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Experimental Modelling of Network-Like Systems

by

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ABSTRACT

Techniques for construction of mathematical models of linear, time-invariant, network-like systems are presented. The problem considered falls in the general category of the identification problem in control system theory. Noiseless measurements are assumed for development of the algorithms.

The thesis covers two areas. The first area is concerned with time-domain techniques for determining the system and input matrices. First, a model adaptive algorithm is derived assuming the entire state input vector is measured. Secondly, a non-linear observer parameter estimator technique is presented for the case where a subset of states and the input vector is measured.

The second area is concerned with extracting the parameter submatrices of the network-like system from the system input and system matrices.

Some theoretical results are presented.

The report is a facsimile of the thesis prepared in partial satisfaction of the requirements for the Degree of Doctor of Philosophy awarded by the Division of Systems Engineering, Case-Western Reserve University, Cleveland, Ohio.

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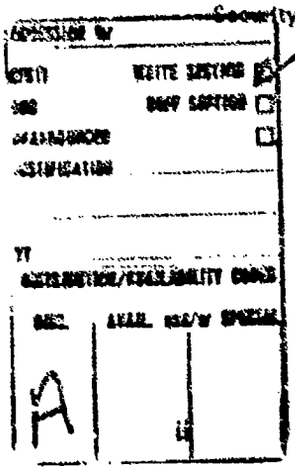

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FOREWORD

This document presents the theoretical results of a study designed to derive techniques for constructing experimental mathematical models of linear, time-invariant, network-like control systems. The study, which is addressed fundamentally to the identification problem in control-system theory, covers two principal areas (1) time-domain techniques for determining system and system input matrices and (2) the extraction of parameter submatrices from system input and system matrices. The report discusses the basic philosophical approaches taken by others in this field toward solving the identification problem, including the model adaptive approach, the black box approach, computational techniques, quasilinearization, the differential approximation, the algebraic realization, the instrumental variable technique, and the algebraic model adaptive technique. While all these approaches are considered, the basic one applied in this study is that of model adaptive identification.

This work, which is directly applicable to various current NWC control-system projects, was performed as part of an academic course pursued during the period September 1966 through February 1971, and was funded in part under the NWC educational fellowship program.

The report is a facsimile of the thesis prepared in partial satisfaction of the requirements for the Degree of Doctor of Philosophy awarded by the Division of Systems Engineering, Case-Western Reserve University, Cleveland, Ohio.

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Notation

$\underline{x}$	-	vector quantity
A	-	matrix
t	-	time, matrix transpose
R_n	-	n-dimensional real vector space
C_n	-	n-dimensional complex vector space
$R(A)$	-	range of a matrix A
$N(A)$	-	null space of a matrix A
$D(A)$	-	domain of a matrix A
$\underline{a}_{*i}$	-	i-th column of matrix A
$\underline{a}_{i*}$	-	i-th row of a matrix A
$r(A)$	-	rank of a matrix A
F_n	-	n-dimensional vector space with elements from a field F
det ( )	-	determinant of a matrix ( )
tr ( )	-	trace of a matrix ( )
$\cdot \rightarrow \frac{d}{dt}$	-	differentiation
$\otimes$	-	Kronecker (direct) product of matrices
$\oplus$	-	direct sum of matrices
$( )^{-1}$	-	inverse of a matrix ( )
$( )^{I}$	-	generalized inverse of a matrix ( )
ϵ	-	is an element of
[,]	-	closed interval
(,)	-	open interval

Notation

$< , >$	-	inner product
$( )^*$	-	complex conjugate transpose of a matrix ( )
$(\vdots)$	-	partition of a matrix A
$( )_{mn}$	-	mxn matrix ( )
$\| \|$	-	vector or matrix norm

CHAPTER I

INTRODUCTION AND BACKGROUND

INTRODUCTION

The general topic of the thesis deals with what has come to be known in the control system literature as the "identification" problem. Given a physical system which is excited with an input function $\underline{u}(t)$, $t\in[0,T]$, the corresponding output function $\underline{y}(t)$, $t\in[0,T]$ is measured. One is interested in using the input-output data to construct a mathematical model which will produce an output which is sufficiently close to the measured output when the model is excited with the measured input.

There are, of course, many variations of the above viewpoint. A broad classification is into stochastic and deterministic identification problems.

In the stochastic identification problem, one allows that the measurements have noise superimposed on the true input-output functions. No noise is assumed present in the deterministic case. Further, within each of the stochastic and deterministic classifications, there are subclassifications into linear and nonlinear identification problems. These subclassifications refer to the general properties of the model. Each linear or nonlinear

identification problem also may be categorized into model types such as static, differential equation, difference equation, transfer function, and pulse transfer function types.

Differential, static and difference equation models may be called either multi-input or multi output models if there are more than one input and output. Transfer function and pulse transfer function models are generally single input-single output models. Additionally, differential equation and transfer function types have a continuous independent variable, generally, time. Difference equation and pulse transfer function types have a discrete independent variable.

From the above discussion, there exists a complex system of classifying the types of models which may be used to try to match a given set of input-output functions. No guarantee exists that a given model can match a given input-output function, hence, considerable insight and knowledge of the tested system may be necessary to choose a model which will match the measured input-output data within the desired error.

BACKGROUND

A convenient method of discussing solutions to the identification problem is to classify some basic philosophical approaches, and then to discuss the various computational techniques applicable to each approach.

The Model Adaptive Approach. - In the model adaptive approach, one assumes a model for the system of known structure. By the term, known structure, it is meant that everything is assumed known but the parameters of the model. That is, if there are system nonlinearities, the general form of the nonlinearity is known but any multiplier or additive values are unknown. Also, the form of the matrices which describe the system are known. A typical example would be a companion matrix where at most n of the n^2 possible matrix elements are unknown. After the assumption of a model, the identification problem then becomes one of adjusting the model parameters in such a fashion that the error between the model response and system response to the same input is sufficiently small in some sense. All available knowledge and physical principles pertinent to the tested system are useful in the model adaptive approach. Such knowledge includes constraints on the parameter values, types of nonlinearities, and system order.

The Black Box Approach. - In the black box approach, one assumes as little as possible about the tested system. The usual expectation is that the system is linear, but test signals of varying amplitude are used to confirm the linear expectation. Most commonly, the test signal used is a sinusoidal function and the results are typically displayed as a frequency response curve of amplitude and phase as a function of input frequency. The single input-single output type of system is the most common target of the black box approach.

Computational Techniques. - Many and varied types of computational techniques have been developed primarily for the model adaptive approach. A brief discussion of the dominant techniques is presented.

Quasilinearization. - Quasilinearization is a computational technique developed primarily by Bellman and his coworkers [IT-1]. Extensions of the technique have been made by Lee [IT 4] and Sage [CT-4]. It is an iterative procedure based on a generalization of the Newton-Raphson technique for finding the roots of a nonlinear equation. Both linear and nonlinear systems are modelled using quasilinearization. If the technique converges, the convergence is quadratic in nature. There is, of course, no a priori guarantee of convergence. Also, the region of convergence

may be large or small. Measurements required are the system input-output measurements.

Differential Approximation. - Differential approximation is a computational technique where knowledge of the system inputs, states and state derivates is required. Bellman [IT-1] and his coworkers were instrumental in developing this technique. It is basically a gradient technique where an integral of a suitable norm of the difference between the known state derivative and the analytical expression (dependent upon known states and unknown parameters) for the state derivative is minimized. Minimization of the integral yields a set of nonlinear algebraic equations in the parameters. It is often used as a starting routine for other more sophisticated techniques. Linear and nonlinear systems may be modeled by this technique.

Algebraic Realization. - Identification using algebraic realization theory was developed by one of Kalman's students at Stanford University [IT-3]. The technique is based on the concept of determining the Markov parameters of the system. The Markov parameters are an infinite sequence of matrices which are related to the system parameters. It is assumed that the system is linear and time invariant. Based on the above assumption, the identification problem is composed of two sub-problems. The first

sub-problem is concerned with linear algebraic operations on the data to determine a finite number of the Markov parameters. Secondly, linear algebraic operations on the Markov parameters are used to determine output, system and input matrices of the system. There is a certain amount of arbitrariness in the second part of the problem. Consequently, the results are not unique, but, vary within a similarity transformation. The required measurements are input and output data.

Instrumental Variable Technique. - Young [IT-10, IT-11] and Wong [IT-12] have developed computational procedures using instrumental variables. The technique is designed primarily for single input-single output linear, time-invariant systems. Noisy input-output data is measured and used in recursive algorithms for real time parameter estimation. To bypass the necessity to determine derivatives of the input and output explicitly, the input and output are passed through a series of "state variable" filters whose outputs constitute the instrumental variables. By proper choice of the filters, the instrumental variables and parameters have the same relationship to each other as do the original system states and parameters. Consequently, by measuring the instrumental variables, the parameters can be computed.

Algebraic Model Adaptive Technique. - Yore[IT-9] presented a thesis which considered two separate but related problems. The first problem was determining the system and input matrices of a linear time-invariant system where the entire state and input vectors were measured. An iterative procedure based on linear algebraic techniques was used to cause a model state to converge to the measured state. The second problem was to assume that the system generating the data was a network-like system, and to devise methods of determining the values of the components contained in the system and input matrices. For example, if $a_{11} = r_1 c_1$ is an element of the system matrix, what is the value of $r_1 c_1$? Yore's work was apparently the first to address the so-called "component parameter" identification for network-like systems. The "component parameter" identification is the second part of the present thesis.

PROBLEM STATEMENT AND ASSUMPTIONS

Yore's [IT-9] work formed the motivation for the present thesis. It was felt that his techniques were useful but could be improved.

Basic Assumptions. - The following assumptions were made:

1. The measured data is noiseless and generated by a network-like system.
2. System topology is known, i.e., it is known what types of components are present and how they are interconnected.
3. Actual parameter values are unknown.
4. The system is linear and time invariant over the measurement interval.

Problem Statement. - Determine techniques and conditions to identify the parameters of the system described in the above assumptions.

CHAPTER II

RESULTS, CONCLUSIONS, AND RECOMMENDATIONS

INTRODUCTION

A summary of results and comparisons with some corresponding results of Yore is presented. Also included in the chapter are conclusions based on the results and recommendations for further research.

SUMMARY OF RESULTS

A chapter-by-chapter summary of the research results is given.

Chapter III, Model Adaptive Identification. - The algorithm which adapts the model to fit the measured state is based on linear algebraic computations. Both input and system matrices can be identified using measurements of the state and normal operating inputs. The algorithm is iterative and proceeds from iteration-to-iteration by the following relations.

$$(1) \quad A_m^{n+1} = A_m^n + \Delta A^n$$

$$(2) \quad B_m^{n+1} = B_m^n + \Delta B^n$$

The matrices A_m^n , B_m^n are the model system and input matrices at the n-th iteration, respectively. Computation of the updating matrices is accomplished by the following discrete time equivalent of the continuous time error differential equation.

$$(3)\quad \underline{e}^n(\overline{k+1}T) = e^{A_m^n T}\left\{\underline{e}^n(kt)+\left(\int_o^T e^{-A_m^n \tau}\, d\tau\right)\left(\Delta A^n \Delta B^n\right)\begin{pmatrix}\underline{x}(k)\\ \underline{u}(k)\end{pmatrix}\right\}$$

The error, $\underline{e}^{n\Delta} = \underline{x} - \underline{x}_m^n$, is defined by the measured state, $\underline{x}$, and the computed model state $\underline{x}_m^n$. In (3) $\underline{u}$ is the input and T is the sampling interval. By taking a sufficient number of samples, ΔA^n, ΔB^n are computed using matrices whose columns are defined by (3).

A problem which could occur is that the model has been chosen of incorrect order. Two possible effects of incorrect order are that, (1) the algorithm does not converge, and (2) the algorithm converges to erroneous A,B. The important consideration here is which of the states are neglected. By neglecting strongly excited states, poor results can be achieved.

Another problem which could occur is that the state and input could be measured with some error. If measurement error occurs, then the algorithm will converge to erroneous A,B matrices, if it converges at all. The magnitudes of these errors depend on the A,B,$\underline{u}$, and of course,

the measurement error. The sample interval and norm of the system matrix are related to the error in identification at the first iteration. Nearly one step convergence can be achieved under certain conditions. Some numerical experiments for the first order free case indicate that convergence is guaranteed for $a_m^o \triangleq 0$ if the following conditions exist:

(4) $|aT|$ - arbitrary $a \leq 0$

(5) $aT < 2.5 \quad a > 0$

Example 3 showed that under certain conditions, the requirement to measure the entire state vector could be relaxed by identifying a model whose state could be related to the measured output.

Chapter IV, Parameter Estimation Using Nonlinear Observers. - Adopting the viewpoint that the system parameter could be collected into a vector, $\underline{p}$, with differential equation, $\dot{\underline{p}} = \underline{o}$, the original linear system could be augmented to form a nonlinear system. Using the Kronecker product, (6) through (9) show the form of the augmented system.

(6) $\dot{\underline{x}} = I \otimes \underline{x}^t \underline{a} + I \otimes \underline{u}^t \underline{b}$

(7) $\dot{\underline{a}} = \underline{o}$

(8) $\dot{\underline{b}} = \underline{o}$

(9) $\underline{y} = C\underline{x}$

The vectors $\underline{a}$, $\underline{b}$ are formed by collecting the rows of the system matrix A and the input matrix B into the vectors $\underline{a},\underline{b}$, respectively. Not all of the states are measured. The output matrix C is of order mxn, and the input, system matrices are of order nxp, nxn, respectively.

A nonlinear observer of dimension $n+n^2+np$ is constructed whose state vector asymptotically approaches the vector $(\underline{x},\underline{a},\underline{b})^t$ under certain conditions. Success of the scheme depends on one's ability to stabilize the observer. Since the observer is nonlinear and, generally, time-varying, stabilization may be difficult. A procedure for stabilization was developed.

Chapter V - Estimation of Parameter Sub-Matrices. Chapter V assumed that the system to be identified was a network-like system. Therefore, the A,B matrices have a special structure. Assumption of the network-like system constrains the elements of A,B. The constraints appear by forcing certain elements of A,B to be either zero or one. Other elements are related by algebraic equations to each other. Thus, information about A,B can be determined prior to actual testing of the system.

Knowledge of the state, input vectors, and A,B matrices may not be sufficient to determine the parameter matrices uniquely. Some results are given which show where additional measurements should be taken to give unique results. Both linear and nonlinear techniques for determining the system parameters are given. It is shown that for a special class of networks, all parameters may be determined from the A,B matrices.

COMPARISON WITH YORE'S RESULTS

The notation in Chapter III is used to compare the results achieved here with those by Yore. Table 1 shows the difference in the algorithms.

It is the author's opinion that Yore unnecessarily restricted his technique by requiring the sample interval be sufficiently small that $e^{AT} \sim I+AT$.

If Yore's time requirement is used for the author's algorithm, then one step convergence is achieved. One step convergence is not achieved for the same sample time using Yore's algorithm.

Two restrictions on the sample time are given by Yore. For the first order autonomous case, it is necessary and sufficient that $|aT| \leq Ln2$ for convergence of the algorithm. For higher order, non diagonal, autonomous case, it is

Table 1. Comparison of Results of Yore and Gates

	YORE	GATES
Autonomous Case	$\Delta AXO = \frac{1}{T} E1$	$\left(\int_0^T e^{-A_m \tau} d\tau\right) \Delta AXO = E1 - e^{A_m T} EO$
Non-Autonomous Case	$(\Delta A \Delta B) \begin{matrix} X & U \\ U & O \end{matrix} = \frac{1}{T} E1$	$\left(\int_0^T e^{-A_m \tau} d\tau\right) (\Delta A \Delta B) \begin{vmatrix} X & O \\ U & O \end{vmatrix} = E1 - e^{A_m T} EO$

First Iteration Results for $A_m^o \doteq (0)$ and Autonomous Case

$$A_m^1 = \frac{1}{T}\left(e^{A\underline{T}}\, XO_m XO^{-1}\right) * \qquad A_m^1 = \frac{1}{T}\left(e^{A\underline{T}_I}\right)$$

* $XO_m \triangleq (\overbrace{\underline{x}_o, \underline{x}_o, \underline{x}_o, \ldots, \underline{x}_o}^{n \text{ - times}})$

sufficient that $n \max_{i,j} |a_{ij}| \leq Ln2$ for the identification to converge. The results of Chapter III show that, at least for the first order case, the sampling time requirements are much less restrictive than Yore presents.

Comparing Yore's results with those in Chapter V on determination of the parameters in the A,B matrices is difficult because the approaches are fundamentally different.

Yore's main effort was characterization of a sufficient number of measurements in terms of his concepts of statically and dynamically independent variables. His results are phrased in terms of the existence of such sets of variables. No procedures or techniques are given to determine which network currents and voltages constitute statically and dynamically independent sets of variables. No direct use of system topology was made.

In contrast, the effort in Chapter V was based exclusively on detailed knowledge of system topology. The main effort was to use the topology, A,B matrices, state, and input vectors to determine the parameter submatrices.

As a byproduct of the computations, the need for additional measurements other than state and input is determined. Direct application of linear algebraic techniques, principally, determination of ranks of various matrices, is required.

CONCLUSIONS

The basic algorithm in Chapter III is effective and works well for the cases in which it is applicable. Its capability of providing nearly one step convergence should prove useful. The two principle effects of measurement noise and/or incorrect model order are, (1) non-uniqueness of results, and (2) unique but erroneous results.

Measurement noise can be significant if the states are weakly excited. The data matrix which must be inverted becomes ill-conditioned. A consequence of the ill-conditioning is that small errors in the data matrix elements cause large errors in the updating matrices. The errors can cause the iterations to oscillate about the correct values of A,B matrices. A filter can be cascaded with the basic algorithm to smooth the fluctuations.

Knowledge of the system structure can be used to develop a nonlinear observer which has on-line potential for combined state and parameter estimation. The observer is best suited for analog or hybrid mechanization because a large number of first order differential equations with relatively widely separated time constants need to be integrated.

The critical problem is insuring that the observer is stable. A complicating factor is the requirement for the

feedback for each parameter equation to decrease to zero while keeping the overall system stable. A technique was developed to stabilize the observer.

Knowledge of the system topology is necessary for unique parameter determination for the network-like case. Straightforward results and techniques have been demonstrated for parameter determination.

RECOMMENDATIONS FOR FUTURE WORK

Future research effort should be directed to answer the following questions. Most work should be directed to the computational aspects of the problem.

1. How can the effects of widely spread eigenvalues be handled computationally?

2. If the measurements are corrupted with noise, how can the results of statistical estimation theory be applied to give "good" estimates of A,B using the algorithm of Chapter 3?

3. How can the effects of incorrect model order be minimized?

4. How can the sample interval for the basic algorithm be chosen for most rapid convergence? Should it be modified from iteration to iteration?

5. How can the model adaptive algorithm be used where only a subvector of the state vector is measured? That is, when can additional operations on output data such as integration be used to identify another model whose parameters are related to the original parameters by known relations?

CHAPTER III

MODEL ADAPTIVE IDENTIFICATION

INTRODUCTION

A model adaptive algorithm is derived under the usual assumption that the pair [A,B] is completely controllable [Kalman, CT-2]. Further, it is assumed that the state vector and input vector are measured perfectly. After the algorithm is developed, the effects of measurement error and incorrect model order are considered. Some uniqueness results and computational rules of thumb are presented.

The physical system is assumed to be governed by the constant coefficient, vector-matrix differential equation (10).

(10) $\dot{\underline{x}}(t) = A\underline{x} + B\underline{u}(t)$, $\underline{x}_o = \underline{x}(0)$ $t\epsilon[0,t_f]$

and that $\underline{x}(t)$ and $\underline{u}(t)$ are n and p vectors measured perfectly on the interval $[0,t_f]$. A, B are not known.

A model of the system is assumed to have the constant coefficient vector-matrix differential equation (11).

(11) $\dot{\underline{x}}_m = A_m\underline{x}_m + B_m\underline{u}$, $x_m(0) \triangleq \underline{x}_0$

The chart records of $\underline{x}(t)$, $\underline{u}(t)$ are divided into n+p+1 samples which are used to determine the A, B matrices.

By assuming a pair $[A_m, B_m]$, integrating (11) and comparing the measured and model states at the sampling instants $t_i, i = 0,1,2,\ldots,n+p+1$, the model is forced to converge to the system represented by (10). The updating scheme is algebraic.

DERIVATION OF ALGORITHM

The method of forcing the model data to converge to the measured data is to define the following quantities. ΔA, ΔB are the updating matrices.

(12) $A \triangleq A_m + \Delta A$

(13) $B \triangleq B_m + \Delta B$

In terms of the defined variables in (12) and (13), equation (10) is rewritten.

(14) $\dot{\underline{x}} = A_m\underline{x} + \Delta A\underline{x} + B_m\underline{u} + \Delta B\underline{u}$

Let $\underline{e}(t) = \underline{x} - \underline{x}_m(t)$ denote the difference between the measured data and the model data, then by subtracting (11) from (14) the error differential equation is (15).

(15) $\dot{\underline{e}} = A_m\underline{e} + \Delta A\underline{x} + \Delta B\underline{u}, \ \underline{e}(0) \triangleq \underline{0}$

Since $\underline{e}$, $\underline{x}$, $\underline{u}$ are known, ΔA, ΔB can be computed. In order to compute ΔA, ΔB (15) is converted to the equivalent discrete time system (16).

$$(16) \qquad \underline{e}(k+1) = e^{A_m T}\left\{\underline{e}(k) + \left(\int_0^T e^{-A_m \tau} d\tau\right)(\Delta A \Delta B)\begin{pmatrix}\underline{x}(k)\\ \underline{u}(k)\end{pmatrix}\right\}$$

where $T = t_{i+1} - t_i$, $i = 0,1.2,\ldots,n+p$ and is the sampling interval.

The error and input-output vectors are formed into the following matrices.

$$(17) \qquad UO = \left(\underline{u}(o),\ldots,\ \underline{u}(n+p-1)\right)$$

$$(18) \qquad XO = \left(\underline{x}(o),\ldots,\ \underline{x}(n+p-1)\right)$$

$$(19) \qquad EO = \left(\underline{e}(o),\ldots,\ \underline{e}(n+p-1)\right)$$

$$(20) \qquad E1 = \left(\underline{e}(1),\ldots,\ \underline{e}(n+p)\right)$$

Using (16), the matrices defined by (17) - (20) are related as follows:

$$(21) \qquad E1 = e^{A_m T}\left\{EO + \left(\int_o^T e^{-A_m \tau} d\tau\right)(\Delta A \Delta B)\begin{pmatrix}XO\\ UO\end{pmatrix}\right\}$$

Assuming the existence of the inverse of $\begin{pmatrix}XO\\ UO\end{pmatrix}$, then, the updating equations to improve the model in an iterative fashion are:

$$(22) \qquad A_m^{k+1} = A_m^{k} + \Delta A$$

$$(23) \qquad B_m^{k+1} = B_m^{k} + \Delta B$$

$$(24) \qquad (\Delta A \Delta B) = \left(\int_0^T e^{-A_m \tau} d\tau \right)^{-1} \left\{ e^{-A_m T} E1 - E0 \right\} \begin{pmatrix} XO \\ UO \end{pmatrix}^{-1}$$

EFFECTS OF MEASUREMENT ERROR

The effects of measurement error on the A,B matrix errors are formulated in the present section.

Assume that the actual state and input vectors are measured with errors $\underline{e}_x$ and $\underline{e}_u$ and define the errors by (25) - (28).

$$(25) \qquad \hat{\underline{x}} = \underline{x} + \underline{e}_x$$

$$(26) \qquad \hat{\underline{u}} = \underline{u} + \underline{e}_u$$

$$(27) \qquad \hat{A} = A + EA$$

$$(28) \qquad \hat{B} = B + EB$$

EA, EB are identification errors induced by $\underline{e}_x$, $\underline{e}_u$. The identification algorithm operates on the apparent system given by (29).

$$(29) \qquad \dot{\hat{\underline{x}}} = \hat{A}\,\hat{\underline{x}} + \hat{B}\,\hat{\underline{u}}, \quad \hat{\underline{x}}(0) = \hat{\underline{x}}_o$$

Subtracting the actual system equation $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{x}(o) = \underline{x}_o$ from (29) yields the equation relating the measurement error to the induced identification errors EA, EB in (30).

$$(30) \qquad \dot{\underline{e}}_x = A\underline{e}_x + B\underline{e}_u + (EA)\hat{\underline{x}} + (EB)\hat{\underline{u}}$$

Estimating EA, EB using the discrete time equivalent of (30) yields (31) which is the relationship of the measurement error and the A,B error.

$$(31)\quad \underline{e}_x(k+1) = e^{AT}\left\{\underline{e}_x(k) + \left(\int_0^T e^{-A\tau}d\tau\right)\left(B\underline{e}_u(k) + (EA\ EB)\begin{pmatrix}\hat{\underline{x}}(k)\\ \hat{\underline{u}}(k)\end{pmatrix}\right)\right\}$$

thus, the errors are given by (32).

$$(32)\quad (EA\ EB) = \left[\left(\int_0^T e^{-A\tau}d\tau\right)^{-1}\left\{e^{-AT}El_x - EO_x\right\} - BEO_u\right]\begin{pmatrix}\hat{XO}\\ \hat{UO}\end{pmatrix}^{-1}$$

El_x, EO_x, $\hat{XO}$, $\hat{UO}$ are defined as in the section on the basic algorithm.

EFFECT OF SYSTEM ORDER ERRORS

The present section provides the formulation of the effect of errors in selecting the model system order.

Assume the system is precisely modeled by the state equation,

$$(33)\quad \begin{pmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix} + \begin{pmatrix}B_1\\ B_2\end{pmatrix}\underline{u}$$

where $\underline{x}_1$ is a r-vector of the assumed model order and $\underline{x}_2$ is an n-r vector of the ignored variables. Now letting $\underline{e} = \underline{x}_1 - \underline{x}_m$ and, $B_1 = B_m + \Delta B$, $A_{11} = A_m + \Delta A_{11}$ as in the

basic algorithm, (33) is transformed into a vector matrix differential equation in the error variable and the sub-vector of the ignored variables and is given in (34).

$$(34)\quad \begin{pmatrix} \dot{\underline{e}} \\ \dot{\underline{x}}_2 \end{pmatrix} = \begin{pmatrix} A_m & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \underline{e} \\ \underline{x}_2 \end{pmatrix} + \begin{pmatrix} \Delta A_{11} \\ A_{21} \end{pmatrix} \underline{x}_1 + \begin{pmatrix} \Delta B_1 \\ B_2 \end{pmatrix} \underline{u}$$

The effect of the ignored variables is that of an additional input and can be considered as an unknown disturbance.

Analysis of (34) indicates that the eigenvalues of the error plus ignored variable system matrix are those of the model system matrix A_m and those of the ignored system matrix A_{22}.

An estimate of the effect of the error in system order on the identification can be obtained by the discrete time equivalent of (34).

Decomposing (34) into two equations,

$$(35)\quad \underline{x}_2(k+1) = e^{A_{22}T}\left\{\underline{x}_2(k) + \left(\int_0^T e^{-A_{22}\tau}\, d\tau\right)\left(A_{21}\underline{x}_1(k) + B_2\underline{u}(k)\right)\right\}$$

$$(36)\quad \underline{e}(k+1) = e^{A_m T}\left\{\underline{e}(k) + \int_0^T e^{-A_m\tau}\, d\tau\, \left[\Delta A_{11}\underline{x}_1(k) + \Delta B_1\underline{u}(k) + A_{12}\underline{x}_2(k)\right]\right\}$$

and writing the matrix equation,

$$(37)\quad E1 = e^{A_m T}\left[EO + \left(\int_o^T e^{-A_m \tau}\, d\tau\right)(\Delta A_{11}\, \Delta B_1)\begin{pmatrix} XO_1 \\ UO_1 \end{pmatrix}\right] + e^{A_m T}\left(\int_o^T e^{-A_m \tau}\, d\tau\right)(X_2)$$

and solving for $(\Delta A_{11}\, \Delta B_1)$, the induced error matrices, for the case where E1 = E0 = (0), i.e., $\underline{x}_m \to \underline{x}_1$

$$(38)\quad (\Delta A_{11}\, \Delta B_1) = \left[\left(\int_o^T e^{-A_m \tau}\, d\tau\right)^{-1}\left(-A_{12} X_2\right)\right]\begin{pmatrix} XO_1 \\ UO_1 \end{pmatrix}^{-1}$$

and the matrix X_2 is the n-r xn+p matrix formed from n+p samples of the solution of (35).

By examining (38), one sees that if the data matrix $\begin{pmatrix} XO_1 \\ UO_1 \end{pmatrix}$ is non-singular, then the identification converges to some (unique) matrices A_m, B_m which, in general, are not the same as the submatrices A_{11}, B_1. If the data matrix is singular, then, the identification diverges or at best yields some arbitrary A_m, B_m depending on whether or not one desires to specify the dependent elements of A_m, B_m.

The important result is that incorrect model order is not too critical to convergence of the identification. Additionally, identification using different inputs can cause different error matrices $\Delta A_{11}, \Delta B$. The difference is

caused by the interaction of measured states, ignored states and input. For example, if the input did not excite the ignored variables strongly, then, $\underline{x}_2(t) \approx \underline{o}$ and different (nearly null) ΔA_{11}, ΔB, matrices will result than if the ignored states are strongly excited. A method of checking for correct system order of the model could be exciting the system with different inputs and comparison of the results. No change in A_m, B_m with different inputs would mean correct order. A change in A_m, B_m with different inputs would mean incorrect order. A change in A_m, B_m with different inputs could mean wrong order, but it could also be the result of another problem.

It is possible that the model could be the correct order but some weakly excited states and numerical difficulties cause some changes in A_m, B_m using different inputs. The problem lies primarily with the computational aspects of the identification. Ill-conditioned data matrices caused by weakly excited states are difficult to handle. None of the example problems worked presented any difficulties of this nature.

INVERSION OF DATA MATRICES

A series of results concerning the conditions for the inversion of the data matrices in the basic algorithm are given under the assumption that the data are measured without error.

The following notation is used:

$$(39) \qquad \begin{pmatrix} XO \\ UO \end{pmatrix} = \begin{vmatrix} X_1 & X_2 \\ U_1 & U_2 \end{vmatrix}$$

where $\begin{pmatrix} XO \\ UO \end{pmatrix}$ is an n+px n+p matrix and X_1, U_2 are nxn and pxp matrices.

Theorem #1. Necessary and sufficient conditions for the inverse of (39) to exist are,

(1) $\det (X_1^t X_1 + U_1^t U_1) \neq 0$

(2) $\det (X_2^t X_2 + U_2^t U_2) \neq 0$

Proof: Since the inverse of $\begin{pmatrix} XO \\ UO \end{pmatrix}$ exists if and only if $\det \begin{pmatrix} XO \\ UO \end{pmatrix} \neq 0$, if a bound on the determinant of (39) can be established, then, the theorem is proved. Let $A = (A_1 A_2) = \begin{pmatrix} X_1 & X_2 \\ U_1 & U_2 \end{pmatrix}$, then, $A_1^t A_1 = X_1^t X_1 + U_1^t U_1$ and $A_2^t A_2 = X_2^t X_2 + U_2^t U_2$. By Wegner's theorem [Bodewig, M-1, p. 71], $\det A = (\det A_1^t A_1) \cdot (\det A_2^t A_2)$. Therefore, $\det A \neq 0$ if and only if the theorem is true.

Theorem #2. Sufficient conditions for (39) to have an inverse are,

(1) $\det X_1 \neq 0$

(2) $\det U_2 \neq 0$

(3) $\det (X_1 - X_2 U_2^{-1} U_1) \neq 0$

(4) $\det (U_2 - U_1 X_1^{-1} X_2) \neq 0$

Proof:

$$\text{Let } D = \begin{Bmatrix} \left(X_1 - X_2 U_2^{-1} U_1\right)^{-1} & -X_1^{-1} X_2 \left(U_2 - U_1 X_1^{-1} X_2\right)^{-1} \\ -U_2^{-1} U_1 \left(X_1 - X_2 U_2^{-1} U_1\right)^{-1} & \left(U_2 - U_1 X_1^{-1} X_2\right)^{-1} \end{Bmatrix}$$

and by multiplication $D\begin{pmatrix} XO \\ UO \end{pmatrix} = \begin{pmatrix} XO \\ UO \end{pmatrix} D = I$, hence, $D = \begin{pmatrix} XO \\ UO \end{pmatrix}^{-1}$

CONVERGENCE AND UNIQUENESS

Theorem #3. Sufficient conditions for the uniqueness of the identification are:

(1) $\underline{e}(t) \triangleq \underline{0}$ $t \in [0, t_f]$

(2) that there exist n+p linearly independent vectors $\begin{pmatrix} \underline{x}(t) \\ \underline{u}(t) \end{pmatrix}$ in the open interval $t\varepsilon(0, t_f)$.

Proof: The error equation (15) is $\dot{\underline{e}}(t) = A_m \underline{e}(t) + (A-A_m)\underline{x}(t) + (B-B_m)\underline{u}(t)$, $\underline{e}(0) \triangleq \underline{0}$ and if $\underline{e}(t) \triangleq 0$ on the closed interval $t\varepsilon[0, t_f]$, then $\dot{\underline{e}}(t) = \underline{0}$ in the open interval $t\varepsilon(0, t_f)$. Consequently, the error equation takes the form $\underline{0} = (A-A_m : B-B_m) \cdot \begin{pmatrix} \underline{x}(t) \\ \underline{u}(t) \end{pmatrix}$ $t\varepsilon(0, t_f)$ under the hypothesis of the theorem.

Let Y be the matrix of the linearly independent samples, then, the matrix equation below is valid:

$$(0) = (A-A_m : B-B_m)Y$$

Since Y is non-singular, then, its multiplier must be the null matrix, i.e., $A = A_m$ and $B = B_m$.

Some insight into the selection of the time interval for the basic identification algorithm can be obtained for the free response case. The more general case is considerably more complex and has not yielded to analysis.

Theorem #4. Let the updating algorithm (21) have the initial value $A_o = (0)$ and consequently the first iteration, A_1, be given by $A_1 = \frac{1}{T}(E1-E0)XO^{-1}$, then,

$$\lim_{T \to 0} (A_1-A) = (0)$$

Proof: The proof proceeds by direct computation. Substituting for E1, E0, since $A_o \triangleq (0)$, the expression for A_1 is ,

$$A_1 = \frac{1}{T}\left(X1-X0_m-(X0-X0_m)\right)XO^{-1} \tag{40}$$

Also, by definition, $X1 = e^{AT}XO$ and substituting into (40) it follows that A_1 is given in (41).

$$A_1-A = \frac{1}{T}(e^{AT}-I) \tag{41}$$

Performing the power series expansion and rearranging A_1-A is given in (42).

$$A_1-A = \frac{A^2}{2}T + \ldots + \frac{A^n T^{n-1}}{n!} + \ldots \tag{42}$$

Now taking the limit as $T \to 0$ yields the theorem.

As a remark, (41) shows precisely why one step convergence is not obtained, but also shows that for T sufficiently small, that A_1 can be very close to A. A bound on the difference A_1-A can be obtained, however, the bound will be conservative.

Theorem #7. Let $A_1 = \frac{1}{T}(e^{AT}-I)$, then, the error A_1-A is given by,

$$(43)\quad \|A_1-A\| \le \|A\|^2\frac{T}{2}\, e^{\|A\|T} = \epsilon,\ T,\ \epsilon>0$$

Proof: Using the series expansion for e^{AT}, it follows that [Pipes, M-7, p.133],

$$(44)\quad A_1-A = \frac{1}{T}\cdot\left(\frac{A^2T^2}{2} + \frac{A^3T^3}{3!} + \ldots\right) = \frac{A^2T}{2}\cdot e^{AT},\ \Theta_0 \le \Theta \le 1$$

To get an upper bound, let $\Theta = 1$, and take the norm of (44).

$$(45)\quad \|A_1-A\| \le \left\|\frac{A^2T}{2}\right\| \|e^{AT}\| \le \frac{\|A\|^2T}{2}\, e^{\|A\|T} = \epsilon$$

Equation (45) can be used as a guide for choosing T to get within a specified error on the first iteration. Let $\epsilon/\|A\|$ be the desired fraction error, then, solving (46) for the number $\|A\|T$ and estimating $\|A\|$ to get an estimate of T which will achieve the desired maximum error fraction $\epsilon/\|A\|$.

$$(46)\quad \|A\|T\ e^{\|A\|T} = 2\epsilon/\|A\|$$

Figure (/) shows $\|A\|T$ vs $\epsilon/\|A\|$

$\|A\| \cdot T$ FOR DESIRED FIRST STEP ITERATION ERROR

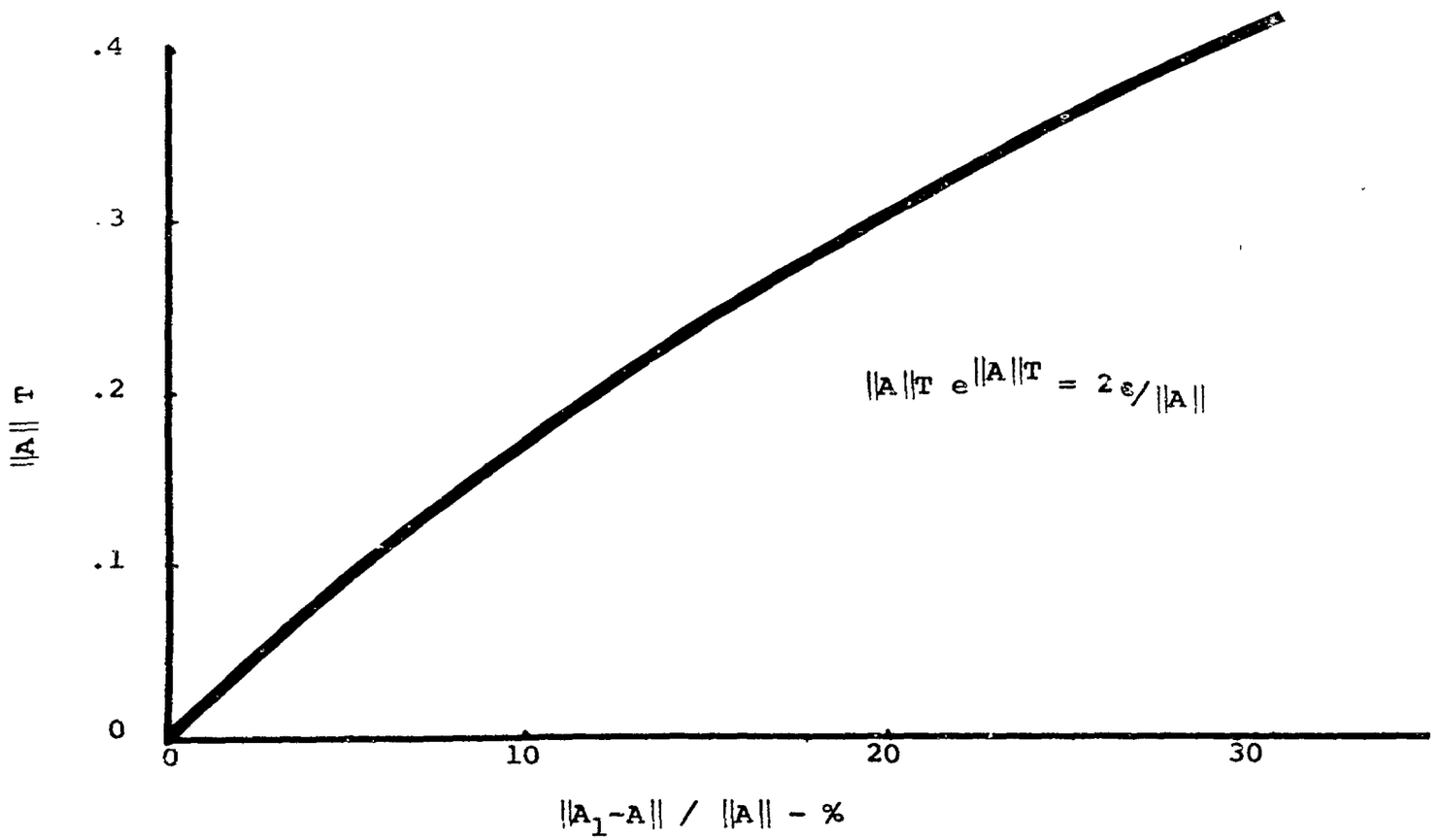


FIGURE 1

A proof that a specific value of aT which guarantees convergence for the first order case (the simple one) has not been achieved. Figures 2 to 4 show the results of some numerical experiments for various values of aT. Figure 2 indicates monotone convergence if aT<0 and the possibility of sustained oscillation if aT>2.5. Figures 3 and 4 verify that the convergence is monotone if aT<0 and oscillatory convergence for aT>0. Based on the numerical results, one should use the following guide for selection of the sampling interval.

Rule 1. $a<0$

T can be arbitrary but should be small to yield to rapid convergence.

Rule 2. $a>0$

$T \leq 2.5/a$ but should also be small to yield rapid convergence.

The following expression for the errors can be derived by simple manipulation of the basic algorithm.

$$(47) \quad \frac{a_{n+1}-a_n}{a_n} = \frac{e^{aT}-e^{a_nT}}{e^{a_nT}-1}, \quad n > 0$$

$$(48) \quad \frac{a_{n+1}}{a_n} = (e^{aT}-1) \frac{e^{aT}-e^{a_nT}(e^{aT}-1)}{e^{a_nT}(e^{aT}-1)_{-1}}, \quad n > 0$$

CONVERGENCE OF FIRST, SECOND ITERATIONS OF FIRST ORDER FREE SYSTEM $\dot{x} = ax$, $x(o) = x_o, a_m(o) \triangleq 0$

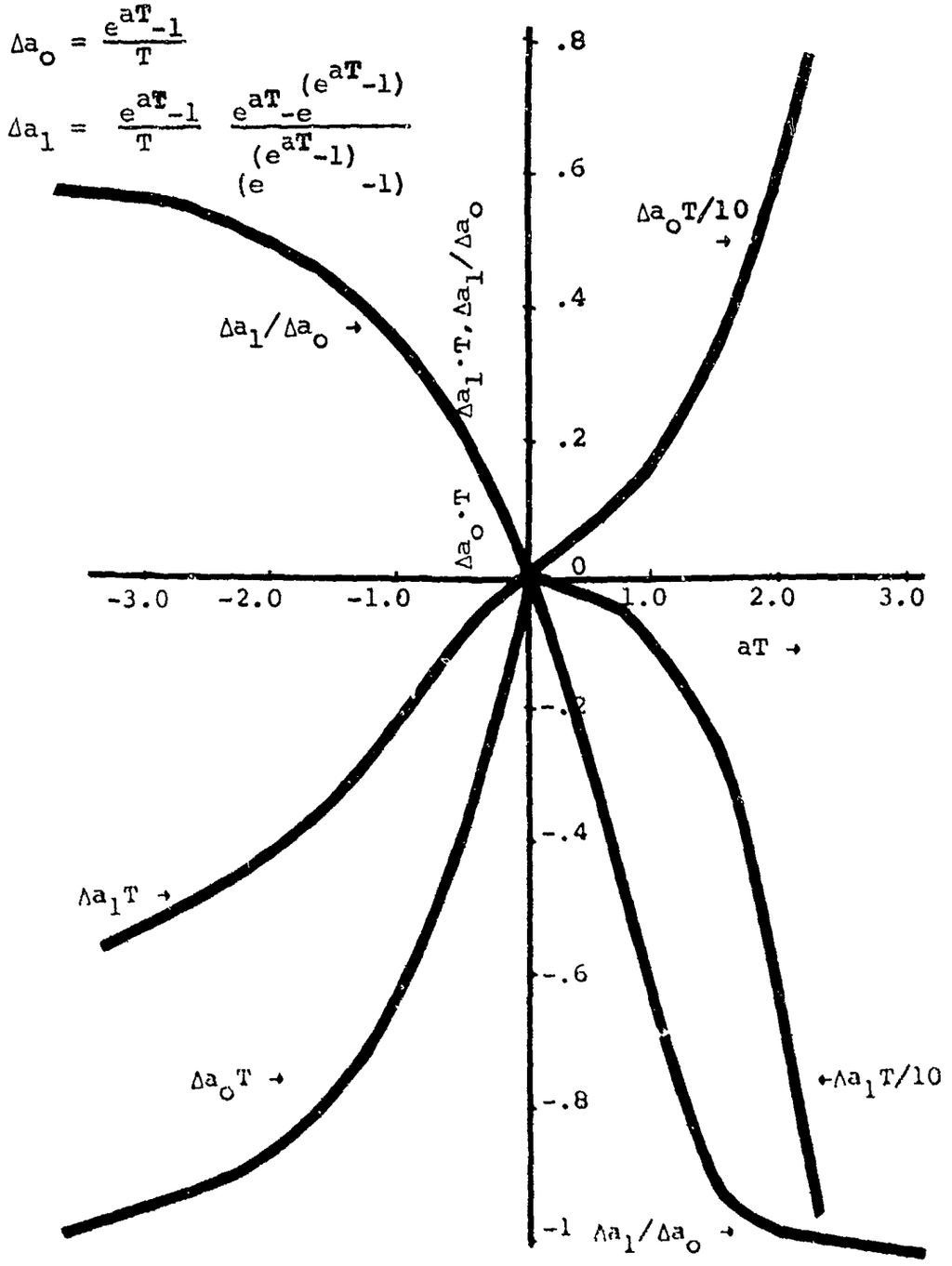


FIGURE 2

CONVERGENCE PROPERTIES FOR FIRST ORDER SYSTEM

$\dot{x} = ax, \quad a < 0$

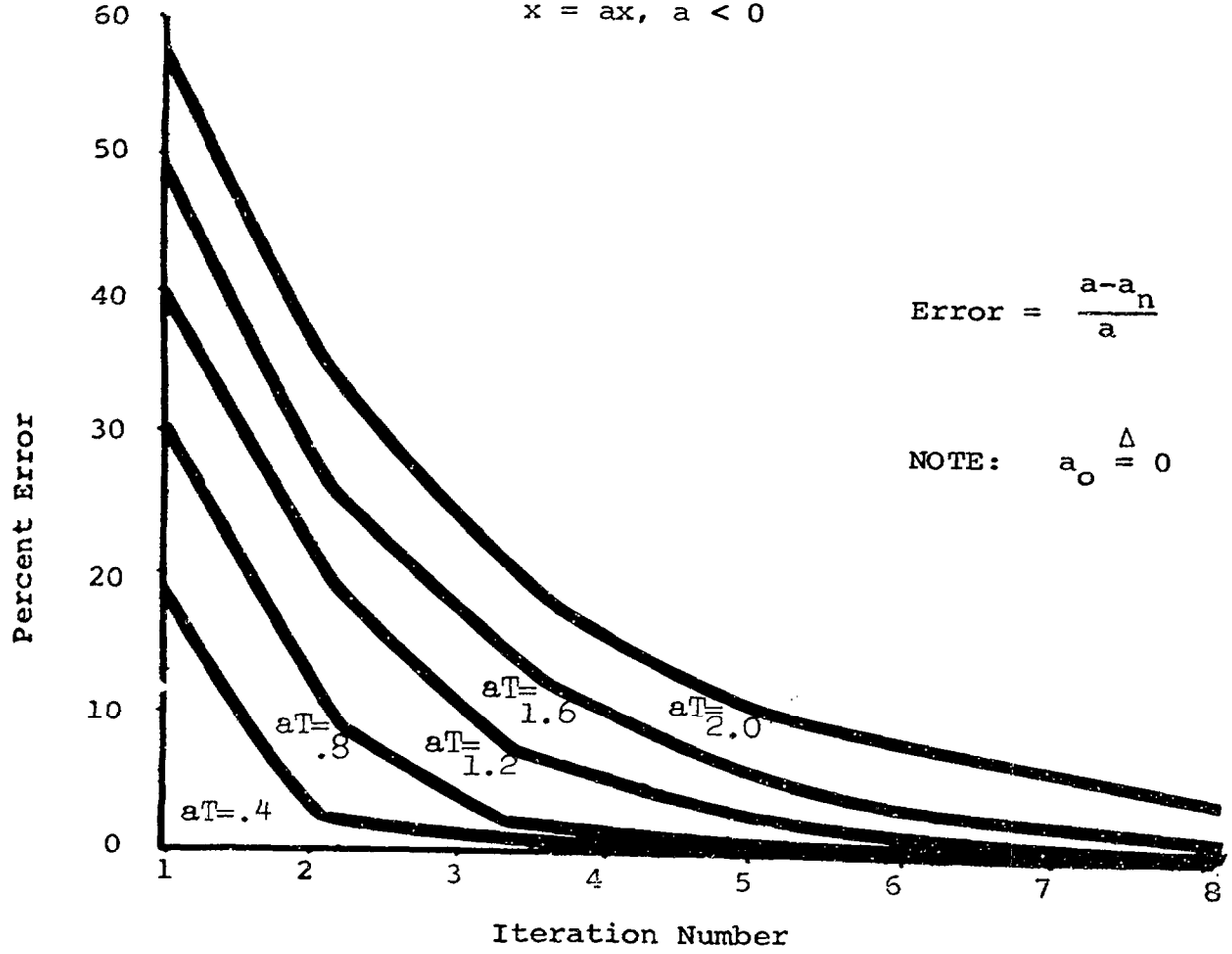


FIGURE 3

CONVERGENCE PROPERTIES FOR
FIRST ORDER SYSTEM

$\dot{x} = ax,\ a > 0$

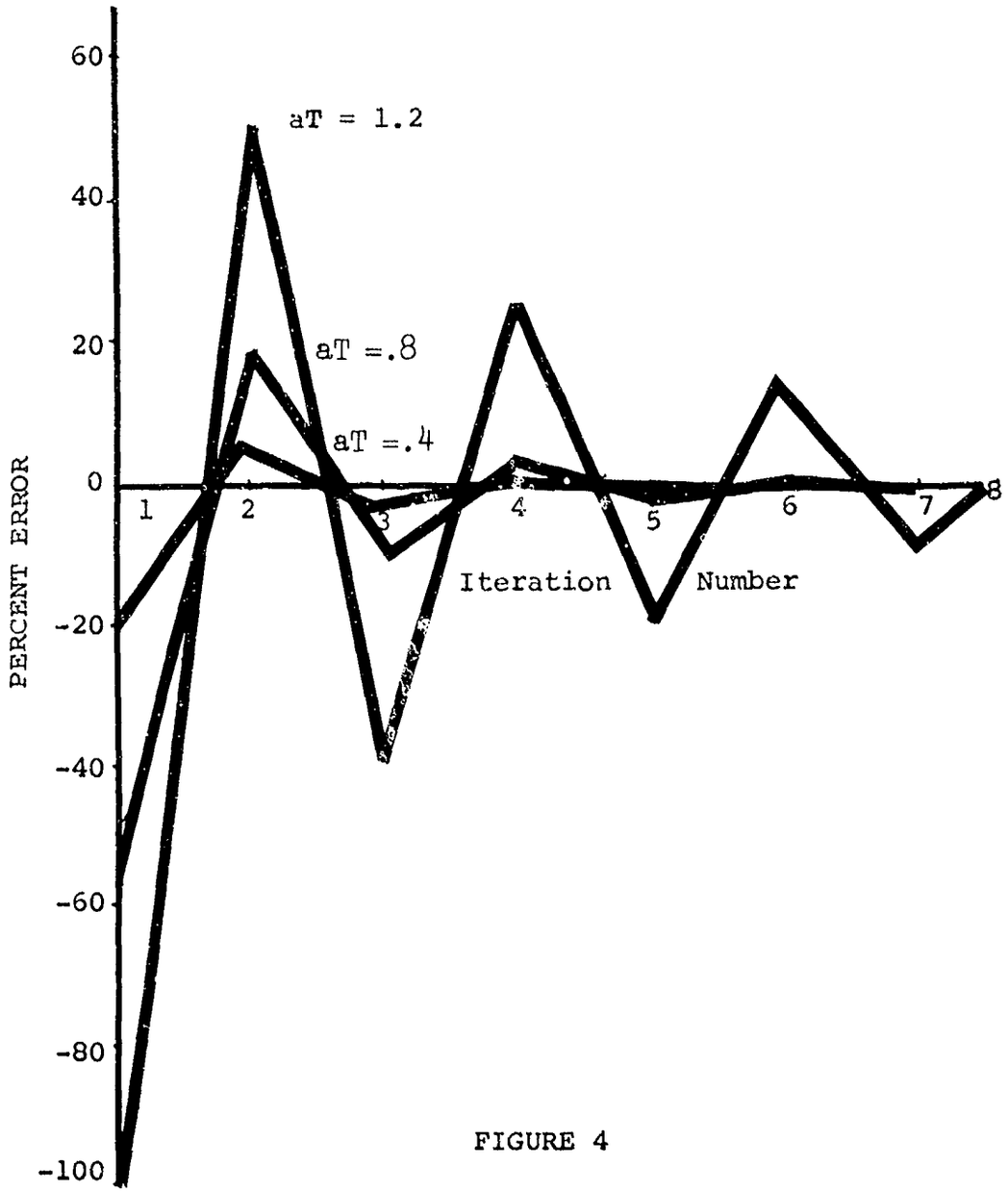


FIGURE 4

NUMERICAL EXAMPLES

Two numerical examples are presented to illustrate the technique. In each case, the initial assumption was that $A_m = (0)$, $B_m = (0)$. Both single input and multi-input systems were identified.

Example #1 - Second order 1 input case: The first example problem attempted was an oscillatory system described by the phase variable model.

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix} \tag{49}$$

$$\underline{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{50}$$

A step input of 2 was used with initial condition $x_1(0) = .278$, $x_2(0) = .536$. Runs were made first assuming $\underline{b}$ was known and then assuming $\underline{b}$ unknown. Convergence to the correct $A, \underline{b}$ was rapid in both cases. A sampling time of .1 second was used.

Table 2. Identification of Second Order Phase Variable System With Known Input Matrix.

ITERATION NUMBER	A_{11}	A_{12}	A_{21}	A_{22}
0	0.0	0.0	0.0	0.0
1	-.1165	1.0375	-3.7528	-2.2839
2	.0039	1.0047	-4.0335	-2.00550
3	.0000	.9996	-4.0014	-2.0017
Correct Value	.0000	1.0000	-4.0000	-2.00000

Table 3. Identification of Second Order Phase Variable System

ITERATION NUMBER	A_{11}	A_{12}	A_{21}	A_{22}	b_1	b_2
0	0	0	0	0	0	0
1	-.1915	.8918	-3.5994	-1.9858	.0495	.8987
2	-.0264	.9900	-3.9945	-2.0177	.0082	.8987
3	-.0068	.9920	-4.0022	-2.0020	.0033	.9996
4	-.0063	.9912	-3.9992	-2.0001	.0032	.9989
Correct Value	.0000	1.0000	-4.0000	-2.0000	.0000	1.0000

Example #2 - Sixth order, 2 input complete circuit: A complete circuit in the sense of Brayton and Moser [FE-3] was chosen for the second example. The network schematic is shown in Figure 5. Inputs to the system were $u_1 = 20.0$, $u_2 = 10.0 + 5 \sin .08\ t$ and the initial conditions were zero. The sampling interval was 0.1 sec.

Again, it was assumed that the A,B matrices were null initially. The numerical results show rapid convergence to the correct values. A general structure for the A,B matrices is shown below.

(51)
$$A = \begin{pmatrix} \frac{-1}{R_1 C_1} & 0 & 0 & 1/C_1 & 0 & 0 \\ 0 & \frac{-1}{R_5 C_2} & 0 & 0 & 1/C_2 & 0 \\ 0 & 0 & 0 & -1/C_3 & -1/C_3 & -1/C_3 \\ -1/L_1 & 0 & 1/L_1 & -R_1/L_1 & 0 & 0 \\ 0 & -1/L_2 & -1/L_2 & 0 & -R_2/L_2 & 0 \\ 0 & 1/L_3 & 1/L_3 & 0 & 0 & -R_3/L_3 \end{pmatrix}$$

(52)
$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/L_1 & 1/L_2 \\ 0 & 0 \end{pmatrix}$$

A COMPLETE CIRCUIT

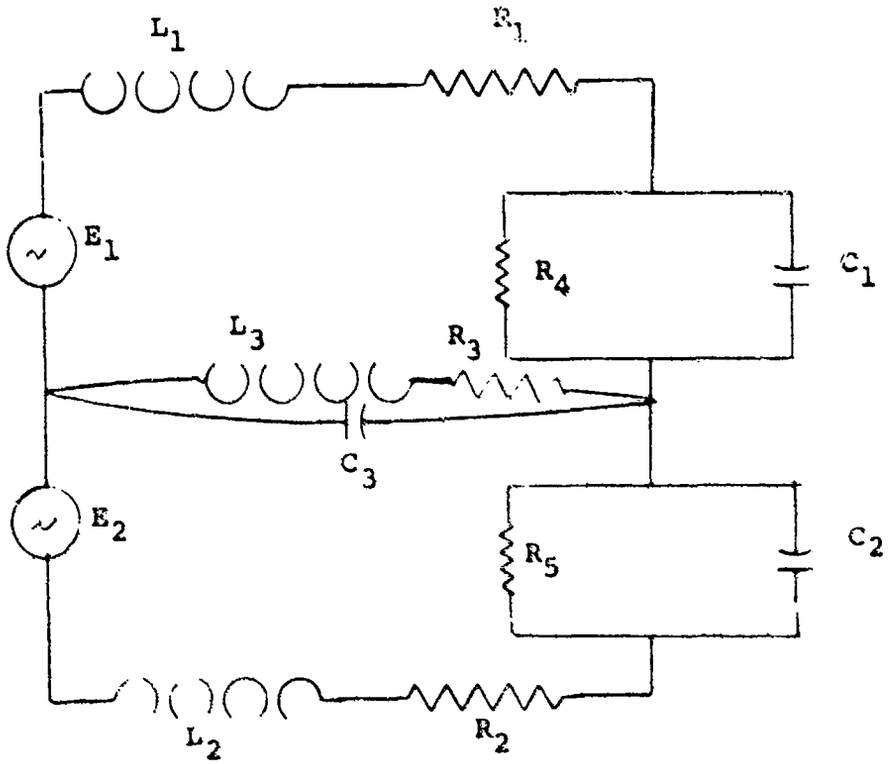


FIGURE 5

Table 4. Identification of Sixth Order, Two Input Complete Circuit

PARAMETER	ITERATION NUMBER 0	1	4	10	CORRECT VALUE
A_{11}	0.0	-6.15	-5.27	-5.01	-5.0
A_{12}	0.0	-2.30	.92	+ .03	0.0
A_{13}	0.0	10.49	-4.11	- .11	0.0
A_{14}	0.0	5.03	10.93	10.02	10.0
A_{15}	0.0	.31	-.04	-.00	0.0
A_{16}	0.0	-1.30	.96	.02	0.0
A_{21}	0.0	-.00	.00	.00	0.0
A_{22}	0.0	-2.18	-2.00	-2.00	-2.0
A_{23}	0.0	- .38	.02	.00	0.0
A_{24}	0.0	.012	-.01	-.00	0.0
A_{25}	0.0	1.40	2.00	2.00	2.0
A_{26}	0.0	.01	-.01	-.00	0.0
A_{31}	0.0	.27	-.03	.00	0.0
A_{32}	0.0	-.05	-.04	-.00	0.0
A_{33}	0.0	-1.53	.24	.01	0.0
A_{34}	0.0	-.66	-1.07	-1.00	-1.0
A_{35}	0.0	.72	1.00	1.00	1.0
A_{36}	0.0	-.48	-1.04	-1.00	-1.0
A_{41}	0.0	-3.23	-11.16	-10.02	-10.0

Table 4. (continued)

PARAMETER	ITERATION NUMBER 0	1	4	10	CORRECT VALUE
A_{42}	0.0	-7.21	1.38	.02	0.0
A_{43}	0.0	29.37	6.64	9.96	10.0
A_{44}	0.0	-7.32	-3.67	-4.00	-4.0
A_{45}	0.0	.97	-.26	-.00	0.0
A_{46}	0.0	-4.10	.49	.00	0.0
A_{51}	0.0	-.05	.05	.00	0.0
A_{52}	0.0	-3.46	-5.08	-5.00	-5.0
A_{53}	0.0	-3.66	-4.83	-5.00	-5.0
A_{54}	0.0	.16	-.03	-.00	0.0
A_{55}	0.0	-4.44	-5.00	-5.00	-5.0
A_{56}	0.0	.13	-.02	-.00	0.0
A_{61}	0.0	.03	-.04	-.00	0.0
A_{62}	0.0	1.84	3.30	3.34	3.34
A_{63}	0.0	1.95	3.19	3.34	3.34
A_{64}	0.0	-.09	.01	-.00	0.0
A_{65}	0.0	.29	.04	.00	0.0
A_{66}	0.0	-6.40	-9.75	-10.00	-10.0
B_{11}	0.0	3.07	-.40	-.00	0.0
B_{12}	0.0	-.31	.24	.00	0.0
B_{21}	0.0	.00	.00	.00	0.0
B_{22}	0.0	.39	.00	-.00	0.0

Table 4. (continued)

PARAMETER	ITERATION NUMBER 0	1	4	10	CORRECT VALUE
B_{31}	0.0	-.36	.07	.00	0.0
B_{32}	0.0	.24	-.02	.00	0.0
B_{41}	0.0	5.85	10.82	10.02	10.0
B_{42}	0.0	-.94	.21	.00	0.0
B_{51}	0.0	.06	-.02	-.00	0.0
B_{52}	0.0	3.87	5.00	5.00	5.0
B_{61}	0.0	-.03	.03	.00	0.0
B_{62}	0.0	.06	-.02	-.00	0.0

Example #3 - Generalization of Algorithm: A possible generalization to allow identification when the entire state is not measured is indicated by an example. The procedure is to define a new state space in which the model parameters may be computed by using output data and the output data integral.

Consider the second order phase variable system in equations (53) - (55).

(53) $\dot{x}_1 = x_2$

(54) $\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + bu$

(55) $y = x_1$

Now define new variables $w_1 = \int_0^t x_1(\tau)d\tau$, $w_2 = \int_0^t x_2(\tau)d\tau$, $v = \int_0^t u(\tau)d\tau$. By integrating (53) - (54) and substituting w_1, w_2, the following state equations in the new states arise.

(56) $\dot{w}_1 = w_2 + x_1(0) = y$

(57) $\dot{w}_2 = a_{21}w_1 + a_{22}w_2 + b_2v + x_2(0)$

The unknown quantities are a_{21}, a_{22}, b_2, $x_2(0)$.

For the special case $x_2(0) = 0$, the model is,

(58) $$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1(0) \\ v \end{pmatrix}$$

The state variables of the above system are known since y, $\int_0^t y(\tau)d\tau$, $x_1(0)$ are known. Application of the basic algorithm is now done straightforwardly.

Example Problem #4: Effects of Model Order Error.

The robustness of the model adaptive algorithm to model order errors is indicated by the present example. A second order model is fitted to data generated by a third order system using a slight modification of the basic algorithm. The modification is that the identification is not stopped when an A,B matrix is determined. That is, the basic algorithm is applied to a data matrix (XO, UO) using n+p measured state and input vectors and an A,B is determined which fits the data. Every vector is then displaced to the left, and a new data vector replaces the last column of the matrices XO, UO. The algorithm is applied to the new data matrix. Previously determined model matrices A,B are used as starting points.

The third order system chosen is a model of a gun turret for use on ships. It is described by the transfer function shown below.

$$(59) \qquad \frac{\Theta}{V} = \frac{12.1}{S\,(S+18)(S+3.5)}$$

Θ is the turret position and V is the voltage applied to the turret drive. A step of three volts was used as the drive input.

Table 5 shows that the basic algorithm is robust in the sense that it converges to a unique answer when a lower order model is fitted to data generated by a higher order system. For the example chosen, the coefficients are close to those obtained by neglecting the most heavily damped pole.

Table 5. Second Order Model Fitted to Third Order Data

Fit No.	Iterations to Convergence	a_{11}	a_{12}	a_{21}	a_{22}	b_1	b_2
1	3	-.001	1.006	-1.182	-2.61	-.000	.532
2	3	-.002	.994	-.151	-3.243	.001	.575
Coefficients obtained by disregarding pole at -18		0.0	1.000	0.0	-3.5	0.0	.535

SUMMARY

A model adaptive identification scheme was developed which yields the system and input matrices of linear, time invariant multi-input systems. Measurements of the state and inputs to the system were assumed.

Solution of linear equations, generation of the model transition matrix and its integral, and integration of the model equations were the primary numerical techniques required.

The primary requirement for unique convergence was the non-singularity of the data matrices.

Some rules of thumb on choice of sampling interval were given.

Several example problems showed that the algorithm is rapidly convergent.

CHAPTER IV

PARAMETER ESTIMATION BY NON-LINEAR OBSERVERS

INTRODUCTION

The previous chapter assumed that all states were measured and developed a model adaptive identification scheme which worked very well for that case. In many cases of engineering interest, it may indeed be possible to measure all these states. However, if the states are not all measured, then a technique is desired for parameter estimation which will work using the measured states which are a subset of the system states. The present chapter discusses parameter estimation using a generalization of linear observers. Preliminary background material is developed for linear observers and the effects of unknown parameters are discussed. Following the background discussion, the theory of non-linear observers/parameter estimators is developed. Finally, a numerical example is presented.

LINEAR OBSERVERS

At the present time, linear observer techniques used for estimating the states of linear systems are well known, principally through the work of Luenberger [SE-2]. In general, observers estimate states with an error which asymptotically approaches zero. The design of these observers

is predicated upon the assumption that one knows the system and the input matrices for the system to be observed. Some things about linear observers which are not well known, however, are the effects of system and input matrix changes on the state estimate. The following brief development shows the effects of parametric changes in the observed system on the observer states. The result of the development shows that the error in a state estimate is dependent not only on the observed system's state and input to the system, but also on the variations in the system and input matrix. It is assumed that the observed system is described by equations (60) and (61) where C is the known output matrix.

(60) $\dot{\underline{x}} = A\,\underline{x} + B\underline{u}$, $\underline{x}(0)$ not known

(61) $\underline{y} = C\,\underline{x}$

Further, assume that the system is erroneously thought to be modeled by equations (62) and (63). Consequently, the observer will be designed using the erroneous system matrix A_m and the erroneous input matrix B_m.

(62) $\dot{\underline{x}}_m = A_m \underline{x}_m + B_m \underline{u}$, $\underline{x}_m(0)$ not known

(63) $\underline{y} = C\underline{x}$

Equation (64) describes the observer.

(64) $\dot{\underline{w}} = F\underline{w} + G\underline{y} + H\underline{u}$

The desired objective is to have the observer state $\underline{w} = T\underline{x} + \underline{e}$ where $\underline{e}$ goes to zero at some satisfactory rate. For the case at hand, T is a non-singular transformation and we would like to take $\hat{\underline{x}} = T^{-1}(\underline{w})$ as the state estimate. If the observer is designed properly, the error will in fact approach zero asymptotically. By performing some algebraic manipulations and subtracting equation (60) from equation (64) the expression for the error in the state estimate is given by equation (65).

$$(65)\quad T^{-1}\dot{\underline{e}} = T^{-1}FT\ T^{-1}\underline{e} + (T^{-1}FT + T^{-1}GC - A_m - \Delta A)\ \underline{x} + (T^{-1}H - B_m - \Delta B)\underline{u}$$

If the state estimate is to asymptotically decay to zero, we need to eliminate the forcing terms and further, ensure that the observer is stable. Since it is thought that equation (62) models the system, we choose the matrices T, F, G, H to satisfy equations (66) and (67).

$$(66)\quad TA_m - FT = GC$$

$$(67)\quad H - TB_m = 0$$

Choosing the observer to satisfy those equations, yields the error in the state estimate shown by equation (68)

$$(68)\quad T^{-1}\dot{\underline{e}} = (A_m - T^{-1}GC)\ T^{-1}\underline{e} - \Delta A\underline{x} - \Delta B\underline{u}$$

It is obvious that the state estimate error is composed of two parts. The first part is the free response resulting from the unknown initial condition $\underline{x}(0)$ since, if it were known, we could set $\underline{e}(0) = \underline{x}(0) - T^{-1}\underline{w}(0) = \underline{0}$ and have no free response term. The second is the forced response caused by the unknown parameter variations ΔA and ΔB. These parameter variations enter in a multiplicative fashion, and their effect on the state estimate depends entirely on how the system is excited and what these parameter variations are. Utilization of the observer to provide the state estimate for the model adaptive identification scheme of the previous chapter will provide a state estimate which is in error by the amount shown in equation (68). Consequently, the linear observer has not been found to be useful for identification. Modification of the way in which the system equation is written and considered yields a non-linear observer whose state vector is the original system state and the parameter vector.

NON-LINEAR OBSERVERS

The last section developed the expression for the error in the state estimate of an observer improperly designed. If the observer for a linear system is designed based on erroneous system and input matrix information, the observer estimates the state of the observed syst .. wit' an

error which is defined by linear differential equation. This error is uncorrectable since it depends on knowledge of the unknown matrices ΔA, ΔB, and the state which is also unknown. A method of circumventing the difficulty of unknown system parameters and system states has been developed using non-linear observers. Essentially, what is done is to rewrite the linear system of equations in non-linear form with an augmented state vector. This new non-linear system has as its states the original state $\underline{x}$ plus a vector of parameters for the system matrix A and for the input matrix B. The parameter vector for the system matrix is called $\underline{a}$ and it is formed by stacking the rows of the system matrix A into a vertical vector. In similar fashion, the parameter vector $\underline{b}$ for the input matrix B is formed by stacking the rows of the B matrix into a vector. Using the Kronecker product the non-linear system then is described by equations (69) through (72). Notice that constraints on the parameters are also included. The constraints are the e_{ij}, d_{ij}, k_{ij}, h_{ij}, terms.

$$(69)\quad \dot{\underline{x}} = I \otimes \underline{x}^t \underline{a} + I \otimes \underline{u}^t \underline{b}$$

$$(70)\quad \dot{\underline{a}} = \underline{0} \qquad e_{ij} < a_{ij} < d_{ij} \quad i,j = 1,2,\ldots,n$$

$$(71)\quad \dot{\underline{b}} = \underline{0} \qquad k_{ij} < b_{ij} < h_{ij} \quad i = 1,2,\ldots,n$$

$$j = 1,2,\ldots,p$$

$$(72)\quad \underline{y} = C\underline{x}$$

Given the non-linear augmented system we construct an observer which will yield the entire augmented state, that is, the original system's state plus the parameter vectors $\underline{a}$, $\underline{b}$. To construct the observer, we build in a structure similar to that in the augmented system and include parameter constraints if they are known. The general form of the observer is shown in equations (73) through (75).

$$(73)\quad \dot{\underline{w}}_1 = F_{11}(t)\underline{w}_1 + I \otimes \underline{w}_1^t \underline{w}_2 + I \otimes \underline{u}^t \underline{w}_3 + G_1(t)\underline{y}$$

$$(74)\quad \dot{\underline{w}}_2 = F_{22}(t)\underline{w}_2 + F_{21}(t)\underline{w}_1 + G_2(t)\underline{y}$$

$$(75)\quad \dot{\underline{w}}_3 = F_{33}(t)\underline{w}_3 + F_{31}(t)\underline{w}_1 + G_3(t)\underline{y}$$

It is desired that the observer state vector correspond to the augmented state vector of the original system within some error. The notation involved is shown in equation (76) through (78).

$$(76)\quad \underline{w}_1 = \underline{x} + \underline{e}_1$$

$$(77)\quad \underline{w}_2 = \underline{a} + \underline{e}_2$$

$$(78)\quad \underline{w}_3 = \underline{b} + \underline{e}_3$$

As in the last section, it is desired to develop the error equations for the non-linear observer. After some algebraic manipulation and subtraction of equations (69) through (72) from equations (73), (74) and (75), it is seen

that the error equation is given by equations (79) through (81).

$$(79)\quad \dot{\underline{e}}_1 = (F_{11} + A)\underline{e}_1 + I \otimes \underline{e}_1{}^t \underline{e}_2 + I \otimes \underline{x}^t \underline{e}_2 + I \otimes \underline{u}^t \underline{e}_3 + (F_{11} + G_1 C)\underline{x}$$

$$(80)\quad \dot{\underline{e}}_2 = F_{22}\underline{e}_2 + F_{21}\underline{e}_1 + F_{22}\underline{a} + (F_{21} + G_2 C)\underline{x}$$

$$(81)\quad \dot{\underline{e}}_3 = F_{33}\underline{e}_3 + F_{31}\underline{e}_1 + F_{33}\underline{b} + (F_{31} + F_3 C)\underline{x}$$

Examining equations (79) through (81) it is seen that the error equations are driven by the state which is unknown. To eliminate the state from the error equations, we introduce the constraints shown by (82) through (84).

$$(82)\quad F_{11} = -G_1 C$$

$$(83)\quad F_{21} = G_2 C$$

$$(84)\quad F_{31} = -G_3 C$$

The final form of the error equation is shown in equations (85) through (87).

$$(85)\quad \dot{\underline{e}}_1 = (A - G_1(t)C)\underline{e}_1 + I \otimes \underline{w}_1{}^t \underline{e}_2 + I \otimes \underline{u}^t \underline{e}_3$$

$$(86)\quad \dot{\underline{e}}_2 = F_{22}(t)\underline{e}_2 - G_2(t)C\underline{e}_1 + F_{22}(t)\underline{a}$$

$$(87)\quad \dot{\underline{e}}_3 = F_{33}(t)\underline{e}_3 - G_3(t)C\underline{e}_1 + F_{33}(t)\underline{b}$$

In general, it may be highly desirable to let the gain matrices G_1, G_2, G_3 be time varying. Therefore, the feedback matrices F_{11}, F_{21}, F_{31} will also be time varying. If the observer is to act as a successful state estimator and parameter estimator, the error equation must eventually go to zero. In general, this means that the error equation must be stable and the matrices $F_{22}(t)$, $F_{33}(t)$ must go to zero in such a fashion that the stability of the error equations is preserved. The matrices F_{22}, F_{33} must go to zero because inspection of (86) and (87) indicates that $\underline{e}_2$, $\underline{e}_3$ are forced by some linear combination of the parameter vectors $\underline{a}$, $\underline{b}$. These forcing terms must eventually be eliminated, and since $\underline{a}$, $\underline{b}$ are constant, the forcing terms can only be eliminated by letting F_{22}, F_{33} eventually approach zero as the state and parameter estimates converge to their correct values. In general, the observer is non-linear and time varying, hence, its stability properties can be critically dependent on the input $\underline{u}$ and the state of the observed system ($\underline{w}_1 = \underline{x} + \underline{e}_1$) thus, the scheme may work for some selection of inputs and not for others. As is common for other identification schemes, all state variables must be excited. Another consideration is that the stability of the observer cannot be assessed prior to the computation. The cause of the problem is that the error equations contain

the unknown matrices A, B, as well as the unknown state.

Selection of the gain matrices G_1, G_2, G_3 must be done with some care. In contrast to the linear observer, where elements of the gain matrix would be non-negative, the gain matrix elements are either positive or negative. A study of the model structure of the identified system is required to determine the proper sign on the gain elements for the parameter vector estimate. For example, suppose a first order system with two parameters a, b is to be identified. The system and observer are described in (88)-(93).

(88) $\dot{x} = -ax + bu \qquad a, b > 0$

(89) $\dot{a} = \dot{b} = 0$

(90) $y = x$

(91) $\dot{w}_1 = w_2 w_1 + w_3 u + g_1(y-w_1) \qquad w_2, w_3 > 0$

(92) $\dot{w}_2 = g_2(y-w_1)$

(93) $\dot{w}_3 = g_3(y-w_1)$

The gain g_1 should be positive to insure that w_1 has negative feedback. Gain g_2 should be negative. The reason for this is that if $y-w_1$ is positive, then, x is greater than w_1. Now assume, temporarily, that $w_3 \triangleq b$. If $x > w_1$ then w_2 is too large and should be decreased, i.e., $\dot{w}_2 < 0$, or $g_2 < 0$. If $y-w_1 < 0$, then, $x < w_1$. When $x < w_1$, then for $w_3 \triangleq b$, w_2 is too small and should be increased, i.e., $\dot{w}_2 > 0$ or, again, $g_2 < 0$.

By similar reasoning, it is clear that g_3 should be positive.

The determination of the gain sign for higher order systems is a generalization of the above example. Each term is considered by itself, i.e., all parameters but one are assumed correct in order to determine the relationship of the sign of $y-w_1$ and the parameter gain sign to achieve the required sign on the parameter estimator rate.

OBSERVER STABILIZATION

Stabilization of the observer is difficult because constant feedback tends to make the parameter estimates diverge from their true values. Such divergence causes observer output error which in turn causes the parameter estimates to be in error and the cycle tends to repeat. To illustrate the point, the error equations associated with the previous example are given below. A parameter feedback term is added in (92), (93).

$$\dot{e}_1 = -(g_1 + w_2)e_1 + e_2 x + e_3 u \tag{94}$$

$$\dot{e}_2 = f_2 e_2 + g_2 e_1 + f_2 a \tag{95}$$

$$\dot{e}_3 = -f_3 e_3 + g_3 e_1 + f_3 b \tag{96}$$

Analysis of (95) and (96) indicates that e_2 and e_3 tend to approach a and b if f_2 and f_3 are positive constants. If $f_2 = f_3 = 0$, then, a equilibrium solution to (94) - (96) is $e_1 = e_2 = e_3 = 0$. The difficulty is the solution is not stable. A choice of f_2 and f_3 must be made such that f_2 and f_3 tend to zero while maintaining the system stability.

A stabilization technique which has worked is to modify the parameter equation so that the parameter rate is zero at the end of some time interval during which the observer has provided acceptably accurate parameter estimates. The parameters are denoted by p_i, $i = 1,2,\ldots,n$ where n is the total number of parameters. One possible parameter equation is (97).

$$(97) \quad \dot{p}_i = -\alpha e^{-t/\tau} p_i, \; p_i(0) = p_{io} \quad i = 1,2,\ldots,n$$

The time solution for p_i is (98)

$$(98) \quad p_i = p_{io} \, \mathrm{EXP}\left(\alpha \tau \, (\mathrm{EXP}(-t/\tau - 1))\right) \quad i = 1,2,\ldots,n$$

There are two adjustable parameters which relate the steady-state value, $p_i(\infty)$ to the initial value P_{io}. The τ parameter primarily controls the rate at which $\dot{p}_i$ approaches zero, and the α parameter primarily controls the ratio $p_i(\infty)/p_{io}$. The exact expression for $p_i(\infty)/p_{io}$ is (99).

$$(99)\quad \frac{p_i(\infty)}{p_{io}} = \mathrm{EXP}(-\alpha\tau) \qquad i = 1,2, \ldots, n$$

Let P_{io} be the correct value of the parameter, then, $p_i(\infty)$ is the approximate value of the parameter.

If $\alpha\tau = .02$, then, $\frac{p_i(\infty)}{P_i} = .98$, and the parameters of the approximate model are only two percentage points different from the parameters of the exact model. Application of the approximate parameter approach to the previous example leads to (100) - (101) as estimators for the a,b parameters.

$$(100)\quad \dot{w}_2 = -\alpha e^{-t/\tau} w_2 + g_2(y-w_1)$$

$$(101)\quad \dot{w}_3 = -\alpha e^{-t/\tau} w_3 + g_3(y-w_1)$$

The estimators are stable for $\alpha > 0$ during the time period for which the feedback is effective. By proper design choice of τ, g_1, g_2, g_3, the observer can be made to give a stable, accurate estimate of the system parameters.

NUMERICAL EXAMPLE

A numerical example to illustrate the technique was worked. Equations (102) - (114) describe the second order system and its fifth order state and parameter estimator.

(102) $\dot{x}_1 = x_2$

(103) $\dot{x}_2 = a_{21}x_1 - a_{22}x_2 + bu$

(104) $y = x_1$

(105) $\dot{a}_{21} = \dot{a}_{22} = \dot{b} = 0$

(106) $a_{21} = 4$

(107) $a_{22} = 2$

(108) $b = 1$

(109) $\dot{w}_1 = w_2 + g_1 (y-w_1)$

(110) $\dot{w}_2 = -w_3w_1 - w_4w_2 + w_5u + g_2 (y-w_1)$

(111) $\dot{w}_3 = -\beta w_3 - 400 (y-w_1)$

(112) $\dot{w}_4 = -\beta w_4 - 1000 (y-w_1)$

(113) $\dot{w}_5 = -\beta w_5 + 50 (y-w_1)$

(114) $\beta = .4 \text{ EXP}(-20t)$

The state-parameter estimator to system correspondence was $w_1 \rightarrow x_1$, $w_2 \rightarrow x_2$, $w_3 \rightarrow a_{21}$, $w_4 \rightarrow a_{22}$, $w_5 \rightarrow b$. Table 6 shows the numerical results.

Table 6. Observer Identification Example.

Time (sec)	$Y-W_1$	W_3	W_4	W_5
0	-4.22×10^{-3}	2.50	3.50	1.50
.4	-5.43×10^{-4}	2.36	4.10	1.19
.8	-7.14×10^{-4}	2.15	3.96	1.09
1.2	-7.31×10^{-4}	2.14	4.01	1.06
1.6	-4.86×10^{-4}	2.13	4.06	1.03
2.0	-2.02×10^{-5}	2.11	4.05	1.02
2.4	-1.85×10^{-5}	2.12	4.06	1.02
2.8	-6.16×10^{-5}	2.11	4.07	1.01
3.2	7.28×10^{-5}	2.09	4.01	1.01
3.6	-1.31×10^{-4}	2.08	4.01	1.01
4.0	-2.83×10^{-5}	2.08	4.02	1.00
4.4	3.35×10^{-5}	2.06	4.00	1.00
4.8	-1.20×10^{-4}	2.06	3.99	1.00
5.2	-1.58×10^{-5}	2.06	3.99	1.00
5.6	1.76×10^{-5}	2.05	3.95	1.00
5.8	-9.93×10^{-5}	2.04	3.96	.99
Correct Value	0	2.00	4.00	1.00

The estimator converged to the correct values rapidly for the choice of initial conditions and parameter stabilization technique. It is felt that the estimation scheme is well suited to an analog mechanization.

SUMMARY

A combined state-parameter estimation technique was developed. The scheme is based on a generalization of the linear observer concept and was shown to converge well for a specific example.

The estimator is multi-linear and can be unstable unless a time varying feedback on the parameter estimation portion is used. A method of stabilizing the observer was presented.

Computationally, the estimator is good because it can be implemented straightforwardly on an analog computer. Feedback gain adjustment is relatively easy. Since no analog-to-digital converters are required, on-line simulation should be a potential use of the technique.

CHAPTER V

ESTIMATION OF PARAMETER SUBMATRICES

INTRODUCTION

The present chapter is concerned with determining the parameter submatrices of network-like physical systems. These parameter submatrices are contained in the system and input matrices of the system which are assumed to be known. A further consideration is determining if enough measurements have been taken to uniquely define the parameter element matrices.

As is well known [Koenig, FE-8] if there are e-elements in the system and if two e-measurements, i.e., e-current measurements and e-voltage measurements in the case of electrical networks, are made, then each pair of measurements defines a parameter such as a resistor or a capacitor. By using the generalization of Kirchoff's voltage laws systematically through the theory of linear graphs, a sufficient number of measurements to estimate the values of the e-parameters is e. The techniques of the present chapter allow further reductions in the number of measurements by making full use of the known topological information concerning the system and the results from the identification methods previously presented.

ASSUMPTIONS AND FORMULATION OF EQUATIONS

The state equations are assumed to be in the form as given in the Appendix and it is further assumed that the network does not have excess dynamic elements, e.g., capacitors in parallel and inductors in series. Mutual inductance is allowed under the last assumption. It is also assumed that the A,B matrices have been determined by some identification technique and that there are no non-state connected sources, i.e., K_{rd}, K_{gd} are null matrices.

Under the above assumptions, the known information is summarized in the following equations.

(115) $\dot{\underline{x}} = A\underline{x} + B\underline{u}$

Where $\underline{x}(t)$, $\underline{u}(t)$, $t \in [0,T]$ are known, and A,B are known,

(116) $$\begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

(117) $$\begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

(118) $(J_{11}) = (C_s)$

(119) $(J_{22}) = (L_s)$

(120) $F_{11} = -T_2(R+T_4^t\, G^{-1} T_4)^{-1}\, T_2^t$

$$(121)\quad F_{12} = T_2(R+T_4^t\, G^{-1}t_4)^{-1}\, T_4^t\, G^{-1}T_3 + T_1$$

$$(122)\quad F_{21} = -T_1^t - T_3^t(G+T_4R^{-1}T_4^t)^{-1}\, T_4R^{-1}T_2^t$$

$$(123)\quad F_{22} = -T_3^t\, (G+T_4R^{-1}T_4^t)^{-1}\, T_3$$

$$(124)\quad H_1 = K_{cd}$$

$$(125)\quad H_2 = K_{cd}$$

The unknowns in (116) - (125) are C_s, L_s, R, G, but it is known that C_s, R, G are diagonal, and L_s is symmetric and positive definite.

SOME BASIC RESULTS

The first results concern the independence of the matrix equations (120) - (123).

Result #1 $\quad F_{12} - F_{21}{}^t = 2T_1$

To see this, use the matrix identity $AB^t(C+BAB^t)^{-1} = (A^{-1} + B^tC^{-1}B)^{-1}\, B^tC^{-1}$ and rewrite (122) as,

$$(126)\quad F_{21}^t = -T_1 - T_2(R+T_4^tG^{-1}T_4)^{-1}T_4^t\, G^{-1}T_3$$

and subtracting F_{21}^t from F_{12}, the result follows.

Result #2 $F_{12}+F_{21}^{t} = 2T_2(R+T_4^{t} G^{-1}T_4)^{-1}T_4^{t}G^{-1}$

The result follows from above.

The next results are concerned with the relationship of the number of resistive and dynamic elements to the singularity of the topology matrices and may be established by simple dimension analysis of these matrices.

Result #3 $\det T_1 \neq 0$ only if $n_c = n_l$

Result #4 $\det T_2 \neq 0$ only if $n_c = n_r$

Result #5 $\det T_3 \neq 0$ only if $n_l = n_g$

Result #6 $\det T_4 \neq 0$ only if $n_r = n_g$

Since the matrices R,G, are diagonal, the results above imply the following results.

Result #7 T_2 is non-singular only if the number of tree capacitors is equal to the number of chord resistors.

Result #8 T_3 is non-singular only if the number of tree conductors is equal to the number of chord inductors.

DETERMINATION OF C_s, L_s

In general, the dynamic element matrices will not be determined by state and input measurements only. Some additional measurements may be required, hence, the purpose of the present section is to develop a technique for using the information available from the state and input measurements to full advantage and to show how to select the additional measurements required to uniquely define C_s, L_s.

The off diagonal blocks A_{12}, A_{22} of the system matrix A, and the input matrices B_1, B_2 are used to determine as many elements of C_s, L_s as possible, and the remaining elements must be specified by a combination of topological information and additional measurements.

Using the constraint between F_{12}, F_{21}^t and noting that $F_{12} = C_s A_{12}$ and $F_{21} = L_s A_{21}$, the three matrix equations available for determining C_s, L_s are,

$$C_s A_{12} - A_{21}^t L_s = 2T_1 \tag{127}$$

$$C_s B_1 = H_1 \tag{128}$$

$$L_s B_2 = H_2 \tag{129}$$

Two sets of equations may be derived from (130) - (131) by considering either L_s or C_s as independent.

$$(130) \qquad C_s(A_{12}B_2 \vdots B_1) = (2T_1B_2 + A_{21}^{t}H_2 \vdots H_1)$$

$$(131) \qquad L_s(A_{21}B_1 \vdots B_2) = (2T_1^{t}B_1 + A_{12}^{t}H_1 \vdots H_2)$$

Now drawing on the diagonal structure of C_s and letting $X = (A_{12}B_2 \vdots B_1)$, $Y = (2T_1B_2 + A_{21}^{t}H_2 \vdots H_1)$, (130) can be written as (132).

$$(132) \qquad C_i \underline{x}_{i_*} = \underline{y}_{i_*} \quad i = 1,2, \ldots , n_c$$

By adding the components of $\underline{x}_{i_*}$, $\underline{y}_{i_*}$, the C_i's can be found from (133).

$$(133) \qquad C_i \sum_{j=1}^{n_\ell + p_c} x_{ij} = \sum_{j=1}^{n_\ell + p_c} y_{ij} \quad i = 1,2, \ldots , n_c$$

Let r be the number of non-zero row sums of X, then,

$$(134) \qquad C_i = \frac{\sum_{j=1}^{n_\ell + p_c} y_{ij}}{\sum_{j=1}^{n_\ell + p_c} x_{ij}} \qquad i = 1,2, \ldots , r$$

The number of elements of C_s which must be specified by additional measurements is n-r.

Since L_s may be non-diagonal, the determination of the elements of its elements is more difficult.

The Kronecker product can be used to advantage, however, by post-multiplying (130) by $(A_{21}B_1 \vdots B_2)^t$ to convert to a system of square matrix equations.

Let $W = (A_{21}B_1 \vdots B_2)(A_{21}B_1 \vdots B_2)^t$ and $Z = (-2T_1{}^tB_1 + A_{12}{}^tH_1 \vdots H_2)(A_{21}B_1 \vdots B_2)^t$, then, (129) is written in the Kronecker product notation as (135).

$$(135) \quad I \otimes W^t \underline{\ell}_s = \underline{z} = \begin{pmatrix} W^t & & 0 \\ & \ddots & \\ 0 & & W^t \end{pmatrix} \underline{\ell}_s$$

where W is a $n_\ell x n$ and $\underline{\ell}_s$ is an n_ℓ^2 - dimensional vector composed of the transposed rows of L_s.

If W^t has rank n_ℓ, then the problem is solved, and the elements of L_s are given by,

$$(136) \quad \underline{\ell}_s = W^{-t} \otimes I \underline{z}$$

The more usual case is that rank $w = r < n_\ell$ and since rank $A \otimes B$ = rank $A \cdot$ rank B, rank $(I \otimes w)$ = rank $(I \otimes w^t) = n_\ell \cdot r$ and $n_\ell^2 - r \cdot n$ components of $\underline{\ell}_s$ must be specified by topological information and additional measurements.

Now drawing on the symmetric structure of L_s, there are $\frac{n_\ell (n_\ell + 1)}{2}$ possible different elements of $\underline{\ell}_s$. Define the reduced matrix and vector D, $\underline{\ell}_s$, $\underline{z}$ of order $n_\ell^2 \times n_\ell (n_\ell+1)$ and $\frac{n_\ell(n_\ell+1)}{2} \times 1$ respectively obtained from $I \otimes W^t$,

$\underline{\ell}_s$ by deleting redundant entries caused by the symmetry of L_s by (137).

(137) $\quad (D)\underline{\ell}_s = \underline{z}$

The equation can be further reduced in order by permuting rows, columns, and components of D, $\underline{\ell}_s$, to $\underline{z}$ to yield the following equation.

(138) $$(D_{r^*r^*} \vdots D_{r^*2}) \left[\begin{array}{c} \underline{\ell}_1 \\ \hline \underline{\ell}_2 \end{array}\right] = \underline{z}_1$$

where r^* = rank D and D is an $r^* \times \frac{n_\ell(n_\ell+1)}{2}$ matrix. Solving (138) for $\underline{\ell}_1$,

(139) $$\underline{\ell}_1 = \underline{z}_1 - D_{r^*r^*}^{-1} (D_{r^*2}) \underline{\ell}_2$$

hence, the vector $\underline{\ell}_2$ must be specified either by topological information or additional measurements in order to define the complete vector uniquely. The number of these measurements is $\frac{n_\ell(n_\ell+1)}{2} - r$.

DETERMINATION OF R, G SPECIAL CASE $T_4 = (0)$

For the special case where there is no tree to cotree resistive coupling, the determination of R, G is greatly simplified.

It is assumed that C_s, L_s have been determined, hence, from that knowledge and knowledge of the system matrix A,

the submatrices $F_{11} = C_s A_{11}$, $F_{22} = L_s A_{22}$ may be determined. Then using the topological knowledge of the system, R, G are defined in (140) and (141).

$$F_{11} = -T_2 R^{-1} T_2^t \tag{140}$$

$$F_{22} = -T_3^t G^{-1} T_3 \tag{141}$$

Converting (140) and (141) to n_r^2 and n_g^2 dimensional vector equations by the Kronecker product after pre and post-multiplying by T_2^t, T_3 as appropriate,

$$-T_2^t T_2 \otimes T_2^t T_2 \underline{x} = \underline{f}^*_{11} \tag{142}$$

$$-T_3 T_3^t \otimes T_3 T_3^t \underline{y} = \underline{f}^*_{22} \tag{143}$$

where $\underline{x}$, $\underline{y}$ are composed of vectors of the rows of R^{-1}, G^{-1}, respectively, and $\underline{f}^*_{11}$, $\underline{f}^*_{22}$ are vectors of the rows of $T_2^t F_{11} T_2$, $T_3 F_{22} T_3^t$, respectively.

The number of components of $\underline{x}$, $\underline{y}$ which must be specified to define $\underline{x}$, $\underline{y}$ uniquely is $n_r^2 - r_2^2$, $n_g^2 - r_g^2$ where r_2, r_3 are the ranks of T_2, T_3 respectively.

APPROXIMATE SOLUTIONS

It may happen that one is interested in the approximate values of R, G, C_s, L_s to serve as either a starting point for more exact solutions, as in the case of successive

approximations, or because approximate values are sufficient. The generalized inverse may be used to great advantage because of the special structure of the topology matrices, i.e., their elements are 0, ± 1 and the construction of the generalized inverse is much easier than for the general case.

By using the matrix identity $(C+B^tA^{-1}B)^{-1} = C^{-1}-C^{-1}B^t(A+BC^{-1}B^t)^{-1}BC^{-1}$, (120, (123) may be rewritten as,

$$F_{11} = -T_2R^{-1}T_2^t + T_2R^{-1}T_4^t(G+T_4R^{-1}T_4^t)^{-1}T_4R^{-1}T_2^t \quad (144)$$

$$F_{22} = -T_3^tG^{-1}T_3 + T_3^tG^{-1}T_4(R+T_4^tG^{-1}T_4)^{-1}T_4^tG^{-1}T_3 \quad (145)$$

Let $\hat{F}_{12} = F_{12} + F_{21}^t$

$$\hat{F}_{12} = -2T_2R^{-1}T_4^t(G + T_4R^{-1}T_4)^{-1}T_3 \quad (146)$$

Therefore, using the generalized inverse,

$$(G^{-1})^I = -(I + T_2^I\underline{\hat{F}_{12}}_2T_3^IT_4)^IT_2^IF_{11}(T_2^t)^I \quad (147)$$

$$(R^{-1})^I = (I+T_2^I\underline{\hat{F}_{12}}_2T_3^IT_2^IF_{11}(T_2^t)^I \quad (148)$$

Using the diagonal structure of R, G, the i-th diagonal elements of (146), (147) are $\frac{1}{g_i}$, $\frac{1}{r_i}$, respectively.

MAIN SOLUTION TECHNIQUE FOR R,G

Since the identification technique provides more information than the A,B matrices, this information is used to help determine R,G. The outline of the scheme is to determine C_s, L_s by the techniques presented, and then use the state equations plus knowledge of the system topology to determine R,G. The remaining information used is the state derivative which is easily computed from $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ where A, $\underline{x}$, B, $\underline{u}$ are known. A prime benefit of this technique is that linear algebraic solutions are required instead of non-linear solutions.

The technique is formulated for the case where the independent variables are the state variables and the tree conductor voltages. The corresponding equations are,

$$(149) \quad (T_2R^{-1})\ (-T_2^t\underline{v}_s - T_4{}^t\underline{v}_g) = T_i\underline{i}_s - C_s\dot{\underline{v}}_s + H_{cd}\underline{i}_{cd}$$

$$(150) \quad -T_3{}^t\underline{v}_g = L_s\dot{\underline{i}}_s - T_1{}^t\underline{v}_s + K_{\ell d}\underline{e}_{bd}$$

The unknowns in (149), (150) are $\underline{v}_g$, R^{-1}. If r_3 = rank T_3, then $n_g - r_3$ additional measurements of components of $\underline{v}_g$ are required to define $\underline{v}_g(t)$ $t\in[0,T]$ uniquely. Assume then that $\underline{v}_g$ has been defined and from the chart records and computation of $\underline{x}$, form the following matrices.

$$(151) \quad V_s = \left(\underline{v}_s(t_c), \ \ldots\ , \underline{v}_s(t_k)\right)$$

$$(152) \quad I_s = \left(\underline{i}_s(t_o), \ \ldots \ , \ \underline{i}_s(t_k)\right)$$

$$(153) \quad y_s = \left(\dot{\underline{v}}_s(t_o), \ \ldots \ , \ \dot{\underline{v}}_s(t_k)\right)$$

$$(154) \quad U_s = \left(\underline{i}_{cd}(t_o), \ \ldots \ , \ \underline{i}_{cd}(t_k)\right)$$

$$(155) \quad V_g = \left(\underline{v}_g(t_o), \ \ldots \ , \ \underline{v}_g(t_k)\right)$$

Using (153) - (155), equation (148) can be expanded to yield the matrix equation (156).

$$(156) \quad T_2 R^{-1}(-T_2{}^t V_s - T_4{}^t V_g) = T_1 I_s - C_s Y_s + H_{cd} U_s$$

The matrix $T_2{}^t V_s - T_4{}^t V_g$ is $n_r$xk. The number of samples k is chosen such that $(-T_2{}^t V_s - T_4{}^t V_g) = W$ has rank n_r and in general,

$$(157) \quad T_2 R^{-1} = (T_1 V_s - C_s Y_s + H_{cd} U_s) W^{-1} = Z$$

Again, using the diagonal structure of R, (157) can be rewritten as,

$$(158) \quad \left(\frac{1}{r_1}\underline{t}_{21}, \ \ldots \ , \ \frac{1}{r_{nr}}\underline{t}_{2nr}\right) = (\underline{z}_1, \ \ldots \ , \ \underline{z}_{nr})$$

(158) is solved for the r_i, i = 1, 2, ... , n_r by forming the column sum,

$$(159)\qquad \frac{1}{r_j}\sum_{k=1}^{n_c} t_{2kj} = \sum_{k=1}^{n_c} z_{kj} \quad j = 1,2, \ldots , n_r$$

and if there are p non-zero column sums in z,

$$(160)\qquad r_j = \frac{\sum_{k=1}^{n_c} t_{2kj}}{\sum_{k=1}^{n_c} z_{kj}} \quad j = 1, 2, \ldots , p \leq n_r$$

The remaining n_r - p diagonal elements of r must be defined by additional measurements. A similar technique may be used if $\underline{i}_r$ is taken as one of the independent variables instead of $\underline{v}_g$.

ALTERNATIVE SOLUTION TECHNIQUE FOR R, G

An alternative technique to solve for the diagonal elements of R,G is to write (120), (121), (122) as a non-linear set of algebraic equations in the form,

$$(161)\qquad f_i(r_1,\ldots,r_{nr},g_1,\ldots,g_{ng})-h_i=0 \quad i=1,2,\ldots,n^2+n_\ell^2+n_c n_\ell$$

and solve using the technique of successive approximations [Rall, M-7]. The algorithm is an iterative one given by,

$$(162)\qquad \underline{x}^{k+1} = \underline{x}^k - J^{-1}(\underline{x}^k)\,(\underline{f}(\underline{x}^k)-\underline{h})$$

and $\underline{x}$ is the vector with components of (161), $\underline{f}(\underline{x})$, $\underline{h}$ are the vector equivalents of (161) and $J(\underline{x}^k)$ is the Jacobian

matrix of (161) evaluated at the k-th iteration. The technique of successive approximations works well for the non-linear equations considered because the types of non-linearities are product and square law types.

The initial starting value is obtained from the approximate solution technique presented earlier. The major difficulty is the algebra involved in writing out the equations, writing out the form of the Jacobian matrix, and eliminating the redundant equations so that J^{-1} exists.

DETERMINATION OF R,G - SPECIAL CASE (COMPLETE CIRCUITS)

For the case where the network-like system is complete in the sense of Brayton and Moser [FE-3] all of the matrices C_s, L_s, R, G may be determined without additional measurements.

A complete circuit is a network which has special properties. The reader is referred to Brayton and Moser [FE-3, p. 4 and 13] for a thorough discussion of complete circuits. The essence of the completeness property is that the state equations can be defined in terms of a potential function. A procedure similar to that for formulating dynamical equations from the Hamiltonian function in classical dynamics can be employed to write the state equations.

For a complete circuit, the state equations have the form,

$$(163)\quad L_j \frac{di_j}{dt} = \frac{\partial P}{\partial i_j} \qquad j = 1,2, \ldots , n_\ell$$

$$(164)\quad C_k \frac{dv_k}{dt} = -\frac{\partial P}{\partial v_k} \qquad k = 1,2, \ldots , n_c$$

For the linear, constant coefficient case P has the form,

$$(165)\quad P = \frac{1}{2} \sum_{j=1}^{n_r} r_j \cdot i_j^2 - \sum_{k=1}^{p_i} e_k i_k + \frac{1}{2} \sum_{h=1}^{n_g} g_h v_h^2 - \sum_{n=1}^{p_b} j_n v_n + \underline{i}_s^t T_1 \underline{v}_s$$

where i_j, $j = 1,2, \ldots , n_r \leq n$, v_h, $h = 1,2, \ldots , n_g \leq n_c$ are components of the state vector $\begin{pmatrix} \underline{v}_s \\ \underline{i}_s \end{pmatrix}$, the e_k's and j_n's are the sources, and the r_j's and g_h's are the resistors and conductors.

Applying (163), (164) to (165), the parameter submatrices have the forms,

$$(166)\quad C_s A_{11} = \begin{pmatrix} g_1 & & & 0 \\ & \ddots & & \\ & & g_{ng_0} & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix}$$

$$(167)\quad C_s A_{12} = T_1$$

$$(168) \quad L_s A_{22} = \begin{pmatrix} r_1 & & & 0 \\ & \ddots & & \\ & & r_{nr_0} & \\ 0 & & & \ddots 0 \end{pmatrix}$$

$$(169) \quad L_s A_{21} = -T_1^t$$

where A_{11} is $n_c x n_c$ and A_{22} is $n_\ell x n_\ell$ and C_s, L_s are both diagonal.

$$(170) \quad C_s B_1 = H_1$$

$$(171) \quad L_s B_2 = H_2$$

The simplicity of the above formulation allows the determination of C_s, L_s, R, G by inspection.

NUMERICAL EXAMPLE

Several example problems are presented illustrating the techniques discussed. Both complete and other types of circuits are considered.

Example Problem #1, Second Order Bridge Circuit

A bridge circuit has some features which make it a good vehicle for illustration. The most important feature for parameter estimation is that there is extensive tree-cotree resistive coupling. An immediate effect of the tree-cotree resistive coupling is that knowledge of the system

states and topology is not sufficient to determine all parameters uniquely.

Figure 6 shows the circuit schematic and its associated graph. The state variables are the inductor current, i_6, and the capacitor voltage, e_5. Equations (172) to (182) define the circuit model and parameter matrices.

The main solution technique using linear equations and the non-linear technique using Newton-Raphson iteration are illustrated.

$$(172)\quad A = \begin{pmatrix} -\frac{1}{C_5}\left(\frac{1}{R_1+R_2} + \frac{1}{R_3+R_4}\right) & -\frac{1}{C_5}\left(\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right) \\ \frac{1}{L_6}\left(\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right) & -\frac{1}{L_6}\left(\frac{R_1 R_2}{R_1+R_2} + \frac{R_3 R_4}{R_3+R_4}\right) \end{pmatrix} = \begin{pmatrix} -388 & .056 \\ .033 & -.570 \end{pmatrix}$$

$$(173)\quad \underline{b} = \begin{pmatrix} 0 \\ \frac{1}{L_6} \end{pmatrix} = \begin{pmatrix} 0 \\ .400 \end{pmatrix}$$

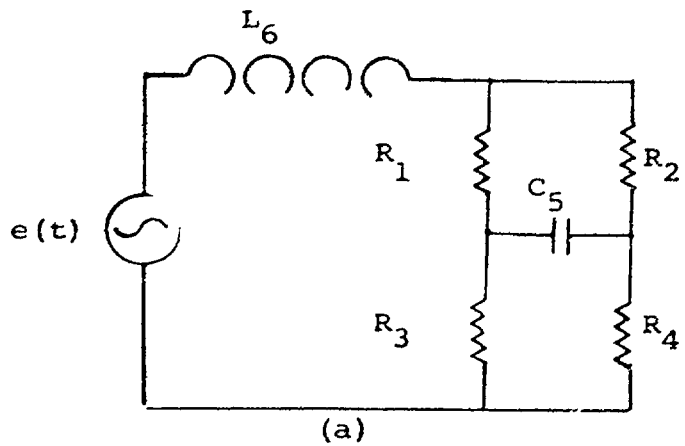
$$(174)\quad T_1 = 0$$

$$(175)\quad T_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

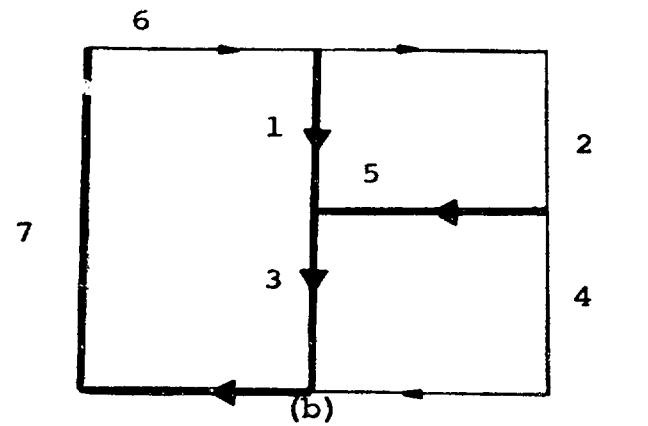
$$(176)\quad T_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$(177)\quad T_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

BRIDGE CIRCUIT



BRIDGE CIRCUIT



SYSTEM GRAPH

tree
cotree

FIGURE 6

(178) $$R = \begin{pmatrix} R_2 & 0 \\ 0 & R_4 \end{pmatrix} = \begin{pmatrix} 2.000 & 0 \\ 0 & 3.000 \end{pmatrix}$$

(179) $$G = \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_3} \end{pmatrix} = \begin{pmatrix} 1.000 & 0 \\ 0 & 1.000 \end{pmatrix}$$

(180) $C_s = C_5 = 1.500$

(181) $L_s = L_6 = 2.500$

(182) $K_{\ell d} = 1$

There are five equations and six unknowns, therefore, one of the parameters must be specified. If a unique solution is desired then additional measurements are necessary for unique specification of parameters.

Alternatively, an arbitrary (positive) value may be assigned. If the latter course is followed then the others will be determined by that specification.

The equations to be solved are (183) to (187).

(183) $$L_6 = \frac{1}{b_2} = 2.50$$

(184) $$-\left(\frac{R_1}{R_2+R_1} - \frac{R_3}{R_3+R_4}\right) = -c_5 \cdot a_{12}$$

$$(185) \qquad \frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_2+R_4} = L_6a_{22}$$

$$(186) \qquad \frac{1}{R_1+R_2} = \frac{1}{R_2+R_4} = c_5a_{11}$$

$$(187) \qquad \frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4} = L_6a_{21}$$

Since the off-diagonal terms are related, it follows that c_5 may be computed from A, $\underline{b}$ alone.

$$(188) \qquad c_5 = \frac{a_{21}}{a_{12}} \cdot L_6 = 1.50$$

For determining the resistive values there are three equations and four unknowns, thus, a resistive value must be measured or specified.

For the purpose of the example, it was assumed that R_4 was known and the technique of successive approximations used to solve the resulting equations. By manipulation, the non-linear equations to be solved are,

$$(189) \qquad \frac{R_1R_2}{R_1+R_2} = \frac{3R_3}{R_2+3} = 1.42$$

$$(190) \qquad \frac{1}{R_1+R_2} + \frac{1}{R_3+R_4} = .58$$

$$(191) \qquad \frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+3} = .08$$

To solve the above equations, a parameter vector $\underline{p} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$ was defined and used in the Newton-Raphson scheme below.

$$(192) \qquad \underline{p}^{K+1} = \underline{p}^k - J^{-1}(\underline{p}^k)\ (\underline{f}(\underline{p}^k)-\underline{h})$$

The Jacobian matrix was,

$$(193) \qquad J(\underline{p}^k) \begin{pmatrix} \frac{R_2^k}{(R_1^k+R_2^k)} & \frac{(1-R_1^k)}{(R_1^k+R_2^k)} & \frac{R_1^k}{R_1^k+R_2^k} & \frac{(1-R_2^k)}{\frac{R_1^k+R_3^k}{R_3+3}} & \frac{(1-R_3^k)}{R_3^k+3} \\ \frac{-1}{(R_1^k+R_2^k)}\,2 & & \frac{-1}{(R_1^k+R_2^k)}\,2 & \frac{-1}{(R_3^k+3)}\,2 & \\ \frac{1}{R_1^k+R_2^k} & \frac{(1-R_1^k)}{R_1^k+R_2^k} & \frac{-R_1^k}{(R_1^k+R_2^k)}\,2 & \frac{1}{R_3^k+3} & \frac{(1-R_3)}{(R_3^k+3)} \end{pmatrix}$$

An initial guess which was 50 percent in error was used to start the Newton-Raphson iteration. The results are shown in Table 7.

Table 7. Newton-Raphson Iterative Solution for Parameter Values

	PARAMETERS		
ITERATION NUMBER	R1	R2	R3
0	1.5	1.0	1.5
1	1.62	1.35	0.76
2	0.96	2.23	0.68
3	0.61	2.16	1.20
4	0.82	2.11	1.13
5	0.96	2.03	1.01
6	0.99	2.01	1.01
7	1.00	2.00	1.00
Correct Values	1.00	2.00	1.00

To illustrate the main solution technique it was assumed that an identification scheme had yielded the A, $\underline{b}$ matrices and the state trajectories. The state derivative was computed by $\dot{\underline{x}} = A\underline{x} + \underline{b}\, u$. An additional voltage measurement was required because the rank of the matrix T_3 was 1. To determine the vector $\underline{v}_g = \begin{pmatrix} e \\ e_3 \end{pmatrix}$ uniquely, the relation (150) is,

$$(194) \qquad -T_3{}^t\underline{v}_g = L_s\dot{\underline{i}}_s + u$$

Substituting for T_3, $\underline{v}_g$, $\underline{i}_s$, L_s, gives the branch voltage relation below.

$$(195) \qquad -(e_1 + e_3) = -L_6\dot{i}_6 + e$$

By measurement of e_1, e_3 can be computed since L_6, $\dot{i}_6$, e, are known.

Equation (156) was used to determine the chord resistors r_2, r_4.

$$(196) \quad \left(-\frac{1}{r_2} \quad \frac{1}{r_4}\right) \begin{vmatrix} e_5(t_1)-e_1(t_1) & e_5(t_2)-e_1(t_2) \\ -e_5(t_1)-e_3(t_1) & -e_5(t_1)-e_3(t_2) \end{vmatrix} = c_5\begin{vmatrix} \dot{e}_5(t_1) & \dot{e}_5(t_2) \end{vmatrix}$$

Samples were taken at $t_1 = 1.0$ sec, $t_2 = 3.2$ sec. Table 8 shows the sampled values and the results for r_2, r_4.

SECOND ORDER BRIDGE CIRCUIT STATE AND STATE DERIVATIVE
RESPONSE TO STEP OF TEN VOLTS

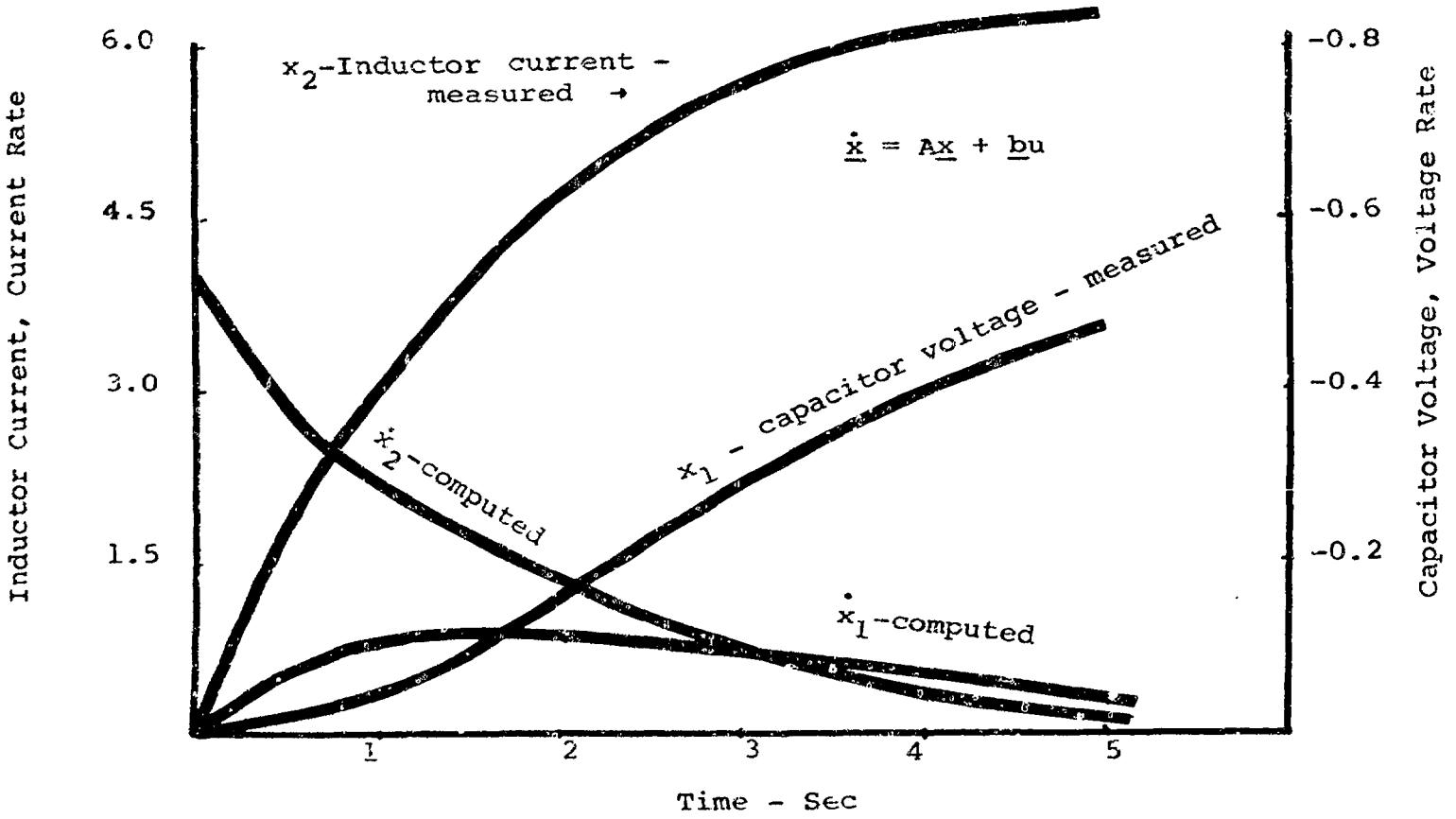


FIGURE 7

MEASURED AND COMPUTED BRANCH VOLTAGE FOR
SECOND ORDER BRIDGE CIRCUIT

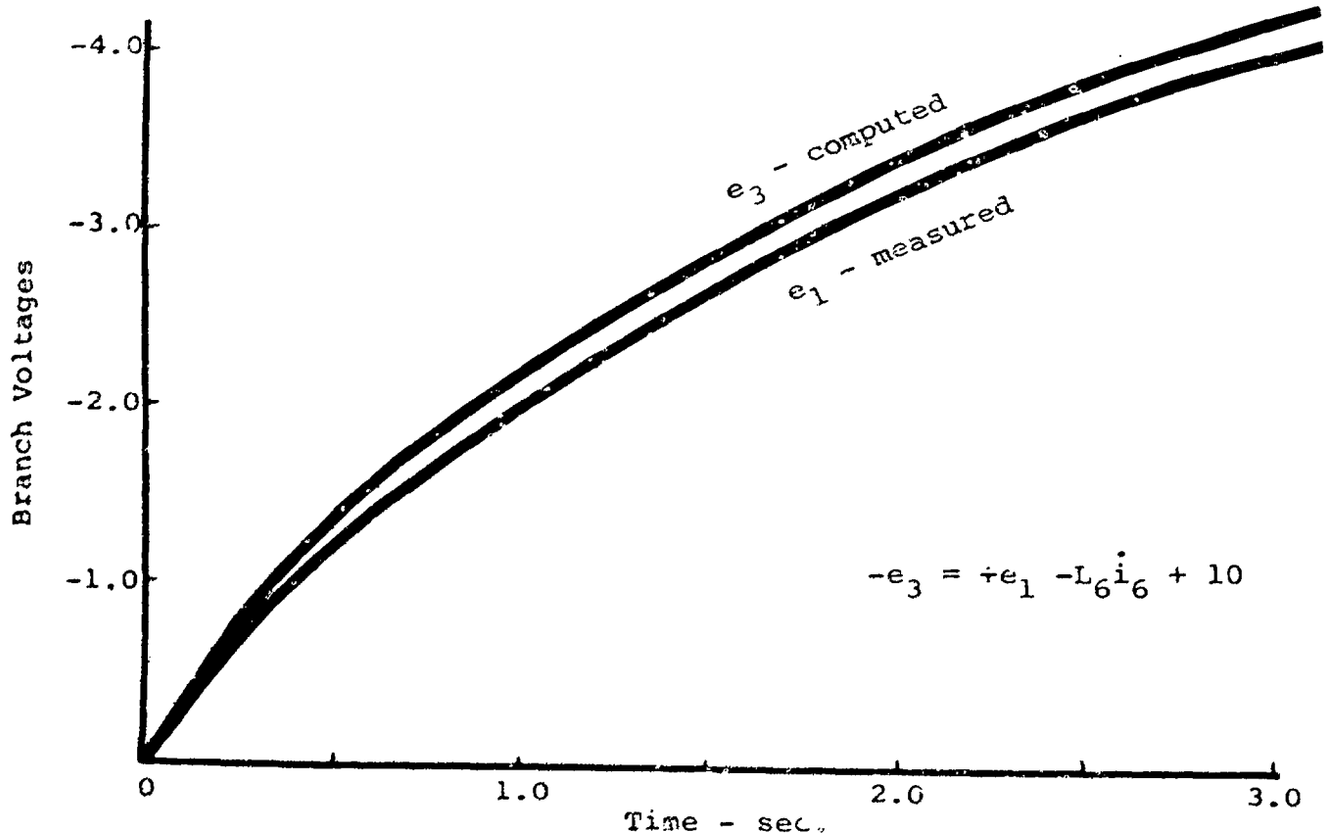


FIGURE 8

Table 8. Computational Results for Chord Resistors

VARIABLE	TIME t = 1.0 sec.	t = 3.2 sec.
$c_5\dot{e}_5$ - Amps	-.208	-.235
e_5 - Volts	-.082	-.444
e - Volts	10	10
e_1 - Volts	-2.059	-4.063
e_3 - Volts	-2.266	-4.293
r_2 - 2 ohms	-	-
r_4 - 3.01 ohms		
Correct Values		
r_2 = 2 ohms		
r_4 = 3 ohms		

Once the chord resistors were known, the branch resistors were easily determined by,

$$(197) \quad r_1 = e_1(t_1)(i_1(t_1))$$

$$(198) \quad r_3 = e_3(t_1)(i_3(t_1))$$

The branch currents were given by the chord currents where,

$$(199) \quad i_1 = -i_6 + i_2 = -i_6 + e_2/v_2 = i_6 + \frac{(e_5-e_1)}{r_2}$$

$$(200) \quad i_3 = -i_6 + i_4 = i_6 + e_4/v_4 = -i_6 + \frac{(-e_5-e_5)}{r_4}$$

At $t_1 = 1.0$ sec, the values of i_1, i_3 were computed below,

$$(201) \quad i_1(1.0) = 1.0$$

$$(202) \quad i_3(1.0) = 1.0$$

The result of the example is that given two state measurements, an additional branch voltage measurement, and the input, that all six of the network parameters can be computed.

Example Problem #2 - Sixth Order, Two-input Complete Circuit.

Example problem #2 illustrates the parameter determination technique for the special case of a complete circuit. The system is the complete circuit for Example #2. Table 9 shows the results of the numerical computation and the corresponding parameters.

Some of the parameters are over-determined, that is, there are more than one equation for some parameters. A useful benefit of the above fact is that computational "noise" can be smoothed out by averaging results. A check on the consistency of the identification is also provided.

Table 10 shows the results of the computation for each parameter when the average is used for the over-determined equations.

Example problem 2 shows that measurement of six states and two inputs allows the computation of 11 parameters. The computation involves simple algebraic techniques. Example problem #1 contrasts with problem #2 in the relatively complex technique of computation plus the need for additional measurements other than state and input.

Table 9. Numerical Results for Complete Circuit.

PARAMETERS	COMPUTED VALUES FROM IDENTIFICATION
$1/R_4C_1$	5.01
$1/C_1$	10.00
$1/R_5C_2$	2.00
$1/C_2$	2.00
$1/C_3$	1.00
$1/C_3$	1.00
$1/C_3$	1.00
$1/L_1$	10.02
$1/1L_1$	9.96
$1/L_1$	10.02
R_1/L_1	4.00
$1/L_2$	5.00
$1/L_2$	5.00
$1/L_2$	5.00
R_2/L_2	5.00
$1/L_3$	3.34
$1/L_3$	3.34
R_3/L_3	10.00

Table 10. Comparison of Actual and Computed Parameters For Complete Circuit.

PARAMETER	COMPUTED VALUE	ACTUAL VALUE
C_1	0.10	0.10
C_2	0.50	0.50
C_3	1.00	1.00
R_1	1.01	1.00
R_2	1.00	1.00
R_3	3.00	3.00
R_4	1.99	2.00
R_5	1.00	1.00
L_1	0.10	0.10
L_2	0.20	0.20
L_3	0.30	0.30

Example Problem #3 - Complete Circuit Model of Second-Order Bridge Circuit.

A pertinent issue for modeling purposes is the existance of a model of the system which will give the same response as that measured but have a different topology and parameter values. Example #3 shows that a model of the circuit in Example #1 exists,which has a different topology and different parameter values, yet has the same A-matrix. The input matrices are different, thus, for the same input, different state responses will be observed.

A multiplicative scaling of the bridge circuit input, however, to provide the input to a complete circuit model of the incomplete circuit yields the identical state trajectory.

The net result is that there exists a second-order complete circuit model of the second-order bridge circuit which has identical stability properties, i.e., has the same A-matrix and the two circuits have identical free responses to identical initial conditions.

Additionally, the forced responses to the same input differ only by a multiplicative constant. Figure 9 shows a schematic of each circuit.

SECOND ORDER BRIDGE CIRCUIT AND
A COMPLETE CIRCUIT MODEL

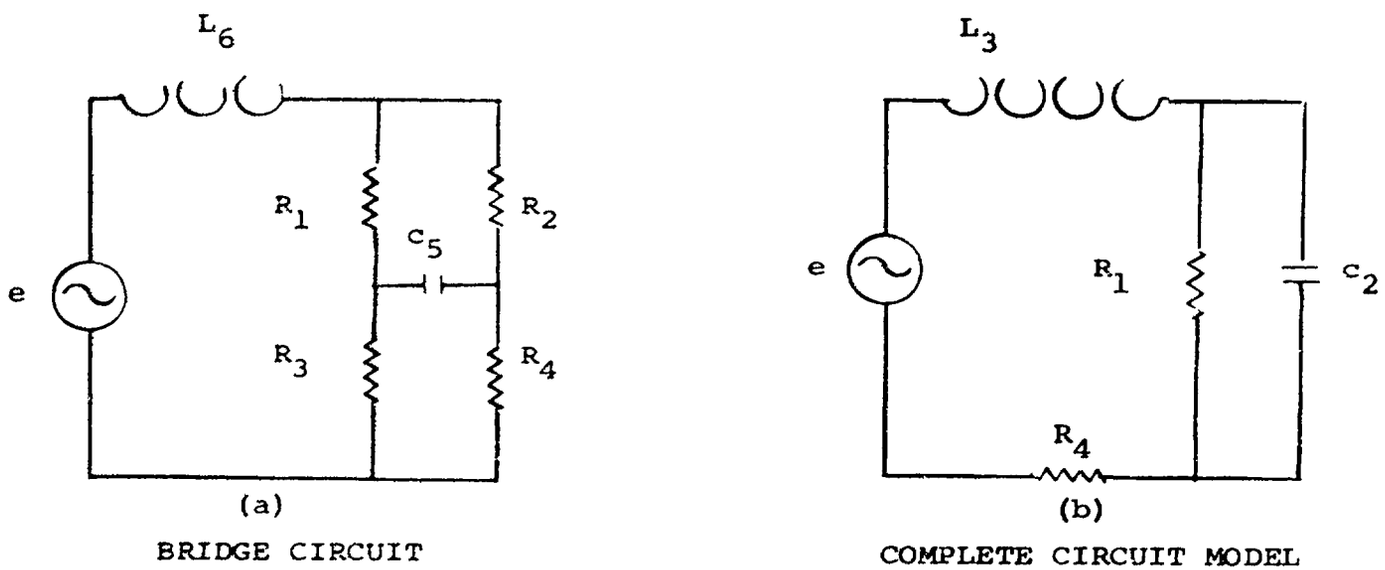


FIGURE 9

The A, $\underline{b}$ matrices of the bridge circuit are,

$$(203)\quad A = \begin{pmatrix} -.388 & -.056 \\ .033 & -.570 \end{pmatrix} \qquad \text{(Bridge Circuit)}$$

$$(204)\quad \underline{b} = \begin{pmatrix} 0 \\ .4 \end{pmatrix} \qquad \text{(Bridge Circuit)}$$

By comparison, the A, $\underline{b}$ of the complete circuit are,

$$(205)\quad A = \begin{pmatrix} \frac{-1}{R_1 C_2} & \frac{-1}{C_2} \\ \frac{1}{L_3} & \frac{-R_4}{L_3} \end{pmatrix} \qquad \text{(Complete Circuit)}$$

$$(206)\quad \underline{b} = \begin{pmatrix} 0 \\ \frac{1}{L_3} \end{pmatrix} \qquad \text{(Complete Circuit)}$$

If the A, $\underline{b}$ matrices were identical, then an inconsistency would arise since there are inconsistent requirements for L_3, i.e., from the a_{21}, b_2 elements.

$$(207)\quad L_3 = \frac{1}{a_{21}} = 30 \text{ h}$$

$$(208)\quad L_3 = 1/b_2 = 2.5 \text{ h}$$

By choosing the complete circuit parameters to satisfy (consistently),

(209) $C_2 = 17.90$ farads

(210) $R_1 = 6.95$ ohms

(211) $L_3 = 30$ henries

(212) $R_5 = (30)\ (.570)$ ohms

then, the complete circuit has the same A-matrix as the bridge circuit. Additionally, if the input e(t) is multiplied by a constant, k, and used to drive the complete circuit, then identical state trajectories for arbitrary initial conditions result. The gain constant must satisfy,

(213) $k = .4L_3$

Alternatively, it may be desired to choose L_3 (the only inconsistently specified parameter) to minimize some error criterion.

Referring to the results of Chapter ^ the state error equation is,

$$(214)\quad \dot{\underline{e}} = \begin{pmatrix} -.388 & -.056 \\ 1/L_3 & -.570 \end{pmatrix} \underline{e} + \begin{pmatrix} 0 & 0 \\ .033-\frac{1}{L_3} & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ .4-\frac{1}{L_3} \end{pmatrix} u, \underline{e}\ (0) = \underline{0}$$

The A, $\underline{b}$ matrices of the bridge circuit are,

$$(203)\quad A = \begin{pmatrix} -.388 & -.056 \\ .033 & -.570 \end{pmatrix} \qquad \text{(Bridge Circuit)}$$

$$(204)\quad \underline{b} = \begin{pmatrix} 0 \\ .4 \end{pmatrix} \qquad \text{(Bridge Circuit)}$$

By comparison, the A, $\underline{b}$ of the complete circuit are,

$$(205)\quad A = \begin{pmatrix} \frac{-1}{R_1 C_2} & \frac{-1}{C_2} \\ \frac{1}{L_3} & \frac{-R_4}{L_3} \end{pmatrix} \qquad \text{(Complete Circuit)}$$

$$(206)\quad \underline{b} = \begin{pmatrix} 0 \\ \frac{1}{L_3} \end{pmatrix} \qquad \text{(Complete Circuit)}$$

If the A, $\underline{b}$ matrices were identical, then an inconsistency would arise since there are inconsistent requirements for L_3, i.e., from the a_{21}, b_2 elements.

$$(207)\quad L_3 = \frac{1}{a_{21}} = 30 \text{ h}$$

$$(208)\quad L_3 = 1/b_2 = 2.5 \text{ h}$$

By choosing the complete circuit parameters to satisfy (consistently),

(209) $C_2 = 17.90$ farads

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then, the complete circuit has the same A-matrix as the bridge circuit. Additionally, if the input e(t) is multiplied by a constant, k, and used to drive the complete circuit, then identical state trajectories for arbitrary initial conditions result. The gain constant must satisfy,

(213) $k = .4L_3$

Alternatively, it may be desired to choose L_3 (the only inconsistently specified parameter) to minimize some error criterion.

Referring to the results of Chapter 3, the state error equation is,

$$(214)\quad \dot{\underline{e}} = \begin{pmatrix} -.388 & -.056 \\ 1/L_3 & -.570 \end{pmatrix} \underline{e} + \begin{pmatrix} 0 & 0 \\ .033-\frac{1}{L_3} & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ .4-\frac{1}{L_3} \end{pmatrix} u, \underline{e}\ (0) \overset{\Delta}{=} \underline{0}$$

One criterion which could be used is to minimize the mean square steady state error to a step and require $L_3 \geq 0$. Solving for the steady state error (u = 10 volts, x_1 = -1.0 volts at steady state).

$$(215) \quad e_{1/ss} = .144 e_{2ss}$$

$$(216) \quad e_{2/ss} = (1.024)\frac{(L_3\ 3.967-9)}{1.570L_3-.144}$$

The mean square error (ϵ^2) is the sum of the squares of the mean error components.

$$(217) \quad \epsilon^2 = e_1^2/_{ss} + e_2^2/_{ss} = (1.024)\frac{(L_3 3.967-9.0)^2}{L_3 1.570-.144}$$

A minimum, $\epsilon^2 = 0$ exists for,

$$(218) \quad L_3(3.967) = 9 \quad \text{or}$$

$$L_3 = 2.27$$

Figure 10 shows the corresponding error trajectory.

COMPARISON OF BRIDGE CIRCUIT AND COMPLETE CIRCUIT MODEL STATES (u=10volts)

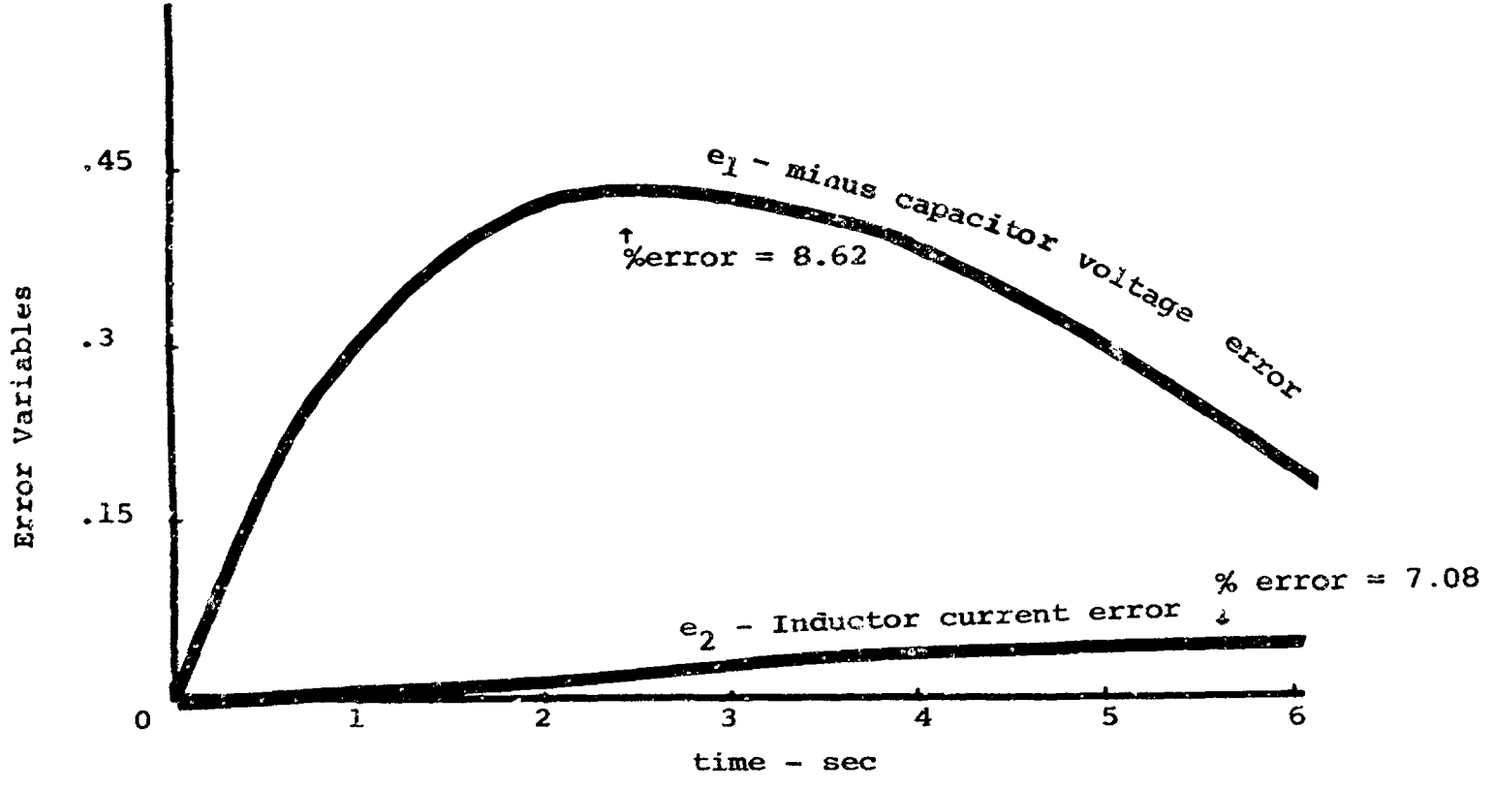

FIGURE 10

SUMMARY

Chapter V has dealt with techniques for extracting the parameter submatrices from the A, B, matrices. System topology plays an important role because its knowledge enables one to determine a priori if knowledge of the state and input is sufficient to determine all parameters uniquely. Such knowledge is important in planning a testing program.

If one tries to determine the parameters from knowledge of the A, B, matrices and system structure only, then a set of nonlinear algebraic equations must be solved. Newton-Raphson iteration is effective for this case.

For the case where additional tree or cotree variables can be measured, then the parameters can be determined by linear algebraic operations.

Example problems illustrating the various techniques were presented.

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APPENDIX

FORMULATION OF NETWORK-EQUATIONS IN STATE FORM

The following development is based on the work by MacFarlane [FE-9] with modifications in notation and form to suit the problem of parameter estimation.

Let T_o be the matrix relating branch and chord variables for a tree of the network which contains the maximal number of capacitive elements and such that the cotree contains the maximal number of inductive elements. The only capacitive elements not in the tree are those which form capacitor only loops with capacitors in the tree, and the only inductive elements which are not in the cotree are those which form inductor only cut sets with inductors in the cotree. Also, the tree is selected to include all voltage sources such that all current sources are in the cotree. If the currents and voltages are selected as above then the vector-matrix equations given below follow.

$$(219)\quad \begin{pmatrix} \underline{i}_b \\ \underline{e}_c \end{pmatrix} = \begin{pmatrix} 0 & T_o \\ -T_o^t & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_b \\ \underline{i}_c \end{pmatrix}$$

where the subscripts b, c denote branch and chord variables respectively.

The branch and chord vectors are partitioned into sub-vectors,

$$(220)\quad \underline{i}_b = \begin{pmatrix} \underline{i}_{bc} \\ \underline{i}_{bg} \\ \underline{i}_{b\ell} \end{pmatrix} \qquad \underline{i}_c = \begin{pmatrix} \underline{i}_{cd} \\ \underline{i}_{cr} \\ \underline{i}_{cc} \\ \underline{i}_{cd} \end{pmatrix}$$

$$(221)\quad \underline{e}_c = \begin{pmatrix} \underline{e}_{c\ell} \\ \underline{e}_{cr} \\ \underline{e}_{cc} \end{pmatrix} \qquad \underline{e}_b = \begin{pmatrix} \underline{e}_{bc} \\ \underline{e}_{bg} \\ \underline{e}_{b\ell} \\ \underline{e}_{bd} \end{pmatrix}$$

where the second subscripts, c, g, ℓ, d, r, refer to capacitive, conductive, inductive, driving, and resistive elements, respectively.

Equation (219) takes the partitioned form in (222) where it assumed the sources are ideal.

$$(222)\quad \begin{pmatrix} \underline{i}_{bc} \\ \underline{i}_{bg} \\ \underline{i}_{b\ell} \\ \hline \underline{e}_{c\ell} \\ \underline{e}_{cr} \\ \underline{e}_{cc} \end{pmatrix} = \left(\begin{array}{ccc|cccc} & & & T_1 & T_2 & T_6 & H_{cd} \\ & 0 & & T_3 & T_4 & 0 & H_{gd} \\ & & & T_5 & 0 & 0 & H_{\ell d} \\ \hline -T_1^t & -T_3^t & -T_5^t & K_{\ell d} & & & \\ T_2^t & -T_4^t & 0 & K_{rd} & & 0 & \\ -T_b^t & 0 & 0 & K_{cd} & & & \end{array}\right) \begin{pmatrix} \underline{e}_{b\ell} \\ \underline{e}_{br} \\ \underline{e}_{bc} \\ \underline{e}_{bd} \\ \underline{e}_{cc} \\ \underline{i}_{cg} \\ \underline{i}_{c\ell} \\ \underline{i}_{cd} \end{pmatrix}$$

The submatrix given by (223) is the matrix for the case where all voltage sources are open-circuited and all current sources are short circuited.

$$(223)\qquad T = \left(\begin{array}{ccc|ccc} & & & T_1 & T_2 & T_6 \\ & 0 & & T_3 & T_4 & 0 \\ & & & T_5 & 0 & 0 \\ \hline -T_1^t & -T_3^t & -T_5^t & & & \\ -T_2^t & -T_4^t & 0 & & 0 & \\ -T_6^t & 0 & 0 & & & \end{array}\right)$$

The component equations are assumed to be of the form,

$$(224a)\qquad \underline{i}_{bc} = C_s \dot{\underline{e}}_{bc}$$

$$(224b)\qquad \underline{i}_{bg} = G\, \underline{e}_{bg}$$

$$(224c)\qquad \underline{e}_{cr} = R\, \underline{i}_{cr}$$

$$(224d)\qquad \left(\begin{array}{c} \underline{e}_{c\ell} \\ \hline \underline{e}_{b\ell} \end{array}\right) = \left(\begin{array}{c|c} L_s & M_i^t \\ \hline M_i & L_e \end{array}\right) \left(\begin{array}{c} \underline{i}_{c\ell} \\ \hline \underline{i}_{b\ell} \end{array}\right)$$

The state variables are defined to be,

$$(225a)\qquad \underline{v}_s \overset{\Delta}{=} \underline{e}_{bc}$$

$$(225b)\qquad \underline{i}_s \overset{\Delta}{=} \underline{i}_{c\ell}$$

The remaining variables are renamed,

(226a) $\underline{i}_{br} \triangleq \underline{i}_r$

(226b) $\underline{i}_{b\ell} \triangleq \underline{i}_e$

(226c) $\underline{e}_{bg} \triangleq \underline{v}_g$

(226d) $\underline{e}_{cc} \triangleq \underline{v}_e$

The subscripts s, r, g, e, denote state, resistive, conductive and excess dynamic element vector, respectively.

By substituting the renamed variables and component equations into (219) and rearranging, the network equations are in (225).

$$(227)\quad \begin{pmatrix} C_s\dot{\underline{v}}_s \\ L_s\dot{\underline{i}}_s + M_i^t\dot{\underline{i}}_e \\ \hline G\underline{v}_g \\ R\underline{i}_r \\ \hline \underline{v}_e \\ \underline{i}_e \end{pmatrix} = \begin{pmatrix} 0 & T_1 & 0 & T_2 & T_6 & 0 \\ -T_1^t & 0 & -T_3^t & 0 & 0 & -T_5^t \\ \hline 0 & T_3 & 0 & T_4 & 0 & 0 \\ -T_2^t & 0 & -T_4^t & 0 & 0 & 0 \\ \hline -T_6^t & 0 & 0 & 0 & 0 & 0 \\ 0 & T_5 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{v}_s \\ \underline{i}_s \\ \hline \underline{v}_r \\ \underline{i}_g \\ \hline C_e\dot{\underline{v}}_e \\ L_e\dot{\underline{i}}_e + M_i\dot{\underline{i}}_s \end{pmatrix}$$

$$\begin{pmatrix} H_{cd} & 0 \\ 0 & K_{\ell d} \\ \hline H_{gd} & 0 \\ 0 & K_{rd} \\ \hline 0 & K_{cd} \\ H_{\ell\ell} & 0 \end{pmatrix} \begin{pmatrix} \underline{i}_{cd} \\ \hline \underline{e}_{bd} \end{pmatrix}$$

where,

$C_s \rightarrow$ diagonal $n_c x n_c$ matrix of tree capacitors

$C_e \rightarrow$ diagonal $n_1 x n_1$ matrix of chord capacitors

$L_s \rightarrow$ symmetric, positive definite $n_\ell x n_\ell$ matrix of chord inductors.

$L_e \rightarrow$ symmetric, positive definite, $n_2 x n_2$ matrix of tree inductors.

$G \rightarrow$ diagonal $n_g x n_g$ matrix of tree conductors

$R \rightarrow$ diagonal $n_r x n_r$ matrix of chord resistors

From the dimension definitions of the component matrices, the topological matrices have dimensions,

$T_1 \rightarrow n_c x n_\ell$

$T_2 \rightarrow n_c x n_r$

$T_3 \rightarrow n_g x n_\ell$

$T_4 \rightarrow n_g x n_r$

$T_5 \rightarrow n_2 x n_\ell$

$T_6 \rightarrow n_c x n_1$

The input matrices have dimensions,

$K_{\ell d} \rightarrow n_\ell x p_c$

$K_{rd} \rightarrow n_r x p_c$

$K_{cd} \rightarrow n_1 x p_c$

$H_{cd} \rightarrow n_c x p_b$

$$H_{gd} \rightarrow n_g x p,$$

$$H_{\ell d} \rightarrow n_2 x p_b$$

By eliminating the excess variables, the equation has the form,

$$J\dot{\underline{x}} = F\underline{x} + H\underline{u} \tag{228}$$

where,

$$J = \begin{pmatrix} C_s + T_6 C_e T_6^t & 0 \\ 0 & L_s + T_5^t M_i + T_5^t L_e T_5 \end{pmatrix} \tag{229}$$

$$F = \left[\begin{pmatrix} 0 & T_1 \\ -T_1^t & 0 \end{pmatrix} + \begin{pmatrix} 0 & T_2 \\ -T_3^t & 0 \end{pmatrix} \begin{pmatrix} G & T_4 \\ -T_4^t & R \end{pmatrix} \begin{pmatrix} 0 & T_3 \\ -T_2^t & 0 \end{pmatrix} \right] \tag{230}$$

$$H = \left[\begin{pmatrix} 0 & T_2 \\ -T_3^t & 0 \end{pmatrix} \begin{pmatrix} G & T_4 \\ -T_4^t & R \end{pmatrix}^{-1} \left(\begin{array}{cc:cc} H_{gd} & 0 & & \\ 0 & K_{rd} & & \\ \hdashline & & H_{cd} & 0 \\ & & 0 & K_{\ell d} \end{array} \right) \right. \tag{231}$$

$$\left. \left(\begin{array}{cc:cc} 0 & 0 & & \\ -M_i^t & H_{cd} & & \\ \hdashline & & T_6 & 0 \\ & & 0 & -T_5^t \end{array} \right) \left(\begin{array}{cc:cc} C_e & 0 & & \\ 0 & L_e & & \\ \hdashline & & 0 & K_{cd} \\ & & H_{\ell d} & 0 \end{array} \right) \right]$$

$$\underline{x} = \begin{pmatrix} \underline{v}_s \\ \underline{i}_s \end{pmatrix} \tag{232}$$

$$\underline{u} = \begin{pmatrix} \underline{i}_{cd} \\ \underline{e}_{bd} \\ \hdashline \dot{\underline{i}}_{cd} \\ \dot{\underline{e}}_{bd} \end{pmatrix} \tag{233}$$

In summary of subscript notation, s → state, g → conductive, r → resistive, e → excess dynamic, cd → chord driver (current source), and bd → branch driver (voltage source).