0. \qquad (14)$$

In order to make the two filter equations (13) and (14) consistent, we eliminate the time derivatives by applying the operator $\frac{\partial}{\partial y}$ on eq. (13) and $\frac{\partial}{\partial x}$ on eq. 14 and subtracting. This yields, after omitting irrelevant terms, the diagnostic filter equation

$$\frac{f^2}{g}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y}\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial y^2}$$

$$+\frac{\partial \omega}{\partial x}\frac{\partial^2 z}{\partial y \partial p} - \frac{\partial \omega}{\partial y}\frac{\partial^2 z}{\partial x \partial p} -$$

$$-\frac{\partial f}{\partial y}\left(\frac{\partial z}{\partial y} + \frac{2f}{g}u\right) = f\Delta^2 z. \qquad (15)$$

After we differentiate eq. (13) and eq. (14) pertaining to x and y respectively and add the results, we obtain as a corresponding dynamic filter equation

$$\frac{d}{dt}\Delta^2 z = \left(\frac{f^2}{g} + \Delta^2 z\right)\frac{\partial \omega}{\partial p} - \left(\frac{\partial \omega}{\partial x}\frac{\partial^2 z}{\partial x \partial p} + \frac{\partial \omega}{\partial y}\frac{\partial^2 z}{\partial y \partial p}\right) +$$

$$+\frac{\partial v}{\partial y}\frac{\partial^2 z}{\partial x^2} - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial y^2} +$$

$$+\frac{\partial f}{\partial y}\left(\frac{\partial z}{\partial x} - \frac{2f}{g}v\right). \qquad (16)$$

We arrive at a diagnostic ω-equation by applying the 2-dimensional Laplace operator on eq. (12), differentiating eq. (16) with respect to p, and eliminating $\frac{\partial}{\partial t}\Delta^2\frac{\partial z}{\partial p}$:

$$\sigma\Delta^2\omega + \left(\frac{f^2}{g} + \Delta^2 z\right)\frac{\partial^2\omega}{\partial p^2} - \frac{\partial^2 z}{\partial x\partial p}\frac{\partial^2\omega}{\partial x\partial p} - \frac{\partial^2 z}{\partial y\partial p}\frac{\partial^2\omega}{\partial yp}$$

$$+ \left(2\frac{\partial\sigma}{\partial x} - \frac{\partial^3 z}{\partial x\partial p^2}\right)\frac{\partial\omega}{\partial x} + \left(2\frac{\partial\sigma}{\partial y} - \frac{\partial^3 z}{\partial y\partial p^2}\right)\frac{\partial\omega}{\partial y} + \left(\Delta^2\sigma - \frac{\partial^2\Delta^2 z}{\partial p^2}\right)\omega$$

$$= \frac{\partial}{\partial p}\left(u\frac{\partial\Delta^2 z}{\partial x} + v\frac{\partial\Delta^2 z}{\partial y}\right) - \Delta^2\left(u\frac{\partial^2 z}{\partial x\partial p} + v\frac{\partial^2 z}{\partial y\partial p}\right)$$

$$- \frac{\partial}{\partial p}\left[\frac{\partial v}{\partial y}\frac{\partial^2 z}{\partial x^2} - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial y^2}\right] - \frac{\partial M}{\partial p} \qquad (17)$$

where M represents the last term of eq. (16).

It is well known that diagnostic filter equations are superior to their prognostic counterparts, i.e., conditions such as

$$\frac{du}{dt} = \frac{dv}{dt} = 0; \frac{dw}{dt} = 0; \frac{d}{dt}\,\mathrm{div}\,V = 0.$$

The solution of eq. (15) and (16) presupposes, of course,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Delta^2\psi, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial\omega}{\partial p} = \Delta^2\chi,$$

$$u_1 = -\frac{\partial\psi}{\partial y},\ v_1 = \frac{\partial\psi}{\partial x},\ u_2 = \frac{\partial\chi}{\partial x},\ v_2 = \frac{\partial\chi}{\partial y}\ .$$

In accordance with predominantly horizontal flow and required separability, we obtain a first approximative determination of ψ_1 and thus of u_1 and v_1 from

$$\frac{f^2}{g}\Delta^2\psi + \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{2\partial^2\psi}{\partial x\partial y}\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2}$$

$$+ \frac{\partial f}{\partial y}\left[\frac{2f}{g}\frac{\partial\psi}{\partial y} - \frac{\partial z}{\partial y}\right] = f\Delta^2 z\ . \qquad (18)$$

The condition that eq. (18) and eq. (15) are elliptic, the latter within an iteration scheme, is

$$\left(\frac{f^2}{g} + \frac{\partial^2 z}{\partial x^2}\right)\left(\frac{f^2}{g} + \frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0 \qquad (19)$$

or approximately $\frac{f^2}{g} + \Delta^2 z > 0$ which is considerably less restrictive than $\frac{f^2}{g} + \frac{\Delta^2 z}{2} > 0$ of the balance eq. (1).

The ellipticity condition for the new ω-equation (17) is that the quadratic form

$$\begin{vmatrix} \sigma & 0 & -\frac{1}{2}\frac{\partial^2 z}{\partial x \partial p} \\ 0 & \sigma & -\frac{1}{2}\frac{\partial^2 z}{\partial y \partial p} \\ -\frac{1}{2}\frac{\partial^2 z}{\partial x \partial p} & -\frac{1}{2}\frac{\partial^2 z}{\partial y \partial p} & \frac{f^2}{g} + \Delta^2 z \end{vmatrix} \qquad (20)$$

be positive definite which results in the inequalities

$$\sigma > 0 \text{ and } \sigma\left(\frac{f^2}{g} + \Delta^2 z\right) - \frac{1}{4}\left[\left(\frac{\partial^2 z}{\partial x \partial p}\right)^2 + \left(\frac{\partial^2 z}{\partial y \partial p}\right)^2\right] > 0 \,. \qquad (21)$$

In order to insure ellipticity in the cyclonic case if $\sigma > 0$ and the inequality (21) does not hold a priori, σ has to be adjusted to a slightly positive value σ_c so that the relation

$$\sigma_c\left(\frac{f^2}{g} + \Delta^2 z\right) - \frac{1}{4}\left[\left(\frac{\partial^2 z}{\partial x \partial p}\right)^2 + \left(\frac{\partial^2 z}{\partial y \partial p}\right)^2\right] = \epsilon_1 > 0 \qquad (22)$$

with ϵ_1 as a small quantity, is fulfilled which is essentially in agreement with procedures applied by Smagorinsky[26] and Krishnamurti.[27] In anticyclonic conditions, $\frac{f^2}{g} + \Delta^2 z < 0$, the height field has to be smoothed so that

$$\frac{f^2}{g} + \Delta^2 z_A = \epsilon_2 > 0 \qquad (23)$$

with ϵ_2 as a small quantity. The smoothing results simultaneously in more favorable

[26] J. Smagorinsky: "On the Inclusion of Moist Adiabatic Processes in Numerical Prediction Models," *Berichte des Deutschen Wetterdienstes* Nr. 38 (Band 5) Bad Kissingen, 1957.

[27] T. N. Krishnamurti.

values σ_A, $\frac{\partial^2 z_A}{\partial x \partial p}$, and $\frac{\partial^2 z_A}{\partial y \partial p}$ so that

$$\sigma_A\left(\frac{f^2}{g} + \Delta^2 z_A\right) - \frac{1}{4}\left[\left(\frac{\partial^2 z_A}{\partial x \partial p}\right)^2 + \left(\frac{\partial^2 z_A}{\partial y \partial p}\right)^2\right] > \epsilon_3 \approx 0 \qquad (24)$$

reasonably holds. Alternatively, in the anticyclonic case, terms involving products of second-order derivatives in eq. (17) and (18) are to be neglected in a numerical integration by over-relaxation for all points for which the inequalities (21) and (19), respectively, are not valid. It should be pointed out again that the inequalities (22) through (24) are necessary in case of initial hyperbolicity in order to adapt ω in eq. (12) to be the greater scale of the horizontal divergence of the filter equation (16) which is rather a smoothed $\frac{\partial \omega}{\partial p}$. Hyperbolicity can only be eliminated or is not present, respectively, if the hydrostatic filter condition is not utilized for the derivation of a continuity equation such as (10), i.e., if w and $\frac{dp}{dt}$ are maintained.

Lateral boundary conditions for eq. (18) are $\psi_L = \frac{g}{f} z_L$ both for a physical boundary and that of a hemispheric grid. For the ω-equation, approximate lateral boundary values ω_L may be determined by applying the regression eq. $\Delta^2 \omega_L = -\lambda(x, y, p)\,\omega_L$ to eq. (17) so that an ordinary differential equation results (mentioned by Reuter[28] as well as by Eliassen and Kleinschmidt[29] and generally preferable to Charney's advective model[30]). In this respect, eq. (17) may be further simplified to be consistent with geostrophic approximations. As usual, $\omega_o = 0$ at the top of the atmosphere, and $\omega_{p_o} \approx -\rho V \cdot \nabla_h \phi_s$ with $\phi_s(x, y)$ as the geopotential of the ground.

It is characteristic of the (x, y, p, t)-system that, initally, u and v are required only at the lower boundary for the computation of ω_{p_o} and that u_2 and v_2 can be easily determined from the Poisson eq. $\Delta^2 \chi = -\frac{\partial \omega}{\partial p}$, to be solved at each p-level. Due to the present insufficiency of observations and restriction to relatively few levels, the emphasis has been essentially on a rather accurate solution $\psi^{(1)}$ of eq. (18) which may be followed by a determination of $\omega^{(1)}$ from eq. (17) by numerical relaxation.

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[28] H. Reuter: "Zum gegenwaertigen Stand der numerischen Wettervorhersage," *Osterreichisches Ingenieur-Archiv*, Band X, Heft 2-3, Springer-Verlage Wien, 1956.

[29] A. Eliassen and E. Kleinschmidt: "Dynamic Meteorology," *Handbuch der Physik*, Bd. XLVIII, Springer Verlag, Berlin, Goettingen. Heidelberg, 1957.

[30] J. G. Charney.

Alternatively, ψ and ω may be determined in conjunction, and the problem of finding a suitable over-relaxation coefficient along the lines of Stuart and O'Neill[31] and O'Brien[32] would practically be confined to the ω-equation. This numerical process is somewhat complicated by the necessity of finding the divergent wind components intermediately. Fortunately, the ψ- and ω-equations are both essentially linear so that initial guess and convergence are not critical.

The inclusion of linear frictional terms in the filter equations does not present difficulties. Nonlinear terms would, however, cause computational instability and would also be incompatible with the requirement of strong smoothing in case of convection since the isobaric divergence equation cannot cope with situations involving effective static stabilities $H_2 < 0$. In these situations, a convective temperature adjustment such as that developed by S. Manabe[33] has to be made. All these difficulties and/or adjustments can be avoided by a general nonhydrostatic system of filter equations developed in Sections 6 and 7.

5. Ramifications of Optimal Hydrostatic Filter Equations. The advantages of eqs. (15) and (17) in relation to Thompson's equations (1) and (2) are their greater versatility in case of strong anticyclonic movements, a better determination of winds including the elimination of the weakness of the balance equation with respect to anticyclonic winds ascertained by Elsaesser,[34] and the agreement with Hollman's analysis[35] regarding the failure of the balance equation in strong anticyclonic situations and the use of the geostrophic instead of the gradient wind. Fjortoft's filter conditions (5), as well as ours, involve, of course, the second total time derivative so that standing gravity-inertia waves of the form $\zeta''=A \sin kx e^{\pm mz} \sin \nu t$ and $\zeta'' = A \sin kx \sin mz \sin \nu t$ are preserved and $\left(\frac{d}{dt} \text{div } V\right)_{t_o=0} \neq 0$. Scale inconsistencies and adjustment difficulties inherent in Fjortoft's method are, however, avoided through the establishment of diagnostic equations and the corresponding imposition of Helmholtz' theorem.

Apart from a better determination of the stream function ψ in the cyclonic case as well, the ω-equation (17) also permits the computation of stronger divergences/convergences and vertical velocities because $\frac{g}{f}\Delta^2 z > \Delta^2 \psi$ so that the vertical influence increases relatively to the horizontal influence in a solution involving

[31] D. W. Stuart and T. H. R. O'Neill: "The Over-Relaxation Factor in the Numerical Solution of the Omega Equation," *Monthly Weather Review*, Vol 95, No. 5, May 1967.

[32] J. J. O'Brien: "The Over-Relaxation Factor in the Numerical Solution of the Omega Equation," *Monthly Weather Review*, Vol 96, No. 2, February 1968.

[33] S. Manabe: "Simulated Climatology of a General Circulation Model with a Hydrologic Cycle," *Monthly Weather Review*, Vol 93, No. 12, December 1965.

[34] H. W. Elsaesser.

[35] G. Hollmann: "Ueber die Grenzen der geostrophischen Approximation und die Einfuehrung nichtgeostrophischer Methoden," *Berichte des Deutschen Wetterdienstes*, Nr. 38 (Band 5), Bad Kissingen, 1957.

a Green's function. A practical demonstration of this has been given by Smagorinsky[36] who utilized the geostrophic vorticity equation including the full term $\left(\frac{f^2}{g} + \Delta^2 z\right)\frac{\partial \omega}{\partial p}$ and, thus, subconsciously an expression approaching equation (16), with resulting vertical velocities on the order of 30 cm sec^{-1}.

The importance of the new filter equations becomes particularly evident if we develop the absolute vorticity in isentropic coordinates in a Taylor series which yields, in view of

$$\frac{d}{dt}\left(\zeta_\theta + f\right) = -\left(\zeta_\theta + f\right) \mathrm{div}_\theta V \tag{25}$$

$$\left(\zeta_\theta + f\right)_t = \left(\zeta_\theta + f\right)_{t_o}\left\{1 - \mathrm{div}\, V_{t_o}\, \Delta t + \frac{1}{2}\left[(\mathrm{div}\, V)^2 - \frac{d}{dt}\mathrm{div}\, V\right]_{t_o} \Delta t^2 + \ldots.\right\} \tag{26}$$

Since $\frac{d}{dt}\mathrm{div}\, V = 0$, within the context of the balance equation and of the order $(\mathrm{div}\, V)^2$, the quadratic term of eq. (26) approaches the linear term in moderate latitudes and under average conditions in about 7 days. The replacement of the balance equation as an initialization method which requires a moderate reprogramming effort becomes thus imperative in multi-level numerical weather prediction.

According to Miyakoda et al,[37] the movement of cyclones and the tendency for deepening or filling are the major problems in short-range forecasts, i.e., for 1 to 2 days. In a 2-week forecast, the life histories of cyclones are also important features of the prediction. This is particularly reflected in eq. (26) although, for medium- to long-range forecasts, there is a partial adjustment for the lack of an initial $\frac{d}{dt}\mathrm{div}\, V = 0$ because of the interaction of the primitive equations, the continuity, and the thermodynamic equation. It is significant and in correspondence with the above-stated conclusions that in the experimental predictions described by Miyakoda et al the intensities of the highs and lows weakened appreciably after 6 or 8 days reflecting the fact that the forecast of eddy kinetic energy was less than the observed. The wiggling in the pattern of geopotential height becomes more pronounced with increasing computation time, and the lack of development of a certain cyclone on the

[36] J. Smagorinsky: "On the Inclusion of Moist Adiabatic Processes in Numerical Prediction Models."

[37] K. Miyakoda et al: "Experimental Extended Predictions with a Nine-Level Hemispheric Model," *Monthly Weather Review*, Vol 97, No. 1, Jan 1969.

2d and 3d days along the middle Pacific polar frontal zone may also be due to inadequacies of initial data or initialization by the balance and associated balance ω-equation. The quasi-stationary modes, or long waves, are more dominant; while the eastward-moving components, the relatively shorter waves, are too small in amplitude which is in agreement with Elsaesser's findings. Significantly, the vertical velocities calculated by the balance ω-equation are weaker than those taken from the prediction computation based on the time-dependent primitive equations.

In Sections 6 and 7, it becomes apparent that the filter conditions expressed by eq. (5) are completely adequate in the hydrostatic system, i.e., that more refined smoothing techniques utilizing measured winds are not required or are inconsistent with the hydrostatic system respectively. Employing the hydrostatic continuity equation amounts indeed implicitly to a smoothing process and a restriction of scale several times the height of a homogeneous atmosphere so that the magnitude of the horizontal divergence does not exceed that of the vertical vorticity. Diffusion terms, such as described by Smagorinsky et al,[38] are thus not due to a lack of resolution inherent in the filter equations. They are rather, as stated by the authors, a consequence of the grid size used which, if it exceeds 30 km significantly, amounts to an additional smoothing of "signal" functions obtainable by our filter equations.

Finally, it should be mentioned that the inequalities (19) and (21), which are consistent with the existence of the continuity equation (10), are a generalization of and in basic agreement with the results obtained by Van Mieghem,[39] that our filter equations are sufficient as far as the process of adaptation described by Yeh Tu-Cheng and Li Mai-Tsun[40] is concerned, and that the sigma-coordinate system developed by Phillips[41] is also subject to the restrictions of the hydrostatic system as soon as the effective static stability H_2 becomes negative.

6. On the Modification of and Initial Conditions for the Differential Equations of Meteorology and Related Problems. The fundamental problem is to determine commensurable smooth fields of u, v and w under consideration of the fact that w cannot be measured, generally speaking. Since the filtering process must be both consistent for all three wind components and not arbitrarily independent of T, p, and ρ, it must be adaptive in nature. Necessary and sufficient conditions for an optimal filtering process are, therefore, a wind *vector* filter operation, complete adapatability with respect to the continuity and thermodynamic equation, and computability, i.e., existence of a unique solution. The quasi-ergodic filter equation

[38] J. Smagorinsky et al: "Numerical Results from a Nine-Level General Circulation Model of the Atmosphere," *Monthly Weather Review*, Vol 93, No. 12, December 1965.

[39] J. M. Van Mieghem: "Hydrodynamic Instability," *Compendium of Meteorology*, American Meteorological Society, Boston, Massachusetts, 1951.

[40] Y. Tu-Cheng and L. Mai-Tsun: "Adaptation Between the Pressure and the Wind Field in Meso- and Small-Scale Motions," AFCRL-68-0275, Air Force Cambridge Research Laboratories, Bedford, Massachusetts, May 1968.

[41] N. A. Phillips: "A Coordinate System Having Some Special Advantages for Numerical Forecasting," *Journal of Meteorology*, Vol 14, Oct., 1956.

$$\frac{d^2 \hat{V}}{dt^2} = 0 \tag{27}$$

which yields three filter conditions involving smooth variables $\hat{u}$, $\hat{v}$, and $\hat{w}$ fulfills the requirements mentioned above which is shown in detail in Section 7. It has to be kept in mind that $\frac{d\hat{V}}{dt}$ is to be expected to be conserved for an infinitesimal part of the trajectory only. In other words: The resulting three diagnostic filter equations are more accurate than eq. (27) implies.

Inspection of the filter conditions and diagnostic filter equations reveals that transitory sound and shear gravity waves are effectively smoothed out. Typical values for the wavelength of pressure waves are 100 m to 1500 m, and shear gravity wave lengths are of the order 1000 m. The existence of these waves accounts for the quasi-isotropic spectra of atmospheric turbulence. According to Charles[42], turbulence due to shear in the vicinity of the ground exhibits frequencies above 2½ cycles/second and is termed "mechanical turbulence" in contrast to "convection turbulence" which is due to hydrostatic instability and typically occurs at frequencies of about ½ cycle/minute. Since the analysis of shear gravity or Helmholtz waves presupposes the existence of wind discontinuities which are filtered out through the use of pressure and temperature fields, a wavelength of 5 km may be considered as the lower limit pertaining to the applicability of the new filter equations. In view of the fact that Anderson[43] selected a 300 m height interval because it was found to be the minimum interval over which meaningful values of both vertical wind shear and lapse rate could be derived from rawinsonde data, minimum grid distances for numerical forecasts would amount to 300 m – 400 m in the vertical and about 2000 m in the horizontal. Isolated pressure and temperature discontinuities incompatible with such three-dimensional grid would have to be smoothed out initially. Indeed, application of the second-order filter condition

$$\frac{d^2 \hat{w}}{dt^2} = 0 \tag{28}$$

and, therefore, abandonment of the hydrostatic assumption is a priori to be expected to reduce the approximate minimum hydrostatic wave length of 40 km (five times the height of a homogeneous atmosphere) by one order of magnitude. In agreement with Pai[44] we can thus conclude that the new filter equations include essentially the vorticity and entropy modes of the hydrodynamic equations, but not the transitory

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[42]B. N. Charles: "A Summary of Available Information on the Vertical Motions of Air in the Troposphere," Sandia Corporation Working Paper TID-4500, Office of Technical Services, Department of Commerce, Washington, D. C., July, 1959.

[43]A. D. Anderson: "Free Air Turbulence," Journ. of Meteorology, Vol. 14, No. 6, 1957.

[44]Shih-I Pai: "Viscous Flow Theory," Vol. II, D. Van Nostrand Company, Princeton, N.J., 1957.

sound and associated shear gravity modes. In fact, in the new filter equations the Coriolis terms are not to be neglected which makes them amenable to meso-scale phenomena. Only small scale phenomena such as individual clouds and tornadoes cannot be incorporated.

There are two possibilities as to the use of filtered wind components $\hat{u}, \hat{v}, \hat{w}$ and corresponding smooth pressure and temperature fields: Either these fields are used as quasi-exact initial values together with a very small grid size, horizontally and vertically, or they would be considered as approximate initial values in conjunction with a greater grid size. In the first case, the available input information would be utilized to its maximum extent, and commensurable smaller scale "noise" would be generated. In the second case, scale-preserving operators would be required, either external ones such as a two-dimensional one mentioned by Shuman[45], or internal ones of the diffusion type. Whether a small-scale integration can be achieved, depends, of course, also on the timely availability of boundary values. We do not adhere to equations involving the familiar Reynolds stresses such as

$$\overline{\frac{\partial \hat{u}}{\partial t}} + \overline{\frac{\partial u}{\partial t}} + \overline{\hat{u}\frac{\partial \hat{u}}{\partial x}} + \overline{\hat{u}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}} + \cdots = -\overline{\frac{1}{\zeta}\frac{\partial p}{\partial x}} + f\hat{v} \qquad (29)$$

since the bars lose their meaning in the absence of an ergodic theorem which has been emphasized by Kampé de Fériet.[46] The statistical operation shown above leads, of course, not to a new prognostic equation. For example, the time average $\overline{\frac{\partial u}{\partial t}}$ in general does not vanish sufficiently for small intervals. Stable statistical averages cannot even be obtained over longer time intervals since, as is well-known, the meteorological generation process is not a stationary one. In fact, only a "potential vorticity" of the form

$$f + \Delta^2 \psi_B \; f_c \frac{\partial}{\partial p}\left[P(p)\frac{\partial \psi_B}{\partial p}\right]$$

with P as a measure of standard static stability and ψ_B as the stream function obtained through the classical balance equation would be sufficiently conserved. For this reason, it would be more profitable in linear statistical forecasting to employ ψ- instead of ϕ-statistics. The use of bars is evidently only strictly possible in the case of homogeneous and isotropic turbulence whence the wind-pressure correlation function vanishes. In addition, incompressibility is assumed. Under these conditions, the Kármán-Howarth equation

[45] F. G. Shuman: "Numerical Methods in Weather Prediction: II Smoothing and Filtering," Monthly Weather Review, Vol. 85, No. 11.

[46] J. Kampé de Fériet: *Atmospheric Diffusion and Air Pollution*, Academic Press, New York, London, 1959 (Discussion, p. 439).

$$\frac{\partial \overline{u^2} f}{\partial t} + 2(\overline{u^2})^{3/2}\left(h' + 4\frac{h}{r}\right) = 2\nu u^2 (f'' + 4f') \tag{30}$$

involves only wind correlations. In view of the above it must be stated that fundamental statistical assumptions associated with the derivation of Reynolds stresses do not hold for the differential equations of meteorology. The stresses indicate only that the generation model is not invariant under a smoothing operation and are only of use in connection with structural and analogy considerations. As soon as the link between the smoothed and perturbation fields is assumed to be negligible, i.e., the generation of subgrid noise is to be prevented, diffusion terms have to be added or filters have to be applied for specified time intervals.

In view of an adaptive, i.e., coherent smoothing process for all variables, the horizontal exchange coefficient, or rather function K, should be the same for $\hat{u}$, $\hat{v}$, and $\hat{T}$. The diagnostic filter equation for $\hat{w}$ eliminates the need of diffusion terms for this variable and identically fulfills the equation of continuity so that diffusion of mass does not occur. According to Obuchow[47] this leads to

$$K = k_1 D_o^{1/3} \ell_3^{4/3} \tag{31}$$

with k_1 as a dimensionless factor of the order of one, ℓ_3 as an observation measure, and D_o as proportional to the squared deformation of the velocity field

$$D_3^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \tag{32}$$

Equation (32) is only an approximation since it has been derived under the assumption of homogeneous turbulence and incompressibility, whence div V = 0. It holds, however, well in its two-dimensional form in the hydrostatic systems, whence $K = k_2 D_2^{1/2} \ell_2^2$. In this form it has been utilized by Smagorinsky et al.[48] The $\ell^{4/3}$-relation first established by Richardson has, therefore, to be considered questionable for regions of convective activity. This has been ascertained by Lettau[49] and is due to strong concurring vertical divergence. Again, only because of the availability of the diagnostic and simultaneously optimal $\hat{w}$-filter equation (63) of Section 7, eqs (31) and (32) in their two-dimensional form can be expected to hold

[47] A. M. Obuchow: "Ueber die Energieverteilung im Spektrum einer turbulenten Stroemung," 1941, published in *Statistiche Theorie der Turbulenz*, Akademie-Verlag, Berlin 1958.

[48] J. Smagorinsky, et al: "Nine-Level General Circulation Model of the Atmosphere."

[49] H. Lettau: *Atmospheric Diffusion and Air Pollution*, Academic Press, New York, London, 1959 (Discussion, p. 116).

reasonably. In case of grid-scale diffusion involving frequencies which are small in comparison with the upper limit of the spectrum, $D_o \approx$ const. According to Obuchow[50] and Richardson eq. (31) reduces then to $k_2 f(\ell)$. This simple expression appears to apply satisfactorily to geostrophic conditions only. As to the formulation of suitable vertical diffusion terms, no attempt will be made here.

The effects of ground friction are not considered in the derivation of the filtering equations carried out in Section 7. Introduction of surface stresses of the form

$$\tau_x = C_D \rho u \sqrt{u^2 + v^2} \quad , \quad \tau_y = C_D \rho v \sqrt{u^2 + v^2} \tag{33}$$

lead, however, to endogeneous nonlinearities with associated non-convergence of the numerical relaxation process. Apparently, quadratic friction terms include small-scale phenomena, such as external sound and shear gravity waves, and are also not strictly compatible with a vanishing vertical velocity at a horizontal surface. In fact, since the variable velocity vectors $\vec{v}(t)$ at the upper and lower boundaries are not available and the vertical velocity at the lower boundary is a function of an approximate horizontal velocity close to the earth's surface and the geopotential of the ground, scales below about 6 km have to be excluded a priori. With a linear frictional term introduced by Guldberg and Mohn and proposed as suitable by Phillips[51] the first equation of motion would read as

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv - ku \tag{34}$$

More explicitly, $k = a(x, y)\, k_2(z)$ though k_2 might be considered constant in a sufficiently deep layer adjacent to the ground. Application of terms (33) in connection with a grid of small length would only lead to the generation of undesirable noise. Terms of the form ku and kv can be fully absorbed in the filtering equations derived in Section 7 although they have been omitted for the purpose of simplicity. Since motion in the friction layer is such that there is horizontal convergence of mass in areas of cyclonic activity and vice versa, as stressed by Eliassen and Kleinschmidt,[52] these terms ought to be included in practical computations.

The incorporation of suitable lower and upper boundaries in our initialization scheme is discussed in Section 7. Pertaining to the whole globe, the problem of lateral boundary conditions does, of course, not present itself. In a quasi-hemispheric grid system, the horizontal wind components have to be assumed to be geostrophic, and an

[50] A. M. Obuchow: "Ueber die Energieverteilung im Spektrum einer turbulenten Stroemung."

[51] N. A. Phillips: Numerical Weather Prediction.

[52] A. Eliassen and E. Kleinschmidt: "Dynamic Meteorology."

approximate vertical velocity has to be computed along the lines described in Section 5. Lateral winds at sizable mountainous shapes have in general to be assumed as geostrophic winds also, at least initially.

The Coriolis parameter has been treated as f = f(x,y) in Section 7. For the purpose of simplicity we have, however, considered g = const. In a more exact form, eq. (43) in Section 7 would read as

$$\frac{d}{dt}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{d\rho}{dt}\cdot\frac{\partial p}{\partial z} + \rho\frac{dg}{dt} = 0 \qquad (35)$$

with $\frac{dg}{dt} = \frac{\partial g}{\partial z}w = -\frac{2g_\infty}{R}w$. This linear correction as well as the nonlinear terms associated with $\frac{df}{dt}$ and $\frac{dk(z)}{dt}$ may be initially neglected in the relaxation process, i.e., prior to obtaining stable numerical results from the simplified as well as linear filtering equations.

It is possible to derive the filtering equations for spherical coordinates or under consideration of map scale factors. For the purpose of simplicity, we have also omitted diffusion terms in the continuity equation for water vapor and terms involving differences between atmospheric and water or snow temperatures in the thermodynamic equation. These complications would not change the basic character of the filtering equations.

The discussion in the preceding paragraphs appears to be relevant for an understanding of the ramifications of the new filtering equations. We have not only succeeded in deriving optimal filter equations, but also constructed a powerful system of four prognostic equations and one diagnostic equation for numerical weather prediction together with initial conditions the details of which are exhibited in Section 7. This system would sufficiently cope with fronts, hurricanes, and instabilities associated with the jet stream. Were it not for the inaccuracy and lack of observations and the restriction of an upper boundary with w = 0, this system, together with well-digestible non-adiabatic inputs, would conceivably make it possible to extend the limit of predictability associated with the (x, y, p, t)-system considerably beyond 2 weeks. This may be comforting to scientists involved in extended forecasting and general circulation simulations such as Miyakoda[53] and Smagorinsky[54] who considered a potential limitation of 2 weeks as rather pessimistic.

[53] K. Miyakoda, et al.

[54] J. Smagorinsky, et al: "Nine-Level General Circulation Model of the Atmosphere."

7. **Generalized Optimal Filter Equations Free of Hydrostatic Limitations.** In Section 6, we have shown that it is necessary to consider pressure, p, a priori as a continuous variable and that pressure kinks have to be smoothed out in order to avoid quasi-infinite pressure gradients in agreement with Haltiner and Martin.[55] Discontinuities of zeroth order involving temperature, or rather virtual temperature, require even stronger smoothing. The application of the differential equations of meteorology is only possible with smoothed variables including consistently filtered winds. For simplicity, we omit the filter symbol Λ in the following derivations.

Since the filter condition

$$\frac{d^2}{dt^2}(\rho V) = 0 \tag{36}$$

implicitly includes a term $\frac{d}{dt}\operatorname{div} V$, a system of non-linear partial differential equations would result which does not permit an equilibrium solution and, consequently, could not be solved by relaxation methods.

As already mentioned (Section 6), the filter condition is

$$\frac{d^2 V}{dt^2} = 0 \tag{37}$$

which has already been applied in Section 4 except for the vertical wind component. If eq. (36) were valid, it would imply a strong identification of ρ with reference to V and, thus, make the differential equations more deterministic, i.e., all invariant under a filter operation, which is not possible.

We now apply eq. (37) with reference to the equations of motion

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \tag{38}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \tag{39}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{40}$$

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[55] G. J. Haltiner and F. L. Martin: *Dynamical and Physical Meteorology*, McGraw-Hill Book Company, New York, Toronto, London, 1957.

with the intermediate result

$$\frac{d}{dt}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{d\rho}{dt}\cdot\frac{\partial p}{\partial x} - \rho\frac{d}{dt}(fv) = 0 \tag{41}$$

$$\frac{d}{dt}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{d\rho}{dt}\cdot\frac{\partial p}{\partial y} + \rho\frac{d}{dt}(fu) = 0 \tag{42}$$

$$\frac{d}{dt}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{d\rho}{dt}\cdot\frac{\partial p}{\partial z} = 0\,. \tag{43}$$

With $F = \frac{dp}{dt}$, under consideration of eq. (38) and (39) with regard to $-\frac{d}{dt}(fv)$ in eq. (41) and $\frac{d}{dt}(fv)$ in eq. (42), respectively, and in view of the continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\operatorname{div} V \tag{44}$$

we arrive at

$$\frac{\partial F}{\partial x} + \frac{\partial p}{\partial x}\operatorname{div} V - \left(\frac{\partial u}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial p}{\partial z}\right)$$

$$+ f\rho\left(\frac{1}{\rho}\frac{\partial p}{\partial y} + fu\right) - v\rho\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right) = 0 \tag{45}$$

$$\frac{\partial F}{\partial y} + \frac{\partial p}{\partial y}\operatorname{div} V - \left(\frac{\partial u}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial p}{\partial z}\right)$$

$$+ f\rho\left(-\frac{1}{\rho}\frac{\partial p}{\partial x} + fv\right) + u\rho\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right) = 0 \tag{46}$$

$$\frac{\partial F}{\partial z} + \frac{\partial p}{\partial z}\operatorname{div} V - \left(\frac{\partial u}{\partial z}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial p}{\partial z}\right) = 0 \tag{47}$$

In the next step, we have to express F as a time-independent function which linearly involves the divergence $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. The thermodynamic equation reads in a very general form

$$d\left(\frac{r_s L}{c_p T}\right) + \left(1 - k + r_s\frac{c}{c_p}\right)\frac{dT}{T} - k\frac{d\rho_L}{\rho_L} = \frac{1}{c_p}\frac{đq}{T} \tag{48}$$

where r_s designates the saturation mixing ratio, $L \approx 600$ cal g^{-1}, $c_p = .2405$ cal g^{-1} $\cdot$ (deg C)$^{-1}$ the specific heat of dry air at constant pressure, $c = 1.0$ cal g^{-1}, the specific heat of water, $k = \frac{c_p - c_v}{c_p} = .2848$ where c_v is the specific heat of dry air for constant volume, ρ_L the density of dry air, and $đq$ non-precipitative heat added to a unit mass of air. With $a = \frac{r_s L}{c_p T}$, eq. (48) may also be written in its time-dependent form

$$a \frac{d \ln r_s}{dt} + (1 - k - a + 4.2\, r_s) \frac{d \ln T}{dt} - k \frac{d \ln \rho_L}{dt} = \frac{1}{c_p T} \frac{đq}{dt} . \tag{49}$$

Elimination of the term $\frac{d \ln r_s}{dt}$ in eq. (49) by means of Smagorinsky's and Collin's relation[56]

$$\frac{d \ln r_s}{dt} = (\gamma - 1) \frac{d \ln T}{dt} - \frac{d \ln \rho_L}{dt} \tag{50}$$

with $\gamma = \frac{L}{1.608\, ART}$ which involves L as the latent heat of condensation and A^{-1} as the mechanical equivalent of heat leads to

$$[1 - k + a\,(\gamma - 2) + 4.2\, r_s] \frac{d \ln T}{dt} - (k + a) \frac{d \ln \rho_L}{dt} = \frac{1}{c_p T} \frac{đq}{dt} . \tag{51}$$

From eq. (50) and (51) follows

$$\frac{d \ln r_s}{dt} = \frac{\gamma k + a - 1 - 4.2\, r_s}{1 - k + a(\gamma - 2) + 4.2\, r_s} \frac{d \ln \rho_L}{dt}$$

$$+ \frac{\gamma - 1}{1 - k + a(\gamma - 2) + 4.2\, r_s} \cdot \frac{1}{c_p T} \frac{đq}{dt} \approx \frac{A}{B} \frac{d \ln \rho_L}{dt} \approx \frac{A}{B} \frac{d \ln \rho}{dt} . \tag{52}$$

Under consideration of

$$\frac{dp}{dt} = R_L \left[(1 + 0.6r) \frac{d\rho}{dt} + (1 + 0.6r)\, \rho \frac{dT}{dt} + 0.6\, \rho\, T \frac{dr}{dt} \right] \tag{53}$$

which follows from the equation of state, in view of

$$\frac{d \ln r}{dt} = \delta \frac{d \ln r_s}{dt} \tag{54}$$

[56] J. Smagorinsky and G. O. Collins: "On the Numerical Prediction of Precipitation," *Monthly Weather Review* 83, 1955.

with

$$\delta = \begin{cases} 0 \text{ if } \operatorname{div} V \leq 0 \text{ or } r < r_s \\ 1 \text{ if } \operatorname{div} V > 0 \text{ and } r = r_s \end{cases} \tag{55}$$

and because of

$$\frac{d\rho_L}{\rho_L} = \frac{d\rho}{\rho} = 0.6\, dr \tag{56}$$

eq. (53) can be formulated as

$$F = \frac{dp}{dt} = - R_L \left(1 + \frac{k \div a}{B}\right) \left(1 + 0.6r + \frac{A}{B}\delta \cdot r\right) T \operatorname{div} V + \frac{R_L \rho}{Ac_p} \frac{\partial q}{dt}$$

$$= - M[r,\delta]\, T \operatorname{div} V + N \frac{\partial q}{dt}$$

$$= - MT \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + N \frac{\partial q}{dt} . \tag{57}$$

Substitution of eq. (57) in eqs. (45) through (47) results in the linear diagnostic filter equations

$$MT\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \left[\frac{\partial (MT)}{\partial x} - \frac{\partial p}{\partial x}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+ \frac{\partial u}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial p}{\partial z} - f\rho\left(\frac{1}{\rho}\frac{\partial p}{\partial y} + fu\right)$$

$$+ v\rho\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial x}\left(N\frac{\partial q}{dt}\right) = 0 \tag{58}$$

$$MT\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z}\right) + \left[\frac{\partial (MT)}{\partial y} - \frac{\partial p}{\partial y}\right]\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+ \frac{\partial u}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial p}{\partial z} + f\rho\left(\frac{1}{\rho}\frac{\partial p}{\partial x} - fv\right)$$

$$- u\rho\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(N\frac{\partial q}{dt}\right) = 0 \tag{59}$$

$$MT\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2}\right) + \left[\frac{\partial (MT)}{\partial z} - \frac{\partial p}{\partial z}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$+ \frac{\partial u}{\partial z}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial p}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial p}{\partial z} - \frac{\partial}{\partial z}\left(N\frac{\mathring{d}q}{dt}\right) = 0 \qquad (60)$$

Equation (60) provides an excellent diagnostic equation and reduces the numerical relaxation work considerably which is only required in eq. (58) and (59). In the form of an ordinary linear differential, eq. (60) appears as

$$MT\frac{\partial^2 w}{\partial z^2} + \frac{\partial (MT)}{\partial z}\frac{\partial w}{\partial z} + MT\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 y}{\partial y \partial z}\right) + \left[\frac{\partial (MT)}{\partial z} - \frac{\partial p}{\partial z}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ \frac{\partial u}{\partial z}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial p}{\partial y} - \frac{\partial}{\partial z}\left(N\frac{\mathring{d}q}{dt}\right) = 0 \qquad (61)$$

With

$$\tilde{F} = \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{1}{MT}\left\{\left[\frac{\partial (MT)}{\partial z} - \frac{\partial p}{\partial z}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right.$$

$$\left. + \frac{\partial u}{\partial z}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial p}{\partial y} - \frac{\partial}{\partial z}\left(N\frac{\mathring{d}q}{dt}\right)\right\} \qquad (62)$$

the solution of eq. (61) is

$$w = \int_{z_1}^{z} \ln \frac{(MT)_z}{(MT)_{z_1}} \left[\int_{z_1}^{z} \ln \frac{(MT)_z}{(MT)_{z_1}} \tilde{F}(z)\, dz + C_1\right] dz + C_2 \qquad (63)$$

At the lower boundary, $w_{B_1} = u_{B_1}\frac{\partial \phi}{\partial x} + v_{B_1}\frac{\partial \phi}{\partial y}$ with $\phi(x, y)$ as the geopotential of the ground. Accordingly,

$$C_2 = u_{B_1}\frac{\partial \phi}{\partial x} + v_{B_1}\frac{\partial \phi}{\partial y}. \qquad (64)$$

Since, at the upper boundary $w_{B_2} = 0$, the other integration constant yields the value

$$C_1 = -\left[\int_{z_1}^{z_2} \ln \frac{(MT)_z}{(MT)_{z_1}} dz\right]^{-1}\left\{\int_{z_1}^{z_2} \ln \frac{(MT)_z}{(MT)_{z_1}}\left[\int_{z_1}^{z} \ln \frac{(MT)_z}{(MT)_{z_1}} \tilde{F}(z)\, dz\right] dz + C_2\right\} \qquad (65)$$

We have to remember that the saturation mixing ratio $r_s = r_s(p, T)$ and $\gamma = \gamma(T)$ and that it is necessary to obtain first a solution of eqs. (58), (59) and (63) with $\delta = 0$ whereupon the criterion (55) is applied. One or two iterations will then yield satisfactory results. Unless δ has some variability along the lines suggested by Smagorinsky,[57] the variable δ should be about 0.8 instead of 1.0 in agreement with numerical simulations.

It is to be expected that the over-relaxation factors ϑ in the iteration scheme

$$u^{(n+1)} = u^{(n)} + \vartheta\, G_1 \left[u^{(n)}\, v^{(n)}, w^{(n)}\right]$$

$$v^{(n+1)} = v^{(n)} + \vartheta\, G_2 \left[u^{(n)}, v^{(n)}, w^{(n)}\right] \qquad (66)$$

with $u^{(1)} = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$, $v^{(1)} = \frac{1}{f\rho}\frac{\partial p}{\partial x}$, $w^{(1)} = 0$

in which G_1 and G_2 represent residuals of eq. (58) and eq. (59), respectively, have to be quite small in small-scale solutions involving strong divergence (convergence) and vertical wind velocities. Since $u^{(1)}$ and $v^{(1)}$ become singular at the equator and convergence is slow in very low latitudes, fine grid solutions are not possible in the vicinity of the equator. Due to the fact that the mass field cannot be accurately determined in the equatorial region, horizontal winds, obtained through the tracking of floating balloons, and additional temperature measurements would facilitate the computation of all desired quantities. The use of diagnostic filtering equations for this purpose has been mentioned by several authors including Mintz[57] though in connection with the more restrictive hydrostatic prediction system.

Utilization of the hydrostatic approximation with respect to height determinations weakens the application of the filtering and associated prognostic equations as far as smaller scales are concerned but still allows the computation of divergences exceeding the vorticity on a constant pressure surface. This is of importance pertaining to the immediate applicability of the new prediction system.

As to the upper boundary condition, the assumption $w = 0$, of course, has to be made for a finite height. In this respect, the condition var w = Min. would provide a good separation criterion. This has to coincide with the criteria var $\frac{\partial u}{\partial z}$ = Min., var $\frac{\partial v}{\partial z}$ = Min., var $\frac{\partial T}{\partial z}$ = Min., var $\frac{\partial \rho}{\partial z}$ = Min., as far as interpolation from a lower to a higher level is concerned, i.e., to an average equilibrium boundary which

[57] Y. Mintz; "The Four Basic Requirements for Numerical Weather Prediction in Global Weather Prediction," ed. by Bruce and Kiely, Holt, Rinehart and Winston, New York, 1970.

[58] J. Smagorinsky: "On the Dynamical Prediction of Large-Scale Condensation by Numerical Methods," Geophysical Monograph No. 5, American Geophysical Union, 1960.

exists at about 20-km height. In addition, linear correlation based on considerable prior work would permit the extrapolation of u, v, p, and T and thus facilitate a w-calculation in virtue of eq. (63).

8. **Concluding Remarks.** It is important to emphasize that initialization and the structure of prognostic equations are intimately related. Furthermore, the determination of the w-field must be consistent with that of the u- and v-fields. If the differential equations of meteorology were to be applied in a completely unmodified form, including all equations of motion, the initial fields would have to be found simultaneously in a grid with very small lengths Δx, Δy, Δz which is impossible. The optimal filter equations are also conditional equations with reference to the wind fields once sufficiently smooth pressure, temperature and humidity fields are given. Since these equations imply a very delicate "balance," independently given fields of all variables excluding the generally not available w-field would lead to the generation of artificially large gravity oscillations as stressed by Phillips.[59] Because of the existence of thermal convection, the hydrostatic differential equations cannot adequately describe the physical processes in sufficient detail and appear to be unsuitable as a basis for parameterization processes including cumulus convection which have been emphasized by Leith.[60] In view of the above-mentioned facts, the non-availability of timely boundary values and stability problems, a practical forecast system of highest predictability is, consequently, one which contains only one diagnostic equation. The existence of such an equation is equivalent with the existence of consistent initial fields of variables. The initialization and prognostic model developed here has a considerably greater predictive ability and versatility than the equations presently in use. Long-range, non-adiabatic effects other than precipitation would also be of greater significance in context with the new system of equations. These equations are also indispensable with respect to the dynamical prediction of medium to large-scale condensation and many research efforts including general circulation simulations. As can be ascertained from a number of statements made by the Committee on Atmospheric Sciences, National Research Council, and listed in the appendix, this research has made contributions to a variety of outstanding problems. Finally, this study also has implications regarding kind, methods, density, and frequency of measurements and indicates the necessity for a greater effort of and cooperation between the National Oceanic and Atmospheric Administration and the Aeronautics and Space Administration and beyond that, internationally.

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[59]N. A. Phillips: "Numerical Weather Prediction."

[60]C. Leith: "A Six Level Model of the Atmosphere" in "Global Weather Prediction," ed. by Bruce and Kiely, Holt, Rinehart and Winston, New York, 1970

APPENDIX

RELEVANT STATEMENTS OF THE COMMITTEE ON

ATMOSPHERIC SCIENCES, NATIONAL RESEARCH COUNCIL

The following statements on outstanding problems in weather prediction pertaining to which contributions have been made in this research have been published in "The Atmospheric Sciences and Man's Needs," National Academy of Sciences, Washington, D. C., 1971.

1. **Synoptic and Planetary Scales, p. 18, 19.** . . . Progress has been limited due to the computational requirements and to the fact that vertical convection and condensation play such important parts in mesoscale phenomena. Although numerical models have successfully simulated convection cells, the dynamics and statistics of convection have not been incorporated in the models of the large-scale circulation; this constitutes one of the most serious limitations on progress in atmospheric prediction. . . .

2. **Changes of Microscale and Mesoscale, p. 19-21.** . . . There are severe mathematical and theoretical difficulties in developing general prediction models for these smaller scales. The time scales are, of course, so short that the quasi-geostrophic relation between wind velocity and pressure is not valid. Furthermore, many of the weather systems are fully three dimensional, so that the hydrostatic approximation which is essential to general circulation theory cannot be applied to the smaller scales. . . .

. . . Other phenomena of the mesoscale and microscale, e.g., vertical convection, clear-air turbulence, and secondary features of the planetary boundary layer, have not been incorporated into the numerical models. The physics of these weather features, especially their interaction with phenomena of other scales, remains largely unknown, and promising ideas are needed. . . .

3. **Recommendations, p. 23-28.** . . . Scientific plans for the U. S. contribution to GARP were outlined in the report of the U. S. Committee for GARP referred to earlier. Bhat report emphasized the following requirements: . . . (c) the conduct of regional field programs and computer modeling experiments to improve the physical and mathematical basis of long-range prediction. . . .

. . . There are, however, several specific problems that have been identified by USC-GARP and that require prompt attention in order that the GARP can be successfully carried out. These are . . . (b) A strong numerical modeling effort is needed to

carry out simulation experiments aimed at design of the global observing system. . . .

. . . Improvements in prediction models and the associated increased understanding of atmospheric processes will also contribute to the necessary basis for virtually every other application considered in this report. . . .

. . . Recommendation I-4. In order to advance understanding of mesoscale phenomena and to improve ability to forecast these phenomena, emphasis should be placed on research on fronts, jet streams, organized convection, and other phenomena of mesoscale.

Increased support should be given to the development of numerical models capable of predicting such features as the weather associated with fronts and organized convection. The largest available computer in the hands of a competent numerical modeling group is needed for this purpose. . . .