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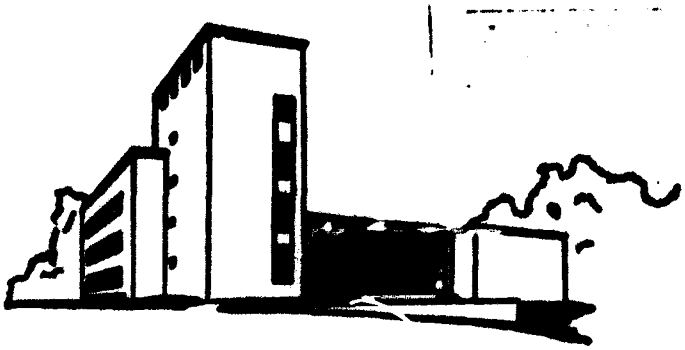

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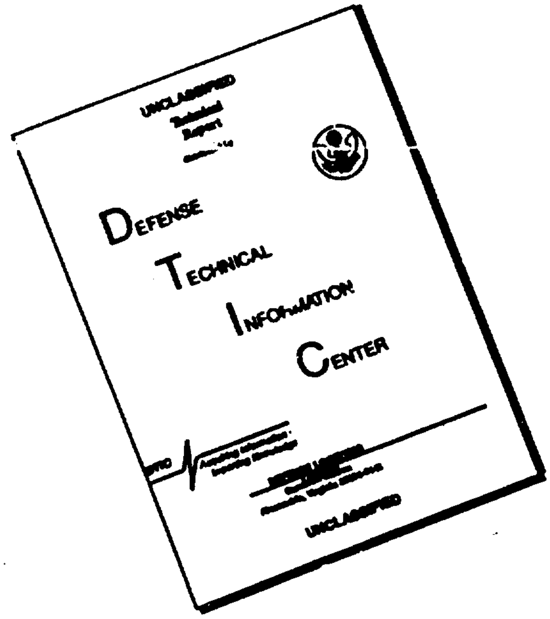

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THE CONTINUOUS MULTIPLE MODULAR DESIGN PROBLEM

by

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and

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13. ABSTRACT

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1. INTRODUCTION

The one-module modular design problem was first stated by David Evans in [3]. In the problem parts are to be grouped into a single module, several of which can then be used to satisfy (or over satisfy) the requirements for parts in a given application. The objective is to minimize the total cost of the parts used for all applications. The formal statement of the problem is:

$$\underset{x,y}{\text{Minimize}} \sum_{i \in I} c_i x_i \sum_{j \in J} d_j y_j$$

Subject to

$$\left.\begin{array}{l} x_i y_j \geq r_{ij} \\ x_i, y_j, c_i, d_j, r_{ij} \geq 0 \end{array}\right\} \text{ for } i \in I,\ j \in J$$

x_i and y_j are integers

where

$I = \{1,2,\ldots,m\}$

$J = \{1,2,\ldots,n\}$

c_i = cost of part i

d_j = demand for application j (an integer)

r_{ij} = number of part i units required in application j (an integer)

x_i = the number of part i units on the module (a decision variable)

y_j = the number of modules needed for the jth application (a decision variable)

The continuous modular design problem is obtained by dropping the integer requirements on x_i and y_j. A simplex-like solution procedure for this problem was given by Shaftel and Thompson in [9]. Several search procedures for the problem were previously given by Evans [3], Charnes and Kirby [1], and, more recently, Passey [6].

Evans in [4] and Rutenberg and Shaftel in [7] have extended the modular design (MD) problem to the case where more than one module can be used. This new problem is called the multiple modular design (MMD) problem. Rutenberg and Shaftel formulated the problem for more than one market with certain other costs and devised a heuristic search procedure for locating good integer solutions. Recently Silverman [10, 11] has presented a search procedure for solving the MMD problem, as formulated by Evans, given a solution to the MD problem.

In the present paper we shall extend the results of our earlier paper [9] to solve the continuous MMD problem. It is felt that solutions to the continuous problem will lead to more efficient solutions of the integer problem. We first sharpen the definition of the MMD problem by relating it to partitions of the MD problem. We then define guillotine partitions and study them in detail. Finally we develop branch and bound and heuristic techniques for finding optimal and good guillotine partitions.

2. THE PRIMAL AND DUAL MMD PROBLEMS

The multiple modular design (MMD) problem for a single market requires that parts be grouped into several modules and various combinations of these modules be used in applications. There is a fixed cost of producing each additional module and certain other costs (such as handling costs) which are a linear function of the number of modules needed in each application.

The MMD has been formally stated [7] as follows:

$$\underset{x,y,p}{\text{Minimize}} \quad \sum_k \Big[\sum_i c_i \, \underset{\sim}{x}_i^{(k)} \; \sum_j d_j \, \underset{\sim}{y}_j^{(k)}\Big] + \sum_k F^{(k)} + \sum_k \sum_j \underset{\sim}{b}_j \, \underset{\sim}{y}_j^{(k)} = g$$

Subject to

$$\sum_k \underset{\sim}{x}_i^{(k)} \underset{\sim}{y}_j^{(k)} \geq \underset{\sim}{r}_{ij} \quad \text{for } i \in I,\ j \in J$$

$$\underset{\sim}{x}_i^{(k)}, \underset{\sim}{y}_j^{(k)}, c_i, d_j, \underset{\sim}{r}_{ij}, \underset{\sim}{b}_j \geq 0 \text{ for } i \in I,\ j \in J,\ k \in K$$

$$\underset{\sim}{x}_i^{(k)}, \underset{\sim}{y}_j^{(k)} \quad \text{integers}$$

where

$$I = \{1,2,\ldots,m\}$$

$$J = \{1,2,\ldots,n\}$$

$$K = \{1,2,\ldots,p\}$$

and p (= the number of modules) is a decision variable;

c_i = cost of part i

d_j = demand for application j

$\underset{\sim}{r}_{ij}$ = number of units of part i required in application j

$\underset{\sim}{b}_j$ = cost of placing a module in application j

$F^{(k)}$ = fixed cost of producing the kth module

$\underset{\sim}{x}_i^{(k)}$ = number of units of part i on module k (a decision variable)

$\underset{\sim}{y}_j^{(k)}$ = number of units of module k needed in application j (a decision variable)

In this paper we shall concentrate on the continuous version of the MMD problem and, therefore will omit the integer restrictions on the x and y variables. For the continuous problem we can make the substitutions

$$x_i^{(k)} = c_i \underset{\sim}{x}_i^{(k)}, \quad y_j^{(k)} = d_j \underset{\sim}{y}_j^{(k)}$$

$$r_{ij} = c_i d_j \underset{\sim}{r}_{ij}, \text{ and } b_j = \underset{\sim}{b}_j / d_j$$

in order to simplify the statement of the problem. Initially we will restrict our attention to the problem where the number, p, of modules has been fixed.

In this case we can drop the fixed charge term in the objective function since it is a constant. The problem then becomes:

$$\underset{x,y}{\text{Minimize}} \ \sum_k [\sum_i x_i^{(k)} \sum_j y_j^{(k)}] + \sum_k \sum_j b_j y_j^{(k)} = g_p \tag{1A}$$

subject to

$$\sum_k x_i^{(k)} \ y_j^{(k)} \geq r_{ij} \tag{2A}$$

$$x_i^{(k)}, \ y_j^{(k)}, \ r_{ij}, \ b_j \geq 0 \tag{3A}$$

Later we will show how to calculate an upper bound on p and give a search procedure for minimizing g_p over all possible values of p. To do this we will have to bring the fixed charge terms back into the objective function.

We shall now derive a simplified primal problem, and then use the Kuhn-Tucker conditions to derive a dual problem similar to the one we found for the single module case in [9].

THEOREM 1. In searching for an optimal solution to problem (A) the linear term in (1A) can be ignored.

PROOF. Let $x^{(k)}$, $y^{(k)}$ for $k \epsilon K$ be feasible for problem (A). If τ_k is any number > 0 then $x^{(k)}/\tau_k$, $y_k \tau_k$ for $k \epsilon K$ is also feasible. By setting $\tau_k = \tau < 1$ for all k the second term of objective function for this solution is reduced while the first term remains constant. By letting $\tau \to 0$ the linear term can be made arbitrarily small, so that problem (A) has no minimizing solution with $y^{(k)} > 0$. However the objective function is nonnegative and hence has an infimum M over the constraint set. If we solve problem (A) without the linear term we can then use the above transformation to obtain a solution with value arbitrarily close to M. Thus we are justified in solving the continuous problem without the linear term.

It should be remarked that for the integer problem there are additional constraints of the form $y_j^{(k)} \geq 1$ so that problem (A) has a minimum solution that is attained within the constraint set. In the integer case the linear term cannot be ignored, and Theorem 1 does not hold for the integer problem.

In order to solve the continuous MMD problem we can (as in [3,9]) isolate one of the set of optimum solutions by requiring that $\sum_{j \in J} y_j^{(k)} = 1$ for each k. But we shall be even more explicit concerning the $x_i^{(k)}$ and $y_j^{(k)}$ variables, as follows.

Suppose that in the kth module we choose to make some of the $x_i^{(k)}$ or $y_j^{(k)}$ variables equal to zero. In order to derive a dual problem that reflects this we wish to record these zero constraints explicitly. Define p subsets $I_1, \ldots, I_p$ and $J_1, \ldots, J_p$ such that $I_k \subset I$, $J_k \subset J$ for $k \in K$ and

$$I \times J = (I_1 \times J_1) \cup (I_2 \times J_2) \cup \ldots \cup (I_p \times J_p)$$

Notice that we do not (as yet) require $(I_h \times J_h) \times (I_k \times J_k) = \emptyset$ for $h \neq k$, i.e., we do not require that the sets $I_k \times J_k$ partition $I \times J$. We now call the following problem the <u>primal problem with p modules</u>:

$$\underset{x}{\text{Minimize}} \sum_{k \in K} \sum_{i \in I_k} x_i^{(k)} = \delta_p \tag{1P}$$

Subject to

$$\sum_{k \in K} x_i^{(k)} y_j^{(k)} - z_{ij} = r_{ij}, \quad i \in I,\ j \in J \tag{2P}$$

$$\sum_{j \in J_k} y_j^{(k)} = 1 \qquad k \in K \tag{3P}$$

$$x_i^{(k)} = 0 \qquad k \in K,\ i \in I - I_k \tag{4P}$$

$$y_j^{(k)} = 0 \qquad k \in K,\ j \in J - J_k \tag{5P}$$

$$z_{ij},\ x_i^{(k)},\ y_j^{(k)} \geq 0 \qquad k \in K,\ i \in I,\ j \in J \tag{6P}$$

where z_{ij} is a surplus variable.

THEOREM 2. An optimal solution to problem (P) exists.

PROOF. Problem (P) is to minimize a continuous function over a compact set and by a well-known theorem of mathematics has an optimal solution.

In this paper we shall indicate some algorithms and some heuristic procedures for finding exact and approximate solutions to the continuous MMD problem.

LEMMA 1. If there are no zero rows or zero columns in the requirements matrix, R, then in any feasible solution, for each i there is at least one k with $x_i^{(k)} > 0$, and for each j there is at least one k with $y_j^{(k)} > 0$.

PROOF. Since $\sum_k x_i^{(k)} y_j^{(k)} \geq r_{ij} > 0$ for a feasible solution, it follows that there is at least one k with $x_i^{(k)} y_j^{(k)} > 0$.

Let λ_{ij} be variables associated with the constraints (2P) and let $g^{(k)}$ be variables associated with constraints (3P). The Kuhn-Tucker conditions associated with the primal problem can be shown to be:

$$\sum_{j \epsilon J_k} \lambda_{ij} y_j^{(k)} = 1 \qquad \text{for } k \epsilon K,\ i \epsilon I_k \tag{1B}$$

$$\sum_{i \epsilon I_k} \lambda_{ij} x_i^{(k)} = g^{(k)} \qquad \text{for } k \epsilon K,\ j \epsilon J_k \tag{2B}$$

$$z_{ij} \lambda_{ij} = 0 \qquad \text{for } i \epsilon I,\ j \epsilon J \tag{3B}$$

$$\lambda_{ij} \geq 0 \qquad \text{for } i \epsilon I,\ j \epsilon J \tag{4B}$$

LEMMA 2. For any pair of dual feasible solutions, $\sum_{i \epsilon I_k} x_i^{(k)} = g^{(k)}$ for $k \epsilon K$.

PROOF. Multiply equation (1B) by $x_i^{(k)}$ and sum over $i \epsilon I_k$; obtaining

$$\sum_{i \epsilon I_k} \sum_{j \epsilon J_k} \lambda_{ij} y_j^{(k)} x_i^{(k)} = \sum_{i \epsilon I_k} x_i^{(k)} \quad \text{for } k \epsilon K.$$

Similarly, multiply (2B) by $y_j^{(k)}$ and sum over $j \epsilon J_k$ and use (3P) to obtain

$$\sum_{j \in J_k} \sum_{i \in I_k} \lambda_{ij} \, y_j^{(k)} \, x_i^{(k)} = g^{(k)} \sum_{j \in J_k} y_j^{(k)} = g^{(k)} \quad \text{for } k \in K$$

Combining these two equations gives the desired result.

LEMMA 3. For any pair of dual feasible solutions $\sum_{k \in K} g^{(k)} = g_p$.

PROOF. This follows from Lemma 2 and (1P).

By analogy to the continuous MD problem (see [9]) and classical linear programming, we shall create a dual problem from constraints (1B), (2B), and (4B) and the objective function. The dual problem is:

$$\underset{\lambda_{ij}}{\text{Maximize}} \quad \sum_i \sum_j \lambda_{ij} \, r_{ij} = f_p \tag{1D}$$

Subject to

$$\sum_{j \in J_k} \lambda_{ij} \, y_j^{(k)} = 1 \qquad \text{for } k \in K,\ i \in I_k \tag{2D}$$

$$\sum_{i \in I_k} \lambda_{ij} \, x_i^{(k)} = g^{(k)} \qquad \text{for } k \in K,\ j \in J_k \tag{3D}$$

$$\lambda_{ij} \geq 0 \qquad \text{for } i \in I,\ j \in J \tag{4D}$$

Constraints (3B) can be interpreted as complementary slackness conditions. They will be forced to hold by the solution procedures we shall present.

Note that for each module, the constraints for fixed k have a similar format to that of the dual problem of the continuous MD problem (see [9]) but restricted to the $I_k \times J_k$ area of the matrix R. Many properties of the continuous MD problem will carry over to the continuous MMD problem.

LEMMA 4. For any pair of feasible primal-dual solutions (whether nonnegative or not) we have

$$g_p - f_p = \sum_i \sum_j \lambda_{ij} \, z_{ij}$$

PROOF. Multiply (2P) by λ_{ij}, sum over all i and j and use (1D) to show:

$$\sum_i \sum_j \sum_k \lambda_{ij} x_i^{(k)} y_j^{(k)} - \sum_i \sum_j \lambda_{ij} z_{ij} = \sum_i \sum_j \lambda_{ij} r_{ij} = f_p \ .$$

As in the proof of Lemma 2 we also have

$$\sum_i \sum_j \sum_k \lambda_{ij} x_i^{(k)} y_j^{(k)} = \sum_k \sum_i x_i^{(k)} = g_p$$

which proves the lemma.

LEMMA 5. (Complementary slackness). For any pair of dual feasible solutions $g_p = f_p$ if and only if $\lambda_{ij} z_{ij} = 0$ for $i \epsilon I$, $j \epsilon J$.

PROOF. The proof follows directly from Lemma 4 and the fact that both λ_{ij} and z_{ij} are nonnegative.

THEOREM 3 (Duality Theorem). A necessary condition that $x_i^{(k)}$, $y_j^{(k)}$ be optimal for problem (P) is that there exists a solution λ_{ij} to problem (D) and that $g_p = f_p$.

PROOF. Using Lemma 1, we can show that the Arrow-Hurwicz-Uzawa constraint qualification [5, p. 102] holds for problem (P). Therefore at the optimum solution to problem (P) there exists a solution to the Kuhn-Tucker problem, since problem (D) together with $\lambda_{ij} z_{ij} = 0$ (which is true if and only if $g_p = f_p$ from Lemma 5) make up the Kuhn-Tucker conditions. These conditions must hold at the optimum because of the constraint qualification [5, pp. 105-106].

The MD problem had a unique solution so the duality theorem for that case was stronger than here. In the MMD problem there are many pairs of solutions satisfying the conditions of Theorem 5, and these correspond to local optima of g_p. Among these, of course, is a global optimum.

3. CHARACTERIZATION OF LOCAL OPTIMA

In the present section we shall give some properties of local optima for the case of a fixed number, p, of modules. It will be shown that a local optimum to the problem can be found by solving p MD problems.

DEFINITION. By a partition of a MMD problem with data matrix R (hereafter called problem R) into p problems with data matrices $R^{(k)}$ (hereafter called problem $R^{(k)}$) for $k \epsilon K$ we mean $R = \sum_{k \epsilon K} R^{(k)}$, i.e.

$$r_{ij} = \sum_{k \epsilon K} r_{ij}^{(k)} \text{ where } r_{ij}^{(k)} \geq 0.$$

An integral partition is one that also satisfies,

$$r_{ij}^{(k)} = c_i \, d_j \, \ell_{ij}^{(k)} \quad \text{where} \quad \ell_{ij}^{(k)} \geq 0 \text{ is an integer}$$

We shall regard each $R^{(k)}$ as defining a MD problem with variables $x_i^{(k)}$, $y_j^{(k)}$ and $\lambda_{ij}^{(k)}$.

THEOREM 4. (A) Given an optimal primal dual solution $x^{(k)}$, $y^{(k)}$ and λ_{ij} to problem R a partition $R = \Sigma R^{(k)}$ can be defined in such a way that $x^{(k)}$, $y^{(k)}$ and λ_{ij} are optimal primal-dual solutions to the MD problem $R^{(k)}$.

(B) Given a partition $R = \Sigma R^{(k)}$, let $x^{(k)}$ and $y^{(k)}$ be optimal primal solutions for the MD problem $R^{(k)}$. Then, if it is possible to choose $\lambda_{ij}^{(k)} = \lambda_{ij}$ for all $k \epsilon K$ as optimal dual solutions to $R^{(k)}$ for all $k \epsilon K$, then $x_i^{(k)}$ and $y_j^{(k)}$ give a local optimum for problem R.

(C) If $g^{(k)}$ is the value of problem $R^{(k)}$ and the condition of (B) holds then

$$g_p = \sum_{k \epsilon K} g^{(k)} .$$

PROOF. (A) From (2P) we have

$$\sum_k x_i^{(k)} \, y_j^{(k)} \geq r_{ij}$$

hence we can choose $r_{ij}^{(k)} \leq x_i^{(k)} \, y_j^{(k)}$ so that $\sum_{k=1}^{p} r_{ij}^{(k)} = r_{ij}$ for each i and j in at least one, and perhaps many different ways. If $z_{ij} = 0$ then $\sum_k x_i^{(k)} \, y_j^{(k)} = \sum_k r_{ij}^{(k)} = r_{ij}$ so $z_{ij}^{(k)} = 0$ for problem $R^{(k)}$. Hence $\lambda_{ij} > 0$ implies $z_{ij}^{(k)} = 0$ and complementary slackness holds. The primal and dual conditions for problem $R^{(k)}$ are as follows:

(1) $x_i^{(k)} y_j^{(k)} \geq r_{ij}^{(k)}$ for $i\epsilon I_k$, $j\epsilon J_k$

(2) $\sum_{j\epsilon J_k} y_j^{(k)} = 1$ for $k\epsilon K$

(3) $\sum_{i\epsilon I_k} x_i^{(k)} = g^{(k)}$ for $k\epsilon K$

(4) $\sum_{j\epsilon J_k} \lambda_{ij} y_j^{(k)} = 1$ for $k\epsilon K$, $i\epsilon I_k$

(5) $\sum_{i\epsilon I_k} \lambda_{ij} x_i^{(k)} = g^{(k)}$ for $k\epsilon K$, $j\epsilon J$

(6) $\sum_{i\epsilon I_k} \sum_{j\epsilon J_k} \lambda_{ij} r_{ij}^{(k)} = f^{(k)}$ for $k\epsilon K$

(7) $x_i^{(k)}, y_j^{(k)}, \lambda_{ij} \geq 0$ for $k\epsilon K$, $i\epsilon I_k$, $j\epsilon J_k$

We have already shown that (1) holds. Equation (2) is the same as (3P); (3) and (6) are true by definition; (4) is (2D); (5) is (3D); and (7) is included in (4P) and (4D). Hence $x_i^{(k)}$, $y_j^{(k)}$ and λ_{ij} are primal-dual solutions to problem $R^{(k)}$.

(B) If for a partition $\sum R^{(k)} = R$ we can choose $\lambda_{ij}^{(k)} = \lambda_{ij}$ for all i and j, then the steps in the proof of (A) are reversible. We omit the rest of the details.

(C) If we sum (6) over k and use the duality theorem for problem $R^{(k)}$ and the partition formula we have

$$\sum_k f^{(k)} = \sum_{k\epsilon K} \sum_{i\epsilon I_k} \sum_{j\epsilon J_k} \lambda_{ij} r_{ij}^{(k)} = \sum_{k\epsilon K} g^{(k)} = g_p = f_p$$

completing the proof of the theorem.

By means of this theorem we see that one way of finding local optima to the MMD problem is to partition R into MD problems $R^{(k)}$ and then use the algorithm of [9] to solve $R^{(k)}$. If it happens that the same dual variables can be used for each problem $R^{(k)}$ then we have found a local optimum to the MMD problem. If we can enumerate all local optima, we can choose the smallest as the global optimum. There are, of course, an infinite number of arbitrary partitions. We shall concentrate on integral partitions, since there are only a finite number of them.

DEFINITION. By a <u>guillotine partition</u> with p pieces of R we shall mean a partition of $I \times J$ into p "rectangular" subsets

$$I \times J = (I_1 \times J_1) \cup \ldots \cup (I_p \times J_p)$$

where

$$(I_h \times J_h) \cap (I_k \times J_k) = \emptyset \text{ for } h \neq k$$

and a partition $R = \sum_{k \in K} R^{(k)}$ where

$$r_{ij}^{(k)} = r_{ij} \text{ if } (i,j) \in I_k \times J_k$$

$$r_{ij}^{(k)} = 0 \text{ if } (i,j) \in I_h \times J_h \text{ and } h \neq k.$$

By a <u>guillotine row partition</u> we mean a guillotine partition with $J_k = J$ for $k \in K$. Similarly by a <u>guillotine column partition</u> we mean a guillotine partition with $I_k = I$ for $k \in K$.

Clearly guillotine partitions are integral and there are only a finite number of them. Also any guillotine partition can be constructed stepwise by row and-or column guillotine partitions applied alternately to a series of sub-problems.

THEOREM 5. Suppose problem R is divided into p rectangular pieces $R^{(k)}$ by a guillotine partition. If $x_i^{(k)}$, $y_j^{(k)}$ and $\lambda_{ij}^{(k)}$ are primal-dual optimal solutions for $R^{(k)}$ then $x_i^{(k)}$, $y_j^{(k)}$ and $\lambda_{ij} = \sum_{k=1}^{p} \lambda_{ij}^{(k)}$ give a local optimum to problem R.

PROOF. Suppose $R^{(k)}$ consists of the matrix $r_{ij}^{(k)}$ defined by

$$r_{ij}^{(k)} = r_{ij} \quad \text{for } (i,j) \in I_k \times J_k$$

$$r_{ij}^{(k)} = 0 \quad \text{for } (i,j) \notin I_k \times J_k$$

where $I_k \subset I$ and $J_k \subset J$ define the rectangular piece $R^{(k)}$. Then if $i \notin I_k$ we have $x_i^{(k)} = 0$ and if $j \notin J_k$ we have $y_j^{(k)} = 0$. Hence $x_i^{(k)} = 0$ if $(i,j) \notin I_k \times J_k$. Therefore $\lambda_{ij}^{(k)} = 0$ if $(k,j) \notin I_k \times J_k$. It follows that if we define $\lambda_{ij} = \sum_{k=1}^{p} \lambda_{ij}^{(k)}$ it will satisfy the dual conditions (2B)-(4B). Hence by Theorem 4 $x_i^{(k)}$ and $y_j^{(k)}$ give a local optimum for problem R.

LEMMA 6. Let $I_k \times J_k$ be a rectangular piece of a guillotine partition and let $g^{(k)}$ be the optimal value of $R^{(k)}$ considered as an MD problem. Then $g^{(k)} \geq \sum_{(i,j)\in I_k \times J_k} r_{ij}$.

PROOF. Since $\lambda_{ij}^{(k)} = 1$ for $(i,j) \in I_k \times J_k$ is always dual-feasible and $g^{(k)} = f^{(k)}$ by the duality theorem, the result is obvious.

DEFINITION. A sub problem $R^{(k)}$ defined on $I_k \times J_k$ is <u>tight</u> if $g^{(k)} = \sum_{(i,j)\in I_k \times J_k} r_{ij}$.

THEOREM 6. Consider the MMD problem with all fixed costs equal to ~~.

(a) For guillotine partition $g_p \geq g_{p+1}$.

(b) Unless all pieces of a guillotine partition with p pieces are tight, there is an additional row or column guillotine partition that can be made such that $g_p > g_{p+1}$.

PROOF. (a) The guillotine partition with p+1 pieces is obtained from that with p pieces by applying a row or column guillotine partition to one of the original pieces. Suppose $I_1 \times J_1$ is to be divided into $(I_{11} \times J_1) \cup (I_{12} \times J_1)$ by a row partition (the proof for column partition is similar). Then the y vector that was optimal for $I_1 \times J_1$ together with the I_{11} components of x will be feasible for $I_{11} \times J_1$. Similarly the same y vector and the I_{12} components of x will be feasible for $I_{12} \times J_1$. Hence (a) follows.

(b) If the $I_1 \times J_1$ piece for the guillotine partition with p pieces is not tight then some cell $(i,j) \in I_1 \times J_1$ has $z_{ij} > 0$. By making the next partition isolate either the row or the column containing (i,j) we can assure that $g_p > g_{p+1}$.

We give next an example of a problem for which the optimum partition is not a guillotine partition. Consider the problem

$$R = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array}$$

and suppose we require $p = 2$. The sum of the elements in R is 27, and the reader may easily verify that every guillotine partition has $g_2 > 27$. However the following, non guillotine, partition achieves $g_2 = 27$:

$$R = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} + \begin{array}{|c|c|c|} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array}$$

$$= R^{(1)} + R^{(2)}$$

The optimum solution to $R^{(1)}$ is

$$x = (3,\ 3,\ 3), \quad y = (0,\ \tfrac{1}{3},\ \tfrac{2}{3}),$$

and the optimum solution to $R^{(2)}$ is

$$x = (3,\ 6,\ 9), \quad y = (\tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3})$$

One optimum dual solution is $\lambda_{ij} = 1$ for all i and j, and there are others.

It is interesting to note that the vertical guillotine partition that defines column 1 as one subproblem and columns 2 and 3 as the other subproblem has a value $g_2 = 27.58$ which is only slightly more than the optimum of 27. We have generally found that guillotine partitions will give local optimum values close to the optimum when they do not themselves provide the optimum.

In the rest of this paper we shall restrict ourselves to guillotine partitions. Other reasons why they are advantageous is that row guillotine partitions restrict the number of different parts that appear on a given module, while column guillotine partitions restrict the number of different modules that appear in a given application. Both of these conditions are probably desirable from a manufacturing point of view, even though they are not specifically included in the objective function.

We intend to take up the question of non-guillotine partitions in another paper.

4. SOLUTION PROCEDURES FOR GUILLOTINE PARTITIONS

Searching successive MD problems of a MMD problem is made easier by the ability to use a type of parametric programming. Given the optimal solution to a MD problem, we can readily determine a good initial feasible solution for any subproblem formed via a guillotine partition, as follows. Let x_i, $i \in I$, y_j, $j \in J$ be the optimal solution to a particular $m \times n$ MD problem. Now assume that we partition the problem into two subproblems made of columns of the original problem (the same reasoning would apply for rows). Let

$$J_k = \{\text{the set of columns which are positive in module } k\},\ k = 1 \text{ or } 2.$$

And $\quad y_j^{(k)} = y_j / \sum_{j \in J_k} y_j \quad j \in J_k$ and $k = 1$ or 2

$$x_i^{(k)} = \max_{j \in J_k} r_{ij}/y_j^{(k)}$$

Let the cells which determine $x_i^{(k)}$ be tight and by increasing the right hand sides of other constraints force these cells to enter the basis until a tree basic solution is formed with $r_{ij}^{(k)*} = x_i^{(k)}\ y_j^{(k)}$ for (i,j) in the tree.

THEOREM 7. (A) The solution described above is primal feasible.

(B) If m and n are > 1 and the original problem is nondegenerate the sum of the objective functions for the initial primal feasible solutions to the subproblems is less than the optimal solution for the original problem.

PROOF. (A) Let $r_{i\tau}/y_\tau^{(k)} = \max_{j \in J_k} r_{ij}/y_j^{(k)}$

then, $x_i^{(k)}\ y_j^{(k)} = r_{i\tau}/y_\tau^{(k)} \cdot y_j^{(k)} \geq r_{ij}/y_j^{(k)} \cdot y_j^{(k)} = r_{ij}$ for all i,j and k.

(B) Let g be the optimal solution to the original problem. And h_1 and h_2 be the objective functions associated with the initial primal feasible solutions of the two subproblems. Then,

$g = \sum_{i \in I} \sum_{j \in J} x_i\, y_j$. We also know that $x_i = \max_{j \in J} r_{ij}/y_j$ for all i,

so that $g = \sum_{i \in I} \sum_{j \in J} y_j \cdot \max_{j \in J} r_{ij}/y_j$.

Now, $$x_i^{(k)}\ y_j^{(k)} = \frac{y_j}{\sum_{j \in J_k} y_j} \cdot \max_{j \in J_k} r_{ij}/y_j^{(k)}$$

$$= \frac{y_j}{\sum_{j \in J_k} y_j} \cdot \sum_{j \in J_k} y_j \max_{j \in J_k} r_{ij}/y_j$$

$$= y_j \max_{j \in J_k} r_{ij}/y_j$$

And, $$h_1 + h_2 = \left[\sum_{i \in I} \sum_{j \in J_1} y_j \max_{j \in J_1} r_{ij}/y_j + \sum_{j \in J_2} y_j \max_{j \in J_2} r_{ij}/y_j \right] .$$

But for m and $n > 1$ (and a nondegenerate problem) there is at least one row in R which contains only one tight cell so that

$$\max_{j \in J} r_{ij}/y_j > \max_{j \in J_k} r_{ij}/y_j .$$

Hence, $g > h_1 + h_2$.

A solution for a subproblem formed in the above manner is usually a very good initial primal feasible solution. It is, of course, an upper bound on the value of the objective function for that subproblem. A lower bound to the objective function value of a subproblem is $\sum_{i \in I_k} \sum_{j \in J_k} r_{ij}^{(k)}$.

It is important to note that the upper bound calculation is the first (and, probably, most time consuming) step of calculating the exact optimum for a subproblem. Because of this, it seems likely that for any branch and bound enumeration procedure it will pay to calculate exact solutions rather than use approximate lower bounds. It is possible to enumerate, explicitly or implicitly, all possible ways of partitioning the problem into p modules (bounds on the value of p will be discussed in the next section). The organization of such an enumeration, not the calculation of lower bounds, is what would make such a complete enumeration difficult. Because of these difficulties we will present some properties which sould make useful tools for obtaining good heuristic solutions.

Given a solution to a particular MD problem, the greatest value by which we can reduce the value of the objective function is $\sum_i \sum_j z_{ij}$. If we form a partition with a subproblem with a single positive column, then the objective function will reduce by at least $\sum_j z_{ij}$ for that row. Since m is generally larger than n, vertical partitions seem the most appropriate. Another rationale for using only vertical partitions can be made by counting the number of new tight cells which would be caused by vertical versus horizontal partitioning. Once a vertical partition is made, subdividing the new subproblems into more vertical partitions is even more appropriate. If a total of n partitions were to be made, n modules each satisfy the needs of exactly one application would yield the optimum value of $\sum_i \sum_j r_{ij}$.

Given the value of p, or an estimate for that value, and using vertical partitioning only, it becomes necessary to intelligently group applications to be satisfied by certain modules. (Note that the techniques discussed here will also apply to horizontal partitioning.) This fact would mean that we would want to choose applications to group whose parts requirements have approximately the same ratios. We would also like to group applications, with requirements which are difficult to fulfill, into the smallest groups. Three possible measures which may be used are:

1). The variance (or absolute difference) along each column is an important measure. Since we are dealing with ratios, we would want a measure of the variance of the normalized row entries. The measure of this type that we shall use is

$$\sum_i \left| r_{ij} / \sum_i r_{ij} - \frac{1}{m} \right| .$$

2). The above value would be useless without some sort of indication of the size order of the entries of a column. Ideally, of course, we would compare each pair of columns to see how closely they fit. An easier, but less informative, method is to compare each column with a given standard order.

The sum of the absolute difference between the order and the standard order will be used as a measure.

3). Given that we make only vertical partitions the number of columns in a given module is likely to be on the order of magnitude of n/p. In order to determine which columns should be in the smaller partitions we look at $\sum_i z_{ij}$ for the single module solution. Columns with larger values of $\sum_i z_{ij}$ would be in partitions which satisfy the least number of applications.

We consider the effectiveness of the three heuristics for a specific example in Section 6.

5. DETERMINING THE NUMBER OF MODULES

As mentioned earlier, n is an upper bound on the value of p while 1 is a lower bound. The solutions for these two cases are quite easy to find; the difference between the two solutions (excluding fixed costs) is exactly $\sum_i \sum_j z_{ij} = Z^{(1)}$ where z_{ij} is the surplus for the case where $p = 1$. Reduction in the value of $Z^{(k)}$ as p increases seems to decrease at least linearly. As more experience in solving MMD problems is obtained, good estimates on p should become available. At present a good estimate (assuming $F^{(p)}$ is an increasing function of p) is the largest value of p such that $F^{(p)} \leq \frac{Z^{(1)}}{2^{p-1}}$.

During a search for the exact value of p, an upper bound which is often better than n, as well as a means for eliminating certain possible values between 1 and n can be achieved. These two procedures are outlined in the next two theorems.

THEOREM 8: Let $Z^{(q)} = \sum_i \sum_j z_{ij}$ when q modules are produced. Then an upper bound on p is the largest integer such that:

$$Z^{(q)} - \sum_{k=q+1}^{p} F^{(k)} > 0$$

where $F^{(k)}$ is the fixed cost of the kth additional module.

PROOF. By definition $Z^{(q)}$ is the maximum we can reduce the value of the objective function by producing additional modules. As soon as the total cost of additional modules surpasses this amount, adding a module cannot be profitable.

THEOREM 9: Given $Z^{(q)}$ and θ, the value of the best minimum solution found so far, calculate c as the largest integer such that

$$\sum_i \sum_j r_{ij} + Z^{(q)} + \sum_{k=1}^{c} F^{(k)} < \theta \ .$$

Then any value of k from (and including) c to q cannot be the optimum.

PROOF. Since $Z^{(q)}$ is a nonincreasing function of q, it is a lower bound on the value of $Z^{(c)}$, $c \leq q$. This implies that a lower bound on the value of the objective function for any value of $c \leq q$ is:

$\sum_i \sum_j r_{ij} + Z^{(q)} + \sum_{k=1}^{c} F^k$. We need only consider values of c which have a lower bound less than the optimum found thus far. Hence the theorem.

6. EXAMPLE

We will give an example problem, enumerate all possible guillotine partitions, and indicate the values of the heuristic measures defined in section 4. The problem we will use is symmetrical so that only vertical partitions need be considered.

R =

1	2	3	1	4
2	9	6	12	10
3	6	16	20	25
1	12	20	25	23
4	10	25	23	36

$\sum_j r_{ij}$ = 11 39 70 81 98

$\sum_i \sum_j r_{ij} = 299$

For $p = 1$ $g^{(1)} = g_1^{(1)} = 367.36$. $Z^{(1)} = 68.36$. For $p = 2$ there are 15 possible vertical guillotine partitions. In the following table 1,2,-3,4,5 means subproblem one has r_{ij} for columns 1 and 2 and zeros in columns 3, 4, and 5, while subproblem 2 has columns 3, 4, and 5 at the value, r_{ij}, and zero's in columns 1 and 2.

Partition	Sub-module 1		Sub-module 2		Total	
	Optimum	Excess	Optimum	Excess	Optimum	Excess
1 - 2,3,4,5	11	0	343.0	55	354.0	55
2 - 1,3,4,5	39	0	299.1	39.1	338.1	39.1
3 - 1,2,4,5	70	0	285.1	56.1	355.1	56.1
4 - 1,2,3,5	81	0	266.9	48.9	347.9	48.9
5 - 1,2,3,4	98	0	247.0	46.0	345.0	46.0
1,2 - 3,4,5	57.4	7.4	276.6	27.6	334.0	35.0
1,3 - 2,4,5	91.1	10.1	262.5	44.5	353.6	54.6
1,4 - 2,3,5	100.7	8.7	245.0	37.9	345.7	46.7
1,5 - 2,3,4	118.8	9.8	226.0	36.0	344.8	45.8
2,3 - 1,4,5	126.9	17.9	221.9	31.9	348.8	49.8
2,4 - 1,3,5	134.6	14.6	198.5	19.5	333.1	34.1
2,5 - 1,3,4	160.2	23.2	185.4	23.4	245.6	46.6
3,4 - 1,2,5	165.6	14.6	179.9	31.9	345.5	46.5
3,5 - 1,2,4	176.4	8.4	153.0	22.0	329.4	30.4
4,5 - 1,2,3	200.3	21.3	145.4	25.4	345.7	46.7

For $p = 2$ then the optimum partition is 3,5 - 1,2,4 $g^{(2)} = 329.4$, $Z^2 = 30.4$. From the enumeration of $p = 2$ we can enumerate all possible solutions for $p = 3$ and $p = 4$. For $p = 3$ the best partitions are 1,2 - 3,5-4, $g^{(3)} = 314.4$, $Z^{(3)} = 15.4$. For $p = 4$ the best partitions are 1,2 - 3-4-5, $g^{(4)} = 306.4$, $Z^{(4)} = 7.4$. Note that the approximation for the reduction in $Z^{(k)}$ given in the last section seems to be very accurate.

Finally, we will show the heuristic measures discussed in section 4.

z_{ij} for the single module case:

	0	1	1.2	4	2
	1	0	6.5	3	8
	1.2	6.5	1.4	.8	0
	4	3	.8	0	7
	2	8	0	7	0
Sum	8.2	18.5	9.9	14.8	17

Normalized value of r_{ij}

	.091	.051	.043	.012	.041
	.182	.231	.086	.148	.102
	.273	.154	.229	.247	.225
	.091	.308	.286	.309	.235
	.364	.256	.357	.284	.368
Ave.	.2	.2	.2	.2	.2
Sum of the absolute difference from the mean	.473	.528	.543	.480	.515

Order (ties are given the average value of their possible positions):

$1\frac{1}{2}$	1	1	1	1
3	3	2	2	2
4	2	3	3	4
$1\frac{1}{2}$	5	4	5	3
5	4	5	4	5

Calculating the sum of the absolute value of the differences between pairs of columns we get

pair	value	pair	value
1,2	7	3,4	2
1,3	5	3,5	2
1,4	7	4,5	4
1,5	3		
2,3	4		
2,4	2		
2,5	6		

Note that the optimum solution for $p = 2$ is 3,5 - 1,2,4 where two rows which are similar along the order and variance measures (3 and 5) with large z_{ij}'s have been grouped into a smaller module. Although these measures are not perfect, they do give an a priori indication of which partitions are reasonable.

As a second example, we will solve the problem presented by Evans in [3], for $p = 2$.

	15	23	44
	13	13	0
	15	17	35
	34	12	22
Sum	77	65	101

For this problem Evans in [4] gives a solution of 270. Silverman [11] gives an optimal solution of 269.06 via a search technique. (Both solutions are for non-guillotine partitions.) We shall show one possible vertical guillotine partion, chosen from the magnitude of $\sum_i z_{ij}$ from the single module solution;

these values are:

column	1	2	3
$\sum_i z_{ij}$	30.3	14.5	51.0

We shall solve the partitions 1, 2 - 3. In this case $g_1^2 = 166.9$, $g_2^2 = 101$, $g^2 = 267.9$. This solution was found in a few minutes by hand via the Shaftel-Thompson algorithm applied to the subproblems.

7. CONCLUSIONS

In this paper we have extended the MD solution of [9] to the MMD problem. We have characterized the local optimum and identified a useful subset of these optimum as elements of a, possibly, exhaustive search. In solving the MMD problem by the methods outlined in this paper several advantages are available, as follows.

(1). The solution is based on solving the MD problem which can be solved quickly. Use of previous solutions for each MD subproblem will start the Shaftel-Thompson MD algorithm at an initially good primal feasible solution.

(2). We have stated conditions that limit the search for p. Also, any branch and bound procedure for searching for an optimum solution to a given p, generates a tree which can be used during the solutions procedure for a new value p'. This means that rather than solving several problems from the beginning we may merely extend the tree for each new value of p.

(3). The solutions for $p = 1$ and $p = n$ can be found quickly and give a good initial upper bound on the value of the objective function.

(4). There exist many good local solutions to the problem so that a heuristic search will tend to find a reasonable solution.

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