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The Response of Systems with a  
Small Stochastic Parameter

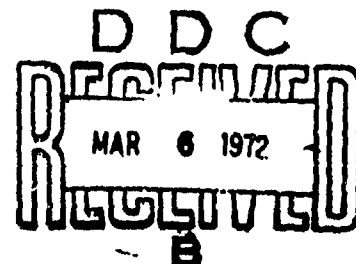
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13. ABSTRACT  This report deals with two problems in the area of stochastic systems. First, the eigenvalue problem which is solved using some techniques of functional analysis. Secondly, these techniques are extended and applied to the problem of forced vibration of a system which has a small stochastic parameter. As an example, the techniques are applied to a string of random density.			

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## 1. Introduction

The class of problems involving stochastic systems can be classified broadly into three categories

- i) Deterministic System - Stochastic Input
- ii) Stochastic System - Deterministic Input
- iii) Stochastic System - Stochastic Input.

Of these three classes, the most highly developed analytical techniques are available for i. A discussion of this topic was recently given by Ames [1]. Most of the techniques for ii and iii are approximate, and are valid for systems with small stochastic parameters. The methods described herein are applicable to the eigenvalue problem, and the inhomogeneous boundary value problem for weakly stochastic systems. Although this may seem restrictive, it does nevertheless encompass a large spectrum of meaningful problems such as free vibration or buckling of a structure whose average properties are known (such as mass produced structures), vibration of stochastic structures forced by either random or deterministic loading, static response of stochastic structures to either random or deterministic loading, stochastic control systems and many others.

## 2. The Eigenvalue Problem

Instead of discussing the general problem immediately, let us consider the following problem. We have a taut string of length  $L$ , and tension  $T$ . The density of this string is

$$\rho(x) = \rho_0 [1 + \epsilon(x)]$$

where  $\epsilon(x)$  is a random variable which is bounded by a small number.

$$P[|\epsilon(x)| > a] = 0$$

and

$$a \ll 1.$$

The equation which governs the small amplitude free motion of this string is

$$T \frac{\partial^2 y}{\partial x^2} - \rho_0 [1 + \epsilon(x)] \frac{\partial^2 y}{\partial t^2} = 0$$

where  $y(x,t)$  is the displacement of the string from its neutral position.

An eigenvalue problem arises when  $y(x,t) = Y(x)e^{i\omega t}$ , and  $y(0,t) = y(L,t) = 0$ .

Namely,

$$\frac{d^2 Y}{dx^2} + \lambda [1 + \epsilon(x)] Y = 0$$

where  $\lambda = \frac{\rho_0}{T} \omega^2$ .

We restrict our attention to the fundamental mode (although the same technique can be applied to any mode), and realize that for each function  $\epsilon(x)$  there is a corresponding  $\lambda$ . That is,  $\lambda$  is a functional of  $\epsilon(x)$  on the interval  $[0,L]$ . It can be shown that  $\lambda$  is a continuous functional, and therefore can be represented as follows

$$\begin{aligned} \lambda = \lambda_0 + & \int_0^L K_1(\xi) \epsilon(\xi) d\xi + \int_0^L \int_0^L K_2(\xi_1, \eta) \epsilon(\xi) \epsilon(\eta) d\xi d\eta \\ & + \dots \int_0^L \dots \int_0^L K_n(\xi_1, \xi_2, \dots, \xi_n) \epsilon(\xi_1) \dots \epsilon(\xi_n) d\xi_n \dots d\xi_1 \dots d\xi_n \\ & + \dots \dots \dots \end{aligned} \quad (1)$$

(c. f. Volterra [2]).

We recall that  $|\epsilon(x)|$  is uniformly bounded by a small number and, as an approximation, truncate the above expression after the first integral, so

$$\lambda \doteq \lambda_0 + \int_0^L K_1(\xi) \epsilon(\xi) d\xi. \quad (2)$$

$\lambda_0$  is the eigenvalue corresponding to  $\epsilon(x) \equiv 0$ ,  $K_1(\xi)$  is evaluated by taking the functional derivative (c. f. [2]). For this problem

$$K_1(\xi) = -\frac{2\pi^2}{L^3} \sin^2 \frac{\pi\xi}{L}.$$

Hence,

$$\lambda \equiv \frac{\pi^2}{L^2} - \frac{2\pi^2}{L^3} \int_0^L \sin^2 \frac{\pi\xi}{L} \epsilon(\xi) d\xi. \quad (3)$$

Since  $\epsilon(\xi)$  is a random variable, so is  $\lambda$ . The problem is completed when the statistical properties of  $\lambda$  are related to those of  $\epsilon(\xi)$ . For the remainder of this report the expected value of a random variable  $u$  is denoted by  $\langle u \rangle$ .

It follows immediately from (3) that

$$\langle \lambda \rangle = \pi^2/L^2 - \frac{2\pi^2}{L^3} \int_0^L \sin^2 \frac{\pi\xi}{L} \langle \epsilon(\xi) \rangle d\xi. \quad (4)$$

In the particular case that

$$\langle \epsilon(\xi) \rangle \equiv 0, \text{ we have}$$

$$\langle \lambda \rangle = \pi^2/L^2$$

and,

$$(\lambda - \langle \lambda \rangle)^2 = \frac{4\pi^4}{L^6} \int_0^L \int_0^L \sin^2 \frac{\pi\xi}{L} \sin^2 \frac{\pi\eta}{L} \langle \epsilon(\xi)\epsilon(\eta) \rangle d\xi d\eta. \quad (5)$$

The quantity  $\langle \epsilon(\xi)\epsilon(\eta) \rangle$  is the autocorrelation of  $\epsilon(x)$  and so we related the mean value and standard deviation of  $\lambda$  to the mean value and autocorrelation of  $\epsilon(x)$ .

This illustrative problem suggests the following general approach.

Suppose

$$M[\epsilon(x)] y(\lambda; + \lambda y(x) = 0$$

where  $M[\epsilon(x)]$  is a linear differential operator which involves the small random variable  $\epsilon(x)$ . The boundary conditions are homogeneous at  $x = 0$  and  $x = L$ . We further suppose the eigenvalue problem has solutions in the neighborhood of  $\epsilon(x) = 0$ . Denote the smallest eigenvalue  $\Lambda$ , and the corresponding value for  $\epsilon(x) = 0$ ,  $\Lambda_0$ . Then by the same reasoning as before we can approximate  $\Lambda$  as follows

$$\Lambda \approx \Lambda_0 + \int_0^L K(\xi) \epsilon(\xi) d\xi.$$

From this it follows that

$$\langle \Lambda \rangle = \Lambda_0 + \int_0^L K(\xi) \langle \epsilon(\xi) \rangle d\xi \quad (6)$$

and if  $\langle \epsilon(\xi) \rangle = 0$

$$(\Lambda - \langle \Lambda \rangle)^2 = \int_0^L \int_0^L K(\xi) K(\eta) \langle \epsilon(\xi) \epsilon(\eta) \rangle d\xi d\eta. \quad (7)$$

The successful application of this method requires calculation of  $K(\xi)$ , but this is not too difficult [2].

### 3. Inhomogeneous Boundary Value Problem

Consider the following class of problems

$$M[\epsilon(x)] y(x) = f(x) \quad (8)$$

where  $M[\epsilon(x)]$  is a linear differential operator depending on the uniformly small function  $\epsilon(x)$ . Furthermore we assume homogeneous boundary conditions at  $x = 0$ ,  $x = L$ . Such problems are easily solved if one knows the Green's function for the problem. The Green's function is, however, dependent on the function  $\epsilon(x)$ . Let  $G_\epsilon(x, \xi)$  be the Green's function corresponding to a particular  $\epsilon(x)$ . Then the solution to (1) for that  $\epsilon(x)$  is

$$y(x) = \int_0^L G_\epsilon(x, \xi) f(\xi) d\xi. \quad (9)$$

but  $G_\epsilon(x, \xi)$  is a functional of  $\epsilon(\eta)$ , and in the case of uniformly small  $\epsilon(\eta)$  can be approximated

$$G_\epsilon(x, \xi) \cong G_0(x, \xi) + \int_0^L K(x, \xi, \eta) \epsilon(\eta) d\eta.$$

Hence the approximate solution to (1) is

$$y(x) = \int_0^L G_0(x, \xi) f(\xi) d\xi + \int_0^L \int_0^L K(x, \xi, \eta) \epsilon(\eta) f(\xi) d\eta d\xi. \quad (10)$$

Where  $G_0(x, \xi)$  is the Green's function corresponding to  $\epsilon(\eta) \equiv 0$ , and  $K(x, \xi, \eta)$  is the functional derivative of the Green's function with respect to  $\epsilon(\eta)$  evaluated at  $\epsilon(\eta) \equiv 0$ . The above formulation is applicable for random  $f(\xi)$  as well, and taking averages we obtain

$$\langle y(x) \rangle = \int_0^L G_0(x, \xi) \langle f(\xi) \rangle d\xi + \int_0^L \int_0^L K(x, \xi, \eta) \langle \epsilon(\eta) f(\xi) \rangle d\eta d\xi \quad (11)$$

and, in case  $\langle f(\xi) \rangle = f(\xi)$  and  $\langle \epsilon(\eta) \rangle \equiv 0$

we obtain

$$\langle y(x) = \langle y(x) \rangle \rangle^2 = \int_0^L \int_0^L \int_0^L \int_0^L K(x, \xi, \eta) K(x, s, \tau) f(\xi) f(s) \langle \epsilon(\eta) \epsilon(\tau) \rangle d\xi ds d\eta d\tau. \quad (12)$$



As an example, consider the following problem

$$\frac{d^2 y}{dx^2} + [1+\varepsilon(x)]y = \delta(x-L/2) \quad (13)$$

$$y(0) = y(L) = 0.$$

Physically, this corresponds to a harmonic point load applied at the center of a taut string whose density is constant plus a small random variation. We further assume that

$$\langle \varepsilon(x) \rangle = 0 \quad (14)$$

and 
$$\langle \varepsilon(x) \varepsilon(y) \rangle = A e^{-b|x-y|}. \quad (15)$$

For equation (13) and the boundary conditions, one obtains

$$K(x, \xi, \eta) \left\{ \begin{array}{ll} \frac{\sin x \sin(L-\xi)}{2\sin L} [\cot(L-\xi) - \cot L] & x < \xi < \eta \\ - \frac{\sin x \sin(L-\xi)}{2\sin L} \cot L & x < \eta < \xi \\ \frac{\sin x \sin(L-\xi)}{2\sin L} [\cot x - \cot L] & \eta < x < \xi \\ \frac{\sin \xi \sin(L-x)}{2\sin L} [\cot(L-x) - \cot L] & \xi < x < \eta \\ - \frac{\sin \xi \sin(L-x)}{2\sin L} \cot L & \xi < \eta < x \\ \frac{\sin \xi \sin(L-x)}{2\sin L} [\cot \xi - \cot L] & \eta < \xi < x \end{array} \right. \quad (16)$$

From eqs. (11), (14), (12), (15) we obtain

$$\langle y(x) \rangle = 0,$$

$$\begin{aligned}
& \frac{(y(x) - \langle y(x) \rangle)^2}{\Lambda} = \\
& \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} [\cot \frac{L}{2} - \cot L] \right\}^2 \frac{2}{b} [x - \frac{1}{b} + \frac{1}{b} e^{-bx}] \\
& + \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} \cot L \right\}^2 \frac{2}{b} [(\frac{L}{2} - x) - \frac{1}{b} + \frac{1}{b} e^{-b(\frac{L}{2} - x)}] \\
& + \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} [\cot x - \cot L] \right\}^2 \frac{2}{b} [\frac{L}{2} - \frac{1}{b} + \frac{1}{b} e^{-\frac{L}{2}}] \\
& + \frac{2}{b^2} (e^{-\frac{L}{2}} - e^{-bx}) (e^{bx} - 1) \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} \cot L \right\} \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} (\cot \frac{L}{2} - \cot L) \right\} \\
& - \frac{2}{b^2} (e^{-bL} - e^{-\frac{L}{2}}) (e^{bx} - 1) \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} \cot L \right\} \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} (\cot x - \cot L) \right\} \\
& + \frac{2}{b^2} (e^{-bL} - e^{-\frac{L}{2}}) (e^{\frac{L}{2}} - e^{bx}) \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} \cot L \right\} \left\{ \frac{\sin x \sin \frac{L}{2}}{2 \sin L} (\cot x - \cot L) \right\}
\end{aligned}$$

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